Rental of a durable good

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Abstract

Should a durable good monopoly facing agents with different length of demand allow only long term contracts? This paper studies an infinite horizon stationary model where a seller rents (or sells) every period of time one good. The number of potential buyers is constant and these buyers value the good differently according to the number of periods they need it (some want the good for only one period, some for two, e.g.). The profit maximizing and efficient allocation mechanism are derived when the monopoly commits to allocate the good for the whole length required by agents. These mechanisms exhibit the (endogenous) cost of committing to allocate the good for several periods, which does not allow the seller to benefit from potential high valuation buyers during these periods. Moreover, both mechanisms are compared and discussed.

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This version is still preliminary.

1 Introduction

Owners of infrastructure must often choose between providing access to consumers who desire to use it for different lengths of time. In this paper, we study this problem in a dynamic context and compare the strategies that would be used by a profit maximizing monopolist and a social welfare maximizing regulator.

We became interested in the problem while studying access to the Internet infrastructure. McKie-Mason and Varian have proposed to use Vickrey auctions, which they call "smart markets", to solve this problem (see McKie-Mason and Varian 1995, 1996, 1997). These auctions allocate resources efficiently and induce participants to reveal their true willingness to pay.[1] However, if some sellers

[1] In a few words, the proposed implementation of "smart markets" consists of allocating access freely in non-congested periods and, at each period of congestion, through standard Vickrey auctions. Customers attach a value to each packet of their messages. If the router faces congestion, packets with the highest valuations are selected up to the capacity. These packets are routed through the node of the network and their sender are required to pay the valuation attached to the first rejected packet.

want to use the network for an extended period of time, smart markets loose their efficiency inducing property, as we have shown in Crémer and Hariton (1999). This problem is practically important for applications such Internet video conferences where users want to be guaranteed a good quality connection over an extended period of time. This guarantee requires no disruption by congestion through the whole use of the service.

The transportation industry faces similar problems. For instance, an airport must arbitrate between leaving or lending slot available for charter flights or committing it to an airline which wants to use it every day for regular flights. In some proposals for liberalizing railroads, the manager of the network will have to arbitrate between freight, which has irregular demand, and passenger trains, which have regular schedule and want a commitment that they will be allowed to use a specific time slot. There is an extensive literature on congestion pricing in transportation industries (see Arnott, de Palma and Lindsey 1999, for an extensive bibliography) but, as far as we know, no paper has studied the problem of consumers with different lengths of use.

Similar problems arise in other network industries. For instance, in the gas and electricity industries, the manager of the transportation network must decide how much capacity to guarantee to a user, knowing that this might prevent him from accepting the request of a future use, who may have a higher willingness to pay.

In order to study this problem, we consider a durable good who can be used by only one agent at the time. At each period where the good is not used, a number of potential users vie for the right to use it. A user is defined by two characteristics: his willingness to pay and the length of time for which he wants to use the good. The owner of the asset must choose one of the users, or decide to leave the good unused in the hope of finding a "better" user in the next period. We simplify the problem by assuming that the desired length of use of a user is public information, but the willingness to pay is private information.

Our problem is clearly an auction problem and, in some sense, a multi-unit auction problem (e.g. Branco 1995, 1996). Let say that the monopolist has two goods to sell (good 1 and good 2, e.g.). In the multi-unit literature, an auctioneer wonders how it should compare separate offers for each of the two goods to offers for the bundle of them. An analogy can be derived with the analysis done in this paper: each of this good is sold in each period (good 1 at period 1 and good 2 at period 2, e.g.) and two consecutive periods correspond to a bundle. Thus, the two frameworks are linked but also exhibit major differences, as explained in more detail in section 6.2. In particular, in our framework, the buyers of good 2 are not yet present when good 1 is sold!

This paper is organized the following way. Section 2 proposes a model of congestion where an auctioneer allocates access to an infrastructure. Section 3 describes the objective of the auctioneer. Section 4 analyzes the benchmark case when there is no asymmetry of information and section 5 the case where asymmetry is present. The profit maximizing mechanism and the efficient mechanism are exhibited and compared in section 6. Finally, section 7 concludes. All proofs are collected in the appendix.

2 The model

In each period $t = 0, \dots, +\infty$, N risk neutral agents compete to use one good. Each category (indexed by $i = 1, \dots, I$) of agents wants the good for only i periods (they are called type i-agents). There are n_i agents (indexed by $j = 1, \dots, n_i$) of each type i, with $N = \sum_{i=1}^{I} n_i$. To simplify the writing, agent j of a type i is called agent ij.

The valuation of agent j of type i for the good is written $v_{ij} \in [\underline{v}_i, \overline{v}_i]$ with $\Delta v_i = \overline{v}_i - \underline{v}_i > 0$ and $\underline{v}_i \geqslant 0$. This valuation is private information of the agent, but not its type, which is assumed to be common knowledge. Moreover, any type i-agent with $i > 1$ only wants the good for i periods, i.e. his willingness to pay for less than i periods is zero. The valuation v_{ij} of agent ij follows a type dependent cumulative distribution function $F^i\left(v_{ij}\right)$, with density $f^i\left(v_{ij}\right) > 0$. Finally, the virtual valuation of an agent of type i and valuation v_{ij} is noted

$$J^i\left(v_{ij}\right) \equiv v_{ij} - \frac{1 - F^i\left(v_{ij}\right)}{f^i\left(v_{ij}\right)},$$

which we assume is increasing in v_{ij}. Furthermore, we assume that the types of all agents, i.e. the v_{ij}s, are independent from each others.

A risk neutral seller sells the use of the good. When selling the good to type i-agent at period t, with $i > 1$, the seller commits to sell it for i periods and not to renegotiate its allocation rule at periods $t+1, \ldots, t+i-1$.

All agents, the buyers and the seller, have the same discount factor δ.

3 The problem of the auctioneer

The objective of the auctioneer is to identify an optimal stationary mechanism for the allocation of the good.

The auctioneer designs a mechanism that can be described by functions $\{p^{ij},\ t^{ij}\}_{i,j}$ where $p^{ij}(v_{ij},\ v_{-ij})$ is the probability that agent ij is given the object and $t^{ij}\left(v_{ij}\right)$ is her paiement. She obtains the good with an expected probability $q^{ij}\left(v_{ij}\right) \equiv \mathrm{E}_{v_{-ij}}\left[p^{ij}\left(v\right)\right]$.

Assume furthermore that the current period t is such that the good is not committed, and denote π the infinite stream of expected profits at any such period. We will consider two different types of owners of the asset. A "firm" maximizes its profits; its "period utility" in a period where the good can be sold is equal to total expected payment it receives

$$\textstyle\sum_{i=1}^{I} \sum_{j=1}^{n_i} \mathrm{E}_{v_{ij}}\left[t^{ij}\left(v_{ij}\right)\right].$$

A "regulator" maximizes social welfare. His period utility when the good can be sold is

$$\textstyle\sum_{i=1}^{I} \sum_{j=1}^{n_i} \mathrm{E}_{v_{ij}}\left[q^{ij}\left(v_{ij}\right)\right].$$

The auctioneer cares about its future expected benefits. It has to take into account that, when it allocates the good at period t to a type i-buyer, it commits itself not to renegotiate this allocation for the i following periods. Period $t+i$ is characterized by the same state for the good than at period t: the good is not committed. The expected future benefits to be collected after period $t+i$ are π, which accounts for $\delta^i\pi$ when discounted in period t. Thus, selling the good to agent ij of valuation v_{ij} yields, on top of the current period profit, a benefit of $\delta^i\pi$.

Using the revelation principle and Belman's principle of optimality, π is the solution of problem $(\mathcal{P}1)$

$$\pi = \max_{\substack{\{p^{ij}(.),t^{ij}(.)\}\ i=1,\ldots,I \\ j=1,\ldots,n_i}} \left[\begin{array}{c} \sum_{i=1}^{I} \sum_{j=1}^{n_i} \left\{ \mathrm{E}_{v_{ij}}\left[X^{ij}\left(v_{ij}\right)\right] + \mathrm{E}_{v}\left[p^{ij}\left(v\right)\delta^i\pi\right] \right\} \\ + \left(1 - \sum_{i=1}^{I} \sum_{j=1}^{n_i} \mathrm{E}_{v}\left[p^{ij}\left(v\right)\right]\right)\delta\pi \end{array} \right] \tag{1}$$

under the constraints[2]

$$\begin{cases} \forall v_{ij}, \forall i, \forall j, & U^i\left(v_{ij}\right) = q^{ij}\left(v_{ij}\right) v_{ij} - t^{ij}\left(v_{ij}\right) \geqslant 0 & \left(IR_{ij}\right) \\ \forall v_{ij}, \forall i, \forall j, & v_{ij} = \arg\max_{\tilde{v}_{ij}} \left\{ q^{ij}\left(\tilde{v}_{ij}\right) v_{ij} - t^{ij}\left(\tilde{v}_{ij}\right) \right\} & \left(IC_{ij}\right) \\ \forall v, \forall i, \forall j, & p^{ij}\left(v\right) \in [0,1] & \left(P_{ij}\right) \\ \forall v, & \sum_{i,j} p^{ij}\left(v\right) \leqslant 1 & \left(P_0\right) \end{cases}$$

where $X^{ij}\left(v_{ij}\right)$ is equal to $t^{ij}\left(v_{ij}\right)$ for a firm and to $v_{ij}q^{ij}\left(v_{ij}\right)$ for a regulator.

The main differences between this problem and the traditional one of an optimal single-unit auction, as studied by Myerson (1981), are of three kinds. First, the set of allocation probabilities is restrained, in the same way as Branco (1995) e.g., because allocation rules cannot display more units than the auctioneer currently has. Second, there are extra terms related to the dynamics of the allocation process. Third, there are in fact two different optimization problems, one for the mechanism itself (allocation probabilities and transfers given π) and another for the level of profits. This is due to the approach taken in this work to study infinite horizon model and stationary mechanisms.

4 Perfect information case

When valuations of potential buyers are public information, the incentive compatibility constraints drop out and problem $(\mathcal{P}1)$ becomes problem $(\mathcal{P}2)$.

Maximizing profits induces the auctioneer to extract as much as possible from agents, making the individual rationality constraint $\left(IR_{ij}\right)$ binding

$$\forall i, \forall v_{ij}, \quad U^i\left(v_{ij}\right) = 0 \quad \Leftrightarrow \quad t^{ij}\left(v_{ij}\right) = v_{ij}q^{ij}\left(v_{ij}\right). \tag{2}$$

The objective function becomes

$$\pi = \max_{\substack{\{p^{ij}(.)\} \\ i=1,\dots,I \\ j=1,\dots,n_i}} \left[\sum_{i=1}^{I} \sum_{j=1}^{n_i} \left\{ \mathrm{E}_v \left[\left[v_{ij} - \delta\left(1-\delta^{i-1}\right)\pi \right] p^{ij}\left(v\right) \right] + \delta\pi \right\} \right] \tag{3}$$

A regulator has the same objective function. The solution of $(\mathcal{P}2)$ is described by the following lemma, proved in the appendix.

Lemma 1. *Let* $\left\{p^{ij*}\left(.\right)\right\}_{i,j}$ *and* π^* *be the solutions of the problem*

$$\begin{cases} W\left(\pi\right) \equiv \max_{\substack{\{p^{ij}(.)\} \\ i=1,\dots,I \\ j=1,\dots,n_i}} \left[\sum_{i=1}^{I} \sum_{j=1}^{n_i} \mathrm{E}_v \left[\left[v_{ij} - \delta\left(1-\delta^{i-1}\right)\pi \right] p^{ij}\left(v\right) \right] \right] + \delta\pi \\ W\left(\pi\right) = \pi \end{cases} \tag{4}$$

under the constraints

$$\begin{cases} \forall v, \forall i, \forall j, & p^{ij}\left(v\right) \in [0,1] & \left(P_{ij}\right) \\ \forall v, & \sum_{i,j} p^{ij}\left(v\right) \leqslant 1 & \left(P_0\right) \end{cases}$$

[2]There are N individual rationality constraints, N incentive compatibility constraints and $(2N+1)$ constraints on the allocation probabilities.

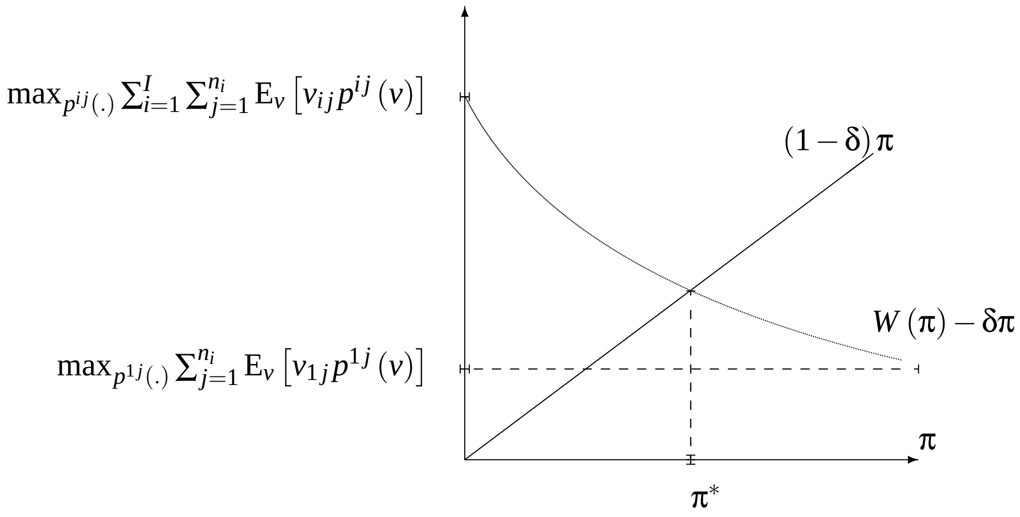

Figure 1: Optimal stationary profit with perfect information

with, $\forall v_{ij}, \forall i, \forall j,$

$$t^{ij*}(v_{ij}) = \mathrm{E}_{v_{-ij}} \left[v_{ij} p^{ij*}(v) \right]. \tag{5}$$

Then $\left\{ p^{ij*}(.)\,,\, t^{ij*}(.) \right\}_{i,j}$ *solves the maximization program* $(\mathcal{P}2)$.

For a given π, the maximization problem is a simple sum of known coefficients multiplied by the probability that the mechanism designer has to choose. Therefore, at given π, the buyer for whom $v_{ij} - \delta\left(1-\delta^{i-1}\right)\pi$ is the highest obtains the good with probability 1. To determine π, we will use the following lemma, also proved in the appendix.

Lemma 2. *The expected per period benefit* $[W(\pi) - \delta\pi]$ *is decreasing and convex, with* $W(0) > 0$.

As shown in figure 1, lemma 2 implies that $[W(\pi) - \delta\pi]$ crosses once the strictly increasing function $(1-\delta)\pi$, for $\pi = \pi^*$. We have proved the following proposition.

Proposition 1. *Under perfect information, the mechanism set by a profit maximizing seller or a social planner allocates the good to (one of) the agents among the highest positive* $\left[v_{ij} - \delta\left(1-\delta^{i-1}\right)\pi^*\right]$. *If* $\left[v_{ij} - \delta\left(1-\delta^{i-1}\right)\pi^*\right]$ *is strictly negative for all* ij*, then the good is not sold in this period.*

This mechanism is further studied in section 6.

5 Asymmetric information case

How will the presence of asymmetric information change the conclusions of the last section? Using Myerson's (1981) methodology, the set of individual rationality and incentive compatibility constraints of problem $(\mathcal{P}1)$ can be rewritten as described in the following lemma.

Lemma 3. *A mechanism is incentive compatible and individually rational if and only if*

$$\forall i, \forall j, \forall v_{ij}, \forall s_{ij}, \qquad \left(v_{ij} - s_{ij}\right)\left[q^{ij}(v_{ij}) - q^{ij}(s_{ij})\right] \geqslant 0 \tag{6}$$

$$\forall i, \forall v_{ij}, \qquad U^i(v_{ij}) = U^i(\underline{v}_i) + \int_{\underline{v}_i}^{v_{ij}} q^{ij}(x)\,dx \tag{7}$$

$$\forall i, \qquad U^i(\underline{v}_i) \geqslant 0 \tag{8}$$

Using lemma 3, the problem $(\mathcal{P}1)$ can be rewritten as described in the following lemma, whose proof is in the appendix.

Lemma 4. *Let* $\left\{p^{ij**}(.)\right\}_{i,j}$ *and* π^{**} *be the solutions of the following problem*

$$\begin{cases} V(\pi) \equiv \max_{\{p^{ij}(.)\}\, \substack{i=1,\dots,I \\ j=1,\dots,n_i}} \left[\sum_{i=1}^{I}\sum_{j=1}^{n_i} \mathrm{E}_v\left[\left[Y^i\left(v_{ij}\right) - \delta\left(1-\delta^{i-1}\right)\pi\right] p^{ij}(v)\right]\right] + \delta\pi \\ V(\pi) = \pi \end{cases} \tag{9}$$

under the constraints

$$\begin{cases} \forall i, \forall j, \forall v_{ij}, \forall s_{ij}, & \left(v_{ij} - s_{ij}\right)\left[q^{ij}\left(v_{ij}\right) - q^{ij}\left(s_{ij}\right)\right] \geqslant 0 & \left(IC'_{ij}\right) \\ \forall v, \forall i, \forall j, & p^{ij}(v) \in [0,1] & \left(P_{ij}\right) \\ \forall v, & \sum_{i,j} p^{ij}(v) \leqslant 1 & \left(P_0\right) \end{cases}$$

where $Y^i\left(v_{ij}\right)$ *is equal to* $J^i\left(v_{ij}\right)$ *for a firm and to* v_{ij} *for a regulator, and* $\forall v_{ij}, \forall i, \forall j$,

$$t^{ij**}\left(v_{ij}\right) = \mathrm{E}_{v_{-ij}}\left[v_{ij} p^{ij**}(v) - \int_{\underline{v}_i}^{v_{ij}} p^{ij**}\left(x, v_{-ij}\right) dx\right]. \tag{10}$$

Then $\left\{p^{ij**}(.),\, t^{ij**}(.)\right\}_{i,j}$ *solves the maximization program* $(\mathcal{P}1)$ *and yields a level* π^{**} *of benefits.*

As in the perfect information case, for a given π, the maximization problem ends up to be a simple sum of known coefficients multiplied by the probability that the mechanism designer has to choose. Thus, the agent with the highest positive coefficient is allocated the object and the following lemma proves the existence of an overall solution.

Lemma 5. *The expected per period benefit* $[V(\pi) - \delta\pi]$ *is decreasing and convex for both cases of function* $Y^i\left(v_{ij}\right)$, *with* $V(0) > 0$.

Thus, depending on its objective function, the solution to the auctioneer allocation problem is given by the following rules.[3]

Proposition 2. *Under asymmetric information, a profit maximizing seller allocates the good to (one of) the agents within the highest positive* $\left[J^i\left(v_{ij}\right) - \delta\left(1-\delta^{i-1}\right)\pi^{**}\right]$. *If* $\left[J^i\left(v_{ij}\right) - \delta\left(1-\delta^{i-1}\right)\pi^{**}\right]$ *is strictly negative for all* ij, *it does not sell the good in this period.*

Proposition 3. *Under asymmetric information, a social planner allocates the good the same way as under public information to (one of) the agents within the highest positive* $\left[v_{ij} - \delta\left(1-\delta^{i-1}\right)\pi^{*}\right]$. *If* $\left[v_{ij} - \delta\left(1-\delta^{i-1}\right)\pi^{*}\right]$ *is strictly negative for all* ij, *he does not sell the good in this period.*

Notice that the first best efficient policy can easily be implemented, even when the regulator does not know buyers' valuations (but still assuming that bidders cannot lie with respect to the number of periods for which they require the good). It is sufficient to organize a second price auction, with the bidders being charged for the rental rate from period 2 to i. This implies that short run buyers, who want the good only for one period, are not charged any rental.

Let now turn to the analysis of the properties of the allocation mechanisms.

[3] If the problem is not regular, one should be able to extend this result by "making it regular", as done in Myerson (1981).

6 Properties of the allocation mechanisms

At first sight, the lesson given by propositions 1 to 3 is that, when comparing two potential buyers, the auctioneer compares their valuations minus, first, their respective cost of inducing truthful revelation (if necessary) and, second, the relative cost of the commitment to deliver the good during the required number of periods. The first cost is embodied in the virtual valuations. The second is related to the additional term $\delta\left(1-\delta^{i-1}\right)\pi$, which is increasing in i, the length of the demand, and in δ, the discount factor. In this section, we discuss in more details the allocation rules.

6.1 Efficient mechanism

Proposition 3 tells that information gathering for the regulator does not interfere with its allocation procedure, meaning that all the discussion can focus on the perfect information case. Indeed, transfers are both benefits for the regulator and costs for bidders and, as monetary transfers are costless, the regulator sums up both terms with equal weight and, eventually, they do not appear in the objective function. Transfers are still meaningful to secure incentive compatibility constraints, which is always the case with the mechanism proposed in proposition 3.

Assuming that there is no asymmetry of information, section 4 shows that profit maximizing and efficient mechanisms are identical. Remember that π^* is the discounted social welfare associated with the infinite repeated allocation of the good, starting from a period in which the good is available. Then, when it has to choose one user, i.e. in a period when the good is available, proposition 1 tells the auctioneer either to allocate the good to (one of) the bidder with the highest

$$v_{ij}+\delta^i\pi^*$$

if this quantity is higher than $\delta\pi^*$ or not to allocate the good if this quantity is smaller than $\delta\pi^*$.

Note that, given that we have assumed that v_{ij} is always positive, if there is at least on buyer of type 1, the good will always be allocated.

Note that $\tilde{v}_{1j}=v_{1j}-(1-\delta)\,\pi^*$. Therefore, given that we have assumed that v_{ij} is always positive, if there is at least on buyer of type 1, the good will always be allocated.

6.1.1 Rental rate and exclusion of bidders with low valuations

Now, choosing the buyer with the greatest $v_{ij}+\delta^i\pi^*$ is equivalent to choosing the one with the highest

$$\tilde{v}_{ij}\equiv v_{ij}-[\pi^*]+\delta^i\pi^*=v_{ij}-\left(1+\delta+\ldots+\delta^{i-1}\right)(1-\delta)\,\pi^*. \tag{11}$$

Thus, the seller rents the good at a price of $(1-\delta)\,\pi^*$ per period, and chooses the agent whose willingness to pay yields the highest profit above the discounted sum of the *rental rate* over the whole duration of its need. Notice that it is not the profit per period which is taken into account.

The condition

$$v_{ij}+\delta^i\pi^*\geq\delta\pi^*$$

is equivalent to

$$\tilde{v}_{ij}\geq-(1-\delta)\,\pi^*.$$

This expression can easily be interpreted. By not renting the good, the auctioneer incurs a cost (a loss of benefits) of $(1-\delta)\pi^*$, the one period rental rate. Thus, the total surplus obtained by renting the good to agent ij must be greater than this loss, otherwise bidders are systematically excluded from the allocation procedure.

6.1.2 Comparative statics related to distribution functions

We now show that if the distribution function of an agent changes so as to increase the auctioneer's welfare, then in some sense which will be made precise, the auctioneer favors short run buyers. Let us call $\tilde{F}^i\left(v_{ij}\right)$ and $F^i\left(v_{ij}\right)$ two cumulative distributions such that $\tilde{F}^i$ first degree stochastically dominates F^i: $\tilde{F}^i\left(v_{ij}\right) \leqslant F^i\left(v_{ij}\right)$ for all v_{ij}. The natural interpretation is that by moving from F^i to $\tilde{F}^i$, bidders with high valuations show up more frequently. As $v_{ij}-\delta\left(1-\delta^{i-1}\right)\pi$ is increasing in v_{ij} for a given π,

$$\mathrm{E}_{v_i\sim\tilde{F}^i}\left[v_{ij}-\delta\left(1-\delta^{i-1}\right)\pi\right] > \mathrm{E}_{v_i\sim F^i}\left[v_{ij}-\delta\left(1-\delta^{i-1}\right)\pi\right].$$

Thus, the valuation the regulator attaches to bidder i is in average higher with the cumulative distribution $\tilde{F}^i$.[4] Clearly, a shift from F^i to $\tilde{F}^i$ improves the welfare of the auctioneer, and we must have $\tilde{W}(\pi) > W(\pi)$ for all π, where we denote by a $\tilde{}$ the quantities corresponding to the distribution $\tilde{F}^i$. From figure 1, it is straightforward that $\tilde{\pi}^* > \pi^*$.[5]

We are now ready to prove the following proposition.

Proposition 4. *Assume that when the distribution function of i-type buyers is F^i, and when the vector of types of other agents is v_{-kl}, agent kl of type v_{kl} gets the good with probability zero, either because it is not allocated or because it is allocated to an agent with a* smaller *type. Then, when the distribution of the valuation i-type buyers is $\tilde{F}^i$, which first degree stochastically dominates F^i, in the same state of nature, agent kl obtains the good with probability zero.*

Note that the proposition puts no constraint on the relationship between i and k. Its proof is straightforward. First, $v_{kl} \leqslant \delta\left(1-\delta^{k-1}\right)\pi^*$ implies $v_{kl} \leqslant \delta\left(1-\delta^{k-1}\right)\tilde{\pi}^*$, which implies that is the good is not allocated to agent kl with F^i, it is not with $\tilde{F}^i$. Second,

$$v_{kl}-\delta\left(1-\delta^{k-1}\right)\pi^* \leqslant v_{rs}-\delta\left(1-\delta^{r-1}\right)\pi^*$$

for some elements v_{rs} of $v{-}kl$ with $r<k$ implies

$$v_{kl}-\delta\left(1-\delta^{k-1}\right)\tilde{\pi}^* \leqslant v_{rs}-\delta\left(1-\delta^{r-1}\right)\tilde{\pi}^*$$

which concludes the proof.

Corollary 1. *If in a state of nature v the good is not allocated when the distribution function of i-type agents is F^i, then it is not when the distribution is $\tilde{F}^i$.*

[4] In the firm's allocation procedure, bidders are valued through virtual valuations minus the cost of allocation commitment. In this case, there is nothing that guarantees *a priori* that bidder i's virtual valuation computed with the cumulative distribution function $\tilde{F}^i$ is in average larger for the firm than the average of F^i.

[5] Formally: $(1-\delta)\tilde{\pi}^* = \tilde{W}(\tilde{\pi}^*)-\delta\tilde{\pi}^* > W(\tilde{\pi}^*)-\delta\tilde{\pi}^*$, and because $(1-\delta)\pi$ is increasing in π while $W(\pi)-\delta\pi$ is decreasing, the result is straightforward.

6.2 Profit maximizing mechanism

6.2.1 Another rental rate

In the presence of asymmetry of information, proposition 2 tells the seller to chooses the buyer ij who maximizes

$$J^i\left(v_{ij}\right) - \delta\left(1-\delta^{i-1}\right)\pi^{**}$$

as long as this quantity is positive. This is equivalent to choosing agent ij so as to maximize

$$\begin{aligned}\tilde{\tilde{v}}_{ij} &\equiv J^i\left(v_{ij}\right) - \left[(1-\delta)\pi^{**}\right] - \delta\left(1-\delta^{i-1}\right)\pi^{**} \\ &= J^i\left(v_{ij}\right) - \left(1+\delta+\ldots+\delta^{i-1}\right)(1-\delta)\pi^{**},\end{aligned} \tag{12}$$

as long as this quantity is at least equal to $-(1-\delta)\pi^{**}$.

The interpretation parallels the one of the efficient solution without asymmetry of information, with the valuation replaced by the virtual valuation which includes the cost of extracting from the buyer information about its willingness to pay.

It may be worthwhile noting that, with asymmetry of information, the rental rate is not deducted from the willingness to pay of the agent, but from the virtual valuation. In particular, the formula is of the form $\left[J^i\left(v_{ij}\right) - \text{Rental}\right]$ and not $\left[J^i\left(v_{ij} - \text{Rental}\right)\right]$.

6.2.2 Exclusion of buyers

As usual in optimal auctions, agents with low valuations may never get the good. If there exists some $v_i^{\min}$ such that $J^i\left(v_i^{\min}\right) = \delta\left(1-\delta^{i-1}\right)\pi^* > 0$, then any agent of type i with $v_{ij} < v_i^{\min}$ is excluded from the market. This phenomenon is nevertheless stronger than what occurs in standard static auctions. Indeed, in a standard case without reservation valuation, virtual valuations must be positive in order for the good to be allocated. Thus, this limit is less stringent than $\delta\left(1-\delta^{i-1}\right)\pi^*$ which is strictly positive for $i > 1$. This is due to the cost of the commitment not to renegotiate the allocation awarded to any agent, even long term ones.

6.2.3 Comparison with optimal multi-unit auctions

In the multi-unit problem, such as the work done by Branco, an auctioneer wonders how it should compare offers for different goods (good 1 and good 2, e.g.) to offers for some bundle of them. An analogy can be derived between the allocation of goods 1 and 2 in the multi-unit framework and the allocation of a good at period 1 and the same good at period 2 in the dynamic context studied in this chapter.

In the multi-unit auction, the seller faces three potential competitors: the buyer who wants the good at the first period (which corresponds to agent of type 1 at $t = 1$ in the dynamic model), the buyer who wants the good at the second period (agent of type 1 at $t = 2$) and the buyer who only values the bundle (that is having the good at both periods, thus agent of type 2 at $t = 1$). The main result of Branco is that one can extend the use of virtual valuations introduced by Myerson (1981) in order to compare those three buyers even if they do not bid for the same object.

The main difference with the multi-unit framework lies in the dynamics of the model developed in this chapter which introduces another type of competition: at $t = 2$, the seller could also sell the good to agent of type 2, willing the good at $t = 2$ and $t = 3$, who appears at $t = 2$. This additional potential buyer complicates the comparison of the bundle and the two stand-alone bids.

Next sections entail to discuss how the regulator and the firm compare short run and long run buyers.

6.3 Short vs Long term buyers: Perfect information case

It is convenient to begin this investigation by considering the choice between two agents v_{ij} and v_{kl} such that the following assumption holds.

Assumption 1. *Agents v_{ij} and v_{kl} have willingness to pay v_{ij} and v_{kl} for the good such that*

$$\begin{cases} v_{ij} = \left(1+\delta+\ldots+\delta^{i-1}\right)v \\ v_{kl} = \left(1+\delta+\ldots+\delta^{k-1}\right)v \end{cases}$$

for some v.

The valuation v represents the identical per period willingness to pay of agents ij and kl. For concreteness, and without loss of generality, assume $i<k$ so that agent ij is the short term buyer and it has the smallest horizon (kl is the long term agent and has the longest horizon).

The regulator is indifferent between renting the good to agents ij and kl if, following (11), $\tilde{v}_{ij} = \tilde{v}_{kl}$, i.e.

$$v_{ij} - (1+\delta+\ldots+\delta^{i-1})(1-\delta)\pi^* = v_{kl} - \left(1+\delta+\ldots+\delta^{k-1}\right)(1-\delta)\pi^* \tag{13}$$

i.e.

$$(1+\delta+\ldots+\delta^{i-1})\left[v-(1-\delta)\pi^*\right] = \left(1+\delta+\ldots+\delta^{k-1}\right)\left[v-(1-\delta)\pi^*\right]$$

i.e.

$$v = (1-\delta)\pi^* \equiv v^*.$$

Such v^* always exists but it may not lie within the support of agents ij or kl valuations.

Moreover, facing agents characterized by valuations equal to v^*, the regulator would decide to allocate the good to agent ij or kl rather than not renting it. Indeed, following the definition proposed in (11), $\tilde{v}_{ij} = \tilde{v}_{kl} = 0 > -(1-\delta)\pi^*$.

Finally, because $(1+\delta+\ldots+\delta^{i-1})$ is smaller than $(1+\delta+\ldots+\delta^{k-1})$, the equal sign in equation (13) is replaced by a greater sign if $v<v^*$ and a smaller sign otherwise. This yields the following proposition.

Proposition 5. *If two agents have the same per period willingness to pay v (assumption 1), the regulator prefers to rent the good to the agent with the shortest horizon if $v<v^*$, where $v^* = (1-\delta)\pi^*$ is the per period rental rate of the good at the optimum. If, on the contrary, $v>v^*$, the regulator would rather allocate the good to the agent with the largest horizon.*

Thus, in the absence of asymmetric information, all what counts for the regulator is to secure its rental rate for each period of allocation. Indeed, as this later commits to its allocation procedure, it prefers to allocate for a longer time if the valuation is higher than the per period average benefits it gets and, otherwise, it would rather allocate the good for a shorter period.

It is possible to get a sense of the critical value v^* by deriving a lower bound on the value of π^*. In order to do so, notice that the regulator could use the following suboptimal allocation policy: pick the bidder of type i such that the expected per period valuation is the largest, i.e. $i \in \arg\max_j \left[\mathrm{E}_{v_j}\left[v_j\right]/\left(1+\delta+\ldots+\delta^{j-1}\right)\right]$. By doing so, the regulator secures an expected per period social welfare of $\mathrm{E}_{v_j}\left[v_j\right]/\left(1+\delta+\ldots+\delta^{j-1}\right)$ at each period and for ever. Thus,

$$\pi^* \geqslant \max_j \left[\frac{\mathrm{E}_{v_j}\left[v_j\right]}{\left(1+\delta+\ldots+\delta^{j-1}\right)(1-\delta)}\right] = \max_j \left[\frac{\mathrm{E}_{v_j}\left[v_j\right]}{1-\delta^j}\right].$$

and the critical value v^* has the following lower bound

$$v^* \geqslant \max_j \left[\frac{\mathrm{E}_{v_j}\left[v_j\right]}{\left(1+\delta+\ldots+\delta^{j-1}\right)}\right].$$

If two agents have the same per period valuation, and this per period valuation is smaller than the maximum expected per period valuation of any type of agents, then agents' valuation is lower than v^* and the auctioneer prefers to allocate the good to the agent with the shortest horizon. Loosely speaking, this shows a trend in the allocation mechanism in favor of short horizon agents which will be founded in the majority of cases.

Let now turn to a more general case. Denote the per period willingness to pay of any agent ij by

$$\hat{v}_{ij} \equiv \frac{v_{ij}}{1+\delta+\ldots+\delta^{i-1}}.$$

Then, following previous analysis, the regulator compares agents ij and kl by comparing $\tilde{v}_{ij}$ and $\tilde{v}_{kl}$, that is

$$\tilde{v}_{ij} \gtrless \tilde{v}_{kl} \quad \text{i.i.f.} \quad \left[\hat{v}_{ij} - v^*\right] \gtrless \frac{1-\delta^k}{1-\delta^i}\left[\hat{v}_{kl} - v^*\right].$$

As i is assumed to be lower than k, the fraction in the right-hand side of the inequality is greater than 1. Whenever $\hat{v}_{kl} > \hat{v}_{ij} > v^*$, the auctioneer favors the long run buyer kl and, if $\hat{v}_{kl} < \hat{v}_{ij} < v^*$, the short run buyer ij is preferred by the auctioneer. In the two remaining cases, $\hat{v}_{ij} > \hat{v}_{kl} > v^*$ and $\hat{v}_{ij} < \hat{v}_{kl} < v^*$, there is no *a priori* systematic preference over one or another agent. In the former one, one could think that because the per period valuation of agent ij is the highest, the auctioneer will favor it but, in fact for this to be case, the difference in valuation must be high enough to compensate the auctioneer for the $k-i-1$ periods following agent ij use of the good that it could have secured with a benefit v_{kl} higher than the rental rate. In the later, the same situation occurs with the auctioneer securing the least loss under the rental rate.

6.4 Short vs Long term buyers: Asymmetric information case

With asymmetry of information, the comparison between two potential buyers of capacity depend not only on their horizon and their valuation, but also on their distribution of types, as it influences virtual valuations. In order to obtain some results, it is assumed, first, that agents ij and kl have the same per period willingness to pay, second, that support of valuations are interrelated in the following way

$$\forall i \in [1, I], \quad \underline{v}_i = \left(1+\delta+\ldots+\delta^{i-1}\right)\underline{v}_1,\ \bar{v}_i = \left(1+\delta+\ldots+\delta^{i-1}\right)\bar{v}_1$$

and, third, that distributions are such that

$$\forall v \in [\underline{v}_1, \bar{v}_1], \quad \Pr\left[v_{ij} < \left(1+\delta+\ldots+\delta^{i-1}\right)v\right] = \Pr\left[v_{kl} < \left(1+\delta+\ldots+\delta^{k-1}\right)v\right].$$

The last two conditions are equivalent to the following assumption.

Assumption 2. *There exists a distribution function $\tilde{\tilde{F}}$, with support in $[\underline{v}_1, \bar{v}_1]$ and an associated density function $\tilde{\tilde{f}}$, such that*

$$\begin{cases} \forall v_{ij} \in [\underline{v}_i, \bar{v}_i], & F^i\left(v_{ij}\right) = \tilde{\tilde{F}}\left(\frac{v_{ij}}{1+\delta+\ldots+\delta^{i-1}}\right) \\ \forall v_{kl} \in [\underline{v}_k, \bar{v}_k], & F^k\left(v_{kl}\right) = \tilde{\tilde{F}}\left(\frac{v_{kl}}{1+\delta+\ldots+\delta^{k-1}}\right). \end{cases}$$

Assumption 2 yields some structure on the virtual valuations functions

$$\begin{aligned} J^i\left(v_{ij}\right) &= v_{ij} - \frac{1-\tilde{\tilde{F}}\left(\frac{v_{ij}}{1+\delta+\ldots+\delta^{i-1}}\right)}{\tilde{\tilde{f}}\left(\frac{v_{ij}}{1+\delta+\ldots+\delta^{i-1}}\right)}.\left(1+\delta+\ldots+\delta^{i-1}\right) \\ &= \left(1+\delta+\ldots+\delta^{i-1}\right)\tilde{\tilde{J}}\left(\frac{v_{ij}}{1+\delta+\ldots+\delta^{i-1}}\right) \end{aligned}$$

where $\tilde{\tilde{J}}(v) \equiv v - \frac{1-\tilde{\tilde{F}}(v)}{\tilde{\tilde{f}}(v)}$ is the virtual valuation computed from the distribution function $\tilde{\tilde{F}}$ and its associated density $\tilde{\tilde{f}}$. If virtual valuations are increasing, so is $\tilde{\tilde{J}}$.

Under assumptions 1 and 2, the auctioneer is indifferent between allocating the good to agents ij and kl if, following (12), $\tilde{\tilde{v}}_{ij} = \tilde{\tilde{v}}_{kl}$, i.e. if v is such that

$$\left(1+\delta+\ldots+\delta^{i-1}\right)\left[\tilde{\tilde{J}}(v) - (1-\delta)\pi^{**}\right] = \left(1+\delta+\ldots+\delta^{k-1}\right)\left[\tilde{\tilde{J}}(v) - (1-\delta)\pi^{**}\right], \tag{16}$$

that is $v = v^{**}$ such that $\tilde{\tilde{J}}(v^{**}) = (1-\delta)\pi^{**}$. It may well be the case that such v^{**} does not exist, that is $\forall v \in [\underline{v}_1, \bar{v}_1]$, $\tilde{\tilde{J}}(v) < (1-\delta)\pi^{**}$. Then, it is assumed that $v^{**} = +\infty$. The following proposition is then a straightforward consequence of equation (16).

Proposition 6. *If two agents have the same per period willingness to pay v (assumption 1) and share the same probability distribution over their respective per period valuation (assumption 2), the profit maximizing seller prefers to rent the good to the agent with the shortest horizon if $v < v^{**}$. If, on the contrary, $v > v^{**}$, the good would rather be allocated to the agent with the largest horizon.*

The comparison arises in the same term than without asymmetric information: the firm wants to secure the average per period revenues $\tilde{\tilde{J}}(v^{**})$. If the per period valuations of both bidders is larger than v^{**}, then the firm gets an "extra benefits" for each period of allocation. Consequently, it prefers to secure this later for as long as possible and allocates the good to the long term bidder. If, on the contrary, both bidders per period willingness to pay is lower than v^{**}, then the regulator incurs a "loss", with respect to its expected per period profits, if it allocated the good to any of the two bidders. In order to limit this loss, the good is given to the shorter horizon bidder.

It is also possible to get some lower bound of π^{**}. As for the perfect information case, the firm could sub-optimally allocate the good to agent of type i with the highest average virtual valuation, i.e. $i \in \arg\max_j \mathrm{E}_{v_j}\left[J^j\left(v_j\right)/\left(1+\delta+\ldots+\delta^{j-1}\right)\right]$ and

$$\pi^{**} \geqslant \max_j \mathrm{E}_{v_j}\left[\frac{\mathrm{E}_{v_j}\left[J^j\left(v_j\right)\right]}{1-\delta^j}\right].$$

Nevertheless, the comparison with v^{**} is not easily possible because of the virtual valuations.

Turning to the more general case with per period willingness to pay $\hat{v}$, the firm compares agents ij and kl by comparing $\tilde{\tilde{v}}_{ij}$ and $\tilde{\tilde{v}}_{kl}$ the following way

$$\tilde{\tilde{v}}_{ij} \gtrless \tilde{\tilde{v}}_{kl} \quad \text{i.i.f.} \quad \left[\tilde{\tilde{J}}\left(\hat{v}_{ij}\right)-\tilde{\tilde{J}}(v^{**})\right] \gtrless \frac{1-\delta^k}{1-\delta^i}\left[\tilde{\tilde{J}}(\hat{v}_{kl})-\tilde{\tilde{J}}(v^{**})\right].$$

As in the absence of asymmetric information, but with a different boundary, the fraction in the right-hand side of the inequality is greater than 1 and whenever $\hat{v}_{kl} > \hat{v}_{ij} > v^{**}$,[6] the firm prefers the long run buyer kl and, if $\hat{v}_{kl} < \hat{v}_{ij} < v^{**}$, the short run buyer ij is favored by the auctioneer. In the two remaining cases, $\hat{v}_{ij} > \hat{v}_{kl} > v^{**}$ and $\hat{v}_{ij} < \hat{v}_{kl} < v^{**}$, there is no *a priori* systematic preference over one or another agent, for the same reasons as in the perfect information case.

Thus, with asymmetry of information, the influence of the horizon of the agents on the optimal allocation has the same general structure than in the efficient case discussed higher, with a different cut-off value. For small per period valuations, short horizon buyers will be preferred. For larger per period valuations, long horizon buyers will be favored.

It is in general impossible to state whether v^{**} is greater or smaller than v^{*}, even in simple case where all the probability distribution functions satisfy properties such as the one described by assumption 2.

Next section compares the allocation strategies of the firm and the regulator.

6.5 Efficient vs Profit maximizing allocation mechanisms

To compare both allocation procedures, this section compares the longest horizon bidder with a strictly positive allocation probability for each mechanism.

6.5.1 Longest horizon bidder to be allocated the good

A bidder is given a strictly positive expected probability to be allocated the good if and only if the highest value the auctioneer attaches to this bidder i is higher than all values this later attaches to the lowest value attached to other bidders. Using the following notations

$$\begin{cases} K^i\left(v_{ij}\right) = v_{ij} - \delta\left(1-\delta^{i-1}\right)\pi^{*}, \\ L^i\left(v_{ij}\right) = J^i\left(v_{ij}\right) - \delta\left(1-\delta^{i-1}\right)\pi^{**} \end{cases}$$

and, recalling that virtual valuations are assumed to be increasing, a bidder of type i has a positive expected allocation probability by the firm or the regulator i.i.f., respectively

$$\begin{cases} K^i\left(\overline{v}_i\right) > \max_k\left[K^k\left(\underline{v}_k\right)\right], \\ L^i\left(\overline{v}_i\right) > \max_k\left[L^k\left(\underline{v}_k\right)\right]. \end{cases} \tag{17}$$

[6] If the problem is regular, then all virtual valuations are increasing as well as $\tilde{\tilde{J}}$.

Notice that $K^1(\bar{v}_1) = L^1(\bar{v}_1) = \bar{v}_1 > 0$. Thus, both maxima must be strictly positive and bidders satisfying (17) will indeed have a strictly positive probability to be allocated the good.

Let R and F be the longest horizon bidders that are given a strictly positive probability to be allocated the good by, respectively, the regulator and firm. That is

$$\begin{cases} R = \sup\left[i | K^i(\bar{v}_i) > \max_k \left[K^k(\underline{v}_k)\right]\right], \\ F = \sup\left[i | L^i(\bar{v}_i) > \max_k \left[L^k(\underline{v}_k)\right]\right]. \end{cases}$$

The lack of structure of the valuations makes it difficult to play with these definitions. For example, if $i < j$, then $K^j(\bar{v}_j)$ is *a priori* not lower than $K^i(\bar{v}_i)$. Indeed, notice that

$$K^j(\bar{v}_j) = (\bar{v}_j - \bar{v}_i) + K^i(\bar{v}_i) - \delta^i\left(1 - \delta^{j-i}\right)\pi^{**}.$$

Thus, $K^j(\bar{v}_j) < K^i(\bar{v}_i)$ requires $\bar{v}_j < \bar{v}_i + \delta^i\left(1 - \delta^{j-i}\right)\pi^{**}$. Thus, one cannot *a priori* think at R as the last bidder such that $K^R(\bar{v}_R) > \max_k\left[K^k(\underline{v}_k)\right] > K^{R+1}(\bar{v}_{R+1})$. The same argument induces that there may exist bidders $i < R$, with shorter demand than bidder R (or F if $i < F$), that are never allocated the good by the regulator (respectively the firm).

Nevertheless, one can still show the two following properties. First, let i be a "long term" bidder which is never allocated the good by the firm, i.e. $F < i$. Then, it must be the case that there exists a bidder $j \leqslant F < i$ such that any bidder of type i has a lower value for the auctioneer than the lowest value provided by bidders j. As virtual valuations are assumed to be increasing, this is equivalent to bidder $\underline{v}_j$ raising more value for the auctioneer than bidder $\bar{v}^i$

$$J^i(\bar{v}_i) - \delta\left(1 - \delta^{i-1}\right)\pi^{**} \leqslant J^j\left(\underline{v}_j\right) - \delta\left(1 - \delta^{j-1}\right)\pi^{**}.$$

As $J^i(\bar{v}_i) = \bar{v}_i$ and $J^j\left(\underline{v}_j\right) = \underline{v}_j - 1/f^j\left(\underline{v}_j\right) < \underline{v}_j$, this implies[7]

$$\begin{aligned} & \bar{v}_i - \delta\left(1 - \delta^{i-1}\right)\pi^{**} < \underline{v}_j - \delta\left(1 - \delta^{j-1}\right)\pi^{**} \\ \text{i.e.} \quad & \bar{v}_i - \underline{v}_j < \delta^j\left(1 - \delta^{i-j}\right)\pi^{**} \leqslant \delta^j\left(1 - \delta^{i-j}\right)\pi^{*}. \end{aligned}$$

Rewriting this last inequality yields $K^i(\bar{v}_i) < K^j\left(\underline{v}_j\right)$. Thus, bidder i is allocated the good with probability zero by the regulator.

Second, bidder R is such that, for all bidders i,

$$\bar{v}_R - \delta\left(1 - \delta^{R-1}\right)\pi^* \geqslant \underline{v}_i - \delta\left(1 - \delta^{i-1}\right)\pi^*$$

i.e.

$$\bar{v}_R - \underline{v}_i + \frac{1}{f^i(\underline{v}_i)} > \bar{v}_R - \underline{v}_i \geqslant \delta^i\left(1 - \delta^{R-i}\right)\pi^*$$

i.e.

$$L^R(\bar{v}_R) > L^i(\underline{v}_i) + \delta^i\left(1 - \delta^{R-i}\right)(\pi^* - \pi^{**}).$$

[7] Recall that the profit maximizing firm gets an expected profit noted π^{**} and the social welfare maximizing regulator gets social welfare π^* such that $\pi^* > \pi^{**}$.

For bidder's type $i \leqslant R$, this yields $L^R(\overline{v}_R) > L^i(\underline{v}_i)$. For bidder's type $i > R$, one cannot secure the same result and bidder R may end up being such that there exists one type i such that $L^R(\overline{v}_R) < L^i(\underline{v}_i)$, that is with no chance to be allocated the good by the firm. Combining both results, bidder R is at most one candidate for the bidder with the longest horizon for the firm allocation mechanism, i.e. $R \leqslant F$.

This two properties are summarized in the following proposition.

Proposition 7. *A profit maximizing firm distorts allocation decisions in favor of bidders with longer horizon, that is the bidder with the longest horizon that is given a strictly positive probability by the efficient allocation mechanism has a shorter horizon than the corresponding bidder under the profit maximizing allocation procedure.*

This result may be helpful in understanding the main differences in terms of allocation between the essential facility being ruled by a profit maximizing firm or a social welfare maximizing regulator. As the firm incurs a cost of gathering information, the value this later attaches to any bidder may end to be lower than the one of the regulator. This makes more plausible for long-run agents, for whom the commitment cost is higher, to get the object when the firm is ruling the allocation. Loosely speaking, by allowing such long term bidders, the firm is putting more pressure on shorter term bidders.

6.5.2 Other elements of comparison

Let assume that bidders' valuation distribution verify assumption 2. Following the analysis done in section 6.3, the regulator prefers bidder ij to bidder kl if $\tilde{v}_{ij} > \tilde{v}_{kl}$, that is when

$$\left[\hat{v}_{ij} - v^*\right] > \frac{1-\delta^k}{1-\delta^i}\left[\hat{v}_{kl} - v^*\right].$$

Then, recalling section 6.4, the firm prefers bidder ij to bidder kl if $\tilde{\tilde{v}}_{ij} > \tilde{\tilde{v}}_{kl}$, that is if

$$\left[\tilde{\tilde{J}}\left(\hat{v}_{ij}\right) - \tilde{\tilde{J}}\left(v^{**}\right)\right] > \frac{1-\delta^k}{1-\delta^i}\left[\tilde{\tilde{J}}\left(\hat{v}_{kl}\right) - \tilde{\tilde{J}}\left(v^{**}\right)\right].$$

Notice that whereas the efficient allocation requires a constant gap between average valuations of two bidders i and j, this is not *a priori* the case for the firm's mechanism. Moreover, when $k > i$, the fraction is higher than 1 while when both are very large, it is equal to 1. It is nevertheless difficult to discuss in more details these comparisons.

7 Conclusion

This chapter develops an analysis of congestion problems based on a new stationary approach of the pricing of capacity on networks. First, it shows that *a priori* well-behaved standard Vickrey allocation mechanisms do not work in the context of goods exhibiting short-term and long-term demands. Then, it develops tractable infinitely repeated allocation models of an identical good, without resale market between agents, when the auctioneer commits not to reallocate a good awarded for several periods. Both optimal and efficient allocation mechanisms are described.

There are five main insights raised by this chapter. The first lesson is that one has to carefully take into account the difficulty raised by the difference in length of customers demands in order to tackle the analysis of congestion problems in a proper way. Second, the rules of the allocation mechanism, either the optimal or the efficient one, are highly different from standard static ones but in the same line of techniques. Third, the optimal mechanism is characterized by, first, the exclusion of customers with low virtual valuations, as usual, and, second, by a cost attached to long term buyers. This cost corresponds to the lack of flexibility induced by the commitment of the seller not to renegotiate its allocation mechanism, i.e. not to rule out long term agents during their consumption of the good were a better offer to show up. Fourth, whereas the allocation procedure of any auctioneer is not based on per period bidders' valuation, the comparison between bidders can be based on their per period willingness to pay. Roughly, when two bidders show up with identical per period valuation that is higher than a threshold, then the auctioneer favors long-term buyers while it favors short-term buyers if the reverse holds. The threshold value is different for the firm and the regulator. Fifth, the firm tends to allocate the good to bidder with longer time horizon than the regulator.

Nevertheless, many problems are left open regarding the primary study of the pricing of the Internet: asymmetry of information with respect with the length of demand, the possibility of queuing or resale, for example.

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A Appendix

A.1 Proof of lemma 1

This follows directly from equation (2) and (3). Thus, $\left\{p^{ij*}\left(v_{ij}\right), t^{ij*}\left(v_{ij}\right)\right\}$ is a solution to the maximization problem in this case. Note that there is no need for the social planner to specify the transfer function $t^{ij}\left(v_{ij}\right)$ to be equal to $t^{ij*}\left(v_{ij}\right)$, as long as it satisfies individual rationality constraints. On the contrary, the firm must set the transfers to $t^{ij*}\left(v_{ij}\right)$. The proof relies only on the linearity of

A.2 Proof of lemma 2

Let define

$$\tilde{W}(\pi, p) \equiv \sum_{i=1}^{I} \sum_{j=1}^{n_i} \mathrm{E}_v\left[\left[v_{ij} - \delta\left(1-\delta^{i-1}\right)\pi\right] p^{ij}(v)\right]$$

where p is the vector made of all p^{ij}. This function is clearly decreasing in π. The convexity results from the fact that function $\tilde{W}$ is linear with respect to π. Simple computations show that $\tilde{W}(\alpha\pi_1 + (1-\alpha)\pi_2, p) = \alpha\tilde{W}(\pi_1, p) + (1-\alpha)\tilde{W}(\pi_2, p)$ for α between 0 and 1. Then, define $p(v|\pi)$ the vector made of probability functions that maximize $W(\pi)$ for a given π. It must be the case that $W(\pi) = \tilde{W}(\pi, p(v|\pi))$ and that $W(\pi) \geqslant \tilde{W}(\pi, p(v|\tilde{\pi}))$ for $\tilde{\pi} \neq \pi$. Combining the linearity of $\tilde{W}$ with respect to π and the relationship between $\tilde{W}(\pi, p)$, $W(\pi)$ and $p(v|\pi)$ yields

$$\begin{aligned}
&\tilde{W}(\alpha\pi_1 + (1-\alpha)\pi_2, p(v|\alpha\pi_1 + (1-\alpha)\pi_2)) \\
&\quad = \alpha\tilde{W}(\pi_1, p(v|\alpha\pi_1 + (1-\alpha)\pi_2)) + (1-\alpha)\tilde{W}(\pi_2, p(v|\alpha\pi_1 + (1-\alpha)\pi_2)) \\
&\quad \leqslant \alpha\tilde{W}(\pi_1, p(v|\pi_1)) + (1-\alpha)\tilde{W}(\pi_2, p(\pi_2)).
\end{aligned}$$

Thus,

$$W(\alpha\pi_1 + (1-\alpha)\pi_2) - \delta[\alpha\pi_1 + (1-\alpha)\pi_2] \leqslant \alpha W(\pi_1) + (1-\alpha) W(\pi_2)$$

and $W(\pi) - \delta\pi$ is convex.

Moreover, $W(0) - 0 = \max_{\{p^{ij}(.)\}_{i,j}} \left[\sum_{i=1}^{I} \sum_{j=1}^{n_i} \mathrm{E}_v\left[\left[v_{ij}\right] p^{ij}(v)\right]\right]$ must be positive if the overall problem has any economic sense: the right-hand side represents social welfare in a static optimal auction with perfect information.

Eventually, $\lim_{\pi\to+\infty}[W(\pi) - \pi] = \max_{\left\{p^{1j}(.)\right\}_j} \left[\sum_{j=1}^{n_1} \mathrm{E}_v\left[\left[v_{1j}\right] p^{1j}(v)\right]\right]$ which must also be positive and lower than $W(0) - 0$.

A.3 Proof of lemma 3

Constraints $\left(IC_{ij}\right)$ are equivalent, for any agent i, all v_{ij} and $\tilde{v}_{ij}$, to

$$\begin{aligned}
U^i\left(v_{ij}\right) &\geqslant q^{ij}\left(\tilde{v}_{ij}\right) v_{ij} - t^{ij}\left(\tilde{v}_{ij}\right) \\
U^i\left(\tilde{v}_{ij}\right) &\geqslant q^{ij}\left(v_{ij}\right) \tilde{v}_{ij} - t^{ij}\left(v_{ij}\right)
\end{aligned}$$

or

$$q^{ij}\left(v_{ij}\right)\left(v_{ij}-\tilde{v}_{ij}\right)\geqslant U^{i}\left(v_{ij}\right)-U^{i}\left(\tilde{v}_{ij}\right)\geqslant q^{ij}\left(\tilde{v}_{ij}\right)\left(v_{ij}-\tilde{v}_{ij}\right) \tag{18}$$

Thus, equation (6) follows. This requires that the expected probability $q^{ij}\left(v_{ij}\right)$ must be non-decreasing as well as $U^{i}\left(v_{ij}\right)$. Dividing (18) by $\left(v_{ij}-\tilde{v}_{ij}\right)$ and taking the limit as $\tilde{v}_{ij}\rightarrow v_{ij}$ yields, almost everywhere and for all i

$$\frac{dU^{i}\left(v_{ij}\right)}{dv_{ij}}=q^{ij}\left(v_{ij}\right)\geqslant 0. \tag{19}$$

Integrating (19) between $\underline{v}_{ij}$ and v_{ij} gives (7). Moreover, $U^{i}\left(v_{ij}\right)$ being non-decreasing and $\left(IR_{ij}\right)$ induce (8).

Conversely, combining (7) and (6) yields (18) which is equivalent to $\left(IC_{ij}\right)$. Moreover, $q^{i}\left(v_{i}\right)$ is non-decreasing, so combining (7) and (8) implies $\left(IR_{ij}\right)$.

A.4 Proof of lemma 4

Lemma 3 sets that the maximization of both auctioneers can be written as maximizing (1) under constraints (6), (7), (8), $\left(P_{ij}\right)$ and $\left(P_{0}\right)$. According to (7), the expected payment of a bidder can be written as

$$t^{ij}\left(v_{ij}\right)=q^{ij}\left(v_{ij}\right)v_{ij}-U^{i}\left(v_{ij}\right)=q^{ij}\left(v_{ij}\right)v_{ij}-\int_{\underline{v}_{i}}^{v_{ij}}q^{ij}\left(x\right)dx-U^{i}\left(\underline{v}_{i}\right)$$

Moreover, standard manipulation of the integral yields

$$\begin{aligned}\mathrm{E}_{v_{ij}}\left[\int_{\underline{v}_{i}}^{v_{ij}}q^{ij}\left(x\right)dx\right]&=\int_{\underline{v}_{i}}^{\overline{v}_{i}}\int_{\underline{v}_{i}}^{v_{ij}}q^{ij}\left(x\right)f^{i}\left(v_{ij}\right)dxdv_{ij}=\int_{\underline{v}_{i}}^{\overline{v}_{i}}\int_{x}^{\overline{v}_{i}}q^{ij}\left(x\right)f^{i}\left(v_{ij}\right)dv_{ij}dx\\&=\int_{\underline{v}_{i}}^{\overline{v}_{i}}q^{ij}\left(x\right)\left[1-F^{i}\left(x\right)\right]dx=\mathrm{E}_{v_{ij}}\left[q^{ij}\left(v_{ij}\right)\frac{1-F^{i}\left(v_{ij}\right)}{f^{i}\left(v_{ij}\right)}\right],\end{aligned}$$

For a profit maximizing seller, the objective is to maximize the bidders' expected payments while preserving individual rationality, so that the optimal mechanism is characterized by $U^{i}\left(\underline{v}_{i}\right)=0$ which gives equation (10). Then, expected payment by agent ij becomes

$$\mathrm{E}_{v_{ij}}\left[t^{ij}\left(v_{ij}\right)\right]=\mathrm{E}_{v_{ij}}\left[\left(v_{ij}-\frac{1-F^{i}\left(v_{ij}\right)}{f^{i}\left(v_{ij}\right)}\right)q^{ij}\left(v_{ij}\right)\right]. \tag{20}$$

Finally, current expected benefits can be rewritten

$$\sum_{i,j}\mathrm{E}_{v_{ij}}\left[\left(v_{ij}-\frac{1-F^{i}\left(v_{ij}\right)}{f^{i}\left(v_{ij}\right)}-\delta\left(1-\delta^{i-1}\right)\pi\right)q^{ij}\left(v_{ij}\right)\right]+\delta\pi.$$

This is the expression given in (9) for a profit maximizing seller.

When dealing with efficiency, transfers are not taken into account in the objective function as far as they respect the incentive compatibility constraint. Thus, the objective function of the social planner can be rewritten

$$\sum_{i,j} \mathrm{E}_{v_{ij}} \left[\left[v_{ij} - \delta \left(1 - \delta^{i-1} \right) \pi \right] q^{ij} \left(v_{ij} \right) \right] + \delta\pi.$$

which corresponds to (9) with $Y\left(v_{ij}\right) = v_{ij}$.

Thus, the auctioneer's problem is to maximize (9) under constraints (6), (7), (8), (P_{ij}) and (P_0). What is left to be proved is that constraints (8) and (7) are all satisfied by the proposed transfer function $t^{i**}(v_i)$. Simple computations show that this is the case. Eventually, the proposed solution of program (9) is a solution to program (1).

Note that, for the regulator, expression (20) about transfers holds but there is no obligation to set $U^i\left(\underline{v}_i\right) = 0$. Nevertheless, the particular transfer function $t^{ij*}\left(v_{ij}\right)$ is a (non-unique) solution to the maximization problem (1).

A.5 Proof of lemma 5

Define the following function

$$\tilde{V}(\pi, p) \equiv \sum_{i=1}^{I} \sum_{j=1}^{n_i} \mathrm{E}_v \left[\left[Y^i\left(v_{ij}\right) - \delta\left(1 - \delta^{i-1}\right)\pi \right] p^{ij}(v) \right]$$

where $Y^i\left(v_{ij}\right)$ corresponds to $J^i\left(v_{ij}\right)$ when maximizing profits or v_{ij} when taking care about efficiency. Then, function $\tilde{V}$ is clearly decreasing in π. The proof for convexity then follows the same line as the proof of lemma 2 because function $\tilde{V}$ is also linear in π. The result in $\pi = 0$ is also a consequence for the problem parameters to be economically meaningful in presence of asymmetric information.

A.6 Proof of proposition 3

What needs to be proved is that the level of final benefits for the regulator is identical with asymmetric information and perfect information. In the presence of asymmetric information, the regulator has to solve problem $(\mathcal{P}1)$. Nevertheless, transfers appear both as benefits for the regulator and costs for bidders, summing up to zero in the objective function. The role of transfers shrinks to securing incentive compatibility constraints, which is always the case with the mechanism proposed in proposition 3. Thus, $\pi^* = \pi^{**}$.