WORKING PAPERS

N° IDEI-841

November 2014

"Capital Investment and Liquidity Management with collateralized debt."

Erwan Pierre, Stéphane Villeneuve and Xavier Warin

INSTITUT D'ECONOMIE INDUSTRIELLE

Toulouse School of Economics

Capital Investment and Liquidity Management with collateralized debt.

Erwan Pierre* Stéphane Villeneuve† Xavier Warin‡

Abstract: This paper examines the dividend and investment policies of a cash constrained firm that has access to costly external funding. We depart from the literature by allowing the firm to issue collateralized debt to increase its investment in productive assets resulting in a performance sensitive interest rate on debt. We formulate this problem as a bi-dimensional singular control problem and use both a viscosity solution approch and a verification technique to get qualitative properties of the value function. We further solve quasi-explicitly the control problem in two special cases.

Keywords: Investment, dividend policy, singular control, viscosity solution
JEL Classification numbers: C61; G35.
MSC Classification numbers: 60G40; 91G50; 91G80.

1 Introduction

In a world of perfect capital market, firms could finance their operating costs and investments by issuing shares at no cost. As long as the net present value of a project is positive, it will find investors ready to supply funds. This is the central assumption of the Modigliani and Miller theorem [17]. On the other hand, when firms face external financing costs, these costs generate a precautionary demand for holding liquid assets and retaining earnings. This departure from the Modigliani-Miller framework has received a lot of attention in recent years and has given birth to a serie of papers explaining why firms hold liquid assets. Pioneering papers are Jeanblanc and Shiryaev [13], Radner and Shepp [21] while more recent studies include Bolton, Chen and Wang [4], Décamps, Mariotti, Rochet and Villeneuve [7] and Hugonnier, Malamud and Morellec [12]. In all of these papers, it is assumed that firms have only access to external equity capital. Should it run out of liquidity, the firm either liquidates or raises new funds in order to continue operations by issuing equity. This binary decision only depends on the severity of issuance costs.
Even if our work is related to these recent papers that incorporate financing frictions into dynamic models of corporate finance to explain why firms hold low-yielding cash reserves, the model we develop here departs from the literature above in the following way. In this

*EDF R&D OSIRIS. Email: erwan.pierre@edf.com
†Toulouse School of Economics (CRM-IDEI), Manufacture des Tabacs, 21, Allée de Brienne, 31000 Toulouse, France. Email: stephane.villeneuve@tse-fr.eu
‡EDF R&D & FiME, Laboratoire de Finance des Marchés de l'Energie (www.fime-lab.org)

paper, we assume that the firm can raise new funds at any time only by issuing collateralized debt. The collateralized debt continuously pays a variable coupon indexed on the firm's outstanding debt that charges a higher interest rate as the firm's debt increases with the covenant that debt value cannot exceed the value of total assets: liquid and productive. The collateralized debt proposed in this paper is somehow similar to the performance-sensitive debt studied in [16] except that the shareholders are here forced to go bankrupt when they are no more able to collateralize a loan with their assets. Therefore, the liability side of the balance sheet of the firm consists in two different types of owners: shareholders and debtholders. Should the firm be liquidated, the debtholders have seniority over shareholders on the total assets. Many models initiated by Black and Cox [3] and Leland [14] that consider the traditional tradeoff between tax and bankruptcy costs as an explanation for debt issuance study firms liabilities as contingent claims on its underlying assets, and bankruptcy as an endogenous decision of the firm management. On the other hand, these models assume costless equity issuance and thus put aside liquidity problems. As a consequence, the firm's decision to borrow on the credit market is independent from liquidity needs and investment decisions. Our model belongs to the class of models that consider endogenous bankruptcy but takes the opposite viewpoint by examining debt issuance of a cash-constrained firm in a model without tax.

From a mathematical point of view, problems of cash management have been formulated as singular stochastic optimal control problems. As references for the theory of singular stochastic control, we may mention the pioneering works of Haussman and Suo [9] and [10] and for application to cash management problems Højgaard and Taksar [11], Asmussen, Højgaard and Taksar [1], Choulli, Taksar and Zhou [5], Paulsen [19] among others. To merge corporate liquidity, capital investment and debt financing in a tractable model is challenging because it involves a rather difficult three-dimensional singular control problem with stopping where the state variables are the book value of equity, the size of productive asset and the outstanding debt while the stopping time is the decision to default. The literature on multi-dimensional control problems relies mainly on the study of leading examples. A seminal example is the so-called finite-fuel problem introduced by Benes, Shepp and Witsenhausen [2]. This paper provides a rare example of a bi-dimensional optimization problem that combines singular control and stopping that can be solved explicitly by analytical means. More recently, Federico and Pham [8] have solved a degenerate bi-dimensional singular control problem to study a reversible investment problem where a social planner aims to control its capacity production in order to fit optimally the random demand of a good. Our paper complements the paper by Federico and Pham [8] by introducing firms that are cash-constrained and indebted[1]. To our knowledge, this is the first time that such a combined approach is used. This makes the problem much more complicated and we do not pretend solving it with full generality, but rather, we pave the way for future developments of these multidimensional singular control models. In particular, we lose the global convexity property of the value function that leads to the necessary smooth-fit property in [8] (see Lemma 8). Instead, we will give properties of the value function (see Proposition 6) and characterize it by means of viscosity solution (see Theorem 1). Furthermore, we will solve explicitly by a standard verification argument the peculiar case of costless reversible

[1] Ly Vath, Pham and Villeneuve [15] have also studied a reversible investment problem in two alternative technologies for a cash-constrained firm that has no access to external funding

investment. A new result is our characterization of the endogenous bankruptcy in terms of the profitability of the firm and the coupon rate.

The remainder of the paper is organized as follows. Section 2 introduces the model with a productive asset of fixed size, formalizes the notion of collateralized debt and defines the shareholders value function. Furthermore, it describes the optimal debt issuance and gives the analytical characterization in terms of free boundary problems. Section 3 built the value function by solving explicitly the free boundary problem associated to the control problem. Section 4 extends the analysis to the case of reversible investment on productive assets.

2 The Model

We consider a firm with a productive asset of fixed size K that is characterized at each date t by the following balance sheet:

K	X_t
M_t	L_t

- K represents the firm's productive assets, assumed to be constant[2] and normalized to one.
- M_t represents the amount of cash reserves or liquid assets.
- L_t represents the volume of outstanding debt.
- Finally, X_t represents the book value of equity.

The productive asset continuously generates cash-flows over time. The cumulative cash-flows process $R = (R_t)_{t\geq 0}$ is modeled as an arithmetic Brownian motion with drift μ and volatility σ which is defined over a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$. Specifically, the cumulative cash-flows evolve as

$$dR_t = \mu\, dt + \sigma\, dB_t$$

where $(B_t)_{t\geq 0}$ is a standard one-dimensionnal Brownian motion with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$.
We allow the firm to increase its cash reserves or cover losses by raising funds in the capital market. In this paper, we assume that firm can issue collateralized loans which enable it to borrow against the value of its assets. Moreover, we assume that the interest payments are variable and modelled by a function α depending only on the volume of debt the firm has issued.

[2]The extension to the case of variable size will be studied in Section 4

Assumption 1 *α is a strictly increasing, continuously differentiable convex function. Furthermore, it is assumed that the collateralized debt is risky, that is*

$$\forall x \geq 0, \alpha'(x) \geq r \text{ and } \alpha(0) = 0 \tag{1}$$

In this framework, the cash reserves evolve as

$$dM_t = (\mu - \alpha(L_{t^-}))dt + \sigma dB_t - dZ_t + dL_t \tag{2}$$

where $(Z_t)_t$ is an increasing right-continuous $(\mathcal{F}_t)_t$ adapted process representing the cumulative dividend payment up to time t and $(L_t)_t$ is a positive right-continuous $(\mathcal{F}_t)_t$ adapted process representing the outstanding debt at time t. Using the accounting relation $1 + M_t = X_t + L_t$, we deduce the dynamics for the book value of equity

$$dX_t = (\mu - \alpha(L_{t^-}))dt + \sigma dB_t - dZ_t. \tag{3}$$

Assumption 2 *The cash reserves must be non negative and the firm management is forced to liquidate when the book value of equity hits zero. Using the accounting relation, this is equivalent to assume the debtholders get back all the assets after bankruptcy.*

The goal of the management is to maximize shareholders value which is defined as the expected discounted value of all future dividend payout. Shareholders are assumed to be risk-neutral and future cash-flows are discounted at the risk-free rate r. The firm can stop its activity at any time by distributing all of its assets to stakeholders. Thus, the objective is to maximize over the admissible control $\pi = (L, Z)$ the functional

$$V(x, l; \pi) = \mathbb{E}_{x,l}\left(\int_0^{\tau_0} e^{-rt} dZ_t\right)$$

where

$$\tau_0 = \inf\{t \geq 0, X_t^\pi \leq 0\}$$

according to Assumption 2. Here x (resp. l) is the initial value of equity capital (resp. debt). We denote by Π the set of admissible control variables and define the shareholders value function by

$$V^*(x, l) = \sup_{\pi \in \Pi} V(x, l; \pi) \tag{4}$$

2.1 Optimal debt issuance

The shareholders optimization problem (4) involves two state variables, the value of equity capital X_t and the value of debt L_t, making its resolution difficult. Fortunately, the next proposition will enable us to reduce the dimension and make it tractable the computation of V^*. Proposition 1 shows that debt issuance is only optimal when the cash reserves are depleted.

Proposition 1 *A necessary and sufficient condition to hold debt is that the cash reserves are depleted, that is*

$$\forall t \in \mathbb{R}^+, L_t M_t = 0 \text{ or equivalently } L_t = (1 - X_t)_+$$

Proof: First, by Assumption 2, it is clear that the firm management must issue debt when cash reserves are nonpositive. Conversely, assume that the level of cash reserves m is strictly positive. We will show that it is always better off to reduce the level of outstanding debt by using the cash reserves. We will assume that the initial value of debt is $l > 0$ and introduce $\pi_t = (L_t, Z_t)$ any admissible strategy. We first assume that $L_0 = l$. Because $m > 0$, we will built a strategy from π as follows:

$$\begin{cases} L_0^\epsilon &= l - \epsilon \text{ for } 0 < \epsilon < \min(m, l) \text{ and } 0 \leq L_t^\epsilon \leq L_t, \\ Z_t^\epsilon &= Z_t + \int_0^t (\alpha(L_s) - \alpha(L_s^\epsilon))\, ds \end{cases}$$

Note that the debt issuance strategy L^ϵ consists in always having less debt that under the debt issuance strategy L and because α is increasing, the dividend strategy Z_t^ϵ pays more than the dividend strategy Z_t. Furthermore, denoting by $\pi^\epsilon = (L_t^\epsilon, Z_t^\epsilon)$, the bankruptcy time under π^ϵ starting from $(x, l - \epsilon)$ and the bankruptcy time under π starting from (x, l) are the same stopping time. Therefore, we will show that it is better off to follow π^ϵ than π.

$$\begin{aligned} V(x, l; \pi^\epsilon) &= \mathbb{E}_{(x,l-\epsilon)}\left(\int_0^{\tau_0^{\pi^\epsilon}} e^{-rs}\, dZ_s^\epsilon\right) \\ &> \mathbb{E}_{(x,l-\epsilon)}\left(\int_0^{\tau_0^{\pi^\epsilon}} e^{-rs}\, dZ_s\right) \\ &= \mathbb{E}_{(x,l)}\left(\int_0^{\tau_0^{\pi}} e^{-rs}\, dZ_s\right) \\ &= V(x, l; \pi) \end{aligned}$$

So if $m > l$, it is optimal to set $l = 0$ by using $m - l$ units of cash reserves while if $m < l$, it is optimal to reduce the debt to $l - m$. In any case, at any time $L_t = (1 - X_t)_+$.
Now, if we have $\Delta L_0 \neq 0$, two cases have to be considered.

- $L_0 = 0$ which is possible only if $m > l$. In that case, we set $L_t^\varepsilon = L_t$ and $Z_t^\varepsilon = Z_t$ for $t > 0$.
- $L_0 > 0$. In that case, we take the same strategy π^ϵ with $0 < \epsilon < \min(m, l + \Delta L_0)$.

$\diamond$

Proposition 1 implies the following dynamics for the book value of equity

$$dX_t = (\mu - \alpha((1 - X_{t^-})^+)dt + \sigma dB_t - dZ_t, \qquad X(0^-) = x \tag{5}$$

We thus define the value function as

$$v^*(x) = V^*(x, (1 - x)_+). \tag{6}$$

2.2 Analytical Characterization of the optimal policy of the firm

Because the level of capital is assumed to be constant, Proposition 1 makes our control problem one-dimensional. Thus, we will follow a verification procedure to characterize the value function in terms of a free boundary problem. We denote by $\mathcal{L}$ the differential operator:

$$\mathcal{L}\Phi = (\mu - \alpha((1-x)^+))\Phi^{'}(x) + \frac{\sigma^2}{2}\Phi^{''}(x) - r\Phi \tag{7}$$

We start by providing a standard result which establishes that a smooth solution to a free boundary problem coincides with the value function v^*.

Proposition 2 *Assume there exists a C^1 and piecewise twice differentiable function w on $(0,+\infty)$ together with a pair of constants $(a,b) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that,*

$$\begin{aligned} \forall x \leq a, &\qquad \mathcal{L}w \leq 0 \ \text{and}\ w(x) = x \\ \forall a \leq x \leq b, &\qquad \mathcal{L}w = 0 \ \text{and}\ w'(x) \geq 1 \\ \forall x > b, &\qquad \mathcal{L}w \leq 0 \ \text{and}\ w'(x) = 1 \end{aligned} \tag{8}$$

$$\text{with } w^{''}(b) = 0 \tag{9}$$

then $w = v^*$.

Proof: Fix a policy $\pi = (Z) \in \Pi$. Let :

$$dX_t = (\mu - \alpha((1 - X_{t^-})^+)dt + \sigma dB_t - dZ_t, \qquad X(0^-) = x$$

be the dynamic of the book value of equity under the policy π. Let us decompose $Z_t = Z_t^c + \Delta Z_t$ for all $t \geq 0$ where Z_t^c is the continuous part of Z.
Let τ_ε the first time when $X_t = \varepsilon$. Using the generalized Itô's formula, we have :

$$\begin{aligned} e^{-r(t\wedge\tau_\varepsilon)}w(X_{t\wedge\tau_\varepsilon}) = w(x) &+ \int_0^{t\wedge\tau_\varepsilon} e^{-rs}\mathcal{L}w(X_s)ds + \int_0^{t\wedge\tau_\varepsilon} \sigma e^{-rs}w^{'}(X_s)dB_s \\ &- \int_0^{t\wedge\tau_\varepsilon} e^{-rs}w^{'}(X_s)dZ_s^c \\ &+ \sum_{0\leq s\leq t\wedge\tau_\varepsilon} e^{-rs}[w(X_s) - w(X_{s^-})] \end{aligned}$$

Because w' is bounded away from zero, the third term is a square integrable martingale. Taking expectation, we obtain

$$\begin{aligned} w(x) = \mathbb{E}_x[e^{-r(t\wedge\tau_\varepsilon)}w(X_{t\wedge\tau_\varepsilon})] &- \mathbb{E}_x\left[\int_0^{t\wedge\tau_\varepsilon} e^{-rs}\mathcal{L}w(X_s)ds\right] \\ + \mathbb{E}_x&\left[\int_0^{t\wedge\tau_\varepsilon} e^{-rs}w^{'}(X_s)dZ_s^c\right] \\ - \mathbb{E}_x&\left[\sum_{0\leq s\leq t\wedge\tau_\varepsilon} e^{-rs}[w(X_s) - w(X_{s^-})]\right] \end{aligned}$$

Because $w' \geq 1$, we have $w(X_s) - w(X_{s^-}) \leq \Delta X_s = -\Delta Z_s$ therefore the third and the fourth terms are bounded below by

$$\mathbb{E}_x \left(\int_0^{t\wedge\tau_\varepsilon} e^{-rs} w^{'}(X_s) dZ_s \right).$$

Furthermore w is positive because w is increasing with $w(0) = 0$ and $\mathcal{L}w \leq 0$ thus the first two terms are positive. Finally,

$$w(x) \geq \mathbb{E}_x \left(\int_0^{t\wedge\tau_\varepsilon} e^{-rs} w^{'}(X_s) dZ_s \right) \geq \mathbb{E}_x \left(\int_0^{t\wedge\tau_\varepsilon} e^{-rs} dZ_s \right)$$

Letting t $\to +\infty$ and $\varepsilon \to 0$ we obtain $w(x) \geq v^*(x)$.
To show the reverse inequality, we will prove that there exists an admissible strategy π^* such that $w(x) = v(x, \pi^*)$. Let (X_t^*, Z_t^*) be the solution of

$$X_t^* = \int_0^t (\mu - \alpha((1 - X_{s^-})^+))ds + \sigma B_t - Z_t^* \tag{10}$$

where,

$$Z_t^* = (x\mathbb{1}_{\{x\leq a\}} + (x-b)^+)1_{\{t=0^-\}} + \int_0^{t\wedge\tau_a^-} 1_{\{X_s^*=b\}} dZ_s^* + a1_{\{t\geq\tau_a\}} \tag{11}$$

with

$$\tau_a = \inf\{t \geq 0, X_{t-}^* \leq a\}$$

whose existence is guaranteed by standard results on the Skorokhod problem (see for example Revuz and Yor [22]). The strategy $\pi^* = (Z_t^*)$ is admissible. Note also that X_t^* is continuous on $[0, \tau_a^-]$. It is obvious that $v(x, \pi^*) = x = w(x)$ for $x \leq a$. Now suppose $x > a$. Along the policy π^*, the liquidation time τ_0 coincides with τ_a because $X_{\tau_a}^* = 0$. Proceeding analogously as in the first part of the proof, we obtain

$$\begin{aligned}
w(x) &= \mathbb{E}_x \left[e^{-r(t\wedge\tau_0)} w(X_{t\wedge\tau_0}^*) \right] + \mathbb{E}_x \left[\int_0^{t\wedge\tau_0^-} e^{-rs} w^{'}(X_s^*) dZ_s^* \right] + \mathbb{E}_x \left[\mathbb{1}_{t>\tau_0} e^{-r\tau_0} (w(X_{\tau_0-}^*) - w(X_{\tau_0}^*)) \right] \\
&= \mathbb{E}_x \left[e^{-r(t\wedge\tau_0)} w(X_{t\wedge\tau_0}^*) \right] + \mathbb{E}_x \left[\int_0^{t\wedge\tau_0^-} e^{-rs} w^{'}(b) dZ_s^* \right] + \mathbb{E}_x \left[\mathbb{1}_{t>\tau_0} e^{-r\tau_0} a \right] \\
&= \mathbb{E}_x \left[e^{-r(t\wedge\tau_0)} w(X_{t\wedge\tau_0}^*) \right] + \mathbb{E}_x \left[\int_0^{t\wedge\tau_0} e^{-rs} dZ_s^* \right],
\end{aligned}$$

where the last two equalities uses, $w(a) = a$ $w^{'}(b) = 1$ and $(\Delta Z^*)_{\tau_0} = a$. Now, because $w(0) = 0$,

$$\mathbb{E}_x \left[e^{-r(t\wedge\tau_0)} w(X_{t\wedge\tau_0}^*) \right] = \mathbb{E}_x \left[e^{-rt} w(X_t^*) \mathbb{1}_{t\leq\tau_0} \right].$$

Furthermore, because w has at most linear growth and π^* is admissible, we have

$$\lim_{t\to\infty} \mathbb{E}_x \left[e^{-rt} w(X_t^*) \mathbb{1}_{t\leq\tau_0} \right] = 0.$$

Therefore, we have by letting t tend to $+\infty$,

$$w(x) = \mathbb{E}_x \left[\int_0^{\tau_0} e^{-rs} dZ_s^* \right] = v(x, \pi^*)$$

which concludes the proof. $\diamond$

Remark 1 *We notice that the proof remains valid when $a = 0$ and $w'(0)$ is infinite which will be the case in section 4.*

The verification theorem allows us to characterize the value function when the firm profitability is lower than the interest rate.

Corollary 1 *If $\mu \leq r$ then it is optimal to liquidate the firm, $v^*(x) = x$.*

Proof: We will show that the function $w(x) = x$ satisfies Proposition 2. To see this, we have to show that $\mathcal{L}w(x)$ is nonpositive for any $x \geq 0$. A straightforward computation gives

$$\text{for } x > 1, \quad \mathcal{L}w(x) = \mu - rx < \mu - r \leq 0,$$

$$\text{for } x \leq 1, \quad \mathcal{L}w(x) = \mu - \alpha(1-x) - rx,$$

Using Equation (1) of Assumption 1, we observe that $\mathcal{L}w(x)$ is nondecreasing for $x \leq 1$ and nonpositive at $x = 1$ when $\mu \leq r$.

$\diamond$

Hereafter, we will assume that $\mu > r$.

3 Solving the free boundary problem

We will now focus on the existence of a function w and a pair of constants (a, b) satisfying Proposition 2. We will proceed in two steps. First we are going to establish some properties of the solutions of the differential equation $\mathcal{L}w = 0$. Second, we will consider two different cases- one where the productivity of the firm is always higher than the maximal interest payment $\alpha(1) \leq \mu$, the other where the interest payment of the loan may exceed the productivity of the firm $\alpha(1) > \mu$.
Standard existence and uniqueness results for linear second-order differential equations imply that, for each b, the Cauchy problem :

$$\begin{cases} rw(x) = (\mu - \alpha((1-x)^+))w^{'}(x) + \dfrac{\sigma^2}{2} w^{''}(x) \\ w^{'}(b) = 1 \\ w^{''}(b) = 0 \end{cases} \tag{12}$$

has a unique solution w_b over $[0, b]$. By construction, this solution satifies $w_b(b) = \frac{\mu - \alpha((1-b)^+)}{r}$. Extending w_b linearly to $[b, \infty[$ as $w_b(x) = x - b + \frac{\mu - \alpha((1-b)^+)}{r}$, for $x \geq b$ yields a twice continuously differentiable function over $[0, \infty[$, which is still denoted by w_b.

3.1 Properties of the solution to the Cauchy Problem

We will establish a serie of preliminary results of the smooth solution w_b of (12).

Lemma 1 *Assume $b > 1$. If $w_b(0) = 0$ then w_b is increasing and thus positive.*

Proof: Because $w_b(0) = 0$, $w_b(b) = \frac{\mu}{r}$ and $\mathcal{L}w_b = 0$, the maximum principle implies $w_b > 0$ on $(0, +\infty)$. Let us define

$$c = \inf\{x > 0, w_b^{'}(x) = 0\}$$

If $c = 0$ then $w_b(0) = w_b^{'}(0) = w_b^{''}(0) = 0$. By unicity of the Cauchy problem, this would imply $w_b = 0$ which contradicts $w_b(b) = \frac{\mu}{r}$. Thus, $c > 0$. If $c < b$, we would have $w_b(c) > 0$, $w_b^{'}(c) = 0$ and $w_b^{''}(c) \leq 0$ and thus $\mathcal{L}w_b(c) < 0$ which is a contradiction. Therefore $w_b^{'}$ is always positive. $\diamond$

Lemma 2 *Assume $b > 1$. We have $w_b^{'} > 1$ and $w_b^{''} < 0$ on $[1, b[$.*

Proof: Because w_b is smooth on $]1, b]$, we differentiate Equation (12) to obtain,

$$w_b^{'''}(b) = \frac{2r}{\sigma^2} > 0$$

As $w_b^{''}(b) = 0$ and $w_b^{'}(b) = 1$, it follows that $w_b^{''} < 0$, and thus $w_b^{'} > 1$ over some interval $]b - \epsilon, b[$, where $\epsilon > 0$. Now suppose by way of contradiction that $w_b'(x) \leq 1$ for some $x \in [1, b - \epsilon]$ and let $\tilde{x} = \sup\{x \in [1, b - \epsilon], w_b^{'}(x) \leq 1\}$. Then $w_b^{'}(\tilde{x}) = 1$ and $w_b'(x) > 1$ for $x \in]\tilde{x}, b[$, so that $w_b(b) - w_b(x) > b - x$ for all $x \in]\tilde{x}, b[$. Because $w_b(b) = \frac{\mu}{r}$, this implies that for all $x \in]\tilde{x}, b[$,

$$w_b^{''}(x) = \frac{2}{\sigma^2}[rw_b(x) - \mu w_b^{'}(x)] < \frac{2}{\sigma^2}[r(x - b + w_b(b)) - \mu] = \frac{2}{\sigma^2}r(x - b) < 0$$

which contradicts $w_b'(b) = w_b'(\tilde{x}) = 1$. Therefore $w_b' > 1$ over $[1, b[$. Furthermore, using Lemma 1,

$$\begin{aligned} w_b^{''}(x) &= \frac{2}{\sigma^2}[rw_b(x) - \mu w_b^{'}(x)] \\ &< \frac{2}{\sigma^2}[rw_b(x) - \mu] \\ &< \frac{2}{\sigma^2}[rw_b(b) - \mu] \\ &= 0. \end{aligned}$$

$\diamond$

The next result gives a sufficient condition on b to ensure the concavity of w_b on $(0, b)$.

Corollary 2 *Assume $b \geq \frac{\alpha(1)}{r}$ and $\mu \geq \alpha(1)$, we have $w_b' > 1$ and $w_b'' < 0$ over $]0, b[$.*

Proof: Proceeding analogously as in the proof of Lemma 2, we define $\tilde{x} = \sup\{x \in [0, b - \epsilon], w_b'(x) \leq 1\}$ such that $w_b'(\tilde{x}) = 1$ and $w_b'(x) > 1$ for $x \in]\tilde{x}, b[$, so that $w_b(b) - w_b(x) > b - x$ for all $x \in]\tilde{x}, b[$. Because $b > \frac{\alpha(1)}{r} > 1$, $w_b(b) = \frac{\mu}{r}$, we have

$$\begin{aligned} w_b''(x) &= \frac{2}{\sigma^2}[rw_b(x) - (\mu - \alpha((1-x)^+))w_b'(x)] \\ &< \frac{2}{\sigma^2}[r(x - b + w_b(b)) - (\mu - \alpha((1-x)^+)] \\ &< \frac{2}{\sigma^2}[r(x-b) + \alpha((1-x)^+)] \end{aligned}$$

Denote by g the function

$$g(x) = \frac{2}{\sigma^2}[r(x-b) + \alpha(1-x)], \qquad x \in [0, 1[.$$

We have $g'(x) = \frac{2}{\sigma^2}[r - \alpha'(1-x)] < 0$ by Assumption 1. Because $g(0) = \frac{2}{\sigma^2}[-rb + \alpha(1)] \leq 0$ if $b \geq \frac{\alpha(1)}{r}$, we have $w_b''(x) < 0$ for $x \in]0, 1]$ which contradicts $w_b'(\tilde{x}) = 1$ and $w_b'(1) > 1$ by Lemma 2. Therefore $w_b' > 1$ over $[0, 1[$, from which it follows $w_b'' < 0$ and w_b is concave on $]0, 1[$. Because Lemma 2 gives the concavity of w_b on $[1, b[$, we conclude. $\diamond$ The next proposition establishes some results about the regularity of the function $b \to w_b(y)$ for a fixed $y \in [0, 1[$.

Lemma 3 *For any $y \in [0, 1[$, $b \to w_b(y)$ is an increasing function of b over $[y, 1]$ and strictly decreasing over $]1, +\infty[$.*

Proof: Consider the solutions H_0^y and H_1^y to the linear second-order differential equation $\mathcal{L}H = 0$ over $[y, \infty[$ characterized by the initial conditions $H_0^y(y) = 1$, $(H_0^y)'(y) = 0$, $H_1^y(y) = 0$, $(H_1^y)'(y) = 1$. We first show that $(H_0^y)'$ and $(H_1^y)'$ are strictly positive on $]y, \infty[$. Because $H_0^y(y) = 1$ and $(H_0^y)'(y) = 0$, one has $(H_0^y)''(y) = \frac{2r}{\sigma^2} > 0$, such that $(H_0^y)'(x) > 0$ over some interval $]y, y + \epsilon[$ where $\epsilon > 0$. Now suppose by way of contradiction that $\tilde{x} = \inf\{x \geq y + \epsilon, (H_0^y)'(x) \leq 0\} < \infty$. Then $(H_0^y)'(\tilde{x}) = 0$ and $(H_0^y)''(\tilde{x}) \leq 0$. Because $\mathcal{L}H_0^y = 0$, it follows that $H_0^y(\tilde{x}) \leq 0$, which is impossible because $H_0^y(y) = 1$ and H_0^y is strictly increasing over $[y, \tilde{x}]$. Thus $(H_0^y)' > 0$ over $]y; \infty[$, as claimed. The proof for H_1^y is similar, and is therefore omitted.

Next, let $W_{H_0^y, H_1^y} = H_0^y (H_1^y)' - H_1^y (H_0^y)'$ be the Wronskian of H_0^y and H_1^y. One has $W_{H_0^y, H_1^y}(y) = 1$ and

$$\begin{aligned} \forall x \geq y, \qquad W'_{H_0^y, H_1^y}(x) =& H_0^y(x)(H_1^y)''(x) - H_1^y(x)(H_0^y)''(x) \\ =& \frac{2}{\sigma^2}[H_0^y(x)(rH_1^y(x) - (\mu - \alpha((1-x)^+))(H_1^y)'(x)) \\ & - H_1^y(x)(rH_0^y(x) - (\mu - \alpha((1-x)^+))(H_0^y)'(x))] \\ =& -\frac{2[\mu - \alpha((1-x)^+)]}{\sigma^2} W_{H_0^y, H_1^y}(x) \end{aligned}$$

Because α is integrable, the Abel's identity follows by integration:

$$\forall x \geq y, \qquad W_{H_0^y, H_1^y}(x) = \exp\left[\frac{2}{\sigma^2}\left(-\mu(x-y) + \int_y^x \alpha((1-u)^+)du\right)\right]$$

Because $W_{H_0^y, H_1^y} > 0$, H_0^y and H_1^y are linearly independent. As a result of this, (H_0^y, H_1^y) is a basis of the two-dimensional space of solutions to the equation $\mathcal{L}H = 0$. It follows in particular that for each $b > 0$, on can represent w_b as :

$$\forall x \in [y, b], \qquad w_b(x) = w_b(y)H_0^y(x) + w_b'(y)H_1^y(x)$$

Using the boundary conditions $w_b(b) = \frac{\mu - \alpha((1-b)^+)}{r}$ and $w_b'(b) = 1$, on can solve for $w_b(y)$ as follows:

$$w_b(y) = \frac{(H_1^y)'(b)\frac{\mu-\alpha((1-b)^+)}{r} - H_1^y(b)}{W_{H_0^y, H_1^y}(b)}$$

Using the derivative of the Wronskian along with the fact that H_1^y is solution to $\mathcal{L}H = 0$, it is easy to verify that:

$$\begin{aligned} \forall b \in [y, 1[, \frac{dw_b(y)}{db} &= \frac{(H_1^y)'(b)\left(\frac{\alpha'(1-b)^+}{r} - 1\right)}{W_{H_0^y, H_1^y}(b)} \\ \forall b \in]1, \infty[, \frac{dw_b(y)}{db} &= \frac{-(H_1^y)'(b)}{W_{H_0^y, H_1^y}(b)} \end{aligned}$$

So $w_b(y)$ is an increasing function of b over $[y, 1]$ and strictly decreasing over $]1, \infty[$. ◇

Corollary 3 *If $b_2 > b_1 > 1$, then $w_{b_2} < w_{b_1}$.*

Proof: Let us define $W = w_{b_1} - w_{b_2}$. Clearly, $W > 0$ on $[b_2, +\infty[$. Moreover, we have $\mathcal{L}W = 0$ on $[0, b_1]$ and $W(0) > 0$ by Lemma 3. Moreover, $w_{b_1}(b_1) = w_{b_2}(b_2)$ and $w_{b_2}(b_2) > w_{b_2}(b_1)$ by Lemma 2. Therefore, the maximum principle implies $w_{b_2} < w_{b_1}$ on $[0, b_1]$. Finally, w_{b_2} is concave and $w_{b_2}'(b_2) = 1$ therefore for $b_1 \leq x \leq b_2$,

$$\begin{aligned} w_{b_2}(x) &\leq w_{b_2}(b_2) + x - b_2 \\ &= \frac{\mu}{r} + x - b_2 \\ &< \frac{\mu}{r} + x - b_1 \\ &= w_{b_1}(x). \end{aligned}$$

◇

3.2 Existence of a solution to the free boundary problem

We are now in a position to characterize the value function and determine the optimal dividend policy. Two cases have to be considered: when the profitability of the firm is always higher than the maximal interest payment ($\mu \geq \alpha(1)$) and when the interest payment exceeds the profitability of the firm ($\mu < \alpha(1)$).

3.2.1 Case: $\mu \geq \alpha(1)$

The next lemma establishes the existence of a solution w_{b^*} to the Cauchy problem (12) such that $w_{b^*}(0) = 0$.

Lemma 4 *There exists $b^* \in]1, \frac{\mu}{r}[$ such that the solution to (12) satisfies $w_{b^*}(0) = 0$.*

Proof: Because $\mu \geq \alpha(1)$, we know from Corollary 2 that $w_{\frac{\mu}{r}}$ is a concave function on $[0, \frac{\mu}{r}]$. Moreover, because $\mu > r$, $w_{\frac{\mu}{r}}(\frac{\mu}{r}) = \frac{\mu}{r}$. Because $w_{\frac{\mu}{r}}$ is strictly concave over $]0, \frac{\mu}{r}[$ with $w_{\frac{\mu}{r}}(\frac{\mu}{r}) = \frac{\mu}{r}$ and $w'_{\frac{\mu}{r}} = 1$, $w_{\frac{\mu}{r}}(x) \leq x$ for all $x \in]0, \frac{\mu}{r}[$. In particular, $w_{\frac{\mu}{r}}(0) < 0$.
Moreover, we have :

$$w_0(0) = \frac{\mu - \alpha(1)}{r} \geq 0.$$

Therefore, Lemma 3 implies $w_1(0) > 0$. Finally by continuity there is some $b^* \in]1, \frac{\mu}{r}[$ such that $w_{b^*}(0) = 0$ which concludes the proof. $\diamond$

The next lemma establishes the concavity of w_{b^*}.

Lemma 5 *The function w_{b^*} is concave on $[0, b^*]$*

Proof: Because $b^* > 1$, Lemma 2 implies that w_{b^*} is concave on $[1, b^*]$ thus $w''_{b^*}(1) \leq 0$.
For $x < 1$, we differentiate the differential equation satisfied by w_{b^*} to get,

$$\frac{\sigma^2}{2} w'''_{b^*}(x) + (\mu - \alpha(1-x))w''_{b^*}(x) + (\alpha'(1-x) - r)w'_{b^*}(x) = 0 \tag{13}$$

Because $w_{b^*}(0) = 0$ we have $w''_{b^*}(0) = -\frac{2}{\sigma^2}(\mu - \alpha(1))w'_{b^*}(0) \leq 0$.
Now, suppose by a way of contradiction that $w''_{b^*} > 0$ on some subinterval of $[0, 1]$. Because w''_{b^*} is continuous and nonpositive at the boundaries of $[0, 1]$, there is some c such that $w'''_{b^*}(c) = 0$ and $w''_{b^*}(c) > 0$. But, this implies

$$w'_{b^*}(c) = -\frac{(\mu - \alpha(1-c))w''_{b^*}(c)}{\alpha'(1-c) - r} < 0$$

which is a contradiction with Lemma 1. $\diamond$

Proposition 3 *If $\mu \geq \alpha(1)$, w_{b^*} is the solution of the control problem (9).*

Proof: Because w_{b^*} is concave on $[0, b^*]$ and $w'(b^*) = 1$, $w' \geq 1$ on $[0, b^*]$. Therefore we have a twice continuously differentiable concave function w_{b^*} and a pair of constants $(a, b) = (0, b^*)$ satisfying the assumptions of Proposition 2 and thus $w_{b^*} = v^*$. $\diamond$

Figure 1 plots some value functions, when $\mu \geq \alpha(1)$, using a linear function for α, $\alpha(x) = \lambda x$ with different values of λ.

Remark 2

When the maximal interest payment is lower than the firm profitability, the value function is concave. This illustrates the shareholders' fear to liquidate a profitable firm. In particular, the shareholders value is a decreasing function of the volatility.

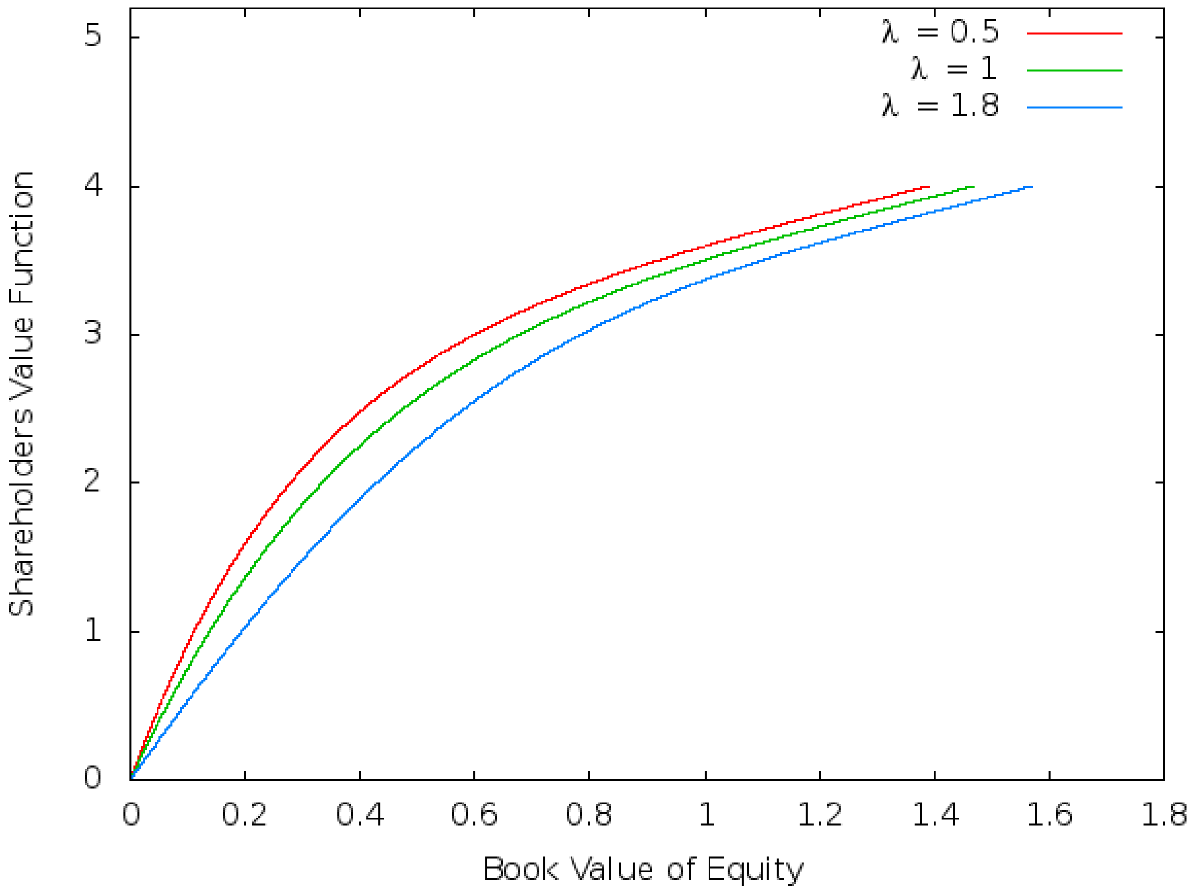


Figure 1: Comparing shareholders value functions with $\mu = 2$, $r = 0.5$, $\sigma = 1$ and $\mu \geq \alpha(1)$ for different values of λ where $\alpha(x) = \lambda x$.

3.2.2 Case: $\mu < \alpha(1)$

We first show that, for all $y \in [0, 1[$, there exists b_y such that w_{b_y} is the solution of the Cauchy Problem (12) with $w_{b_y}(y) = y$.

Lemma 6 *For all $y \in [0, 1[$, we have $w_1(y) > y$.*

Proof: Because α is continuous with $\alpha(0) = 0$ and $\mu > r$, there exists ϵ such that $w_{1-\epsilon}(1-\epsilon) = \left(\frac{\mu-\alpha(\epsilon)}{r}\right) > 1$. Differentiating Equation (12), we observe

$$w'''_{1-\epsilon}(1-\epsilon) = \frac{2}{\sigma^2}(r - \alpha'(\epsilon)) < 0$$

using Equation (1). Therefore $w_{1-\epsilon}$ is convex in a left neighborhood of $1-\varepsilon$. If $w_{1-\epsilon}$ is convex on $(0, 1-\varepsilon)$ then $w_{1-\epsilon}(x) \geq x - (1-\epsilon) + \frac{\mu-\alpha(\epsilon)}{r} > x$ for ε small enough and the result is proved.

If $w_{1-\epsilon}$ is not convex on $(0,1-\varepsilon)$ then it will exist some $\bar{x} < 1-\varepsilon$ such that $w^{''}_{1-\epsilon}(\bar{x}) = 0$, $w^{'''}_{1-\epsilon}(\bar{x}) > 0$ and $w_{1-\epsilon}$ convex on $]\bar{x}, 1-\epsilon]$. Differentiating Equation (12) at $\bar{x}$ gives $w^{'}_{1-\epsilon}(\bar{x}) < 0$. Therefore $w_{1-\epsilon}$ is nonincreasing in a neighborhood of $\bar{x}$. Assume by a way of contradiction that $w_{1-\epsilon}$ is increasing at some point $\hat{x} \in [0, \bar{x}[$. This would imply the existence of $\tilde{x} < \bar{x}$ such that $w^{'}_{1-\epsilon}(\tilde{x}) = 0$, $w^{''}_{1-\epsilon}(\tilde{x}) < 0$ and $w_{1-\epsilon}(\tilde{x}) > 0$ which contradicts Equation (12). Therefore $w_{1-\epsilon}$ is decreasing on $(0, \bar{x})$ and convex on $(\bar{x}, 1-\varepsilon)$ which implies that $w_{1-\epsilon}(x) > x$ for all $x \leq 1-\varepsilon$. To conclude, for any $y < 1$, we can find ε small enough to have $w_{1-\epsilon}(y) > y$ which can be extended to $w_1(y) > y$ by Lemma 3. $\diamond$

Corollary 4 *For all $y \in [0,1[$, there is an unique $b_y \in]1, 1+\frac{\mu}{r}[$ such that $w_{b_y}(y) = y$.*

Proof: By Lemma 1, $w_{1+\frac{\mu}{r}}$ is concave on $]1, 1+\frac{\mu}{r}[$, thus $w_{1+\frac{\mu}{r}}(1) < \frac{\mu}{r} + (1-(1+\frac{\mu}{r})) = 0$. Suppose that there exists c in $[0,1[$ such that $w_{1+\frac{\mu}{r}}(c) > 0$, then there exists $\tilde{x} \in]c, 1[$ such that $w_{1+\frac{\mu}{r}}(\tilde{x}) < 0$, $w^{'}_{1+\frac{\mu}{r}}(\tilde{x}) = 0$, $w^{''}_{1+\frac{\mu}{r}}(\tilde{x}) > 0$ yielding to the standard contradiction with the maximum principle. We thus have $w_{1+\frac{\mu}{r}}(y) < y$ for all $y \in [0, 1+\frac{\mu}{r}]$. Using Lemma 6 and the continuity of the function $b \to w_b(y)$, it exists for all $y < 1$ a threshold $b_y \in]1, 1+\frac{\mu}{r}[$ such that $w_{b_y}(y) = y$. The uniqueness of b_y comes from Corollary 3. $\diamond$

We will now study the behavior of the first derivative of w_{b_y}.

Lemma 7 *There exists $\epsilon > 0$ such that $w^{'}_{b_{1-\varepsilon}}(1-\varepsilon) \geq 1$ and $b_{1-\epsilon} < \frac{\mu}{r}$.*

Proof: Because $\alpha(0) = 0$ and $\mu > r$, it exists $\eta > 0$ such that

$$\forall x \in [1-\eta, 1], \alpha(1-x) + rx - \mu < 0. \tag{14}$$

Moreover $w_{\frac{\mu}{r}}$ is strictly concave on $[1, \frac{\mu}{r}[$ by Lemma 2 and thus

$$\begin{aligned} w_{\frac{\mu}{r}}(1) &\leq w_{\frac{\mu}{r}}(\frac{\mu}{r}) + (1-\frac{\mu}{r}) w^{'}_{\frac{\mu}{r}}(\frac{\mu}{r}) \\ &= 1. \end{aligned}$$

Because by Lemma 2, we have $w'_{\frac{\mu}{r}} > 1$ on $[1, \frac{\mu}{r}[$, there exists $\nu > 0$ such that $\forall x \in [1-\nu, 1], w_{\frac{\mu}{r}}(x) < x$. Let $\varepsilon = \min(\eta, \nu)$. By Corollary 4, it exists $b_{1-\epsilon} \in]1, 1+\frac{\mu}{r}[$ such that $w_{b_{1-\epsilon}}(1-\epsilon) = 1-\epsilon$. We have $w_{b_{1-\epsilon}}(1-\epsilon) > w_{\frac{\mu}{r}}(1-\epsilon)$ and then $b_{1-\epsilon} < \frac{\mu}{r}$ by Corollary 3. Let us consider the function $W(x) = w_{b_{1-\epsilon}}(x) - x$, we have $W(1-\epsilon) = 0$, $W(b_{1-\epsilon}) = \frac{\mu}{r} - b_{1-\epsilon} > 0$. Moreover, W is solution

$$(\mu - \alpha((1-x)^+)W'(x) + \frac{\sigma^2}{2} W''(x) - rW(x) = \alpha((1-x)^+) + rx - \mu. \tag{15}$$

On $[1-\epsilon, 1]$, the second member of Equation (15) is negative due to Equation (14). On $[1, b_{1-\epsilon}]$, it is equal to $rx - \mu$ which is negative because $b_{1-\epsilon} < \frac{\mu}{r}$. Assume by a way of contradiction that there is some $x \in [1-\epsilon, b_{1-\epsilon}]$ such that $W(x) < 0$, then it would exist $\tilde{x} \in [1-\epsilon, b_{1-\epsilon}]$ such that $W(\tilde{x}) < 0, W'(\tilde{x}) = 0$ and $W''(\tilde{x}) > 0$ which is in contradiction with Equation (15). Hence, W is a positive function on $[1-\epsilon, b_{1-\epsilon}]$ with $W(1-\epsilon) = 0$ which implies $w'_{b_{1-\epsilon}}(1-\epsilon) \geq 1$. $\diamond$

Lemma 8 *When $\mu < \alpha(1)$, w_{b_0} is a convex-concave function.*

Proof: According to Corollary 4, there exists $b_0 \in]1, 1 + \frac{\mu}{r}[$ such that $w_{b_0}(0) = 0$ and by Lemma 1, $w'_{b_0} > 0$ on $(0, b_0)$. Using Equation (12), we thus have $w''_{b_0}(0) > 0$ implying that w_{b_0} is strictly convex on a right neighborhood of 0. Because $b_0 > 1$, Lemma 2 implies $w''_{b_0}(x) < 0$ on $[1, b_0[$. If there is more than one change in the concavity of w_{b_0}, it will exist $\bar{x} \in [0, 1[$ such that $w'''_{b_0}(\bar{x}) > 0$, $w''_{b_0}(\bar{x}) = 0$ and $w'_{b_0}(\bar{x}) \geq 0$ yielding the standard contradiction. $\diamond$

Proposition 4 *If $\mu < \alpha(1)$ and $w'_{b_0}(0) \geq 1$, w_{b_0} is the shareholders value function* (4)

Proof: It is straightforward to see that the function w_{b_0} satisfies Proposition 2 when $w'_{b_0}(0) \geq 1$. $\diamond$

Now, we will consider the case $w'_{b_0}(0) < 1$.

Lemma 9 *If $w'_{b_0}(0) < 1$, it exists $a \in]0, 1[$ such that $w_{b_a}(a) = a$ and $w'_{b_a}(a) = 1$.*

Proof: Let $\phi(x) = w'_{b_x}(x)$. By assumption, we have $\phi(0) < 1$ and by Lemma 7, $\phi(1-\varepsilon) > 1$. By continuity of ϕ, there exists $a \in]0, 1[$ such that $w'_{b_a}(a) = 1$. By definition, the function w_{b_a} satisfies $w_{b_a}(a) = a$. $\diamond$

Lemma 10 *w_{b_a} is a convex-concave function on $[a, b_a]$.*

Proof: First, we show that w_{b_a} is increasing on $[a, b_a]$. Because $w'_{b_a}(a) = 1$, we can define $\tilde{x} = \min\{x > a, w'_{b_a}(x) \leq 0\}$. If $\tilde{x} \leq b_a$, we will have $w'_{b_a}(\tilde{x}) = 0$, $w_{b_a}(\tilde{x}) > 0$ and $w''_{b_a}(\tilde{x}) \leq 0$ yielding the standard contradiction. According to Lemma 1, we have $w''_{b_a}(x) < 0$ over $[1, b_a[$ because $b_a > 1$. Proceeding analogously as in the proof of Lemma 8, we prove that w_{b_a} is a convex-concave function because it cannot change of concavity twice. $\diamond$

Lemma 11 *We have $w_{b_a} > 1$ on (a, b_a) with $b_a < \frac{\mu}{r}$.*

Proof: According to Lemma 10, w_{b_a} is convex-concave with $w'_{b_a}(a) = 1$ and $w'_{b_a}(b_a) = 1$, therefore $\forall x \in]a, b_a[, w'_{b_a}(x) > 1$. As a consequence, $w_{b_a}(x) > x$ on $]a, b_a]$ and in particular $w_{b_a}(1) > 1$. Remembering that $w_{\frac{\mu}{r}}(1) < 1$ and using Corollary 3, we have $b_a < \frac{\mu}{r}$. $\diamond$

Proposition 5 *If $w'_{b_0}(0) < 1$, the function*

$$w(x) = \begin{cases} x & \text{for } x \leq a \\ w_{b_a}(x) & \text{for } a \leq x \leq b_a \\ x - b_a + \frac{\mu}{r} & \text{for } x \geq b_a \end{cases}$$

is the shareholders value function (4).

Proof: it is straightforward to check that w satisfies Proposition 2. $\diamond$

Figure 2 plots some value functions, when $\alpha(1) > \mu$, using a linear function for α, $\alpha(x) = \lambda x$ for different values of λ.

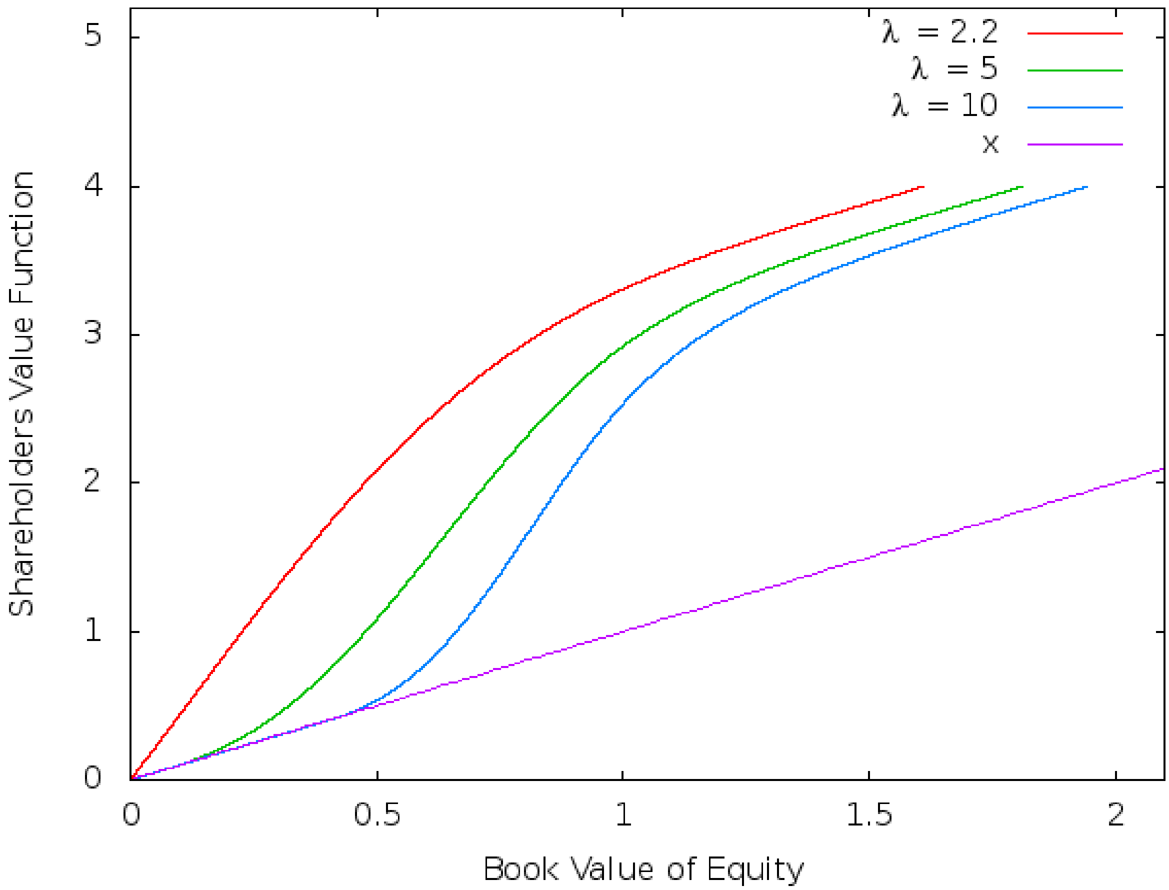


Figure 2: Comparing shareholders value functions with $\mu = 2$, $r = 0.5$, $\sigma = 1$ and $\alpha(1) > \mu$ for different values of λ where $\alpha(x) = \lambda x$.

Remark 3 *When the cost of debt exceeds the firm profitability, the shareholders value is convex for low value of equity. This illustrates the shareholders option to abandon a highly indebted firm. This option value, measured by the threshold a, increases with the cost of debt while the value function decreases with the cost of debt.*

4 Capital investment

In this section, we will extend our model to allow variable investment in the productive assets. We will assume decreasing return to scale by introducing an increasing concave function β with $\lim_{x\to\infty} \beta(x) = \bar{\beta}$ that impacts the dynamic of the book value of equity as follows:

$$\begin{cases} dX_t = \beta(K_t)(\mu dt + \sigma dW_t) - \alpha((K_t - X_t)^+)dt - \gamma|dI_t| - dZ_t \\ dK_t = dI_t = dI_t^+ - dI_t^- \end{cases} \tag{16}$$

where I_t^+ (resp. I_t^-) is the cumulative capital invested (resp. disinvested) in the productive assets up to time t, $\gamma > 0$ is an exogenous proportional cost of investment. We assume that the firm is forced to liquidate when the level of outstanding debt reaches the sum of the liquidation value of the productive assets and the liquid assets, $(1-\gamma)K_t + M_t$. The goal of the management is to maximize over the admissible strategies $\pi = (Z_t, I_t)_{t\geq 0}$ the risk-neutral shareholders value

$$V^*(x,k) = \sup_{\pi} \mathbb{E}_{x,k}\left(\int_0^{\tau_0} e^{-rt}dZ_t\right) \tag{17}$$

where

$$\tau_0 = \inf\{t \geq 0, X_t \leq \gamma K_t\}.$$

By definition, we have

$$\forall k \geq 0, V^*(\gamma k, k) = 0 \tag{18}$$

4.1 Dynamic programming and free boundary problem

In order to derive a classical analytic characterization of V^* in terms of a free boundary problem, we shall appeal to the dynaming programming principle as follows
Dynamic Programming Principle: For any $(x,k) \in S$ where $S = \{(x,k) \in \mathbb{R}_+^2,\ x \geq \gamma k\}$, we have

$$V^*(x,k) = \sup_{\pi} \mathbb{E}\left(\int_0^{\theta} e^{-rt}dZ_t + e^{-r\theta}V^*(X_\theta, K_\theta)\right) \tag{19}$$

where θ is any stopping time.
Let us consider the second order differential operator

$$\mathcal{L}w = \big(\beta(k)\mu - \alpha((k-x)^+)\big)\frac{\partial w}{\partial x} + \frac{\sigma^2\beta(k)^2}{2}\frac{\partial^2 w}{\partial x^2} - rw. \tag{20}$$

The aim of this section is to characterize via the dynamic programming principle the shareholders value as the unique continuous viscosity solution to the free boundary problem

$$F(x,k,V^*,DV^*,D^2V^*) = 0 \tag{21}$$

where

$$F(x,k,w,Dw,D^2w) = \min\left(-\mathcal{L}w, \frac{\partial w}{\partial x} - 1, \gamma\frac{\partial w}{\partial x} - \frac{\partial w}{\partial k}, \gamma\frac{\partial w}{\partial x} + \frac{\partial u}{\partial k}\right).$$

We will first establish the continuity of the shareholders value function which relies on some preliminary well-known results about hitting times we prove below for sake of completeness.

Lemma 12 *Let $a < b$ and $(x_n)_{n\geq 0}$ a sequence of real numbers such that $\lim_{n\to+\infty} x_n = b$ and $\min_n x_n > a$. Let $(X_t^n)_{n\geq 0}$ the solution of the stochastic differential equation*

$$\begin{cases} dX_t^n = \mu_n(X_t^n)dt + \sigma_n dW_t \\ \quad X_0^n = x_n \end{cases}$$

where μ_n and σ_n satisfy the standard global Lipschitz and linear growth conditions. Moreover, $(\sigma_n)_{n\geq 0}$ are strictly positive real numbers converging to $\sigma > 0$ and $(\mu_n)_{n\geq 0}$ is a sequence of bounded functions converging uniformly to μ. Let us define $T_n = \inf\{t \geq 0, X_t^n = a\}$ and $\theta_n = \inf\{t \geq 0, X_t^n = b\}$. We have

$$\lim_{n\to+\infty} \mathbb{P}(\theta_n < T_n) = 1$$

Proof: Let us define the functions $U_n, F_n : I \to \mathbb{R}$, on some bounded interval I containing (a, b) as

$$U_n(y) = \int_0^y \mu_n(z + x_n)dz, \qquad F_n(y) = \int_0^y e^{-\frac{2U_n(z)}{\sigma_n^2}} dz.$$

Because $(\mu_n)_{n\geq 0}$ converges uniformly to μ, we note that $(F_n, U_n)_{n\geq 0}$ converges uniformly to (F, U) where

$$F(y) = \int_0^y e^{-\frac{2U(z)}{\sigma^2}} dz.$$

and

$$U(y) = \int_0^y \mu(z + b)dz$$

Let $Y_t^n = X_t^n - x_n$, $M_t^n = F_n(Y_t^n)$ and $\tau_n = \inf\{t \geq 0, Y_t^n \notin]a_n, b_n[\}$ with $a_n = a - x_n$ et $b_n = b - x_n$. We first show that τ_n is integrable. Because F_n is the scale function of the process Y_t^n, M_t^n is a local martingale with quadratic variation

$$< M^n >_t = \int_0^t \sigma_n^2 e^{-\frac{4U_n(Y_s^n)}{\sigma_n^2}} ds$$

Because

$$E(< M^n >_{t\wedge\tau_n}) \leq \sigma_n^2 t \exp\left(-\frac{4}{\sigma_n^2} \min_{y\in[a_n,b_n]} U_n(y)\right) < +\infty$$

the processes $(M^n_{t\wedge\tau_n})_{t\geq 0}$ and $((M^n_{t\wedge\tau_n})^2 - < M^n >_{t\wedge\tau_n})_{t\geq 0}$ are both martingales. By Optional sampling theorem

$$\mathbb{E}[(M^n_{t\wedge\tau_n})^2 - < M^n >_{t\wedge\tau_n}] = 0$$

which implies

$$\mathbb{E}\left[\int_0^t 1_{[0,\tau_n]}(s)\sigma_n^2 e^{-\frac{4U_n(Y_s^n)}{\sigma_n^2}} ds\right] = \mathbb{E}[F_n^2(Y^n_{t\wedge\tau_n})]$$

and

$$\sigma_n^2 \exp\left(-\frac{4}{\sigma_n^2} \max_{y\in[a_n,b_n]} U_n(y)\right) \mathbb{E}[t \wedge \tau_n] \leq \max_{y\in[a_n,b_n]} F_n^2(y).$$

thus there is a constant $K_n > 0$ such that

$$\forall t \geq 0, \mathbb{E}[t \wedge \tau_n] \leq K_n$$

We conclude by dominated convergence that τ_n is integrable. The martingale property implies

$$\mathbb{E}[F_n(Y^n_{t\wedge\tau_n})] = 0$$

which yields

$$\mathbb{E}[F_n(Y^n_{\tau_n})] = 0,$$

by dominated convergence because

$$\forall t \geq 0, |F_n(Y^n_{t\wedge\tau_n})| \leq \max_{y\in[a_n,b_n]} |F_n(y)|.$$

This is equivalent to

$$F_n(a_n)(1 - p(a_n, b_n)) + F_n(b_n)p(a_n, b_n) = 0$$

with $p(a_n, b_n) = \mathbb{P}(Y^n_{\tau_n} = b_n)$. Hence,

$$p(a_n, b_n) = \frac{-F_n(a_n)}{F_n(b_n) - F_n(a_n)}$$

Moreover,

$$\begin{aligned}\mathbb{P}(\theta_n < T_n) &= \mathbb{P}(X^n_{\tau_n} = b) \\ &= \mathbb{P}(Y^n_{\tau_n} = b - x_n) \\ &= p(a_n, b_n)\end{aligned}$$

Using the uniform convergence of F_n, we have

$$\begin{aligned}\lim_{n\to+\infty} \mathbb{P}(\theta_n < T_n) = &\lim_{n\to+\infty} p(a_n, b_n) \\ =&\frac{-F(a-b)}{F(0) - F(a-b)} \\ =&1\end{aligned}$$

$\diamond$

Lemma 13 *Let $a < b$ and $(x_n)_{n\geq 0}$ a sequence of real numbers such that $\lim_{n\to+\infty} x_n = b$ and $\min_n x_n > a$. Let $(X^n_t)_{n\geq 0}$ the solution to*

$$\begin{cases} dX^n_t = \mu_n(X^n_t)dt + \sigma_n dW_t \\ X^n_0 = x_n \end{cases}$$

with the same assumptions as in Lemma 12. There exist constants A_n and B_n such that

$$\exp\left(-\frac{b - x_n}{\sigma_n^2}(\sqrt{A_n^2 + 2r\sigma_n^2} - A_n)\right) \leq \mathbb{E}[e^{-r\theta_n}] \leq \exp\left(-\frac{b - x_n}{\sigma_n^2}(\sqrt{B_n^2 + 2r\sigma_n^2} - B_n)\right) \tag{22}$$

Proof: Because μ_n are bounded functions, there are two constants A_n and B_n such that $A_n \leq \mu_n(x) \leq B_n$ for all $a < x < b$. We define $\tilde{X}^n_t = x_n + A_n t + \sigma_n W_t$. By comparison, we have $\tilde{X}_t^n \leq X_t^n$ and $\theta_n \leq \tilde{\theta}_n$, with $\tilde{\theta}_n = \inf\{t \geq 0, \tilde{X}_t^n = b\}$. But the Laplace transform of $\tilde{\theta}_n$ is explicit and given by

$$\mathbb{E}[e^{-r\tilde{\theta}_n}] = \exp\left(-\frac{b - x_n}{\sigma_n^2}(\sqrt{A_n^2 + 2r\sigma_n^2} - A_n)\right)$$

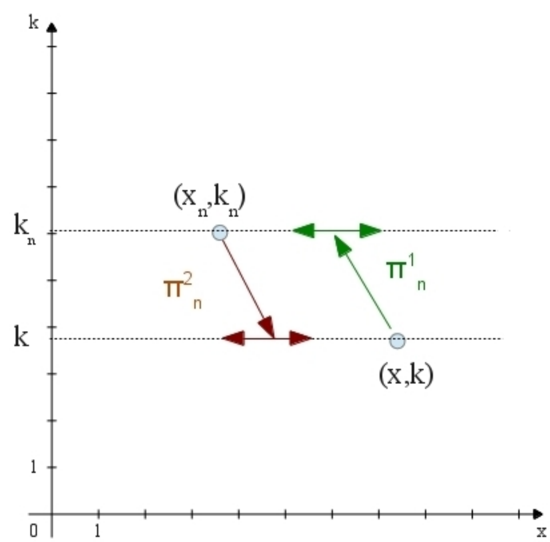
which gives the left inequality of (22). The proof is similar for the right inequality introducing $\bar{X}_t^n = x_n + B_n t + \sigma_n W_t$. $\diamond$

Proposition 6 *The shareholders value function is jointly continuous.*

Proof: Let $(x, k) \in S$ and let us consider (x_n, k_n) a sequence in S converging to (x, k). Therefore, $\{(x_n - \gamma|k - k_n|, k), (x - \gamma|k - k_n|, k_n)\} \in S^2$ for n large enough. We consider the following two strategies that are admissible for n large enough:

- Strategy π_n^1: start from (x, k), invest if $k_n - k > 0$(or disinvest if $k_n - k < 0$) and do nothing up to the minimum between the liquidation time and the hitting time of (x_n, k_n). Denote $(X_t^{\pi_n^1}, K_t^{\pi_n^1})_{t \geq 0}$ the control process associated to strategy π_n^1.
- Strategy π_n^2: start from (x_n, k_n), invest if $k_n - k < 0$(or disinvest if $k_n - k > 0$) and do nothing up to the minimum between the liquidation time and the hitting time of (x, k). Denote $(X_t^{\pi_n^2}, K_t^{\pi_n^2})_{t \geq 0}$ the control process associated to strategy π_n^2.

To fix the idea, assume $k_n > k$. The strategy π_1 makes the process (X, K) jump from (x, k) to $(x - \gamma(k_n - k), k_n)$.



Define

$$\theta_n^1 = \inf\{t \geq 0, (X_t^{\pi_n^1}, K_t^{\pi_n^1}) = (x_n, k_n)\},$$

$$\theta_n^2 = \inf\{t \geq 0, (X_t^{\pi_n^2}, K_t^{\pi_n^2}) = (x, k)\},$$

$$T_n^1 = \inf\{t \geq 0, X_t^{\pi_n^1, x} \leq \gamma K_t^{\pi_n^1, k}\}$$

and

$$T_n^2 = \inf\{t \geq 0, X_t^{\pi_n^2,x_n} \leq \gamma K_t^{\pi_n^2,k_n}\}$$

Dynamic programming principle and $V^*(X_{T_n^1}, K_{T_n^1}) = 0$ on $T_n^1 \leq \theta_n^1$ yield

$$\begin{aligned}
V^*(x,k) &\geq \mathbb{E}\left[\int_0^{\theta_n^1 \wedge T_n^1} e^{-rt} dZ_t^{\pi_n^1} + e^{-r(\theta_n^1 \wedge T_n^1)} 1_{\{\theta_n^1 < T_n^1\}} V^*(X_{\theta_n^1}, K_{\theta_n^1})\Big)\right] \\
&\geq \mathbb{E}\left[e^{-r\theta_n^1} 1_{\{\theta_n^1 < T_n^1\}} V^*(x_n,k_n)\right] \\
&\geq \Big(\mathbb{E}\big(e^{-r\theta_n^1}\big) - \mathbb{E}\big(e^{-r\theta_n^1} 1_{\{\theta_n^1 \geq T_n^1\}}\big)\Big) V^*(x_n,k_n) \\
&\geq \Big(\mathbb{E}\big(e^{-r\theta_n^1}\big) - \mathbb{P}\big(\theta_n^1 \geq T_n^1\big)\Big) V^*(x_n,k_n)
\end{aligned} \tag{23}$$

On the other hand, using $V^*(X_{T_n^2}, K_{T_n^2}) = 0$ on $T_n^2 \leq \theta_n^2$

$$\begin{aligned}
V^*(x_n,k_n) &\geq \mathbb{E}\left[\int_0^{\theta_n^2 \wedge T_n^2} e^{-rt} dZ_t^{\pi_n^2} + e^{-r(\theta_n^2 \wedge T_n^2)} 1_{\{\theta_n^2 < T_n^2\}} V^*(X_{\theta_n^2}, K_{\theta_n^2})\Big)\right] \\
&\geq \mathbb{E}\left[e^{-r\theta_n^2} 1_{\{\theta_n^2 < T_n^2\}} V^*(x,k)\right] \\
&\geq \Big(\mathbb{E}\big(e^{-r\theta_n^2}\big) - \mathbb{E}\big(e^{-r\theta_n^2} 1_{\{\theta_n^2 \geq T_n^2\}}\big)\Big) V^*(x,k) \\
&\geq \Big(\mathbb{E}\big(e^{-r\theta_n^2}\big) - \mathbb{P}\big(\theta_n^2 \geq T_n^2\big)\Big) V^*(x,k)
\end{aligned} \tag{24}$$

The convergence of (x_n, k_n) implies

$$\lim_{n \to +\infty} (x_n - \gamma|k - k_n|, k) = (x,k)$$

from which we deduce using Lemma 12 that

$$\lim_{n \to +\infty} \mathbb{P}(\theta_n^1 \geq T_n^1) = 0 \tag{25}$$

and

$$\lim_{n \to +\infty} \mathbb{P}(\theta_n^2 \geq T_n^2) = 0 \tag{26}$$

Let $\mu_n(X_t^n) = \beta(k_n)\mu - \alpha((k_n - X_t^n)^+)$ and $\sigma_n = \beta(k_n)\sigma$. The function μ_n is bounded by

$$\begin{aligned}
A_n &= \beta(k_n)\mu - \alpha(k_n) \\
B_n &= \beta(k_n)\mu
\end{aligned}$$

thus, according to Lemma 13

$$\exp\left(-\frac{\kappa^n}{\sigma_n^2}(\sqrt{A_n^2 + 2r\sigma_n^2} - A_n)\right) \leq \mathbb{E}[e^{-r\theta_n^1}] \leq \exp\left(-\frac{\kappa^n}{\sigma_n^2}(\sqrt{B_n^2 + 2r\sigma_n^2} - B_n)\right)$$

with $\kappa^n = x - x^n + \gamma|k^n - k|$.
Letting n tend to $+\infty$ and using

$$\begin{array}{ll} \lim_{n\to+\infty} A_n & = \beta(k)\mu - \alpha(k) \\ \lim_{n\to+\infty} B_n & = \beta(k)\mu \\ \lim_{n\to+\infty} \sigma_n & = \beta(k)\sigma \end{array}$$

we obtain

$$\lim_{n\to+\infty} \mathbb{E}(e^{-r\theta_n^1}) = \lim_{n\to+\infty} \mathbb{E}(e^{-r\theta_n^2}) = 1 \tag{27}$$

Finally, we have from (23) and (24),

$$V^*(x,k) \geq \limsup_n V^*(x_n,k_n) \geq \liminf_n V^*(x_n,k_n) \geq V^*(x,k),$$

which proves the continuity of V^*. ◇

We are now in a position to characterize the shareholders value in terms of viscosity solution of the free boundary problem (21).

Theorem 1 *The shareholders value V^* is the unique continuous viscosity solution to* (21) *on S with linear growth.*

Proof: The proof is postponed to the Appendix ◇

4.2 Absence of Investment cost

In this section, we will assume that there is no cost of investment/disinvestment, that is $\gamma = 0$. Let us define

$$H(k)w(x) = (\beta(k)\mu - \alpha((k-x)^+))w'(x) + \frac{\sigma^2\beta(k)^2}{2}w''(x) - rw(x).$$

We will construct a C^2 solution (w,b) to the free boundary problem

$$\max_{k\geq 0} H(k)w(x) = 0 \text{ and } w'(x) \geq 1 \text{ for } 0 \leq x \leq b \tag{28}$$

and

$$\max_{k\geq 0} H(k)w(x) \leq 0 \text{ and } w'(x) = 1 \text{ for } x \geq b \tag{29}$$

which will characterize the shareholders value using a verification theorem similar to Proposition 2 whose proof is omitted. First, proceeding analogously as in the proof of Corollary 1, it is straighforward to prove

Corollary 5 *If $\mu\beta'(0) \leq r$ then it is optimal to liquidate the firm thus $v^*(x) = x$.*

Hereafter, we will assume that $\mu\beta'(0) > r$.

4.2.1 High cost of debt: $\alpha^{'}(0) > \mu\beta^{'}(0)$

Let us define

$$\delta = \frac{2r\sigma^2}{\mu^2 + 2r\sigma^2} \tag{30}$$

and a the unique nonzero solution (if it exists) of the equation

$$\sigma^2(1-\delta)\beta(a) = \mu a. \tag{31}$$

Because β is concave and β' goes to 0 , the existence of a is equivalent to assume

$$\sigma^2\beta'(0) \geq \frac{\mu}{(1-\delta)} \tag{32}$$

Let us define the function w_A for $A > 0$ as the unique solution on $(a, +\infty)$ of the Cauchy problem

$$\mu\beta(x)w_A'(x) + \frac{\sigma^2\beta(x)^2}{2}w_A''(x) - rw_A(x) = 0$$

with $w_A(x) = Ax^\delta$ for $0 \leq x \leq a$ and w_A differentiable at a.

Remark 4 *The Cauchy problem is well defined with the condition w_A differentiable at a. Moreover, it is easy to check, using the definition of a, that the function w_A is also C^2. Because the cost of debt α is high, the shareholders optimally choose not to issue debt but rather adjust costlessly their level of investment.*

Lemma 14 *For every $A > 0$ the function w_A is increasing.*

Proof: Clearly, w_A is increasing and thus positive on $[0, a]$. Let $c = \min\{x > a\,, w_A'(c) = 0\}$. $w_A(c) > 0$ because w_A is increasing and positive in a left neighborhood of c. Thus, according to the differential equation, we have $w_A''(c) \geq 0$ which implies that w_A is also increasing in a right neighborhood of c. Therefore, w_A' cannot become negative. $\diamond$

Lemma 15 *For every $A > 0$, there is some b_A such that $w_A''(b_A) = 0$ and w_A is a concave function on $]a, b_A[$.*

Proof: Assume by a way of contradiction that w_A'' does not vanish. Using Equations (30) and (31), we have

$$\frac{\sigma^2\beta^2(a)}{2}w_A''(a) = -rAa^\delta.$$

Therefore, we equivalently assume that $w_A'' < 0$. This implies that w_A' is stricly decreasing and bounded below by 0 by lemma 14 therefore w_A is an increasing concave function. Therefore, $\lim_{x\to+\infty} w_A'(x)$ exists and is denoted by l. Letting $x \to +\infty$ in the differential equation, we obtain, because β has a finite limit,

$$\frac{\sigma^2\bar{\beta}^2}{2}\lim_{x\to\infty} w_A''(x) = r\lim_{x\to\infty} w_A(x) - \mu\bar{\beta}l.$$

Therefore, either $\lim_{x\to\infty} w_A(x)$ is $+\infty$ from which we get a contradiction or finite from which we get $\lim_{x\to+\infty} w_A''(x) = 0$ by mean value theorem. In the second case, differentiating the differential equation, we have

$$\mu\beta'(x)w_A'(x) + \mu\beta(x)w_A''(x) + \sigma^2\beta'(x)\beta(x)w_A''(x) + \frac{\sigma^2\beta(x)^2}{2}w_A'''(x) - rw_A'(x) = 0 \qquad (33)$$

Proceeding analogously, we obtain that $\lim_{x\to+\infty} w_A'''(x) = 0$ and thus $l = 0$. Coming back to the differential equation, we get

$$0 = r\lim_{x\to\infty} w_A(x)$$

which contradicts that w_A is increasing. Now, define $b_A = \inf\{x \geq a,\, w_A''(x) = 0\}$ to conclude. $\diamond$

Lemma 16 *There exists A^* such that $w_{A^*}'(b_{A^*}) = 1$.*

Proof: For every $A > 0$, we have

$$\mu\beta(b_A)w_A'(b_A) = rw_A(b_A). \qquad (34)$$

Let $A_1 = \frac{\mu\bar{\beta}}{ra^\delta}$. Lemma 14 yields

$$\begin{aligned} w_{A_1}(b_{A_1}) &\geq w_{A_1}(a) \\ &= \frac{\mu\bar{\beta}}{r} \\ &\geq \frac{\mu\beta(b_{A_1})}{r} \end{aligned}$$

Therefore, Equation (34) yields $w_{A_1}'(b_{A_1}) \geq 1$.
On the other hand, let $A_2 = \frac{a^{1-\delta}}{\delta}$. By construction, $w_{A_2}'(a) = 1$ and thus $w_{A_2}'(b_{A_2}) \leq 1$ by concavity of w_A on $(0, b_A)$. Thus, there is some $A^* \in [\min(A_1, A_2), \max(A_1, A_2)]$ such that $w_{A^*}' = 1$. $\diamond$

Hereafter, we denote $b = b_{A^*}$.

Lemma 17 *We have $\mu\beta'(b) \leq r$.*

Proof: Differentiating the differential equation and plugging $x = b$, we get

$$\frac{\sigma^2\beta(b)^2}{2}w_A'''(b) + \mu\beta'(b) - r = 0$$

Because w_{A^*}'' is increasing in a left neighborhood of b, we have $w_A'''(b) \geq 0$ implying the result. $\diamond$

Let us define

$$v = \begin{cases} w_{A^*}(x) & x \leq b \\ x - b + \frac{\mu\beta(b)}{r} & x \geq b \end{cases}$$

We are in a position to prove the main result of this section.

Proposition 7 *The shareholders value is v.*

Proof: We have to check that (v, b) satisfies the free boundary problem (28) and (29). By construction, v is a C^2 concave function on $(0, +\infty)$ satisfying $v' \geq 1$. It remains to check $\max_k H(k)v(x) \leq 0$.
For $x > b$, we have

$$H(k)v(x) = \mu\beta(k) - \alpha((k-x)^+) - \mu\beta(b) - r(x-b).$$

If $k \leq x$, concavity of β and Lemma 17 implies

$$\begin{aligned} H(k)v(x) &= \mu(\beta(x) - \beta(b)) - r(x-b) \\ &\leq (\mu\beta'(b) - r)(x-b) \\ &\leq 0. \end{aligned}$$

If $k \geq x$, we differentiate $H(k)v(x)$ with respect to k and obtain using again concavity of β and convexity of α,

$$\frac{\partial H(k)v(x)}{\partial k} = \mu\beta'(k) - \alpha'(k-x) \leq \mu\beta'(0) - \alpha'(0) \leq 0.$$

Therefore, $H(k)v(x) \leq H(x)v(x) \leq 0$.
Let $x < b$, because v is concave, the same argument as in the previous lines shows that

$$\frac{\partial H(k)v(x)}{\partial k} \leq 0 \text{ for } k \geq x$$

and therefore

$$\max_{k \geq 0} H(k)v(x) = \max_{k \leq x} H(k)v(x).$$

First order condition gives for $0 \leq k < x$

$$\begin{aligned} \frac{\partial}{\partial k}(H(k)v) &= \mu\beta'(k)v'(x) + \sigma^2\beta'(k)\beta(k)v''(x) \\ &= \beta'(k)[\mu v'(x) + \sigma^2\beta(k)v''(x)] \end{aligned}$$

Thus for $0 < x < a$, we have

$$\frac{\partial}{\partial k}(H(k)v) = \beta'(k)A^*x^{\delta-2}\delta[\mu x + \sigma^2\beta(k)(\delta-1)]$$

which gives,

$$\begin{cases} \dfrac{\partial}{\partial k}(H(0)v) > 0 \\ \dfrac{\partial}{\partial k}(H(x)v) < 0 \end{cases}$$

Therefore the maximum $k^*(x)$ of $H(k)v(x)$ lies in the interior of the interval $[0, x]$ and satisfies:

$$\forall 0 < x < a, \beta(k^*(x)) = \frac{\mu x}{\sigma^2(1-\delta)}$$

Hence, for $x \leq a$, we have by construction

$$\begin{aligned}\max_{0\leq k\leq x}\{H(k)v\} &= \frac{\mu^2 x}{\sigma^2(1-\delta)}A^*\delta x^{\delta-1} + \frac{\sigma^2\mu^2x^2}{2\sigma^4(1-\delta)^2}A^*\delta(\delta-1)x^{\delta-2} - rA^*x^\delta \\ &= 0\end{aligned}$$

Now, fix $x \in (a,b)$. We note that $\dfrac{\partial}{\partial k}(H(k)v)$ has the same sign as $\mu v'(x) + \sigma^2\beta(k)v''(x)$ because β is strictly increasing. Moreover, because v is concave and β increasing, we have

$$\min_{0\leq k\leq x} \mu v'(x) + \sigma^2\beta(k)v''(x) = \mu v'(x) + \sigma^2\beta(x)v''(x).$$

Thus, it suffice to prove $\mu v'(x) + \sigma^2\beta(x)v''(x) \geq 0$ for $x \in (a,b)$ or equivalently because β is a positive function that the function ϕ defined as

$$\phi(x) = \mu\beta(x)v'(x) + \sigma^2\beta(x)^2v''(x)$$

is positive. We make a proof by contradiction assuming there is some x such that $\phi(x) < 0$. As $\phi(a) = 0$ by Equation 31 and $\phi(b) > 0$ then there is some $x_1 \in [a,b]$ such that

$$\begin{cases} \phi(x_1) < 0 \\ \phi'(x_1) = 0 \end{cases}$$

Using the differential equation (33) satisfied by v', we obtain

$$\phi'(x_1) = (2r - \mu\beta'(x_1))v'(x_1) - \mu\beta(x_1)v''(x_1) = 0$$

from we deduce

$$\begin{aligned}\phi(x_1) &= \mu\beta(x_1)v'(x_1) + \sigma^2\beta(x_1)^2v''(x_1) \\ &= \mu\beta(x_1)v'(x_1) + \frac{\sigma^2\beta(x_1)}{\mu}(2r - \mu\beta'(x_1))v'(x_1) \\ &= \beta(x_1)v'(x_1)(\mu + \frac{2r\sigma^2}{\mu} - \sigma^2\beta'(x_1))\end{aligned}$$

But $x_1 \geq a$ and thus $\beta'(x_1) \leq \beta'(a)$. Moreover, by definition of a, we have $\sigma^2\beta'(a) \leq \frac{\mu}{(1-\delta)}$. Therefore, Equation (30) yields

$$\begin{aligned}\phi(x_1) &\geq \beta(x_1)v'(x_1)\left(\mu + \frac{2r\sigma^2}{\mu} - \frac{\mu}{1-\delta}\right) \\ &\geq \beta(x_1)v'(x_1)\left(\frac{2r\sigma^2}{\mu} - \mu\frac{\delta}{1-\delta}\right) \\ &\geq \beta(x_1)v'(x_1)\left(\frac{2r\sigma^2}{\mu} - \mu\frac{2r\sigma^2}{\mu^2+2r\sigma^2}\frac{\mu^2+2r\sigma^2}{\mu^2}\right) \\ &= 0\end{aligned}$$

which is a contradiction. ◇

Figure 3 plots some shareholders value functions whith $\alpha'(0) > \mu\beta'(0)$ and $\sigma^2\beta'(0) \geq \frac{\mu}{(1-\delta)}$ for different values of $\beta'(0)$ and using :

- a linear function for α, $\alpha(x) = \lambda x$.
- an exponential function for β, $\beta(x) = \beta_{max}\left(1 - e^{\frac{-\beta'(0)}{\beta_{max}}x}\right)$.

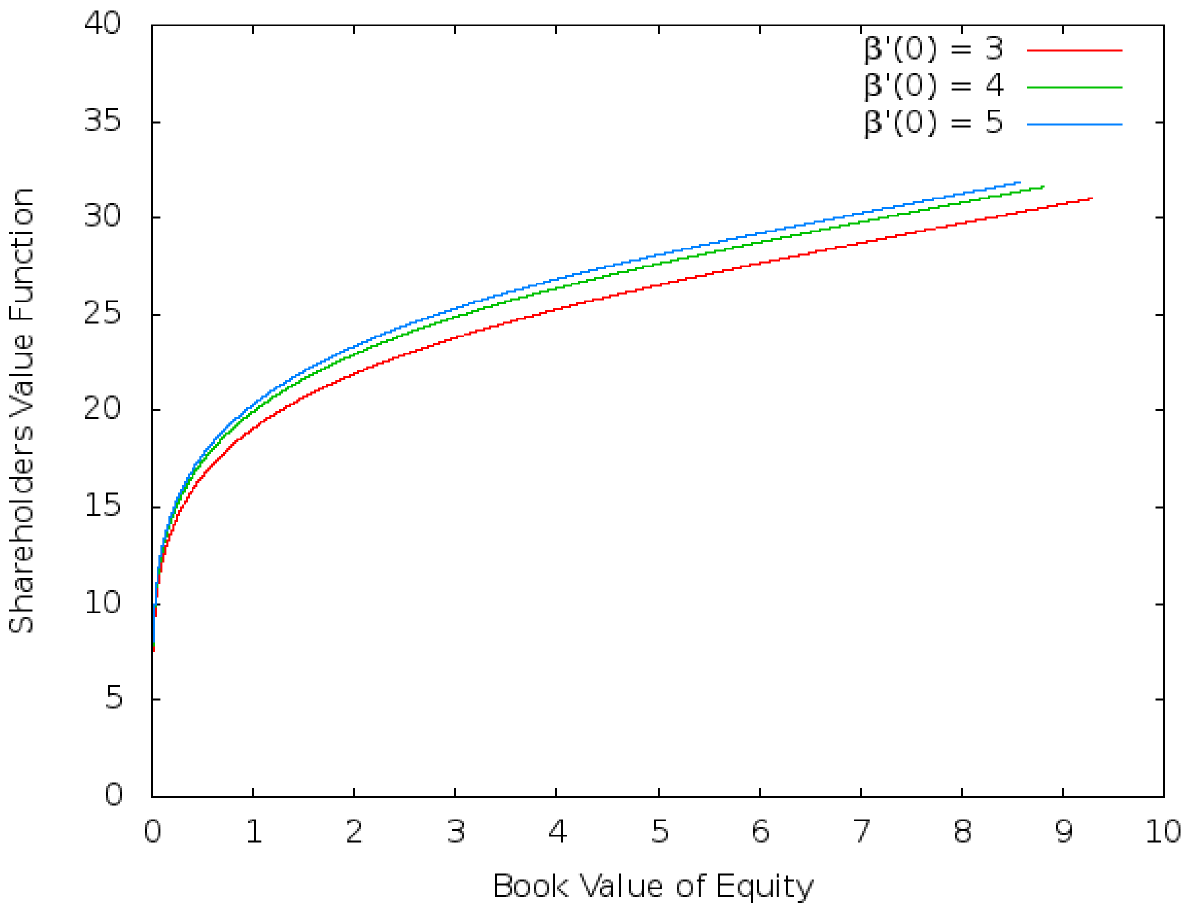


Figure 3: Comparing shareholders value functions with $\mu = 2$, $r = 0.5$, $\sigma = 1$, $\lambda = 10$, $\beta_{max} = 8$, for different values of $\beta'(0)$.

Remark 5 *This result gives us the best strategy in terms of investment/disinvestment. When there is a high cost of debt and when there is no cost of investment/disinvestment, the optimal level of productive assets is given, when* (32) *is fulfilled, by*

$$\forall 0 \leq x \leq a, k(x) = \beta^{-1}\left[\frac{\mu x}{\sigma^2(1-\delta)}\right]$$
$$\forall x \geq a, k(x) = x$$

It shows that the manager shouldn't invest all the cash reserve in productive assets when the book value of equity is low, even if there is no cost of investment/disinvestment . The manager optimally disinvests to lower the volatility of the book value of equity.

Figure 4 plots the optimal level of productive assets for different values of σ. It shows that, for a given level of the book value of equity, the investment level in productive assets is a decreasing function of the volatility.

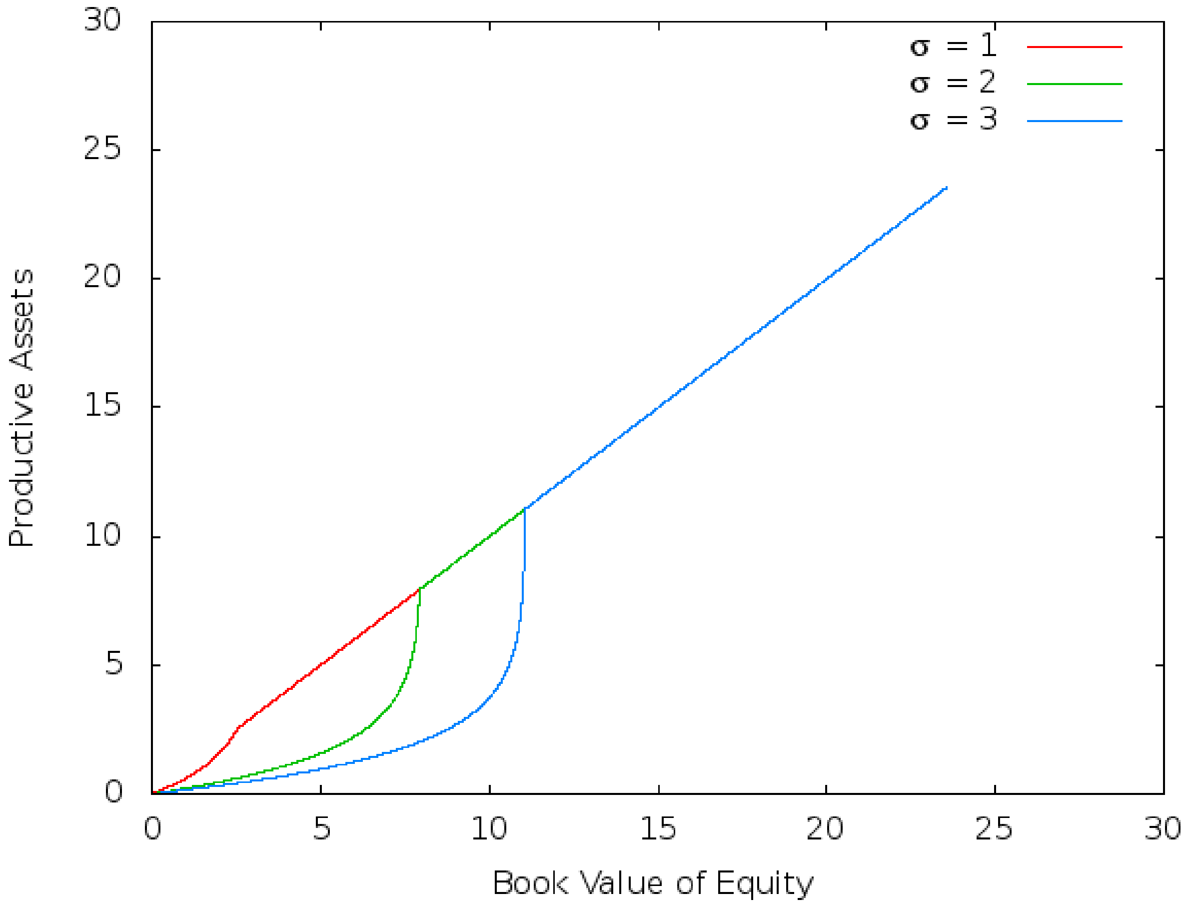


Figure 4: Comparing optimal level of productive assets with $\mu = 2$, $r = 0.5$, $\lambda = 10$, $\beta_{max} = 8$, $\beta'(0) = 5$ for different values of σ.

To complete the characterization of the shareholders value when the cost of debt is high, we have to study the optimal policy when (32) is not fulfilled. We expect that $a = 0$ in that case which means that for all x, the manager should invest all the cash in productive assets. Thus we are interested in the solutions to

$$\mu\beta(x)w'(x) + \frac{\sigma^2\beta(x)^2}{2}w''(x) - rw(x) = 0 \tag{35}$$

such that $w(0) = 0$.

Proposition 8 *Suppose that the functions $x \to \frac{x}{\beta(x)}$ and $x \to \frac{x^2}{\beta(x)^2}$ are analytics in 0 with a radius of convergence R. The solutions w to Equation (35) such that $w(0) = 0$ are given* *by*

$$w(x) = \sum_{k=0}^{\infty} A_k x^{k+y_1}$$

with

$$\forall k \geq 1, A_k = \frac{1}{-I(k+y_1)} \sum_{j=0}^{k-1} \frac{(j+y_1)p^{(k-j)}(0) + q^{(k-j)}(0)}{(k-j)!} A_j$$

where the functions p and q are

$$\begin{cases} p(x) = \dfrac{2\mu x}{\sigma^2 \beta(x)} \\ q(x) = -\dfrac{2rx^2}{\sigma^2 \beta(x)^2} \end{cases}$$

the function I is given by

$$I(y) = \mu\beta'(0)y + \frac{\sigma^2}{2}\beta'(0)^2 y(y-1) - r$$

and y_1 is the positive root of I

$$y_1 = \frac{-\mu + \frac{\sigma^2}{2}\beta'(0) + \sqrt{(\mu - \frac{\sigma^2}{2}\beta'(0))^2 + 2r\sigma^2}}{\sigma^2\beta'(0)}.$$

The radius of convergence of w is at least equal to R.

Proof: This result is given by the Fuchs' theorem [18]. ◇

Note that the solutions of Equation (35) vanishing at zero can be written

$$w_{A_0}(x) = A_0 w_1(x).$$

If the radius of convergence of the Frobenius serie is finite, then the previously defined function w_1 can be extended by use of the Cauchy theorem.

Because $\mu\beta'(0) \geq r$, we have $y_1 < 1$. As a consequence, we have

$$\lim_{x \to 0} w_1'(x) = +\infty \text{ and } \lim_{x \to 0} w_1''(x) = -\infty$$

Thus, proceeding analogously as in Lemma 15, we prove the existence of b such that $w_1''(b) = 0$. Because w_{A_0} is linear in A_0, we choose $A_0 = A^* = \frac{1}{w_1'(b)}$ to get a concave solution w^* to (35) with $w^*(0) = 0$, $(w^*)'(b) = 1$ and $(w^*)''(b) = 0$. We extend $w*$ linearly on $(b, +\infty)$ as usual to obtain a C^2 function on $[0, +\infty[$.

Proposition 9 *The shareholders value is w^*.*

Proof: It suffices to check that w^* satisfies the free boundary problem (28) and (29). By construction w^* is a C^2 concave function on $\mathbb{R}^{+*}$. Because $(w^*)'(b) = 1$, we have

$$\forall x \in]0, b], (w^*)'(x) \geq 1$$

and

$$\forall x \geq b, (w^*)'(x) = 1$$

On $[b, +\infty[$, we have

$$\begin{aligned}
\max_{k\geq 0}\{H_{x,k}w^*\} &= \max_{k\geq 0}\Big[\mu\beta(k) - \alpha((k-x)^+) - \mu\beta(b) + r(b-x)\Big] \\
&= \max\Big[\max_{k\leq x}\mu\beta(k) - \mu\beta(b) + r(b-x), \\
&\qquad \max_{k\geq x}\mu\beta(k) - \alpha(k-x) - \mu\beta(b) + r(b-x)\Big]
\end{aligned}$$

Using β concave increasing, α convexe, $\alpha'(0^+) > \mu\beta'(0^+)$, we have

$$\max_{k\geq 0}\{H_{x,k}w^*\} = \mu\beta(x) - \mu\beta(b) + r(b-x)$$

Then using the concavity of β,

$$\forall x \geq b, \max_{k\geq 0}\{H_{x,k}w^*\} \leq 0$$

It remains to show that for every $x < b$

$$\max_{k\geq 0}\{H_{x,k}w^*\} = 0$$

Using β concave , α convexe, $\alpha'(0) > \mu\beta'(0)$ and w^* concave increasing, we have

$$\forall k > x, \frac{\partial}{\partial k}(H_{x,k}w^*) = (\mu\beta'(k) - \alpha'(k-x))(w^*)'(x) + \sigma^2\beta'(k)\beta(k)(w^*)''(x) \leq 0$$

thus,

$$\max_{k\geq 0}\{H_{x,k}(w^*)\} = \max_{0\leq k\leq x}\{H_{x,k}(w^*)\}$$

Moreover,

$$\begin{aligned}
\forall 0 < k < x, \frac{\partial}{\partial k}(H_{x,k}(w^*)) &= \mu\beta'(k)(w^*)'(x) + \sigma^2\beta'(k)\beta(k)(w^*)''(x) \\
&= \beta'(k)[\mu(w^*)'(x) + \sigma^2\beta(k)(w^*)''(x)]
\end{aligned}$$

We expect

$$\forall x \in]0, b], \forall k \leq x, \frac{\partial}{\partial k}(H_{x,k}(w^*)) \geq 0$$

Notice that $\beta'(k) \geq 0$ and

$$\min_{0\leq k\leq x}\mu(w^*)'(x) + \sigma^2\beta(k)(w^*)''(x) = \mu(w^*)'(x) + \sigma^2\beta(x)(w^*)''(x)$$

because $(w^*)''(x) \leq 0$ and β is increasing. Thus it is enough to prove for every $x < b$,

$$\mu(w^*)'(x) + \sigma^2\beta(x)(w^*)''(x) \geq 0$$

or equivalently, using $\beta \geq 0$,

$$\phi(x) = \mu\beta(x)(w^*)'(x) + \sigma^2\beta(x)^2(w^*)''(x) \geq 0$$

for $x < b$. We make a proof by contradiction assuming the existence of x such that $\phi(x) < 0$. In a neighborhood of 0, we have

$$(w^*)'(x) \sim A^* y_1 x^{y_1 - 1}$$

and

$$(w^*)''(x) \sim A^* y_1(y_1 - 1)x^{y_1 - 2}$$

From which we deduce because $\beta(x)x^{y_1-1} \leq \beta'(0)x^{y_1}$,

$$\lim_{x \to 0} \beta(x)(w^*)'(x) = 0$$

$$\lim_{x \to 0} \beta(x)^2(w^*)''(x) = 0$$

yielding

$$\lim_{x \to 0} \phi(x) = 0$$

But $\phi(b) > 0$ thus there is $x_1 \in]0, b[$ such that

$$\begin{cases} \phi(x_1) < 0 \\ \phi'(x_1) = 0 \end{cases}$$

Using the derivative of Equation (35)

$$\phi'(x_1) = (2r - \mu\beta'(x_1))(w^*)'(x_1) - \mu\beta(x_1)(w^*)''(x_1) = 0$$

from which we deduce :

$$\begin{aligned} \phi(x_1) &= \mu\beta(x_1)(w^*)'(x_1) + \sigma^2\beta(x_1)^2(w^*)''(x_1) \\ &= \mu\beta(x_1)(w^*)'(x_1) + \frac{\sigma^2\beta(x_1)}{\mu}(2r - \mu\beta'(x_1))(w^*)'(x_1) \\ &= \beta(x_1)(w^*)'(x_1)(\mu + \frac{2r\sigma^2}{\mu} - \sigma^2\beta'(x_1)) \end{aligned}$$

Now, remember that $x_1 > 0$ and thus using the concavity of β, we have $\beta'(x_1) \leq \beta'(0)$. Furthermore, $\beta'(0) \leq \frac{\mu^2 + 2r\sigma^2}{\sigma^2\mu}$ when Equation (31) is not fulfilled. Hence,

$$\begin{aligned} \phi(x_1) &\geq \beta(x_1)(w^*)'(x_1)\left(\mu + \frac{2r\sigma^2}{\mu} - \frac{\mu^2 + 2r\sigma^2}{\mu}\right) \\ &\geq 0 \end{aligned}$$

which yields to a contradiction and ends the proof. $\diamond$

Figure 5 plots some shareholders value functions whith $\alpha'(0) > \mu\beta'(0)$ and $\sigma^2\beta'(0) \leq \frac{\mu}{(1-\delta)}$ for different values of $\beta'(0)$ and using :

- a linear function for α, $\alpha(x) = \lambda x$.
- an exponential function for β, $\beta(x) = \beta_{max}\left(1 - e^{\frac{-\beta'(0)}{\beta_{max}}x}\right)$.

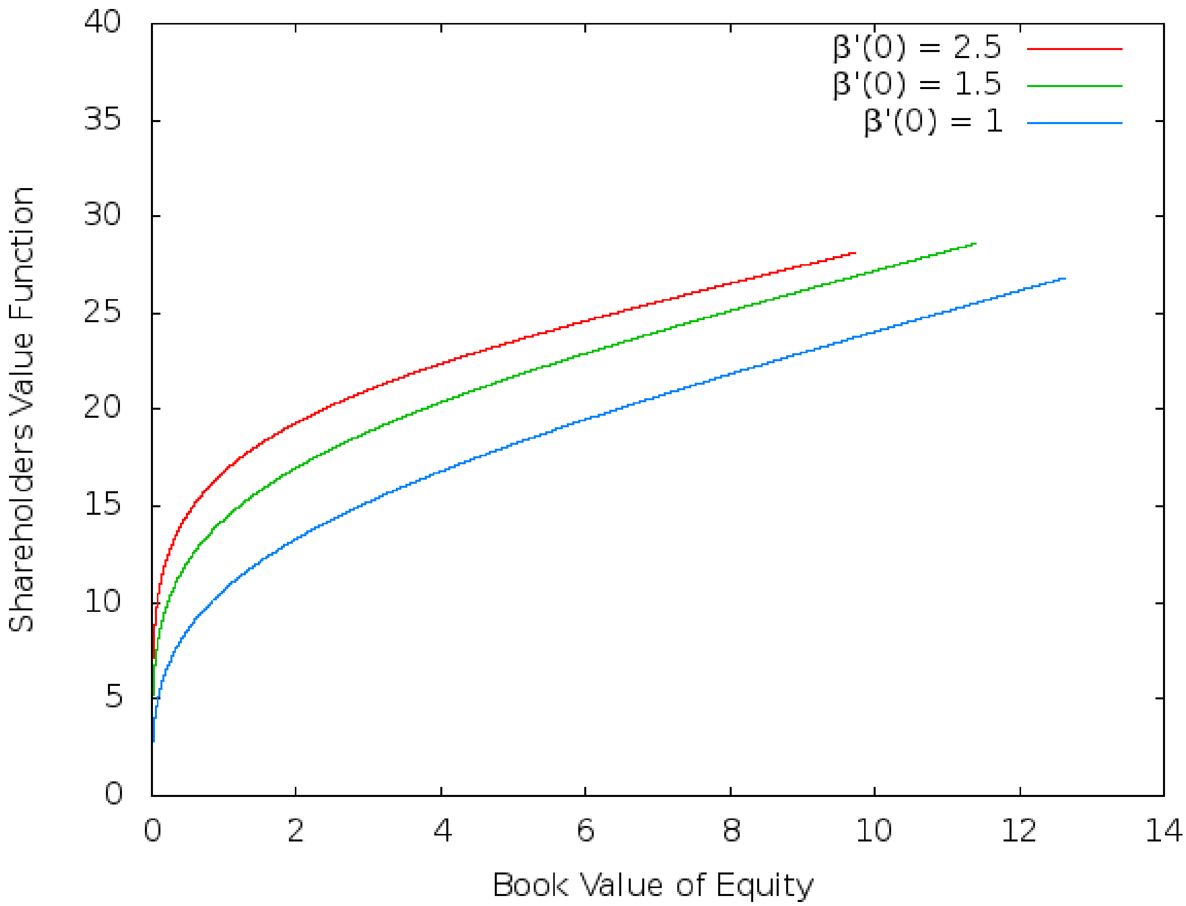


Figure 5: Comparing shareholders value functions with $\mu = 2$, $r = 0.5$, $\sigma = 1$, $\lambda = 10$, $\beta_{max} = 8$, for different values of $\beta'(0)$.

4.2.2 Low cost of debt: $\alpha^{'}(0) \leq \mu\beta^{'}(0)$

When the cost of debt is low, it can be optimal to issue debt in order to increase the investment level. There is a trade-off between the cost of debt α and the investment gain function β. The following lemma gives us a boundary for the cost of the debt when (32) is not fulfilled.

Lemma 18 *If $\alpha'(0^+) < \mu\beta'(0) - \sigma^2\beta'(0)^2(1-r)$ and $\sigma^2\beta'(0) < \frac{\mu}{(1-\delta)}$ then it exists $d > 0$ such that*

$$\forall x \in]0, d[, \max_{k\geq 0}\{H_{x,k}(w^*)\} = \max_{k>x}\{H_{x,k}(w^*)\}$$

Proof: We can show as in Proposition 9 that

$$\forall k < x, \frac{\partial}{\partial k} H_{x,k} w^* \geq 0$$

so,

$$\max_{k \geq 0}\{H_{x,k}(w^*)\} = \max_{k \geq x}\{H_{x,k}(w^*)\}$$

We make a proof by contradiction assuming that in the neighborhood of 0, we have

$$\max_{k \geq 0}\{H_{x,k}(w^*)\} = H_{x,x} w^*$$

thus we have

$$(w^*)'(x) \sim A^* y_1 x^{y_1 - 1}$$

and

$$(w^*)''(x) \sim A^* y_1 (y_1 - 1) x^{y_1 - 2}$$

from which we deduce that in a neighborhood of 0

$$\begin{aligned} \frac{\partial}{\partial k} H_{x,x^+}(w^*) &\sim (\beta'(x) - \alpha'(0^+)) A^* y_1 x^{y_1 - 1} + \beta'(x)\beta(x) A^* y_1 (y_1 - 1) x^{y_1 - 2} \\ &\sim A^* y_1 x^{y_1 - 1}[\mu \beta'(0) - \sigma^2 \beta'(0)^2 (1 - r) - \alpha'(0^+)] > 0 \end{aligned}$$

and there is a contradiction. ⋄

Figure 6 plots the optimal level of productive assets for different values of the cost of debt α.

5 Appendix

5.1 Proof of Theorem 1

Supersolution property. Let $(\bar{x}, \bar{k}) \in S$ and $\varphi \in C^2(\mathbb{R}_+^2)$ s.t. $(\bar{x}, \bar{k})$ is a minimum of $V^* - \varphi$ in a neighbourhood $B_\varepsilon(\bar{x}, \bar{k})$ of $(\bar{x}, \bar{k})$ with ε small enough to ensure $B_\varepsilon \subset S$ and $V^*(\bar{x}, \bar{k}) = \varphi(\bar{x}, \bar{k})$.
First, let us consider the admissible control $\hat{\pi} = (\hat{Z}, \hat{I})$ where the shareholders decide to never invest or disinvest, while the dividend policy is defined by $\hat{Z}_t = \eta$ for $t \geq 0$, with $0 \leq \eta \leq \varepsilon$. Define the exit time $\tau_\varepsilon = \inf\{t \geq 0, (X_t^{\bar{x}}, K_t^{\bar{k}}) \notin \overline{B}_\varepsilon(\bar{x}, \bar{k})\}$. We notice that $\tau_\varepsilon < \tau_0$ for ε small enough. From the dynamic programming principle, we have

$$\begin{aligned} \varphi(\bar{x}, \bar{k}) = V^*(\bar{x}, \bar{k}) &\geq \mathbb{E}\left[\int_0^{\tau_\varepsilon \wedge h} e^{-rt} d\hat{Z}_t + e^{-r(\tau_\varepsilon \wedge h)} V^*(X^{\bar{x}}_{\tau_\varepsilon \wedge h}, K^{\bar{k}}_{\tau_\varepsilon \wedge h})\right] \\ &\geq \mathbb{E}\left[\int_0^{\tau_\varepsilon \wedge h} e^{-rt} d\hat{Z}_t + e^{-r(\tau_\varepsilon \wedge h)} \varphi(X^{\bar{x}}_{\tau_\varepsilon \wedge h}, K^{\bar{k}}_{\tau_\varepsilon \wedge h})\right]. \qquad (36) \end{aligned}$$

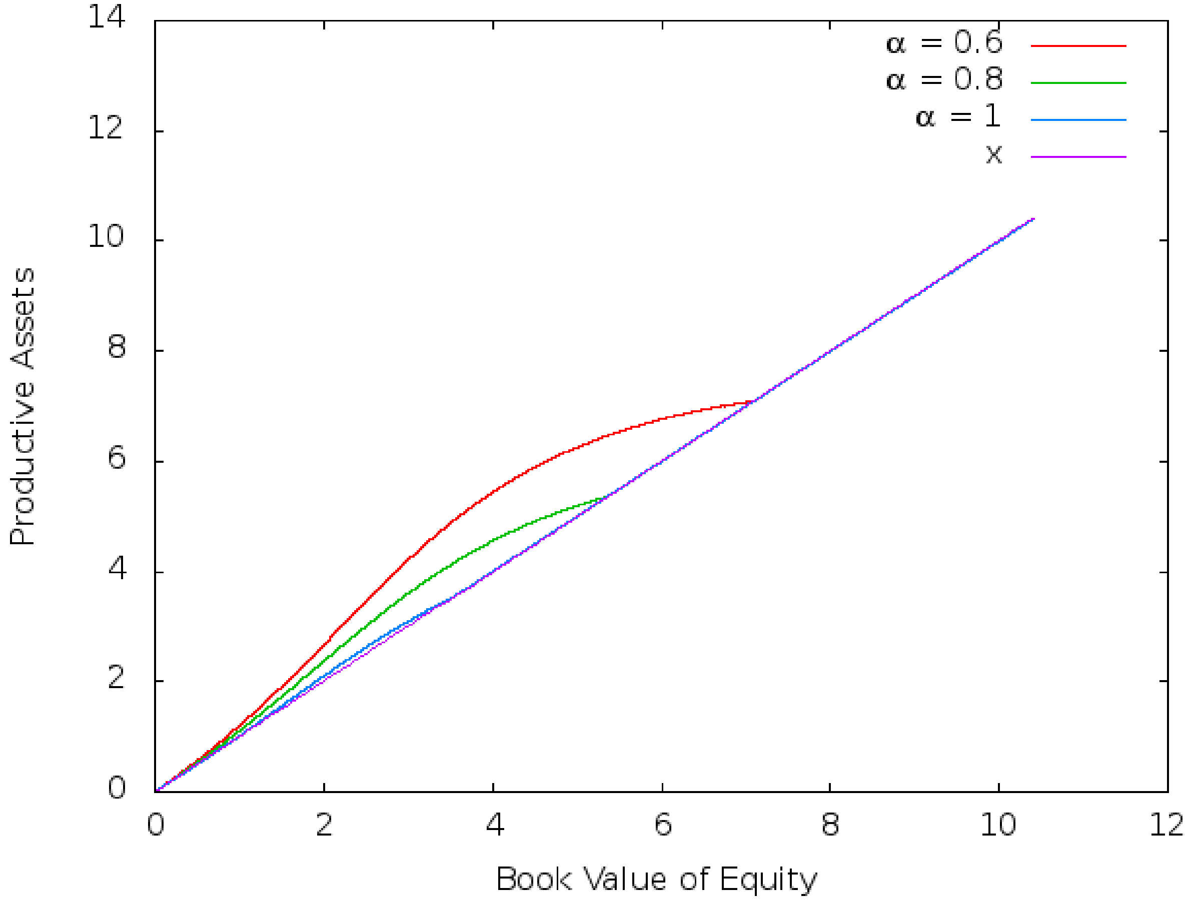


Figure 6: Comparing optimal level of productive assets with $\mu = 2$, $r = 0.5$, $\sigma = 1$, $\lambda = 10$, $\beta_{max} = 8$, $\beta'(0) = 2$, for different values of α.

Applying Itô's formula to the process $e^{-rt}\varphi(X_t^{\bar{x}}, K_t^{\bar{k}})$ between 0 and $\tau_\varepsilon \wedge h$, and taking the expectation, we obtain

$$\begin{aligned}\mathbb{E}\left[e^{-r(\tau_\varepsilon\wedge h)}\varphi(X^{\bar{x}}_{\tau_\varepsilon\wedge h}, K^{\bar{k}}_{\tau_\varepsilon\wedge h})\right] &= \varphi(\bar{x},\bar{k}) + \mathbb{E}\left[\int_0^{\tau_\varepsilon\wedge h} e^{-rt}\mathcal{L}\varphi(X_t^{\bar{x}}, K_t^{\bar{k}})dt\right] \\ &\quad + \mathbb{E}\left[\sum_{0\le t\le\tau_\varepsilon\wedge h} e^{-rt}[\varphi(X_t^{\bar{x}}, K_t^{\bar{k}}) - \varphi(X_{t^-}^{\bar{x}}, K_t^{\bar{k}})]\right]. \quad (37)\end{aligned}$$

Combining relations (36) and (37), we have

$$\begin{aligned}&\mathbb{E}\left[\int_0^{\tau_\varepsilon\wedge h} e^{-rt}(-\mathcal{L})\varphi(X_t^{\bar{x}}, K_t^{\bar{k}})dt\right] - \mathbb{E}\left[\int_0^{\tau_\varepsilon\wedge h} e^{-rt}d\hat{Z}_t\right] \\ &\quad -\mathbb{E}\left[\sum_{0\le t\le\tau_\varepsilon\wedge h} e^{-rt}[\varphi(X_t^{\bar{x}}, K_t^{\bar{k}}) - \varphi(X_{t^-}^{\bar{x}}, K_t^{\bar{k}})]\right] \ \ge\ 0. \quad (38)\end{aligned}$$

⋆ Take first $\eta = 0$. We then observe that X is continuous on $[0, \tau_\varepsilon \wedge h]$ and only the first term of the relation (38) is non zero. By dividing the above inequality by h with $h \to 0$, we conclude that $-\mathcal{L}\varphi(\bar{x}, \bar{k}) \geq 0$.

⋆ Take now $\eta > 0$ in (38). We see that $\hat{Z}$ jumps only at $t = 0$ with size η, so that

$$\mathbb{E}\left[\int_0^{\tau_\varepsilon \wedge h} e^{-rt}(-\mathcal{L}\varphi)(X_t^{\bar{x}}, K_t^{\bar{k}})dt\right] - \eta - (\varphi(\bar{x} - \eta, \bar{k}) - \varphi(\bar{x}, \bar{k})) \geq 0.$$

By sending $h \to 0$, and then dividing by η and letting $\eta \to 0$, we obtain

$$\frac{\partial \varphi}{\partial x}(\bar{x}, \bar{k}) - 1 \geq 0.$$

Second, let us consider the admissible control $\bar{\pi} = (\bar{Z}, \bar{I})$ where the shareholders decide to never payout dividends, while the investment/disinvestment policy is defined by $\bar{I}_t = \eta \in \mathbb{R}$ for $t \geq 0$, with $0 < |\eta| \leq \varepsilon$. Define again the exit time $\tau_\varepsilon = \inf\{t \geq 0, (X_t^{\bar{x}}, K_t^{\bar{k}}) \notin \overline{B}_\varepsilon(\bar{x}, \bar{k})\}$. Proceeding analogously as in the first part and observing that $\bar{I}$ jumps only at $t = 0$, thus

$$\mathbb{E}\left[\int_0^{\tau_\varepsilon \wedge h} e^{-rt}(-\mathcal{L}\varphi)(X_t^{\bar{x}}, K_t^{\bar{k}})dt\right] - (\varphi(\bar{x} - \gamma|\eta|, \bar{k} + \eta) - \varphi(\bar{x}, \bar{k})) \geq 0.$$

Assuming first $\eta > 0$, by sending $h \to 0$, and then dividing by η and letting $\eta \to 0$, we obtain

$$\gamma\frac{\partial \varphi}{\partial x}(\bar{x}, \bar{k}) - \frac{\partial \varphi}{\partial k}(\bar{x}, \bar{k}) \geq 0.$$

When $\eta < 0$, we get in the same manner

$$\gamma\frac{\partial \varphi}{\partial x}(\bar{x}, \bar{k}) + \frac{\partial \varphi}{\partial k}(\bar{x}, \bar{k}) \geq 0.$$

This proves the required supersolution property.

Subsolution Property: We prove the subsolution property by contradiction. Suppose that the claim is not true. Then, there exists $(\bar{x}, \bar{k}) \in S$ and a neighbourhood $B_\varepsilon(\bar{x}, \bar{k})$ of $\bar{x}, \bar{k}$, included in S for ε small enough, a C^2 function φ with $(\varphi - V^*)(\bar{x}, \bar{k}) = 0$ and $\varphi \geq V^*$ on $B_\varepsilon(\bar{x}, \bar{k})$, and $\eta > 0$, s.t. for all $(x, k) \in B_\varepsilon(\bar{x}, \bar{k})$ we have

$$\begin{aligned} -\mathcal{L}\varphi(x,k) &> \eta, &(39)\\ \frac{\partial \varphi}{\partial x}(x,k) - 1 &> \eta, &(40)\\ (\gamma\frac{\partial \varphi}{\partial x} - \frac{\partial \varphi}{\partial k})(x,k) &> \eta. &(41)\\ (\gamma\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial k})(x,k) &> \eta. &(42)\end{aligned}$$

For any admissible control π, consider the exit time $\tau_\varepsilon = \inf\{t \geq 0, (X_t^{\bar{x}}, K_t^{\bar{k}}) \notin B_\varepsilon(\bar{x}, \bar{k})\}$ and notice again that $\tau_\varepsilon < \tau_0$. Applying Itô's formula to the process $e^{-rt}\varphi(X_t^{\bar{x}}, K_t^{\bar{k}})$ between 0 and τ_ε^-, we have

$$
\begin{aligned}
\mathbb{E}[e^{-r\tau_\varepsilon^-}\varphi(X_{\tau_\varepsilon^-},K_{\tau_\varepsilon^-})] &= \varphi(\bar{x},\bar{k}) - \mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-ru}\mathcal{L}\varphi du\right] && (43)\\
&+ \mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-ru}(-\gamma\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial k})dI_u^{c,+}\right] && (44)\\
&+ \mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-ru}(-\gamma\frac{\partial\varphi}{\partial x} - \frac{\partial\varphi}{\partial k})dI_u^{c,-}\right] && (45)\\
&- \mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-ru}\frac{\partial\varphi}{\partial x}dZ_u^c\right] && (46)\\
&+ \mathbb{E}\left[\sum_{0<s<\tau_\varepsilon} e^{-rs}[\varphi(X_s,K_s) - \varphi(X_{s^-},K_{s^-})]\right] && (47)
\end{aligned}
$$

Using relations (39),(40),(41),(42), we obtain

$$
\begin{aligned}
V^*(\bar{x},\bar{k}) &= \varphi(\bar{x},\bar{k}) \\
&\geq \eta\mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-ru}du\right] + \mathbb{E}[e^{-r\tau_\varepsilon^-}\varphi(X_{\tau_\varepsilon^-},K_{\tau_\varepsilon^-})] && (48)\\
&+ \eta\mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-ru}dI_u^{c,+}\right] && (49)\\
&+ \eta\mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-ru}dI_u^{c,-}\right] && (50)\\
&+ (1+\eta)\mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-ru}dZ_u^c\right] && (51)\\
&- \mathbb{E}\left[\sum_{0<s<\tau_\varepsilon} e^{-rs}[\varphi(X_s,K_s) - \varphi(X_{s^-},K_{s^-})]\right] && (52)
\end{aligned}
$$

Note that $\Delta X_s = -\Delta Z_s - \gamma(\Delta I_s^+ + \Delta I_s^-)$, $\Delta K_s = \Delta I_s^+ - \Delta I_s^-$ and by the Mean Value Theorem, there is some $\theta \in]0,1[$ such that,

$$\begin{aligned}
\varphi(X_s, K_s) \;\; - \;\; & \varphi(X_{s^-}, K_{s^-}) = \frac{\partial \varphi}{\partial x}(X_{s^-} + \theta \Delta X_s, K_{s^-} + \theta \Delta K_s)\Delta X_s + \\
& \frac{\partial \varphi}{\partial k}(X_{s^-} + \theta \Delta X_s, K_{s^-} + \theta \Delta K_s)\Delta K_s \\
= \;\; & \frac{\partial \varphi}{\partial x}(X_{s^-} + \theta \Delta X_s, K_{s^-} + \theta \Delta K_s)(-\Delta Z_s - \gamma(\Delta I_s^+ + \Delta I_s^-)) \\
+ \;\; & \frac{\partial \varphi}{\partial k}(X_{s^-} + \theta \Delta X_s, K_{s^-} + \theta \Delta K_s)(\Delta I_s^+ - \Delta I_s^-) \\
= \;\; & -\frac{\partial \varphi}{\partial x}(X_{s^-} + \theta \Delta X_s, K_{s^-} + \theta \Delta K_s)\Delta Z_s \\
+ \;\; & \left(-\gamma \frac{\partial \varphi}{\partial x}(X_{s^-} + \theta \Delta X_s, K_{s^-} + \theta \Delta K_s) + \frac{\partial \varphi}{\partial k}(X_{s^-} + \theta \Delta X_s, K_{s^-} + \theta \Delta K_s)\right)\Delta I_s^+ \\
+ \;\; & \left(-\gamma \frac{\partial \varphi}{\partial x}(X_{s^-} - \theta \Delta X_s, K_{s^-} + \theta \Delta K_s) + \frac{\partial \varphi}{\partial k}(X_{s^-} + \theta \Delta X_s, K_{s^-} + \theta \Delta K_s)\right)\Delta I_s^-
\end{aligned}$$

Because $(X_s + \theta \Delta X_s, K_s + \theta \Delta K_s) \in B_\varepsilon(\bar{x}, \bar{k})$, we use the relations (40),(41),(42) again

$$-(\varphi(X_s, K_s) - \varphi(X_{s^-}, K_{s^-})) \geq (1+\eta)\Delta Z_s + \eta \Delta I_s^+ + \eta \Delta I_s^-$$

Therefore,

$$\begin{aligned}
V^*(\bar{x}, \bar{k}) \;\; \geq \;\; & \mathbb{E}[e^{-r\tau_\varepsilon^-}\varphi(X_{\tau_\varepsilon^-}, K_{\tau_\varepsilon^-})] + \mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-ru} dZ_u\right] \\
+ \;\; & \eta\left(\mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-ru} du\right] + \mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-ru} dI_u^+\right] + \mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-ru} dI_u^-\right] + \mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-ru} dZ_u\right]\right)
\end{aligned}$$

Notice that while $(X_{\tau_\varepsilon}^-, K_{\tau_\varepsilon}^-) \in B_\varepsilon(\bar{x}, \bar{k})$, $(X_{\tau_\varepsilon}, K_{\tau_\varepsilon})$ is either on the boundary $\partial B_\varepsilon(\bar{x}, \bar{k})$ or out of $\bar{B}_\varepsilon(\bar{x}, \bar{k})$. However, there is some random variable α valued in $[0,1]$ such that:

$$\begin{aligned}
(X^{(\alpha)}, K^{(\alpha)}) &= (X_{\tau_\varepsilon^-}, K_{\tau_\varepsilon^-}) + \alpha(\Delta X_{\tau_\varepsilon}, \Delta K_{\tau_\varepsilon}) \\
&= (X_{\tau_\varepsilon^-}, K_{\tau_\varepsilon^-} + \alpha(-\Delta Z_{\tau_\varepsilon} - \gamma \Delta I_{\tau_\varepsilon}^+ - \gamma \Delta I_{\tau_\varepsilon}^-, \Delta I_{\tau_\varepsilon}^+ - \Delta I_{\tau_\varepsilon}^-) \in \partial B_\varepsilon(\bar{x}, \bar{k})
\end{aligned}$$

Proceeding analogously as above, we show that

$$\varphi(X^{(\alpha)}, K^{(\alpha)}) - \varphi(X_{\tau_\varepsilon^-}, K_{\tau_\varepsilon^-}) \leq -\alpha[(1+\eta)\Delta Z_{\tau_\varepsilon} + \eta \Delta I_{\tau_\varepsilon}^+ + \eta \Delta I_{\tau_\varepsilon}^-]$$

Observe that

$$(X^{(\alpha)}, K^{(\alpha)}) = (X_{\tau_\varepsilon}, K_{\tau_\varepsilon}) + (1-\alpha)(\Delta Z_{\tau_\varepsilon} + \gamma \Delta I_{\tau_\varepsilon}^+ + \gamma \Delta I_{\tau_\varepsilon}^-, -\Delta I_{\tau_\varepsilon}^+ + \Delta I_{\tau_\varepsilon}^-)$$

Starting from $(X^{(\alpha)}, K^{(\alpha)})$, the strategy that consists in investing $(1-\alpha)\Delta I_{\tau_\varepsilon}^+$ or disinvesting $(1-\alpha)\Delta I_{\tau_\varepsilon}^-$ depending on the sign of $K^{(\alpha)} - K_{\tau_\varepsilon}$ and payout $(1-\alpha)\Delta Z_{\tau_\varepsilon}$ as dividends leads to $(X_{\tau_\varepsilon}, K_{\tau_\varepsilon})$ and therefore,

$$V^*(X^{(\alpha)}, K^{(\alpha)}) - V^*(X_{\tau_\varepsilon}, K_{\tau_\varepsilon}) \geq (1-\alpha)\Delta Z_{\tau_\varepsilon}$$

Using $\varphi(X^{(\alpha)}, K^{(\alpha)}) \geq V^*(X^{(\alpha)}, K^{(\alpha)})$, we deduce

$$\varphi(X_{\tau_\varepsilon^-}, K_{\tau_\varepsilon^-}) - V^*(X_{\tau_\varepsilon}, K_{\tau_\varepsilon}) \geq (1+\alpha\eta)\Delta Z_{\tau_\varepsilon} + \alpha\eta(\Delta I^+_{\tau_\varepsilon} + \Delta I^-_{\tau_\varepsilon})$$

Hence,

$$\begin{aligned} V^*(\bar{x}, \bar{k}) &\geq \eta\Big(\mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-ru} du\right] + \mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-ru} dI^+_u\right] + \mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-ru} dI^-_u\right] + \mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-ru} dZ_u\right] \\ &+ \mathbb{E}[e^{-r\tau_\varepsilon}\alpha(\Delta Z_{\tau_\varepsilon} + \gamma\Delta I^+_{\tau_\varepsilon} + \gamma\Delta I^-_{\tau_\varepsilon})]\Big) \\ &+ \mathbb{E}[e^{-r\tau_\varepsilon} V^*(X_{\tau_\varepsilon}, K_{\tau_\varepsilon})] + \mathbb{E}\left[\int_0^{\tau_\varepsilon} e^{-ru} dZ_u\right] \end{aligned} \tag{53}$$

We now claim there is $c_0 > 0$ such that for any admissible strategy

$$\begin{aligned} c_0 &\leq \mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-ru} du + \int_0^{\tau_\varepsilon^-} e^{-ru} dI^+_u + \int_0^{\tau_\varepsilon^-} e^{-ru} dI^-_u + \int_0^{\tau_\varepsilon^-} e^{-ru} dZ_u\right] \\ &+ \mathbb{E}\left[e^{-r\tau_\varepsilon}\alpha(\Delta Z_{\tau_\varepsilon} + \gamma\Delta I^+_{\tau_\varepsilon} + \gamma\Delta I^-_{\tau_\varepsilon})\right] \end{aligned} \tag{54}$$

Let us consider the C^2 function, $\phi(x,k) = c_0[1 - \frac{(x-\bar{x})^2}{\varepsilon^2}]$ with,

$$0 < c_0 \leq \min\left\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2\gamma}, \frac{1}{r}, \frac{\varepsilon^2}{\sigma_n^2\bar{\beta}^2}, \frac{\varepsilon}{2d_{max}}\right\}$$

where

$$d_{max} = \sup\left\{\frac{|\beta(k)\mu - \alpha((k-x)^+)|}{\varepsilon}, (x,k) \in B_\varepsilon(\bar{x}, \bar{k})\right\} > 0,$$

satisfies

$$\begin{cases} \phi(\bar{x}, \bar{k}) = c_0 \\ \phi = 0, \qquad \text{pour } (x,k) \in \partial B_\varepsilon \\ \min\left\{1 - \mathcal{L}\phi, 1 - \gamma\dfrac{\partial\phi}{\partial x} + \dfrac{\partial\phi}{\partial k}, 1 - \gamma\dfrac{\partial\phi}{\partial x} - \dfrac{\partial\phi}{\partial k}, 1 - \dfrac{\partial\phi}{\partial x}\right\} \geq 0, \ \text{pour } (x,k) \in B_\varepsilon \end{cases}$$

Applying Itô's formula, we have

$$\begin{aligned} 0 < c_0 &= \phi(\bar{x}, \bar{k}) \leq \mathbb{E}[e^{-r\tau_\varepsilon^-}\phi(X_{\tau_\varepsilon^-}, K_{\tau_\varepsilon^-})] + \mathbb{E}\left[\int_0^{\tau\varepsilon^-} e^{-ru} du\right] \\ &+ \mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-ru} dI^+_u\right] + \mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-ru} dI^-_u\right] + \mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-ru} dZ_u\right] \end{aligned} \tag{55}$$

Noting that $\frac{\partial\phi}{\partial x} \leq 1$ and $\frac{\partial\phi}{\partial k} = 0$, we have

$$\phi(X_{\tau_\varepsilon^-}, K_{\tau_\varepsilon^-}) - \phi(X^{(\alpha)}, K^{(\alpha)}) \leq (X_{\tau_\varepsilon^-} - X^{(\alpha)}) = \alpha(\Delta Z_{\tau_\varepsilon} + \gamma\Delta I^+_{\tau_\varepsilon} + \gamma\Delta I^-_{\tau_\varepsilon})$$

Plugging into (55) with $\phi(X^{(\alpha)}, K^{(\alpha)}) = 0$, we obtain

$$\begin{aligned} c_0 &\leq \mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-ru}du + \int_0^{\tau_\varepsilon^-} e^{-ru}dI_u^+ + \int_0^{\tau_\varepsilon^-} e^{-ru}dI_u^- + \int_0^{\tau_\varepsilon^-} e^{-ru}dZ_u\right] \\ &+ \mathbb{E}\left[e^{-r\tau_\varepsilon}\alpha(\Delta Z_{\tau_\varepsilon} + \gamma\Delta I_{\tau_\varepsilon}^+ + \gamma\Delta I_{\tau_\varepsilon}^-)\right] \end{aligned}$$

This proves the claim (54). Finally, by taking the supremum over π and using the dynamic programming principle, (53) implies $V^*(\bar{x}, \bar{k}) \geq V^*(\bar{x}, \bar{k}) + \eta c_0$, which is a contradiction.

Uniqueness Suppose u is a continuous subsolution and w a continuous supersolution of (21) on S satisfying the boundary conditions

$$u(x,0) \leq w(x,0) \quad u(\gamma k, k) \leq w(\gamma k, k) \text{ for } (x,k) \in S,$$

and the linear growth condition

$$|u(x,k)| + |w(x,k)| \leq C_1 + C_2(x+k) \quad \forall (x,k) \in S,$$

for some positive constants C_1 and C_2. We will show by adapting some standard arguments that $u \leq w$.

Step 1: We first construct strict supersolution of (21) with pertubation of w. Set

$$h(x,k) = A + Bx + Ck + Dxk + Ex^2 + k^2$$

with

$$A = \frac{1 + \mu\bar{\beta}B + \sigma^2\bar{\beta}^2 E}{r} + C_1 \tag{56}$$

and

$$\begin{cases} B &= 2 + \frac{1+C}{\gamma} + \frac{2\mu\bar{\beta}E}{r} \\ C &= \frac{\mu\bar{\beta}D}{r} \\ D &= 2\gamma E \\ E &= \frac{1}{\gamma^2} \end{cases}$$

and define for $\lambda \in [0,1]$ the continuous function on S

$$w^\lambda = (1-\lambda)w + \lambda h.$$

Because

$$\begin{cases} \dfrac{\partial h}{\partial x} - 1 = B + Dk + 2Ex - 1 \geq 1 \\ \gamma\dfrac{\partial h}{\partial x} - \dfrac{\partial h}{\partial k} = \gamma(B + Dk + 2Ex) - (C + Dx + 2k) \geq 1 \\ \gamma\dfrac{\partial h}{\partial x} + \dfrac{\partial h}{\partial k} = \gamma(B + Dk + 2Ex) + (C + Dx + 2k) \geq 1 \end{cases}$$

and

$$\begin{aligned}\mathcal{L}h &= -(\beta(k)\mu - \alpha((k-x)^+))(B + Dk + 2Ex) - \frac{\sigma^2\beta(k)^2}{2}2E + r(A + Bx + Ck + Dxk + Ex^2 + k^2)\\ &\geq (rA - \beta(k)\mu B - \sigma^2\beta(k)^2 E) + (rB - 2\mu\beta(k)E)x + (rC - \mu\beta(k)D)k\\ &\geq 1\end{aligned}$$

we observe that w^λ is a supersolution of

$$F(x, k, u, Du, D^2u) = \lambda$$

Step 2: In order to prove the strong comparison result, it suffice to show that for every $\lambda \in [0, 1]$

$$\sup_S (u - w^\lambda) \leq 0.$$

Assume by a way of contradiction that there exist λ such that

$$\sup_S (u - w^\lambda) > 0. \tag{57}$$

Because u and w have linear growth, we have $\lim\limits_{||(x,k)|| \to +\infty} (u - w^\lambda) = -\infty$.
Using the boundary conditions

$$\begin{aligned} u(x,0) - w^\lambda(x,0) &= (1-\lambda)(u(x,0) - w(x,0)) + \lambda(u(x,0) - (A + Bx + Ex^2)),\\ &\leq \lambda(u(x,0) - (A + Bx + Ex^2)),\\ u(\gamma k, k) - w^\lambda(\gamma k, k) &\leq \lambda(u(\gamma k, k) - (A + (B\gamma + C)k + (D\gamma + E\gamma^2 + 1)k^2)),\end{aligned}$$

and the linear growth condition, it is always possible to find C_1 in Equation (56) such that both expressions above are negative and maximum in Equation (57) is reached inside the domain S.

By continuity of the functions u and w^λ, there exists a pair (x_0, k_0) with $x_0 \geq \gamma k_0$ such that

$$M = \sup_S (u - w^\lambda) = (u - w^\lambda)(x_0, k_0).$$

For $\epsilon > 0$, let us consider the functions

$$\Phi_\epsilon(x, y, k, l) = u(x, k) - w^\lambda(y, l) - \phi_\epsilon(x, y, k, l)$$

$$\phi_\epsilon(x, y, k, l) = \frac{1}{2\epsilon}(|x - y|^2 + |k - l|^2) + \frac{1}{4}(|x - x_0|^4 + |k - k_0|^4)$$

By standard arguments in comparison principle of the viscosity solution theory (see Pham [20] section 4.4.2.), the function Φ_ε attains a maximum in $(x_\epsilon, y_\epsilon, k_\epsilon, l_\epsilon)$, which converges (up to a subsequence) to (x_0, k_0, x_0, k_0) when ε goes to zero. Moreover,

$$\lim_{\epsilon \to +\infty} \frac{(|x_\epsilon - y_\epsilon|^2 + |k_\epsilon - l_\epsilon|^2)}{2\epsilon} \to 0 \tag{58}$$

Applying Theorem 3.2 in Crandall Ishii Lions [6] , we get the existence of symmetric square matrices of size 2 M_ε, N_ε such that:

$$\begin{aligned}(p_\varepsilon, M_\varepsilon) &\in J^{2,+}u(x_\varepsilon, k_\varepsilon),\\ (q_\varepsilon, N_\varepsilon) &\in J^{2,-}w^\lambda(y_\varepsilon, l_\varepsilon),\end{aligned}$$

and

$$\begin{pmatrix} M_\varepsilon & 0 \\ 0 & N_\varepsilon \end{pmatrix} \le D^2\phi_\varepsilon(x_\varepsilon, y_\varepsilon, , k_\epsilon, l_\epsilon) + \varepsilon(D^2\phi(x_\varepsilon, y_\varepsilon, k_\epsilon, l_\epsilon))^2, \tag{59}$$

where

$$p_\varepsilon = D_{x,k}\phi_\epsilon(x_\epsilon, y_\epsilon, k_\epsilon, l_\epsilon) = \left(\frac{(x_\epsilon - y_\epsilon)}{\epsilon} + (x_\epsilon - x_0)^3, \frac{(k_\epsilon - l_\epsilon)}{\epsilon} + (k_\epsilon - k_0)^3\right),$$

$$q_\varepsilon = -D_{y,l}\phi_\epsilon(x_\epsilon, y_\epsilon, k_\epsilon, l_\epsilon) = \left(\frac{(x_\epsilon - y_\epsilon)}{\epsilon}, \frac{(k_\epsilon - l_\epsilon)}{\epsilon}\right).$$

Equation (59) implies[3]

$$\operatorname{tr}\left(\frac{\sigma^2\beta(k_\epsilon)^2}{2}P_\epsilon - \frac{\sigma^2\beta(l_\epsilon)^2}{2}Q_\epsilon\right) \le \frac{3\sigma^2}{2\epsilon}|\beta(k_\epsilon) - \beta(l_\epsilon)|^2 \tag{60}$$

Because u and w^λ are respectively subsolution and strict supersolution, we have

$$\begin{aligned}\min\Big[&-\big(\beta(k_\epsilon)\mu - \alpha((k_\epsilon - x_\epsilon)^+)\big)\Big(\frac{x_\epsilon - y_\epsilon}{\epsilon} + (x_\epsilon - x_0)^3\Big) - \operatorname{tr}\big(\frac{\sigma^2\beta(k_\epsilon)^2}{2}P_\epsilon\big) + ru(x_\epsilon, k_\epsilon),\\ &\frac{x_\epsilon - y_\epsilon}{\epsilon} + (x_\epsilon - x_0)^3 - 1,\\ &\gamma\Big(\frac{x_\epsilon - y_\epsilon}{\epsilon} + (x_\epsilon - x_0)^3\Big) - \Big(\frac{k_\epsilon - l_\epsilon}{\epsilon} + (k_\epsilon - k_0)^3\Big),\\ &\gamma\Big(\frac{x_\epsilon - y_\epsilon}{\epsilon} + (x_\epsilon - x_0)^3\Big) + \Big(\frac{k_\epsilon - l_\epsilon}{\epsilon} + (k_\epsilon - k_0)^3\Big)\Big] \le 0\end{aligned} \tag{61}$$

and

$$\begin{aligned}\min\Big(&-\big(\beta(l_\epsilon)\mu - \alpha((l_\epsilon - y_\epsilon)^+)\big)\frac{x_\epsilon - y_\epsilon}{\epsilon} - \operatorname{tr}\big(\frac{\sigma^2\beta(l_\epsilon)^2}{2}Q_\epsilon\big) + rw^\lambda(y_\epsilon, l_\epsilon),\\ &\frac{x_\epsilon - y_\epsilon}{\epsilon} - 1, \gamma\frac{x_\epsilon - y_\epsilon}{\epsilon} - \frac{k_\epsilon - l_\epsilon}{\epsilon}, \gamma\frac{x_\epsilon - y_\epsilon}{\epsilon} + \frac{k_\epsilon - l_\epsilon}{\epsilon}\Big) \ge \lambda\end{aligned} \tag{62}$$

We then distinguish the following four cases:

- Case 1. If $\frac{x_\epsilon - y_\epsilon}{\epsilon} + (x_\epsilon - x_0)^3 - 1 \le 0$ then we get from (62), $\lambda + (x_\epsilon - x_0)^3 \le 0$ yielding a contradiction when ϵ goes to 0.

[3] We refer to Pham page 81 for details

- Case 2. If $\gamma\Big(\frac{x_\epsilon-y_\epsilon}{\epsilon}+(x_\epsilon-x_0)^3\Big)-\Big(\frac{k_\epsilon-l_\epsilon}{\epsilon}+(k_\epsilon-k_0)^3\Big)\leq 0$ then we get from (62) $\lambda+\gamma\Big((x_\epsilon-x_0)^3-(k_\epsilon-k_0)^3\Big)\leq 0$ yielding a contradiction when ϵ goes to 0.

- Case 3. If $\gamma\Big(\frac{x_\epsilon-y_\epsilon}{\epsilon}+(x_\epsilon-x_0)^3\Big)+\Big(\frac{k_\epsilon-l_\epsilon}{\epsilon}+(k_\epsilon-k_0)^3\Big)\leq 0$, then we get from (62) $\lambda+\gamma\Big((x_\epsilon-x_0)^3+(k_\epsilon-k_0)^3\Big)\leq 0$ yielding a contradiction when ϵ goes to 0.

- Case 4. If

  $$-\big(\beta(k_\epsilon)\mu-\alpha((k_\epsilon-x_\epsilon)^+)\big)\Big(\frac{x_\epsilon-y_\epsilon}{\epsilon}+(x_\epsilon-x_0)^3\Big)-\mathrm{tr}(\frac{\sigma^2\beta(k_\epsilon)^2}{2}P_\epsilon)+ru(x_\epsilon,k_\epsilon)\leq 0$$

  From

  $$-\big(\beta(l_\epsilon)\mu-\alpha((l_\epsilon-y_\epsilon)^+)\big)\frac{x_\epsilon-y_\epsilon}{\epsilon}-\mathrm{tr}(\frac{\sigma^2\beta(l_\epsilon)^2}{2}Q_\epsilon)+rw^\lambda(y_\epsilon,l_\epsilon)\geq\lambda$$

  we deduce

  $$\begin{aligned}\frac{x_\epsilon-y_\epsilon}{\epsilon}\big(\mu(\beta(l_\epsilon)-\beta(k_\epsilon))+\alpha((k_\epsilon-x_\epsilon)^+)-\alpha((l_\epsilon-y_\epsilon)^+)\big)&\\ -\mathrm{tr}(\frac{\sigma^2\beta(k_\epsilon)^2}{2}P_\epsilon)+\mathrm{tr}(\frac{\sigma^2\beta(k_\epsilon)^2}{2}Q_\epsilon)&\\ -\big(\beta(k_\epsilon)\mu-\alpha((k_\epsilon-x_\epsilon)^+)\big)(x_\epsilon-x_0)^3&\\ +r(u(x_\epsilon,k_\epsilon)-w^\lambda(y_\epsilon,l_\epsilon))\leq-\lambda&\end{aligned}$$

  Using (60) we get,

  $$\begin{aligned}&\frac{x_\epsilon-y_\epsilon}{\epsilon}\big(\mu(\beta(l_\epsilon)-\beta(k_\epsilon))+\alpha((k_\epsilon-x_\epsilon)^+)-\alpha((l_\epsilon-y_\epsilon)^+)\big)\\ &-\big(\beta(k_\epsilon)\mu-\alpha((k_\epsilon-x_\epsilon)^+)\big)(x_\epsilon-x_0)^3+r(u(x_\epsilon,k_\epsilon)-w^\lambda(y_\epsilon,l_\epsilon))\leq-\lambda+\frac{3\sigma^2}{2\epsilon}|\beta(k_\epsilon)-\beta(l_\epsilon)|^2\end{aligned}$$

  By sending ε to zero and using the continuity of u, w_i^γ, α and β we obtain the required contradiction: $rM\leq-\lambda$.

This ends the proof.

References

[1] Asmussen, A., Højgaard, B., Taksar, M.: Optimal risk control and dividend distribution policies. Example of excess-of loss reinsurance for an insurance corporation. Finance and Stochastics, **4**, 299-324 (2000)

[2] Benes, V.E., Shepp, L.A., Witsenhausen, H.S.: Some solvable stochastic control problems.*Stochastics*, 4, 38-83 (1980)

[3] Black, F., and J. Cox, 1976, Valuing Corporate Securities: Some Effects of Bond Identures Provisions, *Journal of Finance*, 31, 351-367 (1976)

[4] Bolton, P., Chen, H., Wang, N.: A unified theory of Tobin's q, corporate investment, financing, and risk management, *Journal of Finance*, (2011)

[5] Choulli, T.,Taksar, M., Zhou, X.Y.: A diffusion model for optimal dividend distribution for a company with constraints on risk control. SIAM Journal of Control and Optimization, **41**, 1946-1979 (2003)

[6] Crandall M.G, Ishii H and Lions P.L: User's guide to viscosity solutions of second order Partial differential equations, *Bull.Amer.Soc.* 27, 1-67 (1992).

[7] Décamps, J.P., Mariotti, T., Rochet, J.C. and Villeneuve, S: Free cash-flows, Issuance Costs, and Stock Prices, *Journal of Finance*, 66, 1501-1544 (2011).

[8] Federico,S,. Pham, H.: Characterization of the optimal boundaries in reversible investment problems, *SIAM Journal of Control and Optimization*, 52, 21802223 (2014).

[9] Haussman U.G., Suo W.: Singular optimal stochastic controls I: existence, *SIAM Journal of Control and Optimization*, 33, 916-936 (1995).

[10] Haussman U.G., Suo W.: Singular optimal stochastic controls II: dynamic programming, *SIAM Journal of Control and Optimization*, 33, 937-959 (1995).

[11] Hojgaard, B., Taksar, M.: Controlling risk exposure and dividends pay-out schemes: Insurance company example, *Mathematical Finance* **9**, 153-182 (1999).

[12] Hugonnier, J., Malamud, S. and Morellec, E.: Capital supply uncertainty, cash holdings and Investment, forthcoming *Review of Financial Studies*

[13] Jeanblanc-Picqué, M., Shiryaev, A.N.: Optimization of the flow of dividends. *Russian Mathematics Surveys*, 50, 257-277 (1995)

[14] Leland, H.E.: Corporate debt value, bond covenants, and optimal capital structure. *Journal of Finance* 49, 1213-1252, (1994)

[15] Ly Vath V., Pham, H. and Villeneuve, S. :A mixed singular/switching control problem for a dividend policy with reversible technology investment. *Annals of Applied Probability*, Vol 18 N3, p1164-1200,(2008).

[16] Manso, G., Strulovici B. and Tchistyi A.: Performance-sensitive debt, *The Review of Financial Studies*, 23 1819-1854 (2010)

[17] Miller, M.H., Modigliani, F.: Dividend policy, growth and the valuation of shares, *Journal of Business*, 34, 311-433 (1961)

[18] Nakhl H. : Partial differential equations with Fourier series and boundary value problems, Upper Saddle River, NJ: Pearson Prentice Hall (2005)

[19] Paulsen, J.: Optimal dividend payouts for diffusions with solvency constraints. Finance and Stochastics, **7**,457-474 (2003)

[20] Pham H.: *Continuous-time Stochastic Control and Optimization with Financial Applications*, Springer (2009)

[21] Radner, R., Shepp, L.: Risk versus profit potential: A model for corporate strategy, *Journal of Economic Dynamics and Control* 20, 1373-1393 (1996)

[22] Revuz, D., Yor, M.: *Continuous Martingales and Brownian Motion*, Springer 3rd ed. (1999).