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"Integrating profitability prospects and cash management"

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Integrating profitability prospects and cash management*

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Abstract: We develop a bi-dimensional dynamic model of corporate cash management in which shareholders learn about a firm's profitability and weigh the costs and benefits of holding cash. We explicitly characterize the optimal payout policy. We explain how the evolution of the strength of shareholders' beliefs about profitability and changes in corporate cash management are intertwined. The model predicts that both cash target levels and target dividend payout ratios are increasing in profitability prospects. This yields novel insights into the relationship between profitability prospects, precautionary cash savings, dividend policy and the dynamics of firm value.

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1 Introduction

Liquidity and profitability are two key dimensions of a firm's payout policy. For instance, firms with high and stable profitability are expected to pay high dividends because they have limited reinvestment needs. Growing and cash-constrained firms are expected to pay low dividends because their cash inflows may not significantly exceed efficient reinvestment needs. This intuition is confirmed through an extensive series of empirical studies that develop the idea that the trade-off between the advantages and the costs of cash retention (e.g., flotation costs savings versus the agency costs of free cash flow) evolves as profit accumulates. This idea, referred to as "the life-cycle hypothesis for dividends" proves relevant to explaining a firm's payout practices.[1]

The growing theoretical literature on the dynamics of corporate cash remains different from these considerations because existing models neglect to include profitability issues and focus mainly on liquidity issues.[2] This paper develops a dynamic model of corporate cash management in which shareholders weigh the costs and benefits of holding cash while learning about firm's profitability. We explore how the evolution of the strength of shareholders' beliefs about the firm's profitability and changes in corporate cash management are intertwined. The model provides a unified explanation of a series of facts about payout practices and a theoretical underpinning for recently documented regularities, as well as novel implications regarding the relationship between profitability prospects, precautionary cash savings, dividend policy and the dynamics of firm value.

To start, consider a new, closely held business. Shareholders are cash constrained and decide corporate policy. They must cope with both a profitability concern (the risk of running a project that is not profitable) and a liquidity concern (the risk of being forced to liquidate a profitable project because it cannot meet its short-term operating costs). The issue is how to characterize an optimal corporate policy that specifies at all times whether to continue the project, the cash target inventory and the dividend payout ratio. More generally, our model focuses on cash-constrained corporations that do not precisely know their long-term profitability and learn about it from their realized earnings. Such corporations account for a significant portion of the economy. In addition to start-up firms, other examples of such corporations are knowledge-based companies, firms newly restructured as a consequence of

[1]See, for instance, Fama and French (2001), Grullon, Michaely and Swaminathan (2002), DeAngelo, DeAngelo (2006), DeAngelo, DeAngelo and Stulz (2006), Leary and Michaely (2011).

[2]We review the literature below.

major investments and firms newly listed on stock exchanges.[3] Faced with uncertainty about profitability, firms key challenge is to define a corporate policy ensuring that at any date, the optimal amount of cash is available to avoid a liquidity shortage and to compensate for the costs of holding cash.

Merging corporate liquidity and profitability knowledge in a single model is challenging because it involves a difficult bi-dimensional optimization problem. Our paper overcomes this difficulty by expressing the firm's value as a value function of a bi-dimensional Bayesian adaptative control problem that we solve in a quasi-explicit form. The cash reserves with unknown profitability parameter and the belief about the firm's profitability (i.e., the firm's profitability prospects) are the two state variables of the model. We prove the existence and uniqueness of an optimal control policy. It consists of paying out cash when cash reserves reach a dividend boundary function. We show that this boundary function continuously increases with profitability prospects and is characterized by an ordinary differential equation. This mathematical contribution leads to a unique feature of our model: the dividend payout ratio of the firm is an increasing function of the beliefs about the firm's profitability. A rich set of implications follows, which cannot be obtained focusing solely on liquidity or profitability issues. We highlight our primary findings here.

The model predicts a positive relationship between cash target levels and profitability prospects. Precisely, dividends are distributed each time the flow of cash inside a firm reaches a new maximum value, which depends on the beliefs about the firm's profitability. When cash is paid out, shareholders become more confident about the relevance of their project, have more to lose from negative shocks on earnings, and, accordingly, increase the future cash target level. The model predicts that a firm whose project has unclear profitability becomes well-established by reaching its cash target levels several times.[4] Shareholders of a well-established firm have strong beliefs about the firm's profitability regardless of the level of cash reserves. The model predicts that firms with unclear profitability and firms with well-

[3]Moyen and Platinakov (2013) document that from 1995-1999, on average, 29.7 percent of U.S. businesses each year were either three years old at most or newly restructured as a consequence of major investments. Pastor and Veronesi (2003) stress the increasing number of firms newly listed on the major U.S. stock exchanges between the years 1980 and 2000. Financing constraints are common for young firms, and external finance can be extremely expensive for knowledge-based companies whose profitability is typically difficult to ascertain. A classic example is the funding crisis Intel faced in the early 1980s; see, for instance, Passov (2003).

[4]Formally, in the model, *well-established profitability* means that the conditional variance of the profitability parameter is low, while *unclear profitability* refers to a large conditional variance of the profitability parameter.

established profitability can have high and similar optimal cash reserves. Nevertheless, their payout policies are very different. Well-established firms pay higher dividends and have a more stable dividend policy than do firms facing more uncertainty about profitability. Intuitively, once the firm is well established, shareholders do not expect changes in the firm's growth rate. Cash target levels remain high, the firm has fewer reinvestment needs (the free cash flow problem is more severe), and payouts, as a proportion of available earnings, are large and stable. Thereby, our model contributes to our understanding of dividend smoothing and provides a possible rationale for empirical studies that document that firms that smooth the most tend to pay more dividends and are more subject to agency costs of free cash flow (see, in particular, Leary and Michaely (2011)). The implications regarding the link between optimal cash reserves and profitability prospects are novel and in line with Fama and French (2002), who emphasize that past profitability impacts a firm's cash position.

Other interesting implications follow. The model predicts that the amount of dividends and the value of the firm will exhibit a positive relationship and that firms will not decrease dividends. The model also implies that dividend increases yield a decline in the risk of profitability in the sense that beliefs about profitability are higher after dividend payments. The first two implications are well documented and are usually explained using asymmetry of information and signaling theory.[5] The third implication is novel and in line with the maturity hypothesis formulated by Grullon, Michaely and Swaminathan (2002), who document that dividend changes are related to risk changes. In our model, these results follow from the positive relationship between cash holdings and profitability prospects together with the property that for a fixed level of cash reserves, profitability prospects are higher for a firm that has previously reached its cash target levels several times. Thus, our model provides a rationale for recently documented regularities, as well as a unified explanation of a series of stylized facts that are usually explained with different theories.

Finally, we emphasize that liquidity and profitability issues generate different non-linearities between shareholders' value and the firm's fundamentals. The model predicts that the value of well-established firms is increasing and concave in cash reserves, whereas the value of firms facing more uncertainty about their profitability may exhibit convexity in cash reserves. It follows a non-monotone relationship between firm's volatility and cash reserves. Intuitively, an increase in earnings affects the beliefs about a firm's profitability. If prior beliefs are initially positive, an increase in earnings confirms these prior beliefs. Liquidity

[5]See the pioneering papers by Bhattacharya (1979), Miller and Rock (1985), John and Williams (1985). See Allen and Michaely (2003), DeAngelo, DeAngelo and Skinner (2008) for comprehensive surveys of payout policy.

issues dominate profitability issues, and a firm's volatility decreases with cash reserves. To the contrary, if beliefs about the firms' profitability are initially negative, an increase in earnings challenges these prior beliefs. Profitability issues may prevail, in which case the firm's volatility increases with cash reserves. These results are novel and derive from the shareholders' problem that integrates both liquidity and profitability issues in a bi-dimensional setting.

Relationship to the literature Most dynamic models in corporate finance treat profitability and liquidity separately. Our model is linked to these two strands of the literature. Dynamic corporate models with only liquidity issues study how financial frictions impact the value of cash holdings inside a firm.[6] The prototype of this approach is a model in which the cumulative net cash flow generated by the firm follows an arithmetic Brownian motion. The constant drift represents the firm's profitability per unit of time, and the Brownian shock is interpreted as a liquidity shock. Shareholders have no access to external financing, which creates a precautionary demand for cash. Agency costs of free cash flow create a cost of carrying cash. This results in a unique optimal dividend policy that requires paying shareholders 100 % of earnings beyond an endogenous constant cash target level. Other key insights are that the firm value is increasing and concave in the current level of cash inside the firm; the firm's value is decreasing in the volatility of cash flow; the volatility of stock prices is decreasing in the level of stock prices. Pioneering models include Jeanblanc and Shiryaev (1995), Radner and Shepp (1996) and Kim, Mauer and Sherman (1998). These studies have been extended in a number of directions.[7] A common feature of these contributions is that shareholders do not face uncertainty about the firm's profitability. Only liquidity issues matter. The dividend policy is insensitive to cash flow in the sense that either 0% or 100% of earnings are paid to shareholders.

In their simplest versions, dynamic corporate models with only profitability issues consider that the profitability per unit of time of the firm's project follows a geometric Brownian motion minus a constant that is interpreted either as an exogenous cost (in real option models, see Dixit and Pindyck (1994)) or as the interest on a debt after tax deduction (in

[6] This growing literature is motivated by the dramatic increase in the cash holdings of companies. According to Bates, Kahle and Stulz (2009), listed U.S. industrial firms increased their average cash-to-assets ratios from 10.5% in 1980 to 23.2% in 2006.

[7] For instance, Décamps, Mariotti, Rochet and Villeneuve (DMRV) (2011) study the interaction between cash management, agency costs, issuance costs and stock price; Bolton, Cheng and Wang (2011) extend the model to the case of flexible firm size in order to study the dynamic patterns of corporate investment; Bolton, Chen and Wang (2013), Hugonnier, Malamud and Morellec (2014) introduce capital supply uncertainty and the necessary time needed to secure outside funds into the analysis.

corporate contingent-claim models pioneered by Leland (1994)). In these models, shareholders can issue equity at no cost; their only concern is to decide when to liquidate. They do so when the profitability reaches a sufficiently low threshold. Cash reserves are useless, and it is optimal to immediately pay out any gain to shareholders and to fund any loss by issuing more equity at the market price. Other insights are that the firm value is increasing and convex in the profitability; the firm's value is increasing in the volatility of the profitability; and the volatility of stock prices is increasing in the level of stock prices. [8] Other studies develop corporate models with learning on profitability. For instance, in Pastor and Veronesi (2003), the profitability process is assumed to revert to an unknown mean whose value shareholders learn over time. The model is designed in such a way that the net cash flow reinvested in the firm always remains positive, which avoids both liquidity and liquidation issues. They find that the stock price increases with uncertainty about average profitability and decreases over time as the firm learns about its average profitability. More recently, Moyen and Platinakov (2013) develop a model in which shareholders update their beliefs about firms' quality (high or low) in a dynamic Tobin's q framework. They find evidence that firms with unclear quality are more sensitive to earnings in their investment decisions than are well-established firms. In these models, cash-constrained firms become well established as time passes by learning about their profitability. Unlike in our model, cash management plays no role and firms do not default in these models. Our learning technology is also different. We borrow it from Décamps, Mariotti and Villeneuve (DMV) (2005, 2009); Klein (2009), who study the optimal decision to invest in a project whose profitability is not perfectly known. As in DMV (2005), the decision maker (in our setting, shareholders) observes the values taken by an arithmetic Brownian motion whose drift is unknown and updates her belief about the profitability of the project from these observations. We study the optimization of a dividend flow of a financially constrained firm when there is an uncertainty about the firm's profitability, while DMV (2005) studies the value of waiting to invest when there is an uncertainty about the profitability of the investment project and no financing constraints.

Thus, treated separately, profitability and liquidity issues lead to different nonlinearities between shareholder's value and the firm's fundamentals, different properties of the dynamics of the value of the firm, and different dividend policies, and none of these policies

[8] Leland (1994) extends a number of insightful contributions. An important example is the study by Leland and Toft (1996), in which unconstrained shareholders continuously roll over maturing bonds. Some recent studies build on Leland and Toft (1996) and address the issue of the liquidity of the secondary market for corporate bonds. See, for instance, He and Xiong (2012) and He and Milbradt (2014).

reports main stylized facts about payout practices. Integrating liquidity and profitability issues in a single model is challenging because it requires solving a dynamic bi-dimensional optimization problem. Very few papers address this issue. Murto and Terviö (2014) introduce costly external financing in a real option setting. The profitability per unit of time of the firm's project follows a geometric Brownian motion minus an exogenous cost. This leads to a bi-dimensional control problem that is numerically investigated. Décamps, Gryglewicz, Morellec and Villeneuve (DGMV) (2015) assume that the firm's operating cash flow is proportional to profitability, dynamics governed by a Geometric Brownian Motion. The ratio of cash holdings to profitability is the state variable of the firm's problem. DGMV (2015) find a constant dividend payout ratio that depends on the long-term prospects of the firm. Unlike in our model, in Murto and Terviö (2014) and DGMV (2015), the uncertainty regarding profitability does not result from shareholders' belief about the relevance of the firm's project.

Our study is more closely related to Gryglewicz (2011), who considers that cash-constrained shareholders learn about the firm's profitability. In his model, the profitability per unit of time is a random variable whose value shareholders learn over time. There are no frictions inside the firm, so holding cash is not costly. It follows that any dividend policy that maintains the reserves above a critical level is optimal. Gryglewicz (2011) studies how this framework impacts the optimal capital structure that results from the trade-off between tax shields and bankruptcy costs.

A final related paper is DeMarzo and Sannikov (2014), who study a dynamic contracting model with learning about the firm's profitability. Still, the approach is very different. Asymmetric information arises endogenously because by shirking, an entrepreneur can distort the beliefs of investors about the project's profitability. The paper studies the relationship between incentives and learning.[9] In the implementation of the optimal contract, cash inside the firm is accumulated at no cost. It is shown that above a critical level of cash, liquidation is first-best, despite moral hazard.

We do not model hidden actions. In our model, information is incomplete but symmetric between stakeholders, and holding cash is costly for the firm. The novelty of our analysis is studying the evolution of the trade-off between costs and benefits of holding cash due to learning on the firm's profitability. The optimal policy is unique, the dividend payout ratio changes with the beliefs about the profitability, and liquidation cannot be first-best. These

[9] Other recent papers on dynamic contracting with learning are Prat and Jovanovic (2013) and He, Wei and Yu (2014).

features yield new and different financial implications.

The paper is organized as follows. We lay out the model in Section 2. Section 3 studies benchmarks in which shareholders face profitability and liquidity concerns separately. Section 4 solves the model in a closed form and presents the optimal corporate policies. Section 5 develops the model implications. Section 6 concludes. All the proofs are in the Appendix.

2 The model

2.1 Learning

A firm has a single investment project that generates random cash flows over time. The cumulative cash flow process $\{R_t; t \geq 0\}$ follows an arithmetic Brownian motion with unknown drift Y and known variance σ

$$dR_t = Y\,dt + \sigma\,dB_t, \quad t \geq 0,$$

where $\{B_t; t \geq 0\}$ is a standard Brownian motion independent of Y. The process $\{B_t; t \geq 0\}$ and the random variable Y are defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ with filtration $\{\mathcal{F}_t; t \geq 0\}$ satisfying the usual condition of right continuity and completion by $\mathbb{P}$-negligible sets.

The firm is held by risk-neutral shareholders who do not fully know the project's long-run profitability Y but observe the cumulative cash flow process $\{R_t; t \geq 0\}$. The filtration $\{\mathcal{F}_t^R; t \geq 0\}$, generated by $\{R_t; t \geq 0\}$, corresponds to the information available to the shareholders. Their belief Y_t about the project's profitability Y at any time t is summarized by the conditional expectation of Y given the available information $\mathcal{F}_t^R$; that is,

$$Y_t = \mathbb{E}[Y \mid \mathcal{F}_t^R].$$

We also say that Y_t represents the firm's profitability prospects at time t. We assume that the project's profitability Y can take two possible values and that a probability of $\frac{1}{2}$ on a realization of Y yields an expected project's profitability equal to zero. This assumption implies that Y takes either of the two values $-\mu < 0 < \mu$ and that the belief process $\{Y_t, \mathcal{F}_t^R; t \geq 0\}$ satisfies the filtering equation[10]

$$dY_t = \frac{1}{\sigma}(\mu^2 - Y_t^2)dW_t, \tag{1}$$

[10]See, for instance, Liptser and Shiryaev (1977).

where the so-called innovation process $\{W_t, \mathcal{F}_t^R; t \geq 0\}$ is a Brownian motion given by

$$dW_t = \frac{1}{\sigma}(dR_t - Y_t dt). \tag{2}$$

It follows from (1)-(2) that the present value of future cash flows is finite:

$$\mathbb{E}\left[\int_0^\infty e^{-rs}\, dR_s\right] = \mathbb{E}\left[\int_0^\infty e^{-rs} Y_s\, ds\right] \leq \frac{\mu}{r}.$$

Finally, we note that the cumulative cash flow process is a sufficient statistic for the Bayesian updating. Precisely, a direct application of the Itô's formula yields the relation

$$dR_t = d\phi(Y_t), \tag{3}$$

where ϕ is the increasing function defined on $(-\mu, \mu)$ by $\phi(y) = \dfrac{\sigma^2}{2\mu} \ln\left(\dfrac{\mu + y}{\mu - y}\right)$.

2.2 The shareholders' problem

Risk-neutral shareholders discount future payments at the risk-free interest rate $r > 0$. The model builds on the standard trade-off *costs versus benefits of holding cash*. The firm accumulates cash because of precautionary motives in a costly external financing environment. External financing involves high costs, and issuing new shares is never a profitable option for the firm, which is then liquidated as soon as it runs out of cash. Nevertheless, accumulating cash is costly. Because of internal frictions, the rate of return of cash inside the firm is lower than the cost of capital. Specifically, we consider that investment proceeds in liquid assets are entirely wasted, such that the cost of holding cash per unit of time is $r\,dt$. This allows us to pragmatically account for the agency cost of free cash flow, first emphasized by Easterbrook (1984) and Jensen (1986).[11] Shareholders can reduce these costs by deciding to distribute cash.[12] This, in turn, defines the optimal dividend policy in our setting. On top of this trade-off, shareholders are not aware of the profitability, positive or negative, of

[11] Other studies (see, for instance, Bolton, Chen and Wang (2011), DMRV (2011)) assume a cost of carrying cash $(r - \lambda)\,dt$ with $\lambda \in (0, r]$. Considering $\lambda = r$ yields a fully explicit solution in the complete information benchmark and eases the analysis of our model.

[12] This assumption is standard in models with cash (see, for instance, Bolton, Chen and Wang (2011) or DMRV (2011), among many other articles). This assumption relates to a series of studies that show how investor protection laws and firm-specific governance rules provide investors with the opportunity to reduce the agency costs of free cash flow by deciding/forcing managers to pay out cash (see, for instance, La Porta et al. (2000), Dittmar, Mahrt-Smith and Servaes (2003), Kalcheva and Lins (2007). See also Allen and Michaely (2003), and DeAngelo, DeAngelo and Skinner (2008)).

the firm's project. Thus, shareholders face both a profitability concern (the risk of running a project that is not profitable) and a liquidity concern (the risk of being forced to liquidate a profitable project).

Formally, at each date, shareholders decide when and how to distribute dividends and whether to continue the project. For simplicity, we assume that the liquidation value of the project is equal to 0. The cash reserves X_t at time t correspond to the cumulative net cash flow that is reinvested in the firm and evolve according to

$$dX_t \;=\; dR_t \;-\; dL_t, \tag{4}$$

where $L = \{L_t; t \geq 0\}$ corresponds to the cumulative dividend process chosen by shareholders. Formally, the process L belongs to the set $\mathcal{L}$ of admissible policies.[13] Using (3), we re-write Equation (4) under the form

$$X_t = \phi(Y_t) - \phi(y) + x - L_t \tag{5}$$

where $X_0 = x$ and $Y_0 = y$. Note that as long as the control L is not activated, there is a one-to-one relationship between cash reserves and profitability prospects.

The firm ceases its activity for two possible reasons: (i) it cannot meet its short-term operating costs by drawing cash from its reserves; and/or (ii) shareholders strategically decide to liquidate for profitability reasons because their belief that the drift parameter Y is strongly negative. In that case, shareholders take the remaining cash reserves as dividends. Thus, equation (4) represents the dynamics of the cash reserves up to the liquidation time τ_0 defined as

$$\tau_0 = \inf\{t \geq 0 \,|\, X_t = 0\}. \tag{6}$$

Given a dividend policy $L \in \mathcal{L}$, current cash reserves $x \in [0, \infty)$ and a current belief $y \in (-\mu, \mu)$, the value of the firm is the expected present value of all future dividends, that is,

$$V(x, y; L) = \mathbb{E}_{(x,y)}\left[\int_0^{\tau_0} e^{-rt} dL_t\right], \tag{7}$$

with $\Delta L_{\tau_0} = \max(X_{\tau_0^-}, 0)$. The case $\Delta L_{\tau_0} = X_{\tau_0^-} > 0$ corresponds to a strategic liquidation. Accordingly, we write $\tau_0 = \inf\{t \geq 0 \,|\, (\Delta L)_t = X_{t^-}\}$. The shareholders' problem is to find the optimal value function defined as the supremum of (7) over all admissible dividend

[13] We say that L belongs to the set $\mathcal{L}$ of admissible policies if and only if L is $\mathcal{F}^R$-adapted, right-continuous, and nondecreasing with $L_{0-} = 0$ and if the associated cash reserves $\{X_t, \mathcal{F}^R_t; t \geq 0\}$ satisfy $X_t \geq 0$, $e^{-rt}X_t$ integrable and, $\lim_{t\to\infty} e^{-rt}X_t = 0$ a.s.

policies

$$V^*(x,y) \;=\; \sup_{L \in \mathcal{L}} V(x,y;L), \tag{8}$$

The next lemma shows that the shareholders' problem (8) is well defined and derives a useful equivalent formulation.

Lemma 1 *The value function V^* is finite and can be written under the form*

$$V^*(x,y) \;=\; x + \sup_{L \in \mathcal{L}} \mathbb{E}_{(x,y)} \left[\int_0^{\tau_0} e^{-rs} \left(Y_s - rX_s\right) \, ds \right] < x + \frac{\mu}{r}. \tag{9}$$

Equation (9) explicates the linkage between the firm's value, the belief process and the cost of holding cash. For a given payout policy L, the firm's value is equal to the current amount of cash reserves, plus the present value of cash flows up to the liquidation time, minus the present value of the cost of accumulating cash up to the liquidation time.

3 Benchmarks

Two polar cases are worth mentioning: (i) the case where shareholders face only a profitability concern, and (ii) the case where shareholders face only a liquidity concern. These benchmarks provide natural bounds to the value function V^*.

3.1 First-best benchmark

Let us assume that shareholders can issue and repurchase shares at no cost whenever they want. Under this assumption, shareholders face only a profitability concern. We shall say that shareholders have deep pockets. Accumulating cash does not bring any benefit, while agency costs reduce the rate of return on internally held cash. Intuitively, an optimal policy is to distribute all of the firm's initial cash reserves x as a special payment at date 0, to hold no cash beyond that time, and to liquidate when the belief about the firm's profitability is too low. In this first-best environment, the firm's value is equal to the sum of current cash reserves plus the option value to liquidate the firm. Accordingly, we get:

Proposition 1 *Suppose that shareholders have deep pockets. Then the firm's value corresponds to the value function $\hat{V}$ of the stopping problem*

$$\forall (x,y) \in [0,\infty) \times (-\mu,\mu), \quad \hat{V}(x,y) \equiv x + \sup_{\tau \in \mathcal{T}^R} \mathbb{E}_y \left[\int_0^{\tau} e^{-rs} Y_s ds \right]$$

where $\mathcal{T}^R$ is the set of $\mathcal{F}^R$-stopping times. We have that $V^* \leq \hat{V}$, computations are explicit and lead to

(i) (Firm value)

$$\begin{cases} \hat{V}(x,y) = x, \quad -\mu < y \leq y^*, \\ \hat{V}(x,y) = x + \frac{y}{r} - \frac{h_1(y)}{h_1(y^*)}\frac{y^*}{r} \quad y^* \leq y < \mu, \end{cases} \tag{10}$$

where

$$h_1(y) = (\mu + y)^\gamma (\mu - y)^{1-\gamma}, \;\; y^* = \frac{-\mu}{1-2\gamma} < 0, \tag{11}$$

$$\textit{and where } \gamma \textit{ is the negative root of the equation } x^2 - x - \frac{r\sigma^2}{2\mu^2} = 0. \tag{12}$$

(ii) (Optimal policy) *Distributing all initial cash reserves x at time 0, holding no cash beyond that time, and liquidating at the stopping time $\tau^* = \inf\{t \geq 0 \,:\, Y_t = y^*\}$ is an optimal policy.*

The inequality $V^* \leq \hat{V}$ implies that $V^*(x,y) = \hat{V}(x,y) = x$ for all $x \geq 0$ and $y \leq y^*$. That is, if the belief about the firm's profitability is lower than the first-best liquidation threshold y^*, then the firm is liquidated whatever the amount of cash within the firm. We will observe how liquidity concerns impact the required profitability prospects below which shareholders liquidate the firm. Observe also that the mapping $y \longrightarrow \sup_{\tau \in \mathcal{T}^R} \mathbb{E}_y \left[\int_0^\tau e^{-rs} Y_s ds\right]$ is convex over $(-\mu, \mu)$ and represents the option value to liquidate the firm for profitability reasons. Finally, let us emphasize that because shareholders can issue and repurchase shares at no cost, the logic of Miller and Modigliani (1961) applies and there is a large degree of freedom in designing the payout process.

3.2 Complete information benchmark

Another useful benchmark is the complete information setting in which shareholders face only a liquidity concern. This corresponds to the case where $y = -\mu$ or $y = \mu$ in the main problem (8).[14] We shall denote with V^- and V^+ the associated value functions.

When $y = -\mu$, the firm's profitability is negative, and it is optimal for shareholders to take the initial cash reserves and to liquidate the firm at time $t = 0$. Accordingly, we deduce from (9) that $V^-(x) = x$, $\forall x \in \mathbb{R}_+$. When $y = \mu$, we revert to the classical case studied by many authors. Informally, any excess of cash over a constant dividend boundary x^*_μ is paid out to shareholders such that the cash reserves process is reflected back each time it reaches x^*_μ. The Hamilton-Jacobi-Bellman (HJB) equation

$$\forall x \geq 0 \quad \max\left(-rV(x) + \mu V'(x) + \frac{\sigma^2}{2}V''(x), 1 - V'(x)\right) = 0 \tag{13}$$

[14]Technically, $-\mu$ and μ are absorbing barriers for the belief process.

characterizes the pair (V^+, x^*_μ). The function V^+ is C^2 on $(0,\infty)$, increasing and concave. The marginal value of cash, $V^{+\prime}(x)$, is strictly greater than 1 for $x \in (0, x^*_\mu)$, and equal to 1 for $x \in [x^*_\mu, \infty)$. Thus, when cash holdings exceed x^*_μ, the firm places no premium on internal funds, and it is optimal to make a lump sum payment $x - x^*_\mu$ to shareholders. Previous articles[15] have studied the HJB Equation (13), whose solution is explicit. In our setting, this leads to

Proposition 2 *Suppose that there is no uncertainty about the firm's profitability. For any* $(x, y) \in [0,\infty) \times (-\mu, \mu)$, $V^-(x) \le V^*(x, y) \le V^+(x)$, *computations are explicit and lead to*

(i) (Firm value)

- *The value function V^+ is given by*

$$\begin{cases} V^+(x) = \dfrac{k(x)}{k'(x^*_\mu)} \quad 0 \le x \le x^*_\mu, \\ V^+(x) = x - x^*_\mu + V^+(x^*_\mu), \quad x \ge x^*_\mu, \end{cases}$$

where

$$k(x) = e^{\alpha_+ x} - e^{\alpha_- x}, \;\; x^*_\mu = \frac{1}{\alpha_+ - \alpha_-} \ln\left(\frac{\alpha_-}{\alpha_+}\right)^2, \quad (14)$$

and where $\alpha_- < 0 < \alpha_+$ are the roots of the equation $\frac{1}{2}\sigma^2 x^2 + \mu x - r = 0.$

- *The value function V^- is given by $V^-(x) = x$, $\forall\, x \in \mathbb{R}_+$.*

(ii) (Optimal policy)

- *The process $L^* = \{L^*_t; t \ge 0\}$ defined by*

$$L^*_t = (x - x^*_\mu)^+\, \mathbb{1}_{t=0} \;+\; L^{x^*_\mu}_t\, \mathbb{1}_{t>0} \quad (15)$$

is an optimal policy for problem (8) with $y = \mu$. In Equation (15), $L^{x^*_\mu}$ denotes the solution to the Skohorod problem at x^*_μ for the drifted Brownian motion $\mu\, t \,+\, B_t$.

- *The process $L^* = \{L^*_t; t \ge 0\}$ defined by $L^*_t = x\, \mathbb{1}_{t=0}$ is an optimal policy for problem (8) with $y = -\mu$.*

Drawing on these two benchmarks, Proposition 3 below establishes the first properties of the value function V^*. Our model encompasses convexity and concavity properties, which

[15] See, for instance, Jeanblanc and Shiryaev (1995)

hold separately in the standard corporate contingent-claim models and corporate cash models. Keeping constant the belief y (resp. the cash reserves x), we find that firm value is increasing and concave in cash reserves (resp. increasing and convex in beliefs). Proposition 3 also confirms the intuition that when shareholders are increasingly confident that the drift parameter is μ, then V^* behaves as V^+.

Proposition 3 *The following holds.*

(i) For any $y \in [y^*, \mu)$, the mapping $x \longrightarrow V^*(x, y)$ is increasing and concave on $[0, \infty)$.

(ii) For any $x \in [0, \infty)$, the mapping $y \longrightarrow V^*(x, y)$ is increasing and convex on $(-\mu, \mu)$.

(iii) For any $x \in [0, \infty)$, the mapping $y \longrightarrow V^*(x, y)$ can be continuously extended at $y = \mu$ by posing $V^*(x, y) = V^+(x)$.

4 Model solution

In stochastic control theory, the route to obtaining value functions and optimal control policies involves three steps: (i) apply the dynamic programming principle to derive a free boundary problem whose solution gives a candidate value function; (ii) identify a candidate control strategy; and (iii) prove a verification theorem asserting that the candidates in (i) and (ii) are actually the optimal control and the value function. Assertion (i) requires regularity properties of the value function that are difficult to establish. In particular, in a bi-dimensional setting, a major difficulty is to prove the existence, uniqueness and regularity of the value function through the free boundary that divides the plan $\mathbb{R}^2$ into action and inaction regions. A related crucial issue is to prove the smoothness of the free boundary in $\mathbb{R}^2$. This latter property is instrumental in the construction of optimal policies. Most of the time, these technical difficulties prevent obtaining a characterization of optimal control policies. We overcome these technical difficulties and obtain a complete description of the optimal shareholders' policy. Thereby, we add to the mathematical literature a new solvable bi-dimensional Bayesian adaptative control problem that combines singular control and stopping with a value function that satisfies C^2 smooth fit across a free boundary that is neither linear nor quadratic but characterized by an ordinary differential equation.[16] The

[16]The literature on bi-dimensional control problems relies mainly on the study of leading examples. Another (scarce) solvable control problem in which the free boundary is neither linear nor quadratic is the so-called finite fuel problem introduced by Benes, Shepp and Witsenhausen (1980) and developed by Karatzas, Ocone, Wang and Zervos (2000).

characterization of the dividend boundary function allows us to explain in Section 5 how liquidity and profitability concerns are intertwined and to develop a rich set of implications.

The next section is heuristic and leads to a free boundary problem that should satisfy the value function V^*.

4.1 Heuristic discussion

We know that the value function V^* satisfies the boundary condition $V^*(0, y) = 0$ and the inequality $V^*(x, y) \geq x$ for all $(x, y) \in (0, \infty) \times (-\mu, \mu)$. We also know that $V^*(x, y) = x$ for all $y \leq y^*$ where y^* is the belief below which deep-pocketed shareholders liquidate the firm for profitability reasons. In order to proceed further, we assume in this section that V^* is as smooth as necessary, and we derive some properties that V^* should satisfy.

Dynamic programming. Let us fix some pair $(x, y) \in (0, \infty) \times (-\mu, \mu)$. Let us consider the policy that consists of abstaining from paying dividends for $t \wedge \tau_0$ units of time and, then, in applying the optimal policy associated with the resulting couple $(x + \int_0^{t\wedge\tau_0} Y_s \, ds + \sigma dW_s, y + \int_0^{t\wedge\tau_0} \frac{1}{\sigma}(\mu^2 - Y_s^2) dW_s)$. This policy must yield no more than the optimal policy:

$$\begin{aligned} 0 &\geq \mathbb{E}_{(x,y)} \left[e^{-r(t\wedge\tau_0)} V^* \left(x + \int_0^{t\wedge\tau_0} Y_s \, ds + \sigma dW_s, y + \int_0^{t\wedge\tau_0} \frac{1}{\sigma}(\mu^2 - Y_s^2) dW_s \right) \right] \\ &\quad - V^*(x, y) \\ &= \mathbb{E}_{(x,y)} \left[\int_0^{t\wedge\tau_0} e^{-rs} \left(\mathcal{A}V^*(X_s, Y_s) - rV^*(X_s, Y_s) \right) ds \right]. \end{aligned} \tag{16}$$

The last equality follows from the Itô's formula where $\mathcal{A}$ denotes the partial differential operator defined by

$$\mathcal{A}V(x, y) = \frac{1}{2\sigma^2}(\mu^2 - y^2)^2 V_{yy} + \frac{1}{2}\sigma^2 V_{xx} + (\mu^2 - y^2) V_{xy} + y V_x. \tag{17}$$

Letting t go to zero in (16) yields

$$\mathcal{A}V^*(x, y) - rV^*(x, y) \leq 0$$

for all $(x, y) \in (0, \infty) \times (-\mu, \mu)$.

Dividend boundary. Fix some $(x, y) \in (0, \infty) \times (-\mu, \mu)$. The policy that consists of making a payment $\varepsilon \in (0, x)$ and then immediately executing the optimal policy associated with cash reserves $x - \varepsilon$ must yield no more than the optimal policy:

$$V^*(x, y) \geq V^*(x - \varepsilon, y) + \varepsilon.$$

Subtracting $V^*(x-\varepsilon,y)$ from both sides of this inequality, dividing through by ε and letting ε approach 0 yields

$$V_x^*(x,y) \geq 1 \tag{18}$$

for all $(x,y) \in (0,\infty) \times (-\mu,\mu)$. Thus, from assertion (i) of Proposition 3, the inequality $V_x^*(x,y) > 1$ holds for any $x \in (0, x^*(y))$, where $x^*(y) = \inf\{x,\, V_x^*(x,y) = 1\}$. It follows that for any fixed profitability prospect y, any excess of cash above $x^*(y)$ should be paid out to shareholders. Therefore, the optimal cash policy should not be characterized by a constant threshold, as in the previous literature, but by the dividend boundary function $y \longrightarrow x^*(y)$. Finally, from Proposition 3, we expect to retrieve as a limit result the optimal level of cash reserves derived in the complete information benchmark: the dividend boundary function should satisfy $\lim\limits_{y\longrightarrow\mu} x^*(y) = x_\mu^*$.

Therefore, our heuristic discussion leads us to consider the *moving boundary problem*: find a smooth function V and a positive function b defined on $[-\mu,\mu]$ that solve the variational system

$$\begin{aligned}
b(\mu) &= x_\mu^* && (19)\\
V(0,y) &= 0 \quad \forall y \in (-\mu,\mu), && (20)\\
\mathcal{A}V(x,y) - rV(x,y) &= 0 \quad \text{on the domain } \{(x,y),\ 0<x<b(y),\ -\mu<y<\mu\}, && (21)\\
V_x(x,y) &= 1, \text{ for } x > b(y), && (22)\\
V_x(b(y),y) &= 1,\, V_{xy}(b(y),y) = 0. && (23)
\end{aligned}$$

We expect that the pair (V^*, x^*) coincides with the solution (V,b) (if any) to the system (19)-(23) on the domain $\{(x,y),\ 0 \leq x \leq b(y),\ -\mu < y < \mu\}$. Note that Equation (23) reflects the requirement that $V(.,y)$ is twice continuously differentiable at the boundary function b. In particular, differentiating equation $V_x(b(y),y) = 1$ with respect to y and using $V_{xy}(b(y),y) = 0$ leads to $V_{xx}(b(y),y) = 0$. Clearly, this does not imply that the solution to (19)-(23), if any, is of class C^2 jointly in both variables across the boundary b.

4.2 Solution to the free boundary problem (19)-(23)

We prove the existence and uniqueness of a twice continuously differentiable solution to the free moving boundary problem (19)-(23). From Equation (5), as long as the control L is not activated, the process

$$Z_t = \phi(Y_t) - X_t \tag{24}$$

is constant, which yields a one-to-one mapping between cash reserves and belief. Then, considering the change of variable (24) allows us to re-state problem (19)-(23) in the (z, y)-space and to solve it quasi-explicitly. We obtain the following result.

Proposition 4 *The free boundary problem (19)-(23) has a unique solution* (V, b).

(i) The boundary function b is defined by $b = \max(0, g)$, where g is the unique solution to the ordinary differential equation

$$g'(y) = f(g(y), y) \tag{25}$$

with the boundary condition

$$g(\mu) = x^*_\mu, \tag{26}$$

where,

$$f(x, y) \;=\; \frac{\sigma^2}{\mu^2 - y^2} \frac{yy^* + \left(\mu - r\sigma^2 \left(\frac{y^*}{\mu}\right)^2 \frac{1}{\mu}\right) \phi^{-1}\left(-\frac{\mu}{y^*}\,x\right)}{yy^* \;+\; \mu\phi^{-1}\left(-\frac{\mu}{y^*}\,x\right)} \tag{27}$$

on the domain $\{(x, y) \in [0, \infty) \times (-\mu, \mu) \,|\, x > \frac{y^*}{\mu}\phi(y\frac{y^*}{\mu})\}$, and where x^*_μ is defined in (14). Moreover, the function g is increasing, the threshold $y^{**} \equiv g^{-1}(0)$ is well defined and strictly larger than y^*.

(ii) The function V is positive, twice continuously differentiable on the domain $(0, \infty) \times (-\mu, \mu)$ and defined by the relation.

$$\begin{cases} V(0, y) = 0, \quad \forall -\mu < y < \mu, \\ V(x, y) = A\left(\phi(y) - x\right)\left(h_1(y) - e^{\frac{2}{\sigma^2}\frac{\mu^2}{y^*}(\phi(y)-x)} h_2(y)\right), \;\; \forall\, 0 \le x \le b(y), \\ V(x, y) = x - b(y) + V(b(y), y), \quad \forall\, x \ge b(y), \end{cases} \tag{28}$$

where

$$A(z) \;=\; \frac{\sigma^2}{4}\left(\frac{y^*}{\mu}\right)^2\left(\frac{1}{\mu}\right)^2\left(h_1'(k(z))e^{-\frac{2}{\sigma^2}\frac{\mu^2}{y^*}z} - h_2'(k(z))\right) \;\; \text{with} \;\; k(z) \equiv (\phi - b)^{-1}(z), \tag{29}$$

$h_2(y) = (y + \mu)^{1-\gamma}(\mu - y)^\gamma$ for any $y \in (-\mu, \mu)$, and where h_1 and y^* are defined in (11).

In the next section, we show that the function V defined in (25)-(29) coincides with the optimal value function V^* for problem (8). We can see Equation (25)-(29) as a generalization of assertion (i) of Proposition 2 to an incomplete information setting. It is important to stress the inequality $y^{**} \equiv g^{-1}(0) > y^*$ in assertion (i) of Proposition 4. As we will see, this implies that liquidation never occurs at the first-best level y^*.

4.3 Solution to the shareholders' problem

We prove that the function V solution to (19)-(23) coincides with the optimal value function V^* for problem (8). First, we show that V is an upper bound for V^* (this result is stated and proven in the appendix). Then, we construct an admissible policy, the value of which coincides with V. This, in turn, establishes that $V^* = V$, and thereby provides the optimal dividend policy.

Our estimation is that the optimal cash reserves process is reflected along the free boundary function b in a horizontal direction in the (x, y)-plan. We formalize this using a 2-dimensional version to Skorohod's lemma established by Burdzy and Toby (1995). Specifically, there exists a unique continuous process $\{L^{b(.)};\ t \geq 0\}$ defined on $(\Omega, \mathcal{F}^R, \mathbb{P})$ such that for $\mathbb{P}$-$a.e.$ $\omega \in \Omega$,

- $$(\phi(Y_t(\omega)) - \phi(y) + x - L_t^{b(.)}(\omega), Y_t(\omega)) \in [0, b(y)] \times [y^{**}, \mu), \quad \forall t \in [0, \tau_0], \tag{30}$$

  where $\tau_0 = \inf\{t \geq 0 \,|\, \phi(Y_t) - \phi(y) + x - L_t^{b(.)} = 0\}$,

- $$L_0^{b(.)}(\omega) = 0, \text{ and } t \longrightarrow L_t^{b(.)}(\omega) \text{ is nondecreasing on}$$
  $$\{t \geq 0 \,:\, \phi(Y_t) - \phi(y) + x - L_t^{b(.)} = b(Y_t)\}, \tag{31}$$

- $$t \longrightarrow L_t^{b(.)}(\omega) \text{ is constant on}$$
  $$\{t \geq 0 \,:\, (\phi(Y_t(\omega)) - \phi(y) + x - L_t^{b(.)}(\omega), Y_t(\omega)) \in (0, b(y)) \times (y^{**}, \mu)\}. \tag{32}$$

Conditions (30)-(32) ensure that the policy $L^{b(.)}$ is admissible such that the process

$$X_t^{b(.)} \equiv \phi(Y_t) - \phi(y) + x - L_t^{b(.)} \tag{33}$$

is reflected in a horizontal direction whenever $X_t^{b(.)} = b(Y_t)$. Bringing together our results, we obtain the following Theorem.

Theorem 1 *The following holds:*

(i) **(Firm value)** *The value function V^* for (8) coincides with the unique solution V to the free moving boundary problem (19)-(23) and is given by equations (25)–(29).*

(ii) **(Optimal policy)** *Let $L^{b(.)}$ be the solution to the Skorohod's problem (30)-(32). The process $L^* = \{L_t^*; t \geq 0\}$ is defined by*

$$L_t^* = \left((x - b(y))^+ \mathbb{1}_{y \geq y^{**}} + x \mathbb{1}_{y \leq y^{**}}\right) \mathbb{1}_{t=0} + L_t^{b(.)} \mathbb{1}_{t>0}, \tag{34}$$

is an optimal policy for problem (8).

Equation (34) defines the optimal cash management policy of shareholders who face profitability and liquidity concerns and thus generalizes the policy (15) to an incomplete information setting. A striking difference from the complete information case is that the firm's cash target level is not a constant but a function of the belief about the firm's profitability. Below a critical level of belief, the project is not undertaken. We describe the optimal cash management of the firm as follows:

- If cash reserves/belief (x, y) satisfies $y < y^{**}$, then whatever the amount of cash x, shareholders do not run the project and we have $V(x, y) = x$ for all $y \leq y^{**}$ and for all $x \geq 0$.
- If cash reserves/belief (x, y) satisfies $x > b(y)$, then the amount $x - b(y)$ is paid out to shareholders.
- If cash reserves/belief (x, y) satisfies $0 < x < b(y)$, then no cash is paid out.

5 Model implications

Theorem 1 explains how the trade-off between the costs and benefits of cash retention evolves when shareholders learn about profitability prospects. It provides novel predictions as well as a theoretical underpinning for a series of documented regularities. Furthermore, the quasi-explicit nature of the model solution allows simple numerical illustrations that yield additional results.

5.1 Cash target levels and profitability prospects

Theorem 1 directly yields two novel implications for the link between corporate cash policy and profitability prospects.

- *There is a positive relationship between cash target levels and profitability prospects.*
- *Once we condition on the level of cash reserves, profitability prospects are higher if the firm previously reached its cash target levels several times.*

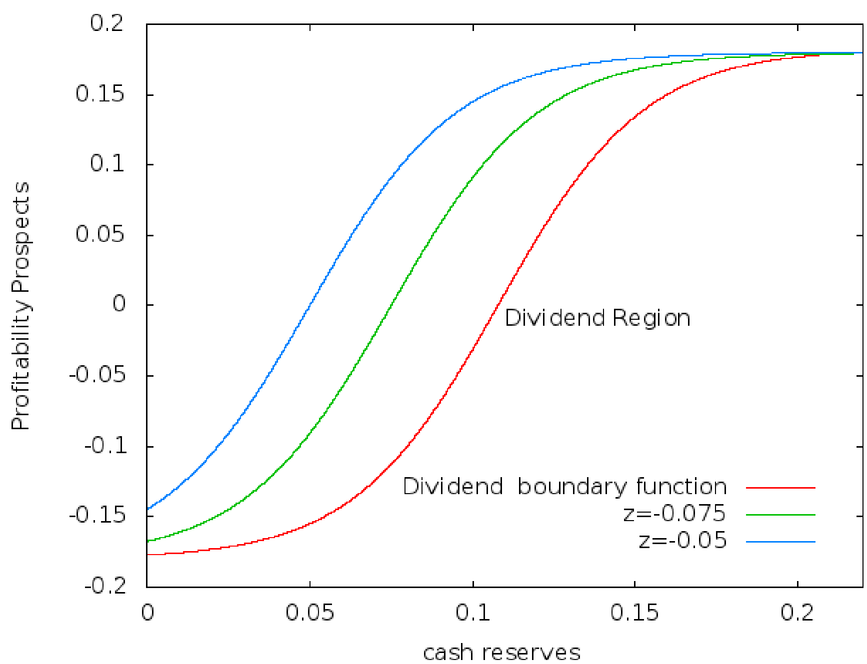
In corporate cash models with known profitability, the optimal amount of cash reserves is a constant boundary. In our study, the uncertainty about the firm's profitability impacts the corporate cash policy, which, in terms of cash target levels, changes as the firm learns about its profitability. Proposition 4 shows that the dividend boundary function b is increasing. Therefore, the model predicts a positive relationship between cash target levels and profitability prospects. Intuitively, a firm has more to lose from liquidity constraints when profitability prospects are high than it does when they are low. This leads shareholders to accumulate more cash when their belief about the firm's profitability becomes more positive. A numerical implementation of the model yields more insights on this property below.

Our model exhibits a path-dependence property of the shareholders' belief about the firm's profitability and predicts that beliefs are higher as the firm reaches its cash target levels several times. To see this point, consider (33) and define $Z_t \equiv \phi(Y_t) - X_t^{b(.)} = L_t^{b(.)} + z$ with $z \in (\phi(y^{**}), \infty)]$. The process $\{Z_t;\, t \geq 0\}$ is increasing and takes a new value each time the firm reaches its cash target level, that is, whenever $X_t^{b(.)} = b(Y_t)$. For a given realization z of $\{Z_t;\, t \geq 0\}$, any cash reserves/belief (x, y) satisfies the relation $\phi(y) - x = z$. Thus, keeping x constant, the higher the value of z is, the higher the shareholders' beliefs y will be. Hence, dividends are distributed each time the total flow of cash into the firm reaches a new maximum value that depends on the current belief. At that time, cash is paid out, the firm increases its cash target level, and each amount of cash reserves is associated with a more positive belief after paying out cash.

We interpret a realization z of $\{Z_t;\, t \geq 0\}$ as a measure of the strength of shareholders' beliefs about the quality of the firm's project. Indeed, for a given $z \in (\phi(y^{**}), \infty)]$, the shareholders' estimates about the firm's profitability evolve in the interval $[\phi^{-1}(z), k(z)]$, where $k(z) = (\phi - b)^{-1}(z)$ as defined in (29). The lower bound $\phi^{-1}(z)$ corresponds to the belief about the firm's profitability when cash reserves are depleted. It is the lowest belief that shareholders may have about the profitability of the firm. The upper bound $k(z)$ corresponds to the belief about the firm's profitability Y at the cash target level $x_z \equiv b(k(z))$. The functions ϕ^{-1} and k are increasing and tend to μ when z tends to ∞. When the firm reaches its cash target levels several times, shareholders' beliefs become more positive, and the range over which beliefs evolve shrinks, as illustrated in Table 1 below. It follows that the conditional variance $\mathrm{Var}[Y|\mathcal{F}_t^R] = \mu^2 - Y_t^2$, is low for a high realization of the process $\{Z_t;\, t \geq 0\}$. We shall refer to $\mathrm{Var}[Y|\mathcal{F}_t^R]$ as the risk of profitability. Thus, after a series of successes, the process $\{Z_t;\, t \geq 0\}$ increases, and the firm becomes well-established in the sense that shareholders' beliefs are strongly positive and remain high for all levels of cash flows. In particular, shareholders of a well-established firm do not infer from high cash flows that growth will be much higher in the future; similarly, they do not infer from low cash flows a sharp decline in profitability prospects. These observations motivate the following definition.

Definition *The process* $\{Z_t;\, t \geq 0\}$ *with* $Z_t = L_t^{b(.)} + z$ *defines the strength of shareholders' beliefs about the quality of the firm's project. We define the firm quality index as a realization of the process* $\{Z_t;\, t \geq 0\}$, *and the risk of profitability as the conditional variance* $Var[Y|\mathcal{F}_t^R]$. *Accordingly, for a given level of cash reserves, the higher the quality index is, the lower the risk of profitability is. A well-established firm is a firm with a high quality index.*

A numerical implementation of the model illustrates the above definition and gives additional insights on the positive relationship between cash holdings and profitability prospects. Figure 1 plots in the (x, y)-plan the dividend boundary function b and the curves $z = \phi(y) - x$ for $z = -0.075$ and $z = -0.05$. We consider the baseline parameter values $r = 0.06$, $\sigma = 0.09$ and $\mu = 0.18$.[17] If shareholders know that $\mu = 0.18$, they will target cash level $x_\mu^* = 0.2176$. In our bi-dimensional setting, information about profitability is incomplete, and each quality index z defines a cash target level $x_z \equiv b(k(z))$, the set of which forms the dividend boundary function. Shareholders agree to run the project if the expected losses do not exceed $y^{**} = -0.1771$. Table 1 complements Figure 1 and gives for a given index z the values of lower and upper bounds of shareholders' belief ($\phi^{-1}(z)$ and $k(z)$) and the associated cash target level x_z.



[17] To ease comparisons, we use the baseline parameters considered in studies in which shareholders face only a liquidity concern; see, for instance, Bolton, Chen and Wang (2011). The parameters r, μ and σ are annualized, σ and μ are expressed in millions of dollars.

Figure 1. The dividend boundary function is in red. The curves $z = \phi(y) - x$ are for the quality indexes $z = -0.075$ and $z = -0.05$. The parameters are $r = 0.06$, $\sigma = 0.09$, $\mu = 0.18$. For those parameters, $x_\mu^* = 0.2176$ and $y^{**} = -0.1771$.

z	$\phi^{-1}(z)$	$k(z)$	x_z
-0.105	-0.1766	0.1591	0.1677
-0.05	-0.144	0.1797	0.2157
0	0	0.1799	0.2172
0.05	0.1448	0.1799	0.2173
0.105	0.1766	0.1799	0.2175

Table 1. For a given quality index z, $\phi^{-1}(z)$ is the lower bound of the shareholders' belief, $k(z)$ is the upper bound of the shareholders' belief, and x_z is the cash target level. The parameters are $\mu = 0.18$, $r = 0.06$, $\sigma = 0.09$. The dividend boundary in the complete information case is $x_\mu^* = 0.2176$.

We obtain the following implications.

- Well-established firms have been successful and have paid dividends several times in the past.
- Well-established firms as well as firms with a high risk of profitability can have high and similar cash target levels.

The path dependence property of beliefs implies that a firm becomes well-established after having established sufficiently high performance records. Thus, well-established firms have paid out dividends several times in the past and have a high level of cash reserves. Our numerical implementation shows that well-established firms and firms that are more uncertain about the relevance of their projects can have similarly high cash target levels. Specifically, in Table 1, the quality indexes $z = -0.05$ $z = 0$, $z = 0.05$, $z = 0.105$ yield almost the same cash target levels, which are close to x_μ^*, the dividend boundary of the complete information case. Nevertheless, the support of the distribution of beliefs remains large for $z = -0.05$ and $z = 0$, whereas it shrinks for $z = 0.105$. The model suggests that firms learning about the relevance of their projects build cash reserves. Cash target levels become high but do not allow one to assess the strength of shareholders' beliefs about the firm's profitability.

5.2 Cash flow sensitivity of dividends

We push the analysis further and provide new insight on how much cash firms pay out (relative to their earnings). We show that well-established firms pay large dividends (relative to their earnings), and have a stable dividend policy, whereas firms with a high risk of profitability pay low dividends and have a dividend policy that is more sensitive to their performance records.

In previous corporate cash models, the amount of dividends per unit of cash available for distribution is constant. In particular, all models with only liquidity issues predicted that at the cash target level, 100% of cash available for distribution is paid to shareholders. Our bi-dimensional model generates more realistic behaviors because the amount of dividends per unit of cash available for distribution depends on the belief about the firm's profitability through an explicit measure, which we refer to as *the cash flow sensitivity of dividends*.[18]

Specifically, suppose that the firm is at its target level of cash $x = b(y)$ and consider what happens after a realization of a shock on the cash flow. To account for the sign of the change in cash flow over a small period of time h, we consider a $\sqrt{h}$ Euler approximation of the model,

$$X_h = x + \sigma\sqrt{h}W_1,$$

and

$$Y_h = y + \frac{\mu^2 - y^2}{\sigma}\sqrt{h}W_1,$$

where W_1 is a standard Gaussian variable. Therefore, the cash available for distribution at time h is

$$\begin{aligned} X_h - b(Y_h) &= X_h - (b(y) + b'(y)(Y_h - y)) \\ &= X_h - x - b'(y)\frac{\mu^2 - y^2}{\sigma}\sqrt{h}W_1 \\ &= \varepsilon(y)(X_h - x), \end{aligned}$$

with

$$\varepsilon(y) = \left(1 - b'(y)\frac{\mu^2 - y^2}{\sigma^2}\right) = \frac{r\sigma^2\left(\frac{y^*}{\mu}\right)^2 \frac{1}{\mu}\phi^{-1}\left(-\frac{\mu}{y^*}b(y)\right)}{yy^* + \mu\phi^{-1}\left(-\frac{\mu}{y^*}b(y)\right)},$$

[18]We borrow the terminology of Almeida, Campello and Weisbach (2004), who define *the cash flow sensitivity of cash* as the firm's propensity to save cash out from cash flows. In Almeida, Campello and Weisbach (2004), the cash flow sensitivity of cash provides a measure of the effect of financial constraints on corporate policies.

where the last equality follows from (27). Thus, if at the cash target level, the variation of earnings $X_h - x$ is positive, then the amount of dividends paid to shareholders is $X_h - b(Y_h)$ such that the function ϵ represents the (target) dividend payout ratio of the firm. The definition is below.

Definition *The cash flow sensitivity of dividends is defined by the function ϵ, where, for all $y \in [y^{**}, \mu]$,*

$$\varepsilon(y) = \frac{r\sigma^2 \left(\frac{y^*}{\mu}\right)^2 \frac{1}{\mu}\phi^{-1}\left(-\frac{\mu}{y^*}b(y)\right)}{yy^* \, + \, \mu\phi^{-1}\left(-\frac{\mu}{y^*}\, b(y)\right)}. \tag{35}$$

The function ϵ models the dividend payout ratio of the firm. It measures how much cash firms pay out relative to their earnings. The function ϵ is continuously increasing and maps $[y^{**}, \mu]$ into $[0, 1]$.

In the complete information benchmark, when $y = \mu$, Equation (35) yields $\epsilon(\mu) = 1$. That is, in the absence of uncertainty about the project's profitability, 100% of cash available for distribution is paid to shareholders. Figure 2 and Table 2 provide numerical implementations of (35).

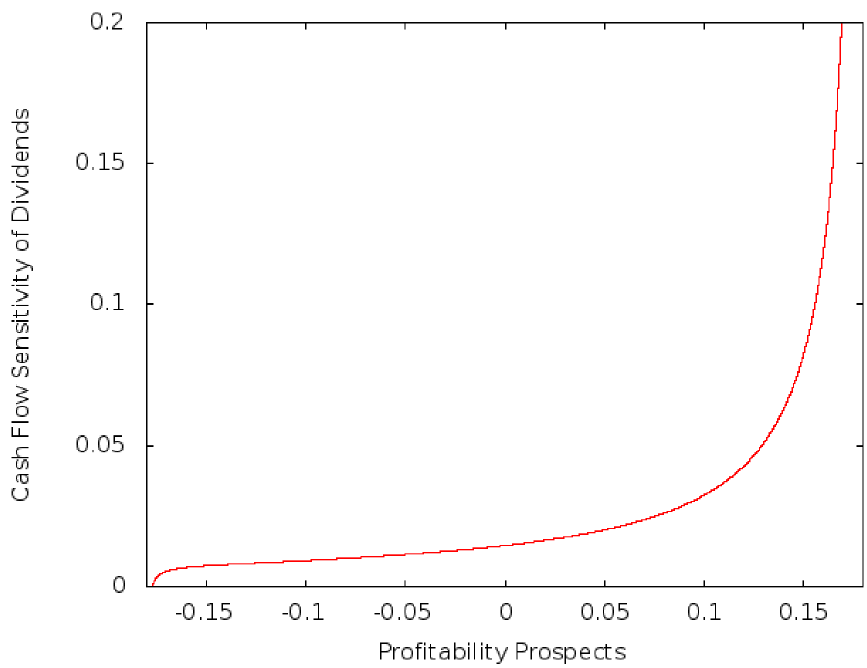


Figure 2. The cash flow sensitivity of dividends $\epsilon(.)$. The parameters are $r = 0.06$, $\sigma = 0.09$, $y^{**} = -0.1771$, $\mu = 0.18$.

z	x_z	$k(z)$	$\epsilon(k(z))$
-0.105	0.1677	0.1591	0.1137
-0.102	0.1832	0.1705	0.2204
-0.1	0.1891	0.1732	0.2853
-0.05	0.2157	0.1797	0.9218
0	0.2172	0.1799	0.9915
0.05	0.2172	0.1799	0.9991
0.105	0.217	0.1799	0.9999

Table 2. For a given quality index z, $\epsilon(k(z))$ is the percentage of cash available for distribution when target cash level is x_z and associated shareholders' belief is $k(z)$. The parameters are $\mu = 0.18$, $r = 0.06$, $\sigma = 0.09$.

Numerical implementations suggest that firms with a low quality index z have a low target cash level and pay virtually no dividends. Intuitively, profitability prospects are bad, but not bad enough to liquidate the project. It is optimal to target a low level of cash reserves and to reinvest available cash in the firm in the hope that the project is actually a good project. When the quality index improves, target cash levels and target dividend payout ratios increase. Cash target levels can be high, and payout ratios can be dramatically different. This point is illustrated in Table 2. At target cash level $x_z = 0.1891$, shareholders receive 28.53% of cash available for distribution. Instead, at target cash level $x_z = 0.2157$, shareholders receive 92.18% of cash available for distribution. Table 2 also illustrates that the same increase in the quality index has less of an impact on the dividend policy of well-established firms than on the dividend policy of firms with a high risk of profitability. Specifically, when the quality index increases from $z = -0.105$ to $z = -0.05$, dividend payments, as a percentage of earnings, increase from 11.37% to 92.18%. However, an increase in the quality index from $z = 0.05$ to $z = 0.105$ does not impact the dividend policy, which remains stable at a high percentage of available earnings. Below, we relate this finding to the so-called smoothing of dividends first documented by Lintner (1956), who stresses that firms are primary concerned with the stability of dividends. Finally, the increase of ϵ in the beliefs together with the path-dependence property of beliefs imply that firms should rarely reduce the amount of their dividends. In summary, we obtain the following implications:

- *There is a positive relationship between profitability prospects and dividend payout ratios.*

- *Well-established firms pay high dividends and have a stable dividend policy.*
- *Firms with a high risk of profitability pay low dividends and have a less stable dividend policy than well-established firms do.*
- *Firms should rarely reduce their dividends.*

Thus, integrating learning on profitability with the free cash flow retention problem allows the explanation of several patterns in the payout policies of corporations. The trade-off between the cost and benefit of cash retention evolves as time passes, implying changes in dividend policies. The driver of this evolution is knowledge about the firm's profitability. As shareholders become confident about the relevance of the firm's project, they increase cash target levels because they have more to lose from liquidity constraints. Cash targets can be high, while the risk of profitability remains large and dividends correspond to a small fraction of available cash. After having established a sufficiently high performance record, the firm becomes well established and the risk of profitability declines. Shareholders do not expect changes in the firm's growth rate. Cash target levels remain high, the free cash flow problem is more severe, and payouts, as a proportion of available earnings, are large and stable. A measure of the strength of beliefs about a given firm can result from the number of analysts following the firm and/or the dispersion of the analysts' opinions; the greater the number of analysts there are and the less dispersed their opinions are, the stronger the strength of the beliefs is. Accordingly, we expect that analysts' should agree and have positive opinions of firms having a high level of cash reserves and paying a large amount of dividends. Analysts' opinions should be more dispersed for firms having high levels of cash reserves but paying low dividends.

These findings are consistent with Leary and Michaely (2011), who document that dividend smoothing is more prevalent for mature firms that hold high level of cash and pay high dividend levels, whereas firms facing the most uncertainty and that have a more disperse analyst forecasts smooth less.[19] Grullon, Michaely and Swaminathan (2002) document that a firm that becomes mature experiences a decline in risk and increases dividends. In these empirical studies, the underlying idea is that the free cash flow problem evolves as time goes by and causes changes in dividend policies.[20] Our learning model formalizes this idea and shows that corporate cash policy is an integral part of the process that leads a firm to

[19]There is a large volume of literature on dividend smoothing, but no consensus about why firms smooth dividends. Leary and Michaely (2011) discuss the theoretical attempts for explaining stability of dividend policy and explain that their findings challenge existing theories.

[20]This relates to the life-cycle hypothesis for dividends developed by DeAngelo, DeAngelo and Stulz (2006).

becoming well-established with low uncertainty about its growth rate and demonstrating a stable dividend policy. Finally, the fact that dividends do not decrease is consistent with the managers' reluctance to reduce dividends and their preference to raise them conservatively in response to earnings, as documented, for instance, by Brav, Graham, Harvey and Michaely (2005). A classical explanation for this phenomenon is that maintaining dividends signals to shareholders the firm's financial health. In our setting, the mechanism that leads to this result is different and relies on the interaction between knowledge about profitability and the trade-off cost versus benefit of holding cash. Thus, our model provides a theoretical underpinning for recently documented regularities, as well as a unified explanation of a series of stylized facts that are usually explained using different theories. The next section studies the consequences of our findings on the dynamics of the firm value.

5.3 Firm dynamics

We saw in the previous section that for a given index $z \in [\phi(y^{**}), \infty)$, payment occurs at time $\tau_z \equiv \{t \geq 0 \,|\, X_t^{b(.)} = x_z\}$ with $x_z = b(k(z))$, and liquidation at time $\tau_0 = \{t \geq 0 \,|\, X_t^{b(.)} = 0\}$. We deduce that the firm value process $\{V_t^*,\ t \geq 0\}$ with $V_t^* \equiv V^*(X_t^{b(.)}, Y_t)$ can be written on time interval $[0, \tau_0 \wedge \tau_z]$ as a function of the cash reserves process $\{X_t^{b(.)},\ t \geq 0\}$. The following holds:

Proposition 5 *For a given index $z \in [\phi(y^{**}), \infty)$, the firm value process $\{V_t^*,\ t \geq 0\}$ satisfies, for any $t \in [0, \tau_0 \wedge \tau_z]$, the relation*

$$V_t^* = V_{[z]}^*(X_t^{b(.)})$$

where

$$V_{[z]}^*(x) \equiv V^*(x, \phi^{-1}(x+z)) = A(z)(h_1(\phi^{-1}(x+z)) - e^{\frac{2}{\sigma^2}\frac{\mu^2}{y^*}z} h_2(\phi^{-1}(x+z))). \tag{36}$$

For each $z \in [\phi(y^{**}), \infty)$ and $t \in [0, \tau_0 \wedge \tau_z]$, the firm value dynamics are given by

$$dV_{[z]}^*(X_t^{b(.)}) = rV_{[z]}^*(X_t^{b(.)})\, dt \ + \ \sigma\, V_{[z]}^{*\prime}(X_t^{b(.)})\, dW_t. \tag{37}$$

Moreover,

(i) For any $z \in [\phi(y^{**}), \infty)$, the mapping $x \longrightarrow V_{[z]}^*(x)$ is increasing on $x \in [0, x_z]$.

- *For any $z \in [\phi(y^{**}), 0)$, the mapping $x \longrightarrow V_{[z]}^*(x)$ is convex on $x \in [0, z]$,*
- *For large quality index z, the mapping $x \longrightarrow V_{[z]}^*(x)$ is concave on $x \in [0, x_z]$.*

(ii) For any $x \in [0, \infty)$, the mapping $z \longrightarrow V^*_{[z]}(x)$ is increasing on $[\phi(y^{**}), \infty)$.

Proposition 5 relates firm value[21] and strength of shareholders' beliefs about the quality of the firm's project. For a given quality index z, the model is unidimensional, and the firm value is an increasing function of the level of its cash reserves. Assertion (i) emphasizes that this function is neither concave nor convex. This is a unique feature of our analysis. In models with only a liquidity concern (such as Bolton, Chen and Wang (2011) or DMRV (2011)), firm value is concave in its cash reserves. The concavity property holds in our setting for sufficiently well-established firms. Intuitively, the risk of profitability is less pronounced for these firms, so their value reacts less to changes in cash flows when past performance has been good. However, when the quality index is too low, profitability issues dominate, and we find that firm value can be convex in its cash reserves. Intuitively, when cash reserves and beliefs are low, the value of the firm does not react very much to increases in cash flows. Still, increases in cash flows increase the beliefs about profitability and the firm's value becomes more sensitive to subsequent increases in cash flows. Figure 2 plots the firm value as a function of its cash reserves for the indexes $z = -0.1$ and $z = 0$. It shows a change in the concavity of firm value for the index $z = -0.1$. Figure 2 also illustrates that the firm value is increasing in the quality index as stated in Proposition 5. Proposition 5 leads to the following testable implications.

- *Keeping the level of liquid assets constant, the more often the firm has paid dividends, the higher the firm's equity value.*
- *Firms with lower equity values pay less in dividends.*
- *Dividend increases yield an increase in the firm's equity value.*

The first implication is new and follows from assertion (ii) of Proposition 5 together with the path dependence property of beliefs. It interprets the level of cash reserves as the amount of the firm's liquid assets. The second and third implications are well documented in the literature. In our setting, they follow from Proposition 5 together with the analysis of the cash flow sensitivity of dividends. The fact that firms with lower equity values pay less in dividends is consistent with Leary and Michaely (2011). The positive relationship between dividend increases and the firm's equity value is often associated with the signaling theory. As we already emphasized, the mechanism that leads to these results is different in our model.

[21] Equivalently, in our simple model, the firm's equity value.

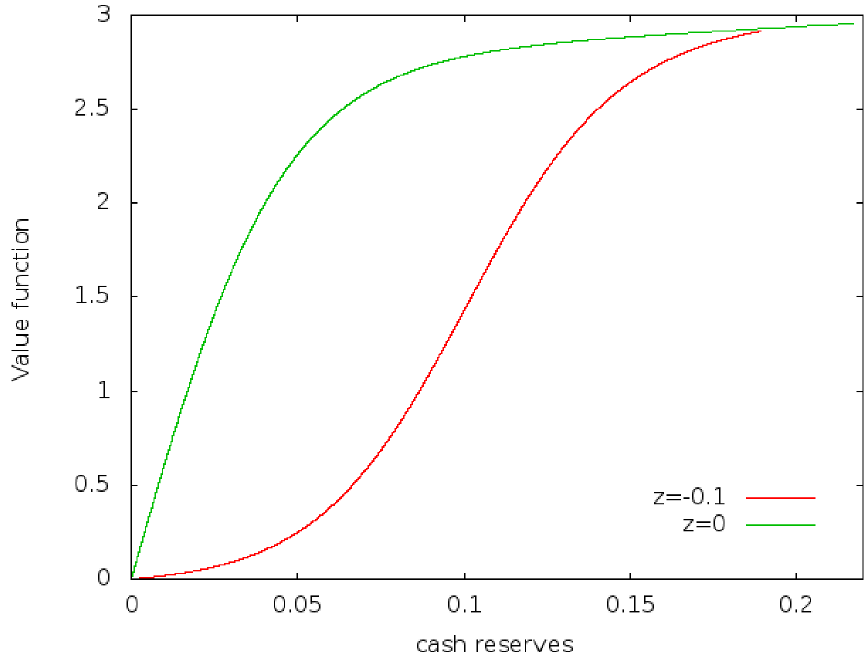


Figure 2. The firm value as a function of the cash reserves for indexes $z = -0.1$ and $z = 0$. The parameters are $r = 0.06$, $\sigma = 0.09$, $\mu = 0.18$.

We deduce from (37) that the features of the volatility of firm value as a function of the level of cash reserves relies on the concavity properties of $V^*_{[z]}$. In the corporate cash literature with known profitability, the firm value is increasing and concave in the cash flow, which implies a negative relationship between firm volatility and the level of cash reserves. In addition, on dividend payment dates, the volatility of the firm coincides with the volatility of cash flows. Considering profitability issues modifies these results. Together with assertion (i) of Proposition 5, Equation (37) implies a non-monotone relationship between firm's volatility and cash reserves. Furthermore, an easy computation shows that $\sigma V'_{[z]}(x_z) = \sigma + V^*_y(x_z, k(z))\dfrac{\mu^2 - k(z)^2}{\sigma}$, which implies that the volatility of the firm is larger than the volatility σ of cash flows at dividend payment dates. Proposition 5, together with a numerical implementation of the model (see Figures 4 and 5), yield the predictions.

- *The firm's volatility is an inverse U-shaped function of the cash reserves when the risk of profitability is large, and it becomes a decreasing function of the cash reserves as the risk of profitability declines.*

- *On dividend payment dates, the strength of shareholders' beliefs about the firm's profitability explains the difference between the volatility of the firm and the volatility of* *cash flows.*

Thus, Figures 4 and 5 illustrate that features of the volatilities of firms depend on the linkage between profitability and liquidity issues. Intuitively, when a firm with a low quality index is in a situation of cash crisis, an increase in earnings challenges the belief that the project is bad. This leads to an increase in the volatility of the firm. If the firm overcomes the cash crisis, cash reserves increase, liquidity concerns progressively prevail over profitability concerns, and the volatility of the firm becomes decreasing in the cash reserves. For well-established firms, liquidity issues always dominate over profitability issues and the volatility of the firm is always decreasing in the level of cash reserves.

Firms with a low quality index are more likely to default. Our numerical implementation suggests that the volatility of bankrupt firms can be extraordinarily high (110% for $z = -0.02$), which is consistent with the literature on corporate bankruptcy. For instance, Campbell Hilscher and Szilagyi (2008) document a mean annualized volatility of bankrupt firms of 106%, a mean profitability (measured as the net income over the market value of total assets) that is slightly negative, and an average liquid position of 0.044, which is too weak to postpone bankruptcy.[22]

The next section discusses the impact of an increase of volatility parameter σ on the firm's value and on corporate cash policy.

[22]The firm's liquid position is measured as the stock of cash and short-term investments over the market value of total assets. In our model, it corresponds (at default) to $\lim_{x \longrightarrow 0} \frac{x}{V_{[z]}(x)}$, which is equal to 0.028 when $z = -0.02$.

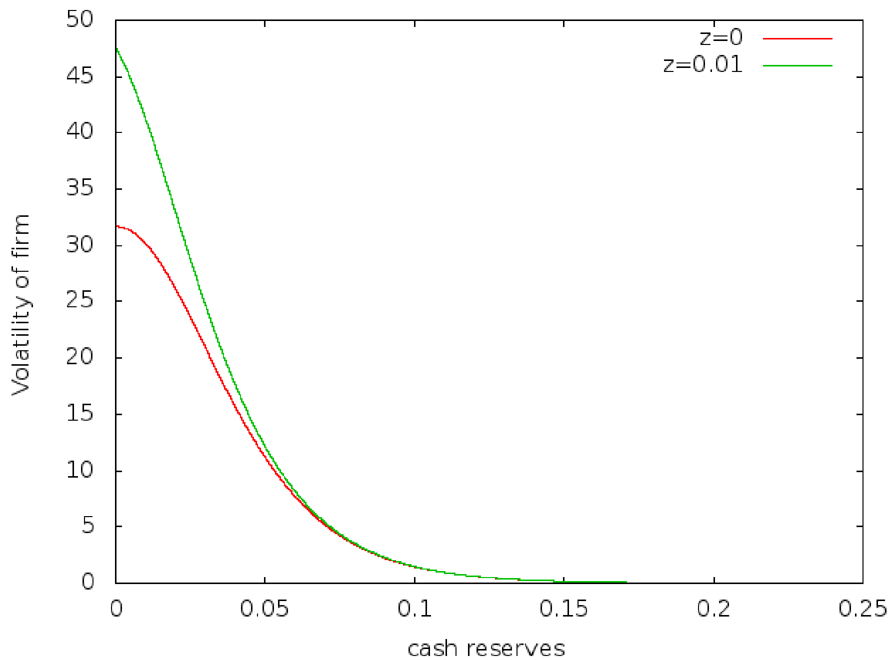


Figure 4. The volatility of firm as a function as a function of the cash reserves for indexes $z = 0$ and $z = 0.01$. The parameters are $r = 0.06$, $\sigma = 0.09$, $\mu = 0.18$.

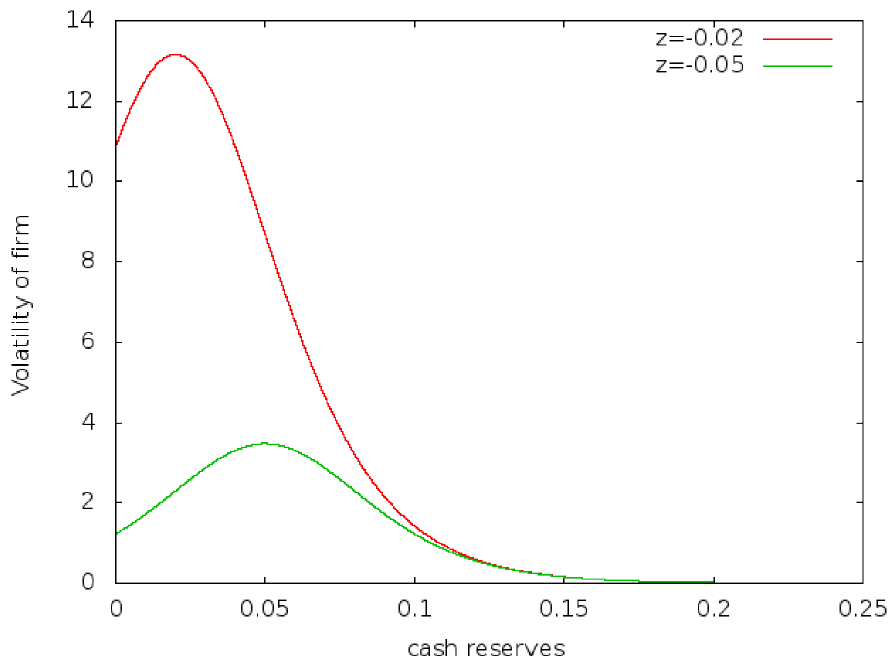


Figure 5. The volatility of firm as a function as a function of the cash reserves for indexes $z = -0.02$ and $z = -0.05$. The parameters are $r = 0.06$, $\sigma = 0.09$, $\mu = 0.18$.

5.4 Risk of liquidity

The volatility of the cash flow σ provides a measure of the risk of liquidity. In models in which the firm's profitability is known, the firm value is decreasing in the volatility of the cash flow σ, whereas the dividend boundary increases in σ.[23] We show that uncertainty about the firm's profitability impacts this latter result.

Proposition 6 *The following holds:*

(i) The firm value V^* is decreasing in the volatility σ.

(ii) The dividend boundary function b is non-monotone in the volatility σ.

Intuitively, a positive shock on σ has two consequences: it increases the risk of a negative liquidity shock, and it impedes learning. This latter point is consistent with Moyen and Platinakov (2012), who stress that firms in volatile economic environments have more difficulty in learning about their long-run prospects. Each of these two effects suggests that the firm's value is decreasing in the volatility σ. Two opposite effects explain the impact of a positive shock on σ on the dividend boundary b. On the one hand, the increase in the risk of a negative liquidity shock leads shareholders to increase corporate cash savings; on the other hand, because learning is more difficult and because holding cash is costly, shareholders may be less prone to incur the cost of accumulating cash following a positive shock on σ. This latter effect dominates when the strength of shareholders' beliefs about the relevance of the project is weak. Formally, an increase in σ increases both the first-best liquidation threshold y^* and the constant dividend boundary x^*_μ. Because the function b is continuous in the beliefs, it is clearly possible to find $\sigma_1 < \sigma_2$ such that the curves b_{σ_1} and b_{σ_2} cross, as shown in Figure 6.[24] Proposition 6 yields the prediction

- *Firms with weak profitability prospects and a high volatility of cash flow accumulate less cash than do firms with a lower volatility of cash flow and otherwise similar characteristics. We should observe the opposite for well-established firms.*

Our analysis provides new insights into the impact of the volatility of cash flow on risk management policies. A risk management policy that decreases the volatility of cash flow is value enhancing. This result holds regardless of the profitability prospects. However, our model suggests that such a policy can have opposite effects in terms of cash management. If beliefs

[23] See, for instance, DMRV (2011).

[24] For clarity, we index the function b by the volatility parameter σ.

about the firm's profitability are low, reducing the volatility of the cash flow improves learning, which leads the firm to increase its cash target level. If beliefs about profitability are not too low, reducing the volatility of cash flows leads the firm to reduce its cash target level.

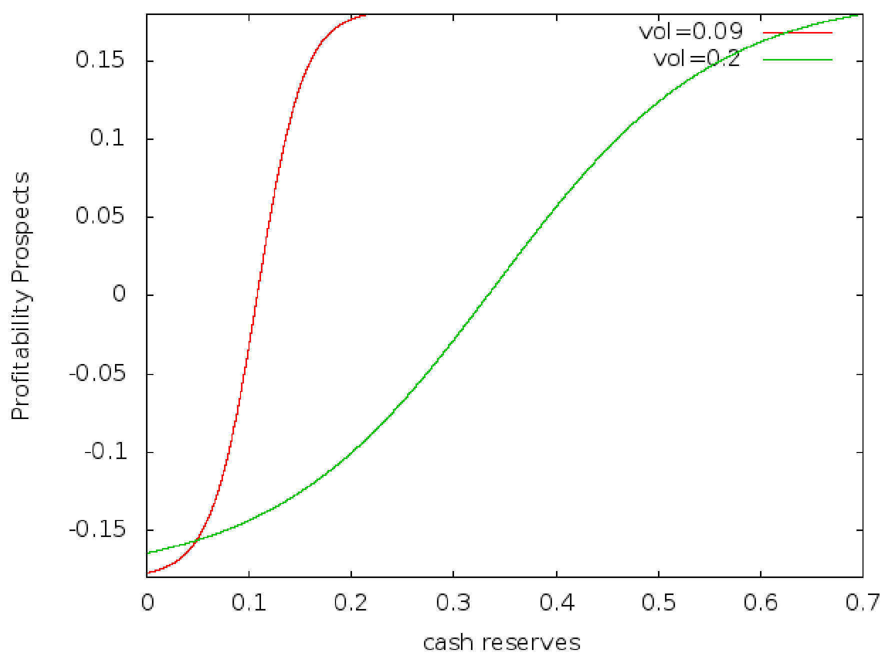


Figure 6. The dividend boundary functions for volatilities of cash flow $\sigma_1 = 0.09$ and $\sigma_2 = 0.2$. The parameters are $r = 0.06$, $\mu = 0.18$. With these parameters, we have $y^*_{\sigma_1} = -0.1773$, $y^*_{\sigma_2} = -0.1679$, $x^*_{\mu,\sigma_1} = 0.2176$ and $x^*_{\mu,\sigma_2} = 0.6980$.

We conclude this section by further discussing the impact of the cost of keeping cash inside the firm on liquidation and dividend policies.

5.5 Cost of carrying cash, liquidation and dividend policies

In our framework, liquidation occurs when the firm's cash reserves fall to 0. The intuition is that in absence of exogenous shocks, shareholders anticipate everything, and, optimally, the firm holds no more cash at the liquidation date. Two remarks are worth emphasizing. First, a cash crisis can trigger liquidation, whereas the risk of profitability is very low. Second, because $y^{**} = b^{-1}(0) > y^*$, liquidation is never first-best. The threshold y^{**} corresponds to the minimum profitability prospects required by shareholders to run the project. In particular, a negative exogenous shock that leads the belief about the firm's profitability below y^{**} triggers liquidation even if cash reserves are abundant. Intuitively, the threshold y^{**} is strictly larger than the first-best liquidation threshold y^* because shareholders require higher profitability to incur the cost of holding cash. Because we aimed at obtaining a quasi-explicit formula, we considered in our study that cash is not remunerated at all inside the firm. Thus, we considered the largest possible cost of carrying cash. Still, a numerical implementation of the model[25] highlights a small gap between the thresholds $y^* = -0.1773$ and $y^{**} = -0.1771$, suggesting that the cost of holding cash does not greatly impact the first-best liquidation threshold. The picture is dramatically different for dividend policies, as shown in the next Proposition.

Proposition 7 *Assume that there is no cost of keeping cash inside the firm such that the cash reserves process evolves according to*

$$dX_t = rX_t dt \,+\, dR_t \,-\, dL_t.$$

(i) (Firm value) *The value function V^* for problem (8) satisfies*

$$V^*(x,y) = \mathbb{E}_{(x,y)}[e^{-r(\tau_0(0)\wedge\tau^*)} X_{\tau_0(0)\wedge\tau^*}(0)] = x + \mathbb{E}_{x,y}\left[\int_0^{\tau_0(0)\wedge\tau^*} e^{-rs} Y_s \, ds\right]$$

, where $X(0)$ denotes the cash reserves process associated with policy $L = 0$, $\tau_0(0) = \inf\{t \geq 0 \,|\, X_t(0) = 0\}$ and $\tau^* = \inf\{t \geq 0 \,|\, Y_t = y^*\}$.

(ii) (Optimal policy)

(1) Paying no dividends and liquidating at $\tau_0(0) \wedge \tau^*$ is an optimal policy for problem (8).

(2) If the initial cash reserves/ belief couple (x, y) satisfies $x \geq \phi(y) - \phi(y^*)$, then any payout policy L such that $X_t(L) \geq \phi(Y_t) - \phi(y^*)$ is optimal and yields the first-best benchmark $V^*(x,y) = \hat{V}(x,y)$.

In other words, if shareholders face no cost of holding cash, they accumulate cash up to a critical curve defined by the relation $x = c(y) \equiv \phi(y) - \phi(y^*)$, beyond which inefficient liquidation can be avoided. The cash policy is indeterminate in the sense that any payout policy that maintains the level of cash beyond the critical curve c allows liquidation to be triggered at the first-best level y^*.[26] In particular, if at date $t = 0$, the cash reserves/belief

[25] We use the baseline parameter values $r = 0.06$, $\sigma = 0.09$ and $\mu = 0.18$.

[26] In different models, Gryglewicz (2011), DeMarzo and Sannikov (2014) and Murto and Tervio (2014) also derive the existence of such a critical curve when holding cash is not costly.

pair (x, y) satisfies $x \geq c(y)$, then there is an uncountable number of payout policies that deliver the first-best firm value $\hat{V}$. This relates to Miller and Modigliani's dividend irrelevance Theorem, which holds for an all-equity firm in a frictionless world in which retained cash is invested in zero-NPV assets.[27] We show that the irrelevance of the dividend policy holds for a cash-constrained firm that learns about its profitability provided that the level of cash inside the firm remains at all times above a critical amount that allows it to sustain any liquidity shock. The absence of the cash retention problem allows shareholders to optimally maintain the level of cash beyond the critical level. On the contrary, when accumulating cash is costly, cash must be disgorged before the critical level is reached. There is uniqueness of the optimal payout policy, and the first-best value of the firm is never attained. Last, it is worth noting that $\lim_{y \longrightarrow \mu} c(y) = \lim_{y \longrightarrow \mu} \phi(y) - \phi(y^*) = \infty$, which suggests that withdrawing the cost of carrying cash from the analysis yields important qualitative and quantitative differences regarding the cash target levels of well-established firms. In particular, the analysis of the cash flow sensitivity of dividends cannot be continued, and the differences in dividend policies between established firms and firms subject to significant risk of profitability cannot be explained.

6 Conclusion

We develop a dynamic model of corporate cash management in which shareholders weigh the costs and benefits of holding cash and learn about the firm's profitability by observing earnings. The model links the evolution of the strength of shareholders' beliefs to changes in corporate cash policy. It provides a unified explanation for a series of stylized facts, a theoretical underpinning for recently documented regularities and novel implications.

In particular, the model predicts that well-established firms have high target levels, pay high dividends, and have more stable dividend policies than do firms facing more uncertainty about their profitability and having otherwise similar characteristics. These latter firms pay low dividends but can still have high cash target levels. The model predicts a positive relationship between dividend increases and the firm's equity value, a negative relationship between risk of profitability and dividend payments, and a non-monotone relationship between firm's volatility and cash reserves.

The richness of the analysis relies on a technical contribution. We succeed in solving a bi-dimensional optimization problem in a quasi-explicit form; this solution allows us to

[27] See, for instance, DeAngelo, DeAngelo and Skinner (2008).

account for knowledge about profitability in a setting in which the firm faces both external and internal frictions. Specifically, the trade-off between the benefits and costs of holding cash evolves as time passes, and the driver of this evolution is learning about the firm's profitability. This is a unique feature of our model from which all our findings result.

We conduct modeling choices to maintain the tractability of the model. We model the agency cost of free cash flow in a pragmatic way and focus on a particular learning technology. More importantly, we assume infinite issuance costs such that security issuance is not possible. It is interesting to note that even in a setting in which the cost of external financing is very high and fixed once for all, the interaction between liquidity and profitability issues generates different dividend policies and different behaviors of a firm's volatility. Another challenging extension is allowing for the possibility of debt and distinguishing dividends from stock repurchases, which have become an important form of payment. These extensions would necessitate introducing additional elements into the model, such as tax shields offered for interest expense, tax-advantaged repurchases or adverse selection problems. Building a tractable model that explains the dynamics of the interdependence between capital structure and equity payout policies remains an important objective for future work.

7 Appendix

Proof of Lemma 1. For all $t > 0$ and all admissible control L, we have

$$e^{-r(t\wedge\tau_0)}X_{t\wedge\tau_0} = x + \int_0^{t\wedge\tau_0} e^{-rs}dX_s - r\int_0^{t\wedge\tau_0} e^{-rs}X_s\,ds.$$

Because X_t is nonnegative before τ_0 and Y is bounded, we obtain

$$\begin{aligned} e^{-r(t\wedge\tau_0)}X_{t\wedge\tau_0} + \int_0^{t\wedge\tau_0} e^{-rs}dL_s &= x + \int_0^{t\wedge\tau_0} e^{-rs}dR_s - r\int_0^{t\wedge\tau_0} e^{-rs}X_s\,ds \\ &\leq x + \frac{\mu}{r} + \sigma\int_0^{t\wedge\tau_0} e^{-rs}dB_s. \end{aligned} \tag{38}$$

Applying the optional sampling theorem to the $\mathcal{F}^B$- martingale $M_t = \int_0^t e^{-rs}dB_s$, we have $\mathbb{E}[M_{t\wedge\tau_0}] = 0$. Taking expectations, we get for all $(x,y) \in [0,\infty)\times(-\mu,\mu)$

$$V(x,y;L) \leq x + \frac{\mu}{r} < \infty.$$

Thus, the value function V^* is finite. Then, we get (9) from Equations (2) and (38).

Proof of Proposition 1. Let us define

$$\Gamma(y) = \sup_{\tau\in\mathcal{T}^R}\mathbb{E}_y\left[\int_0^\tau e^{-rs}Y_s ds\right].$$

It follows from Dayanik and Karatzas (2003) (Corollary 7.1) that the optimal value function Γ is C^1 on $[0,\infty)$. Furthermore, from Villeneuve (2007) (Theorem 4.2. and Proposition 4.6) a threshold strategy $\tau^* = \inf\{t \geq 0 \,|\, Y_t = y^*\}$ is optimal. This allows us to use a standard verification procedure and write the value function Γ in terms of the free boundary problem:

$$\begin{cases} \frac{1}{2\sigma^2}(y+\mu)^2(\mu-y)^2\Gamma''(y) - r\Gamma(y) + y = 0, \; y \geq y^*, \\ \Gamma(y^*) = 0, \;\; \Gamma'(y^*) = 0. \end{cases}$$

Standard computations yield (10). We now show that the value function V^* of problem (8) is bounded above by $\hat{V}$. Because the cash reserves are positive for admissible controls, it follows from (9) that,

$$x + \sup_L \mathbb{E}_{(x,y)}\left[\int_0^{\tau_0} e^{-rs}Y_s\,ds\right]$$

is an upper bound for V^*. From Equation (2), any admissible control L acts on $\mathbb{E}_{(x,y)}\left[\int_0^{\tau_0} e^{-rs}Y_s\,ds\right]$ by modifying only the $\mathcal{F}^R$-stopping time τ_0. Thus,

$$\sup_L \mathbb{E}_{(x,y)}\left[\int_0^{\tau_0} e^{-rs}Y_s\,ds\right] \leq \sup_{\tau\in\mathcal{T}^R}\mathbb{E}_y\left[\int_0^\tau e^{-rs}Y_s ds\right],$$

which implies that the function V^* is bounded above by $\hat{V}$. To conclude the proof, it remains to pin down a policy that yields the value function $\hat{V}$ when the shareholders have deep pockets. For instance, consider the policy $\hat{L}_t = x\mathbb{1}_{t=0} + lt$. According to this policy, the firm distributes all its cash reserves at time 0 and pays a constant dividend flow $l > 0$ afterwards. In order to maintain cash reserves constant and equal to zero after time 0, deep-pocketed shareholders issue or repurchase new shares to offset earnings $(Y_t - l)dt + \sigma dW_t$ and decide strategically when to liquidate. Such a strategy yields them the value

$$x + \sup_L \mathbb{E}_{(x,y)} \left[\int_0^{\tau_0} e^{-rs} Y_s \, ds \right],$$

the result follows.

Proof of Proposition 2. The computation of the value functions V^+ and V^- are standard (see, for instance, Jeanblanc and Shiryaev (1995)) and thus the proof is omitted. To show the inequality $V^*(x,y) \le V^+(x)$, we use Equation (9) and the fact that the process Y_t is bounded by μ to get

$$V^*(x,y) \le x + \sup_L \mathbb{E}_x \left[\int_0^{\tau_0} e^{-rs} (\mu - rX_s) \, ds \right].$$

Proceeding analogously as in the proof of Lemma 1, we can show that

$$V^+(x) = x + \sup_L \mathbb{E}_x \left[\int_0^{\tau_0} e^{-rs} (\mu - rX_s) \, ds \right]$$

which ends the proof.

Proof of Proposition 3.

Proof of assertion (i). Fix $y \in (-\mu, \mu)$ and $0 \le x_1 \le x_2$. For any admissible policy L, the policy

$$\hat{L}_t = (x_2 - x_1)\mathbb{1}_{t=0} + L_t$$

is also admissible. The dynamics of the cash reserves starting from x_2 associated with policy $\hat{L}$, coincides with the dynamics of the cash reserves starting from x_1 associated with policy L. Therefore,

$$\begin{aligned} V^*(x_2, y) &\ge V(x_2, y, \hat{L}) \\ &= V(x_1, y, L) + x_2 - x_1. \end{aligned}$$

Taking the supremum over L gives

$$V^*(x_2,y) \geq V^*(x_1,y) + x_2 - x_1,$$

which yields that $x \to V^*(x,y)$ is increasing. The proof of concavity uses a standard argument of ϵ-optimality (see, for instance, Hoojgard and Taksar (2004)). Precisely, let L^i be an admissible policy and $X^{x_i,y}(L^i)$ the resulting cash reserves process starting from x_i. Without loss of generality, we can assume that $L_t^i = 0$ for all $t \geq \tau_0(L^i)$ where $\tau_0(L^i)$ denotes the liquidation time associated with policy L^i. Thus, for any $\lambda \in (0,1)$, for any $0 < x_1 < x_2$ and $y \in (-\mu,\mu)$, the cash reserves process starting from $x_3 = \lambda x_1 + (1-\lambda)x_2$ associated with policy $L^3 \equiv \lambda L^1 + (1-\lambda)L^2$ has the liquidation time $\tau_0(L^3) = \tau_0(L^1) \vee \tau_0(L^2)$. Then, it follows from (4) and (7) that

$$V(x_3,y;L^3) = \lambda V(x_1,y;L^1) + (1-\lambda)V(x_2,y;L^2).$$

For any $\epsilon > 0$, we can choose L^i, such that $V(x_i,y;L^i) \geq V^*(x_i,y) - \epsilon$. Since the policy L^3 is suboptimal, we have

$$V^*(x_3,y) \geq \lambda V(x_1,y;L^1) + (1-\lambda)V(x_2,y;L^2),$$

leading to

$$V^*(x_3,y) \geq \lambda V^*(x_1,y) + (1-\lambda)V^*(x_2,y) - \epsilon.$$

Concavity follows from the arbitrariness of ϵ.

Proof of assertion (ii). Stochastic Differential Equation (1) implies a non-crossing property of the beliefs' paths. That is, if $-\mu < y_1 \leq y_2 < \mu$ then, $Y_t^{y_1} \leq Y_t^{y_2}$ a.s for every t. Therefore, for any admissible policy L, policy $\hat{L}_t \equiv L_t + \int_0^t (Y_s^{y_2} - Y_s^{y_1})\,ds$ is also admissible. Moreover, cash reserves process $X^{x,y_2}(\hat{L})$ coincides with cash reserves process $X^{x,y_1}(L)$. However policy $\hat{L}$ pays more, thus $V(x,y_2,\hat{L}) \geq V(x,y_1,L)$. Proceeding analogously as in the proof of assertion (i), we then deduce that $V(x,y_2) \geq V(x,y_1)$, and thus that the mapping $y \longrightarrow V^*(x,y)$ is increasing over $(-\mu,\mu)$.

The proof of convexity of the value function with respect to the belief process is more involved. Let us define a probability $\mathbb{Q}$ on $(\Omega,\mathcal{F})$ by its projection on $\mathcal{F}_t$ for every t,

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \exp\left(-\frac{Y}{\sigma}B_t - \frac{1}{2}\left(\frac{Y}{\sigma}\right)^2\right).$$

Girsanov's Theorem implies that the process $K_t = B_t + \frac{Y}{\sigma}t$ is a $\mathcal{F}_t$ Brownian motion independent of Y under $\mathbb{Q}$ and that the natural filtration of K_t coincides with the natural filtration of R_t.[28] Moreover, the process

$$H_t(Y) = \exp\left\{\frac{Y}{\sigma}K_t - \frac{1}{2}(\frac{Y}{\sigma})^2 t\right\}.$$

is a $\mathcal{F}_t$ martingale under $\mathbb{Q}$. For any admissible policy L, we have by monotone convergence

$$\begin{aligned}
\mathbb{E}\left[\int_0^{\tau_0} e^{-rs}\,dL_s\right] &= \lim_{n\to+\infty}\mathbb{E}\left[\int_0^{\tau_0\wedge n} e^{-rs}\,dL_s\right] \\
&= \lim_{n\to+\infty}\mathbb{E}^{\mathbb{Q}}\left[H_{\tau_0\wedge n}(Y)\int_0^{\tau_0\wedge n} e^{-rs}\,dL_s\right] \\
&= \frac{y+\mu}{2\mu}\lim_{n\to+\infty}\mathbb{E}^{\mathbb{Q}}\left[H_{\tau_0\wedge n}(\mu)\int_0^{\tau_0\wedge n} e^{-rs}\,dL_s\right] \\
&+ \frac{\mu-y}{2\mu}\lim_{n\to+\infty}\mathbb{E}^{\mathbb{Q}}\left[H_{\tau_0\wedge n}(-\mu)\int_0^{\tau_0\wedge n} e^{-rs}\,dL_s\right] \\
&= \frac{y+\mu}{2\mu}\lim_{n\to+\infty}\mathbb{E}\left[\int_0^{\tau_0\wedge n} e^{-rs}\,dL_s|Y=\mu\right] \\
&+ \frac{\mu-y}{2\mu}\lim_{n\to+\infty}\mathbb{E}\left[\int_0^{\tau_0\wedge n} e^{-rs}\,dL_s|Y=-\mu\right] \\
&= \frac{y+\mu}{2\mu}\mathbb{E}\left[\int_0^{\tau_0} e^{-rs}\,dL_s|Y=\mu\right] \\
&+ \frac{\mu-y}{2\mu}\mathbb{E}\left[\int_0^{\tau_0} e^{-rs}\,dL_s|Y=-\mu\right].
\end{aligned}$$

Therefore, the value function associated with policy L is a linear function of y. Because V^* is a supremum over L of linear functions of y, it is convex in y.

Proof of assertion (iii). From the proof of assertion ii), we get for all admissible policy L,

$$\mathbb{E}\left[\int_0^{\tau_0} e^{-rs}\,dL_s\right] \geq \frac{y+\mu}{2\mu}\mathbb{E}\left[\int_0^{\tau_0} e^{-rs}\,dL_s|Y=\mu\right].$$

Taking the supremum with respect to L, we obtain

$$V^*(x,y) \geq \frac{y+\mu}{2\mu}V^+(x).$$

Therefore, we have

$$\liminf_{y\to\mu} V^*(x,y) \geq V^+(x).$$

[28]See, for instance, DMV (2005).

However, Proposition 2 implies

$$\limsup_{y \to \mu} V^*(x, y) \leq V^+(x),$$

thus the result.

Proof of Proposition 4.

We consider the change of variable (24) and we define

$$U(z, y) \equiv V(\phi(y) - z, y). \tag{39}$$

Then we re-state problem (19)-(23) in the (z, y)-space. This leads to:

$$V_x(x, y) = -U_z(z, y),$$
$$\mathcal{A}V(x, y) - rV(x, y) = \frac{1}{2\sigma^2}(\mu^2 - y^2)^2 U_{yy}(z, y) - rU(z, y).$$

The condition $V(0, y) = 0$ becomes

$$U(\phi(y), y) = U(z, \psi(z)) = 0,$$

where

$$\psi(z) \equiv \phi^{-1}(z) = \mu \frac{e^{\frac{2\mu}{\sigma^2} z} - 1}{e^{\frac{2\mu}{\sigma^2} z} + 1}. \tag{40}$$

Thus, we reformulate problem (19)-(23) as follows: find a smooth function U and a positive function b defined on $[-\mu, \mu]$ that solve the variational system:

$$b(\mu) = x^*_\mu, \tag{41}$$

$$U(z, \psi(z)) = 0 \quad \forall z \in \mathbb{R}, \tag{42}$$

$$\frac{1}{2\sigma^2}(\mu^2 - y^2)^2 U_{yy}(z, y) - rU(z, y) = 0$$
$$\text{on the domain } \{(z, y), \phi(y) - b(y) < z < \phi(y), \ -\mu < y < \mu\}, \tag{43}$$

$$U_z(\phi(y) - b(y), y) = -1, \tag{44}$$

$$U_{zy}(\phi(y) - b(y), y) = 0. \tag{45}$$

We observe that Equation (43) implies that, for any y with $b(y) > 0$,

$$U(z, y) \ = \ A(z)h_1(y) + B(z)h_2(y), \tag{46}$$

where γ and h_1 are defined in (11)-(12) and $h_2(y) = (y + \mu)^{1-\gamma}(\mu - y)^\gamma$ for any $y \in (-\mu, \mu)$. There are three unknown functions A, B and b. Using (40), we re-write Equation (42) under the form

$$A(z)e^{\frac{2\mu}{\sigma^2}\gamma z} \ + \ B(z)e^{\frac{2\mu}{\sigma^2}(1-\gamma)z} \ = \ 0. \tag{47}$$

Taking the derivative of (47) with respect to z yields

$$A(z)\frac{2\mu}{\sigma^2}\gamma e^{\frac{2\mu}{\sigma^2}\gamma z} + B(z)\frac{2\mu}{\sigma^2}(1-\gamma)e^{\frac{2\mu}{\sigma^2}(1-\gamma)z} = -A'(z)e^{\frac{2\mu}{\sigma^2}\gamma z} - B'(z)e^{\frac{2\mu}{\sigma^2}(1-\gamma)z}. \tag{48}$$

Equations (47) and (48) lead to

$$A(z) = \frac{\sigma^2}{2\mu(1-2\gamma)}\{A'(z) + B'(z)e^{\frac{2\mu}{\sigma^2}(1-2\gamma)z}\}, \tag{49}$$

$$B(z) = -\frac{\sigma^2}{2\mu(1-2\gamma)}\{A'(z)e^{-\frac{2\mu}{\sigma^2}(1-2\gamma)z} + B'(z)\}. \tag{50}$$

Equations (44) and (45) lead to

$$A'(\phi(y)-b(y)) = -\frac{1}{2\mu(1-2\gamma)}h_2'(y), \tag{51}$$

$$B'(\phi(y)-b(y)) = \frac{1}{2\mu(1-2\gamma)}h_1'(y). \tag{52}$$

Plugging (51)-(52) into (49)-(50) yields

$$A(\phi(y)-b(y)) = \frac{\sigma^2}{4\mu^2(1-2\gamma)^2}\{h_1'(y)e^{\frac{2\mu}{\sigma^2}(1-2\gamma)(\phi(y)-b(y))} - h_2'(y)\}, \tag{53}$$

$$B(\phi(y)-b(y)) = -\frac{\sigma^2}{4\mu^2(1-2\gamma)^2}\{h_1'(y) - h_2'(y)e^{-\frac{2\mu}{\sigma^2}(1-2\gamma)(\phi(y)-b(y))}\}. \tag{54}$$

Taking the derivative of (53) and using the relation $y^* = \frac{1}{2\gamma-1}\mu$ together with (40), we obtain[29] from (51),

$$b'(y) = \frac{\sigma^2}{\mu^2-y^2}\frac{yy^* + \left(\mu - r\sigma^2\left(\frac{y^*}{\mu}\right)^2\frac{1}{\mu}\right)\psi\left(\frac{\mu}{y^*}b(y)\right)}{yy^* + \mu\psi\left(\frac{\mu}{y^*}b(y)\right)},$$

for any y such that $b(y) > 0$. The proof of Proposition 4 relies on the next lemma.

Lemma 2 *There exists a unique function g satisfying*

$$g'(y) = f(g(y), y) \tag{55}$$

with the boundary condition

$$g(\mu) = x_\mu^*, \tag{56}$$

where,

$$f(x,y) = \frac{\sigma^2}{\mu^2-y^2}\frac{yy^* + \left(\mu - r\sigma^2\left(\frac{y^*}{\mu}\right)^2\frac{1}{\mu}\right)\phi^{-1}\left(-\frac{\mu}{y^*}x\right)}{yy^* + \mu\phi^{-1}\left(-\frac{\mu}{y^*}x\right)} \tag{57}$$

[29]Of course, considering Equations (54) and (52) yields the same result.

on the domain $\{(x,y) \in [0,\infty) \times (-\mu,\mu) \,|\, x > \frac{y^*}{\mu}\phi(y\frac{y^*}{\mu})\}$*, and where* x^*_μ *is defined in (14). Moreover, the function* g *is increasing,* $y^{**} \equiv g^{-1}(0)$ *is well defined and strictly larger than* y^**.*

Proof of Lemma 2. First, the denominator of (57) is strictly positive if and only if $x > l_1(y)$ where

$$l_1(y) = \frac{y^*}{\mu}\phi(y\frac{y^*}{\mu}).$$

Thus, f satisfies a local Lipschitz condition with respect to x in $\mathcal{D}$, where $\mathcal{D} = \{(x,y) \in \mathbb{R} \times (-\mu,\mu) \,|\, x > l_1(y)\}$. Thus, for any (x,y), there exists a unique solution $g_{x,y}$ to (55) defined on a maximal interval $I \subset (-\mu,\mu)$ passing through (x,y). Remark that f does not satisfy the Lipschitz condition on an open domain containing $-\mu$ or μ.

Second, the numerator of (57) is strictly positive if and only if[30] $x > l_2(y)$ where,

$$l_2(y) = \frac{y^*}{\mu}\phi\left(\frac{yy^*}{\mu - r\sigma^2(\frac{y^*}{\mu})^2\frac{1}{\mu}}\right).$$

The function l_2 is continuously increasing on $[-\mu,\mu]$ and satisfies the inequality $l_2(y) > l_1(y)$ for any $y \in (0,\mu]$. Furthermore, $l_1(0) = l_2(0) = 0$ and $l_1(\mu) < l_2(\mu) = x^*_\mu$. To see the last equality, note that the threshold x^*_μ can be written under the form $x^*_\mu = 2\frac{y^*}{\mu}\phi(y^*)$. Then, a computation shows that $2\phi(y^*) = \phi(\frac{y^*}{1-r\sigma^2(\frac{y^*}{\mu})^2\frac{1}{\mu^2}})$ which implies $l_2(\mu) = x^*_\mu$.

We deduce from the above observations, that any solution g to (55) entering in the domain $\{(x,y) \in \mathcal{D} \,|\, l_1(y) < x < l_2(y)\}$ remains in this domain. Since l_2 is bounded above by x^*_μ on $[-\mu,\mu]$, it follows also that any solution g to (55) defined on a maximal interval I and passing through $(x_0,y_0) \in \{(x,y) \in \mathcal{D} \,|\, x \geq x^*_\mu\}$ is strictly increasing and satisfies $g(y) > l_2(y)$ for all $y \in I$.

Now, let $(y_n)_{n\geq 0}$ an increasing sequence converging to μ. For each $n \in \mathbb{N}$ there exists a unique solution $g_{x^*_\mu,y_n}$ to (55) satisfying $g_n(x^*_\mu) = y_n$. Let us consider the sequence of functions $(g_n)_{n\geq 0}$ defined by the relations

$$\begin{cases} g_n(y) = g_{x^*_\mu,y_n}(y) & \forall y \in (0,y_n], \\ g_n(y) = x^*_\mu & \forall y \in [y_n,\mu]. \end{cases}$$

Our previous remarks on the solutions to (55) together with a standard non crossing property yield that, $(g_n)_{n\geq 0}$ is a decreasing sequence of increasing functions defined on $(0,\mu]$ and

[30] Note that, using (11), we have $\mu - r\sigma^2(\frac{y^*}{\mu})^2\frac{1}{\mu} = \mu(1 - r\sigma^2\frac{1}{\mu^2+2r\sigma^2}) > 0$.

bounded above by x^*_μ. Thus, it admits a pointwise limit g defined on $(0,\mu]$. The function g is bounded above by x^*_μ and satisfies $g(\mu) = x^*_\mu$. We show below that g satisfies (55).

By construction, for each $n \in \mathbb{N}$, for any $y \in (0,\mu)$ one has

$$g_n(y) = x^*_\mu - \int_y^\mu f(g_n(s), s)\, \mathbb{1}_{s \leq y_n}\, ds.$$

A direct computation shows that, for any fixed $y > 0$, the mapping $x \longrightarrow f(x,y)$ is continuously increasing over $\{x \,|\, x \geq l_2(y)\}$. We deduce that, for any $y \in (0,\mu)$,

$$\int_y^\mu \lim_{n\longrightarrow\infty} f(g_n(s), s)\, \mathbb{1}_{s\leq y_n}\, ds = \int_y^\mu f(g(s), s)\, ds \leq \int_y^\mu f(x^*_\mu, s)\, ds < \infty,$$

where the last inequality comes from the fact that the mapping $s \longrightarrow f(x^*_\mu, s)$ is continuous over $(0,\mu)$ with $\lim_{s\longrightarrow\mu} f(x^*_\mu, s) = \frac{1}{2}\frac{1-r\sigma^2(\frac{y^*}{\mu})^2\frac{1}{\mu^2}}{r(\frac{y^*}{\mu})^2} < \infty$. It results from the dominated convergence Theorem that,

$$g(y) = x^*_\mu - \int_y^\mu f(g(s), s)\, ds. \tag{58}$$

Thus, g is defined and increasing on $(0,\mu]$, satisfies the ode (55) and the condition $g(\mu) = x^*_\mu$. A standard extension argument ensures that g is defined on a maximal interval $I \subset (-\mu,\mu)$ as well.

We show that $y^{**} \equiv g^{-1}(0)$ is well defined and satisfies $y^{**} > y^*$. Take the solution g_{0,y^*} to (55) defined on a maximal interval $I \subset (-\mu,\mu)$ passing through $(0,y^*)$. A computation shows that the function $v_1(y) = \frac{y^*}{\mu}(\phi(y^*) - \phi(y))$ defined on $(-\mu,\mu)$ satisfies $v_1(y^*) = g_{0,y^*}(y^*) = 0$ together with the inequality $v_1'(y) < f(v_1(y), y)$ for any $y \in (-\mu, 0]$. We deduce that $g_{0,y^*}(y) > v_1(y)$ for all $y \in (y^*, 0]$. From the Cauchy-Lipschitz Theorem it follows that $g_{0,y^*} > g_{v_1(0),0}$ on a maximal interval, where $g_{v_1(0),0}$ is the solution to (55) passing through $(v_1(0), 0)$.

Now, let us consider the function $v_2(y) = \frac{y^*}{\mu}(\phi(y\frac{y^*}{\mu}) + \phi(y^*)) \geq l_2(y)$ on $[0,\mu]$. Computations shows that $v_2(0) = v_1(0)$, $v_2(\mu) = x^*_\mu$ and $v_2'(y) \leq f(v_2(y), y)$ for any $y \in [0,\mu]$. We deduce that $g_{0,y^*}(y) \geq v_2(y)$ for any $y \in [0,\mu]$. It follows that $g_{0,y^*} > g_{v_1(0),0} \geq g$ which implies that $y^{**} \equiv g^{-1}(0) > y^*$.

Finally, we show that g is the unique solution to $g'(y) = f(g(y), y)$ satisfying the boundary condition $g(\mu) = x^*_\mu$. Suppose the contrary, let g and $\tilde{g}$ be two solutions to (55) with

$g(\mu) = \tilde{g}(\mu) = x^*_\mu$ and $\tilde{g}(y) > \tilde{g}(y)$ over $(0, \mu)$. The functions g and $\tilde{g}$ satisfy (58). It follows that,

$$\tilde{g}(y) - g(y) \;=\; \int_y^\mu f(g(s), s) - f(\tilde{g}(s), s)\, ds. \tag{59}$$

The right hand side of (59) is strictly positive whereas its left hand side is negative because the mapping $x \longrightarrow f(x, y)$ is increasing for any fixed $y > 0$, thus, a contradiction.

We now conclude the proof of Proposition 4.

Let us consider the function g solution to (55)-(57) and let us define $b \equiv \max(0, g)$. The function b is continuously increasing and coincides with g on $[y^{**}, \mu]$. Using (55), (56), it is easy to see that $\phi^{'}(y) \geq b^{'}(y)$ for any y such that $b(y) > 0$ with equality only at y^{**}. Because ϕ is strictly increasing, $\phi - b$ is strictly increasing and continuous on $(-\mu, \mu)$ thus invertible. As a consequence, the function A defined by Equation (29) is well defined everywhere.

The change of variables (24) together with (39), (43), (46), (53) and (54) implies that, for any $(x, y) \in \{(x, y),\, 0 < x < b(y),\, -\mu < y < \mu\}$, we have

$$\begin{aligned} V(x, y) \;&=\; U(\phi(y) - x, y) \;=\; A(\phi(y) - x)h_1(y) + B(\phi(y) - x)h_2(y) \\ &=\; A\left(\phi(y) - x\right)\left(h_1(y) - e^{\frac{2}{\sigma^2}\frac{\mu^2}{y^*}(\phi(y)-x)}h_2(y)\right), \end{aligned} \tag{60}$$

where

$$A(z) \;=\; \frac{\sigma^2}{4}\left(\frac{y^*}{\mu}\right)^2\left(\frac{1}{\mu}\right)^2\left(h_1'(k(z))e^{-\frac{2}{\sigma^2}\frac{\mu^2}{y^*}z} - h_2'(k(z))\right). \tag{61}$$

It is easy to check that $V(x, y)$ is positive on the domain $\{(x, y),\, 0 < x < b(y),\, -\mu < y < \mu\}$. Finally, Equation (22) implies that

$$V(x, y) = x - b(y) + V(b(y), y), \quad \forall\, x \geq b(y).$$

From the above expression it is clear that V is twice continuously differentiable on any open set in $(0, \infty)\times(-\mu, \mu)$ away from the set $\{(x, y),\; x = b(y)\}$. The uniqueness of V follows from the uniqueness of g established in Lemma 2. We show that V is of class C^2 jointly in both variables across the boundary b. Using (60), we observe that the function $\nu(y) = V(b(y), y)$ is differentiable. Because V_x is continuous across the boundary b by construction, we deduce from the differentiability of the boundary b that V_y is continuous across b. Now, let us define $\eta(y) = V_y(b(y), y)$. By continuity of V_y, we use (60) to compute η and we observe that η is differentiable. Then, because V_{xy} is continuous across the boundary b by construction, we deduce from the differentiability of the boundary b that V_{yy} is continuous across b as well.

Finally, using again (60), it is easy to show that the mapping $x \longrightarrow V(x, y)$ is increasing and concave on $[0, \infty)$ for any fixed $y \in (-\mu, \mu)$.

Proof of Theorem 1. We will prove that the solution V to (19)-(23) is an upper bound for V^*. This relies on the following verification lemma.

Lemma 3 *Assume there exists a C^2 function V defined on $[0, \infty) \times (-\mu, \mu)$ that satisfies $V(0, .) = 0$, $x \to V(x, y)$ concave on $[0, \infty)$ for any fixed $y \in (-\mu, \mu)$ and*

$$\max(\mathcal{A}V - rV, 1 - V_x) \leq 0,$$

then $V \geq V^*$.

Proof of Lemma 3. Let us consider an admissible policy L and let us write $L_t = L_t^c + L_t^d$, where L_t^c is the continuous part of L_t and L_t^d is the pure discontinuous part of L_t . We recall the dynamics of cash reserves process and belief process:

$$\begin{aligned} dX_t &= Y_t \, dt + \sigma dW_t - dL_t, \\ dY_t &= \frac{\mu^2 - Y_t^2}{\sigma} \, dW_t. \end{aligned}$$

Using the generalized Itô's formula for a function V that satisfies the assumptions of Lemma 3, we can write:

$$\begin{aligned} e^{-r(t\wedge\tau_0)} V(X_{t\wedge\tau_0}, Y_{t\wedge\tau_0}) &= V(x, y) + \int_0^{t\wedge\tau_0} e^{-rs} \left(\mathcal{A}V(X_{s-}, Y_s) - rV(X_{s-}, Y_s)\right) \, ds \\ &+ \int_0^{t\wedge\tau_0} e^{-rs} V_x(X_{s-}, Y_s) \sigma dW_s \\ &+ \int_0^{t\wedge\tau_0} e^{-rs} V_y(X_{s-}, Y_s) \frac{\mu^2 - Y_t^2}{\sigma} dW_s \\ &- \int_0^{(t\wedge\tau_0)} e^{-rs} V_x(X_s, Y_s) \, dL_s^c \\ &+ \sum_{s \leq t\wedge\tau_0} e^{-rs} (V(X_s, Y_s) - V(X_{s-}, Y_s)), \end{aligned}$$

By assumption, the second term of the right-hand side is negative and, because V has bounded first derivatives, the two stochastic integrals are centered square integrable martingales. Taking expectations and using that $L_t - L_{t^-} = X_{t^-} - X_t$, we get

$$\begin{aligned} \mathbb{E}\left[e^{-r(t\wedge\tau_0)} V(X_{t\wedge\tau_0}, Y_{t\wedge\tau_0})\right] &\leq V(x, y) - \mathbb{E}\left[\int_0^{t\wedge\tau_0} e^{-rs} V_x(X_s, Y_s) \, dL_s\right] \\ &+ \mathbb{E}\left[\sum_{s \leq t\wedge\tau_0} e^{-rs} (V(X_s, Y_s) - V(X_{s-}, Y_s) - V_x(X_{s-}, Y_s)(X_s - X_{s-}))\right]. \end{aligned}$$

By concavity, $V(X_s,Y_s)-V(X_{s-},Y_s)-V_x(X_{s-},Y_s)(X_s-X_{s-})<0$. Therefore, using $V_x\geq 1$, we get

$$V(x,y)\geq \mathbb{E}\left[e^{-r(t\wedge\tau_0)}V(X_{(t\wedge\tau_0)},Y_{(t\wedge\tau_0)})\right]+\mathbb{E}\left[\int_0^{t\wedge\tau_0}e^{-rs}\,dL_s\right].$$

In order to get rid of the first term of the right-hand side, we use again the concavity of V with respect to x and the condition $V(0,.)=0$ to obtain

$$\mathbb{E}\left[e^{-r(t\wedge\tau_0)}V(X_{(t\wedge\tau_0)},Y_{(t\wedge\tau_0)})\right]\leq C\mathbb{E}[e^{-rt}X_t\mathbb{1}_{\tau_0>t}].$$

Because L is admissible, we let t tend to infinity to obtain

$$V(x,y)\geq \mathbb{E}\left[\int_0^{\tau_0}e^{-rs}\,dL_s\right],$$

which ends the proof of Lemma 3.

Let us show that the solution V to (19)-(23) satisfies the assumptions of Lemma 3.

Lemma 4 *The solution V to (19)-(23) satisfies $V_x(x,y)\geq 1$ for any $(x,y)\in[0,\infty)\times(-\mu,\mu)$. The mapping $x\longrightarrow V(x,y)$ is concave on $[0,\infty)$ for any fixed $y\in(-\mu,\mu)$.*

Proof of Lemma 4. First, we show by differentiating equation (28) that $V_{xx}(0,y)<0$ for any $y\in(-\mu,\mu)$. Using equality $h_1(y)=e^{\frac{2\mu^2}{\sigma^2y^*}\phi(y)}h_2(y)$, we obtain that $V_{xx}(0,y)$ has the same sign as $2A^{'}(\phi(y))+A(\phi(y))\frac{2\mu^2}{\sigma^2y^*}$ which has the same sign as $A^{'}(\phi(y))-B'(\phi(y)e^{\frac{-2\mu^2}{\sigma^2y^*}\phi(y)}$ by (49). From equations (61) and (49), we observe that $A^{'}(z)-B'(z)e^{\frac{-2\mu^2}{\sigma^2y^*}z}$ has the same sign as

$$2k'(z)\left(h_1^{''}(k(z))e^{-\frac{2}{\sigma^2}\frac{\mu^2}{y^*}z}-h_2^{''}(k(z))\right)-\frac{2}{\sigma^2}\frac{\mu^2}{y^*}\left(h_1^{'}(k(z)e^{-\frac{2}{\sigma^2}\frac{\mu^2}{y^*}z}-h_2^{'}(k(z))\right). \tag{62}$$

Thus, $V_{xx}(0,y)$ is negative if $h_1^{''}(k(z))e^{-\frac{2}{\sigma^2}\frac{\mu^2}{y^*}z}-h_2^{''}(k(z))<0$. Indeed, $y^*<0$, h_1 is decreasing and h_2 and k are increasing. The functions h_i, $i=1,2$, satisfy $\frac{(\mu^2-y^2)^2}{2\sigma^2}h_i^{''}-rh_i=0$ thus, the sign of $h_1^{''}(k(z))e^{-\frac{2}{\sigma^2}\frac{\mu^2}{y^*}z}-h_2^{''}(k(z))$ is the sign of $h_1(k(z))e^{-\frac{2}{\sigma^2}\frac{\mu^2}{y^*}z}-h_2(k(z))$. Evaluated at $z=\phi(y)$, we obtain

$$h_1(k(\phi(y)))e^{-\frac{2}{\sigma^2}\frac{\mu^2}{y^*}\phi(y)}-h_2(k(\phi(y)))<h_1(y)e^{-\frac{2}{\sigma^2}\frac{\mu^2}{y^*}\phi(y)}-h_2(y)=0,$$

where the last inequality comes from the inequality $k(\phi(y))>y$.

Second, we show that $V_{xx}(x,y)<0$ for any $x\in[0,\infty)$ and $y\in(-\mu,\mu)$. Differentiating twice (21) with respect to x, we find that

$$\mathcal{A}V_{xx}-rV_{xx}=0 \text{ on the set } \{x<b(y)\}.$$

Therefore the process $e^{-rt}V_{xx}(X_t, Y_t)$ is a martingale up to the exit time of the set $\{x < b(y)\}$. Applying the Itô's formula and using the properties $V_{xx}(0, y) < 0$ and $V_{xx}(b(y), y) = 0$, we obtain by denoting $\tau_0 = \inf\{t \geq 0,\ X_t = 0\}$ and $\tau_1 = \inf\{t \geq 0,\ X_t = b(Y_t)\}$,

$$V_{xx}(x, y) = \mathbb{E}\left[e^{-r\tau_0\wedge\tau_1}V_{xx}(X_{\tau_0\wedge\tau_1}, Y_{\tau_0\wedge\tau_1})\right] < 0$$

which proves that the mapping $x \longrightarrow V(x, y)$ is concave on the set $\{x < b(y)\}$. Because V is linear in x outside $\{x < b(y)\}$, we conclude that $x \longrightarrow V(x, y)$ is concave and satisfies $V_x \geq 1$ on $[0, \infty)$ because $V_x(x, y) = 1$ for all $x \geq b(y)$. Thus, Lemma 4 is proven.

Let us show that $\mathcal{A}V - rV < 0$ on the set $\{x > b(y)\}$. We have $V(x, y) = x - b(y) + \nu(y)$ on the set $\{x > b(y)\}$. Because $V_{xy}(b(y), y) = 0$ on the set $\{x > b(y)\}$, we deduce the equalities $V_y(x, y) = V_y(b(y), y)$ and $V_{yy}(x, y) = V_{yy}(b(y), y)$. Therefore, using the fact that V is twice differentiable across b, we obtain that, on the set $\{x > b(y)\}$,

$$\begin{aligned}(\mathcal{A}V - rV)(x, y) &= \frac{1}{2\sigma^2}(\mu^2 - y^2)^2 V_{yy}(b(y), y) + y - rV(x, y)) \\ &= -r(x - b(y)) \\ &< 0.\end{aligned}$$

Thus, the solution V to (19)-(23) satisfies the assumptions of Lemma 3 and is therefore an upper bound for V^*. To prove Theorem 1, it remains to show that the upper-bound V is attainable. It is easy to check that, the process $L^* = \{L^*_t; t \geq 0\}$ with

$$L^*_t = \left((x - b(y))^+ \mathbb{1}_{y \geq y^{**}} + x\mathbb{1}_{y \leq y^{**}}\right) \mathbb{1}_{t=0} + L^{b(.)}_t \mathbb{1}_{t>0},$$

where $L^{b(.)}$ is defined by (30)-(32), satisfies the relation $V(x, y) = V(x, y; L^*)$ for all $(x, y) \in [0, \infty) \times (-\mu, \mu)$. The proof of Theorem 1 is complete.

Proof of Proposition 5. A direct application of the Itô's formula yields (37). We deduce from assertion (ii) of Proposition 3 that the mapping $z \longrightarrow V^*(x, \psi(x + z))$ is increasing in z, where ψ is defined in (40). This proves assertion (ii) of Proposition 5. Let us prove assertion (i). Taking the derivative with respect to x in the explicit formula (36), we get

$$(V^*_{[z]})'(x) = A(z)\psi'(x + z)\{h_1'(\psi(x + z)) - e^{\frac{2}{\sigma^2}\frac{\mu^2}{y^*}z}h_2'(\psi(x + z))\} \tag{63}$$

where ψ is defined in (40). Observing that $A(z)$ is non positive, h_1 is decreasing and h_2 is increasing, we conclude that $V^*_{[z]}$ is strictly increasing in x.

We deduce from (63) that

$$
\begin{aligned}
(V^*_{[z]})^{''}(x) &= \psi^{''}(x+z)A(z)\{h_1'(\psi(x+z)) - e^{\frac{2}{\sigma^2}\frac{\mu^2}{y^*}z}h_2'(\psi(x+z)\} \\
&+ (\psi^{'}(x+z))^2 A(z)\{h_1^{''}(\psi(x+z)) - e^{\frac{2}{\sigma^2}\frac{\mu^2}{y^*}z}h_2^{''}(\psi(x+z)\} \\
&= \psi^{''}(x+z)A(z)\{h_1'(\psi(x+z)) - e^{\frac{2}{\sigma^2}\frac{\mu^2}{y^*}z}h_2'(\psi(x+z)\} \\
&+ \frac{2\sigma^2 r(\psi'(x+z))^2}{(\mu^2-\psi(x+z)^2)^2}A(z)\{h_1(\psi(x+z)) - e^{\frac{2}{\sigma^2}\frac{\mu^2}{y^*}z}h_2(\psi(x+z)\}
\end{aligned} \tag{64}
$$

Because $A(z) \leq 0$ and $h_1(\psi(x+z)) - e^{\frac{2}{\sigma^2}\frac{\mu^2}{y^*}z}h_2(\psi(x+z) \leq 0$, it follows from the last equality that $V^*_{[z]}$ is convex on the domain where $\psi^{''}(x+z)$ is positive. But, ψ is a convex function on $(-\infty, 0)$, thus $V^*_{[z]}$ is convex when both $z < 0$ and $x \leq -z$. Thus, for any fixed $z \in (\phi(y^{**}), 0]$, the value function $V^*_{[z]}$ is convex for low cash reserves (at least on the domain $\{0 \leq x \leq -z\}$) and, in turn, the volatility of stock prices increases in the level of cash reserves.

Using the relations $h_1'(y) = h_1(y)(\frac{\mu^2}{y^*} - y)\frac{1}{\mu^2-y^2}$, $h_2'(y) = h_2(y)(-\frac{\mu^2}{y^*} - y)\frac{1}{\mu^2-y^2}$ and $\mu^2 - \psi^2(z) = \sigma^2\psi'(z)$, we deduce from (64) that $(V^*_{[z]})^{''}(x) < 0$ if and only if

$$
\begin{aligned}
&\frac{h_1(\psi(x+z))}{h_2(\psi(x+z))}e^{-\frac{2}{\sigma^2}\frac{\mu^2}{y^*}z}(-\frac{2}{\sigma^4}\psi(x+z)(\frac{\mu^2}{y^*} - \psi(x+z)) + \frac{2r}{\sigma^2}) \\
&> -\frac{2}{\sigma^4}\psi(x+z)(-\frac{\mu^2}{y^*} - \psi(x+z)) + \frac{2r}{\sigma^2},
\end{aligned}
$$

or, equivalently, using the definitions of the functions h_1, h_2 and ψ

$$
e^{\frac{2}{\sigma^2}\frac{\mu^2}{y^*}x} > \frac{-\frac{1}{\sigma^2}\frac{\psi(x+z)}{y^*}(-\mu^2 - y^*\psi(x+z)) + r}{-\frac{1}{\sigma^2}\frac{\psi(x+z)}{y^*}(\mu^2 - y^*\psi(x+z)) + r}.
$$

Taking the limit when z tends to ∞, we get the condition

$$
e^{\frac{2}{\sigma^2}\frac{\mu^2}{y^*}x} > \frac{-\frac{1}{\sigma^2}\frac{\mu}{y^*}(-\mu^2 - y^*\mu) + r}{-\frac{1}{\sigma^2}\frac{\mu}{y^*}(\mu^2 - y^*\mu) + r}. \tag{65}
$$

Using the relation $y^* = \frac{-\mu}{1-2\gamma}$, we check that the right-hand-side of (65) is positive and lower than 1. To conclude the proof of assertion (i), it is sufficient to show that (65) is satisfied for $x = x^*_\mu$, which is easy to establish using the relation $x^*_\mu = 2\frac{y^*}{\mu}\phi(y^*)$.

Proof of Proposition 6. The proof of assertion (ii) is in the main text. The proof of assertion (i) follows standard arguments developed, for instance, in DMRV (2011). We sketch the proof for sake of completeness. Along the optimal policy, the dynamics of cash reserves is continuous. Therefore, applying Itô formula and taking expectations, we obtain for $x \le b(y)$,

$$\begin{aligned}\frac{\partial V^*}{\partial \sigma}(x,y) &= \mathbb{E}\left[e^{-r\tau_0}\frac{\partial V^*}{\partial \sigma}(X_{\tau_0}, Y_{\tau_0})\right] \\ &+ \mathbb{E}\left[\int_0^{\tau_0} e^{-rs}\frac{\partial^2 V^*}{\partial x \partial \sigma}(X_s, Y_s)\, dL_s\right] \\ &- \mathbb{E}\left[\int_0^{\tau_0} e^{-rs}\mathcal{A}(\frac{\partial V^*}{\partial \sigma})(X_s, Y_s)\, ds\right].\end{aligned}$$

Let us consider each term of the right-hand side of this equality. Because $V^*(0,y) = 0$ for all $\sigma > 0$, we have $\frac{\partial V^*}{\partial \sigma}(0,y) = 0$. Therefore, the first term disappears. Because the optimal dividend policy is singular and is activated only when $x = b(y)$, the second term is

$$\mathbb{E}\left[\int_0^{\tau_0} e^{-rs}\frac{\partial^2 V^*}{\partial x \partial \sigma}(b(Y_s), Y_s)\, dL_s\right].$$

Now, using the smooth-fit principle and the fact that $\frac{\partial^2 V^*}{\partial x^2}(b(y),y) = 0$ for all y, we obtain that

$$\frac{\partial^2 V^*}{\partial x \partial \sigma}(b(y),y) = 0.$$

Finally, differentiating Equation (17) yields

$$\mathcal{A}(\frac{\partial V^*}{\partial \sigma})(x,y) = -\sigma\frac{\partial^2 V^*}{\partial x^2} + \frac{(\mu^2 - y^2)^2}{\sigma^3}\frac{\partial^2 V^*}{\partial y^2}.$$

We then deduce from Proposition 3 that $\mathcal{A}(\frac{\partial V^*}{\partial \sigma})(x,y) \ge 0$, and, in turn, that $\frac{\partial V^*}{\partial \sigma}(x,y) \le 0$.

Proof of Proposition 7. We assume here that the cash reserves are remunerated within the firm at the risk-free rate r. Under this assumption, the cash reserves associated with admissible policy L evolve according to

$$dX_t = rX_t\, dt + dR_t - dL_t.$$

For any policy L, we define the policy L^* as follows:

$$L^*_t = L_t \text{ for } t < \tau^* = \inf\{t \ge 0,\ Y_t = y^*\} \text{ and } (\Delta L^*)_{\tau^*} = X_{\tau^*_-}.$$

Applying Dynamic programming principle and using $V^*(0,y) = 0$ for any y, we have

$$\begin{aligned}V^*(x,y) &= \sup_{L\in\mathcal{L}} \mathbb{E}_{x,y}\left[\int_0^{(\tau_0(L)\wedge\tau^*)-} e^{-rs}\, dL_s + e^{-r\tau^*}V^*(X_{\tau^*_-}, y^*)\mathbb{1}_{\tau_0(L)>\tau^*}\right] \\ &= \sup_{L\in\mathcal{L}} \mathbb{E}_{x,y}\left[\int_0^{(\tau_0(L)\wedge\tau^*)-} e^{-rs}\, dL_s + e^{-r\tau^*}X_{\tau^*_-}\mathbb{1}_{\tau_0(L)>\tau^*}\right] \\ &= \sup_{L\in\mathcal{L}} \mathbb{E}_{x,y}\left[\int_0^{\tau_0(L)\wedge\tau^*} e^{-rs}\, dL^*_s\right].\end{aligned}$$

Now, proceeding analogously as in the proof of Lemma 1, we get

$$V^*(x,y) = x + \sup_{L\in\mathcal{L}} \mathbb{E}_{x,y}\left[\int_0^{\tau_0(L)\wedge\tau^*} e^{-rs} Y_s\, ds\right].$$

We deduce from Proposition 1 and from the almost sure inequality $\tau_0(L)\wedge\tau^* \leq \tau_0(0)\wedge\tau^*$ valid for all policy L, that

$$\mathbb{E}_{x,y}\left[\int_0^{\tau_0(L)\wedge\tau^*} e^{-rs} Y_s\, ds\right] \leq \mathbb{E}_{x,y}\left[\int_0^{\tau_0(0)\wedge\tau^*} e^{-rs} Y_s\, ds\right].$$

Therefore,

$$V^*(x,y) = x + \mathbb{E}_{x,y}\left[\int_0^{\tau_0(0)\wedge\tau^*} e^{-rs} Y_s\, ds\right].$$

Finally, assume that $x > \phi(y) - \phi(y^*)$, then we have

$$\begin{aligned} X_t(L) &= x + \phi(Y_t) - \phi(y) + \int_0^t rX_s(L)ds - \int_0^t dL_s \\ &\geq \phi(Y_t) - \phi(y^*) + \int_0^t rX_s(L)ds - \int_0^t dL_s. \end{aligned}$$

Thus, any policy L with $L_t \leq (x - \phi(y) + \phi(y^*))\mathbb{1}_{t=0} + \int_0^t rX_s ds$ satisfies $\tau^* < \tau_0(L)$ $a.s.$ It follows that $V^*(x,y) = \hat{V}(x,y)$. The proof of Proposition 7 is complete.

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