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Local Identification in Empirical Games of Incomplete Information

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Abstract

This paper studies identification for a broad class of empirical games in a general functional setting. Global identification results are known for some specific models, for instance in some standard auction models. We use functional formulations to obtain general criteria for local identification. These criteria can be applied to both parametric and nonparametric models, as well as models with asymmetry among players and affiliated private information. A benchmark model is developed where the structural parameters of interest are the distribution of private information and an additional dissociated parameter, such as a parameter of risk aversion. Criteria are derived for some standard auction models, games with exogenous variables, games with randomized strategies, such as mixed strategies, and games with strategic functions that cannot be derived analytically.

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1 Introduction

The problem of identification in empirical games has received considerable attention but has only found answers in the context of certain models. Laffont and Vuong (1996) provide nonparametric identification results for some specific first-price sealed-bid auction models with risk-neutral bidders. Guerre, Perrigne and Vuong (2000) examine the particular case of the independent private value (IPV) paradigm and detail how to obtain identification.[1] Athey and Haile (2002) extend previous results to second-price, ascending (English), and descending (Dutch) auctions.

In some models nonparametric identification is not possible. In order to obtain identification, a parametric assumption may be necessary and sufficient. For instance, Donald and Paarsch (1996) present parametric identification results for an IPV auction model with risk aversion, but Campo et al. (2002) show that there is no nonparametric identification for this model. Campo et al. (2002) also show that semiparametric identification can be achieved through additional restrictions.

This paper examines identification in a broad class of games. In order to obtain general identification criteria, we apply a local identification principle. This approach has a long history in econometrics, with notable contributions by, among others, Koopmans, Rubin and Leipnik (1950), Wald (1950), Fisher (1959, 1961, 1966), Wegge (1965) and Rothenberg (1971). In a substantial contribution, Fisher (1966) provides a unified treatment of the theory of identification for simultaneous equation models, and successfully develops Generalized Rank and Order Conditions for local identification. Rothenberg (1971) builds on this work by developing criteria for local identification for more general parametric models. More recently, Chesher (2003) uses quantile functions to provide local identification conditions in nonseparable models. In the literature, the desirability of global identification is clearly understood, but at the same time, the difficulty of obtaining these results is often emphasized.

A promising direction that has rarely been explored is to find a general framework to

[1] They also provide an optimal estimation procedure.

study identification for any structural model of games. This paper develops such a framework and obtains criteria applicable to a wide range of models, by following the local identification approach. We allow for nonparametric models by using a general functional setting.

In general, we consider J games, each game indexed by j, $j = 1, ..., J$. In each game j there are L_j players, each player indexed by l, $l = 1, ..., L_j$. We consider games as independent. Thus, for ease of notation, we define our setting for one game with I players, with each player indexed by i.[2] For each player i, we have an observable x_i, as the result of a transformation φ_i of: (1) an unobservable ξ_i, which represents private information and has a joint distribution F_θ for all possible values of private information, or (2) an additional structural parameter λ_i.(e.g., a parameter of risk aversion). So we have

$$x_i = \varphi_i \left(\xi_i, F_\theta, \lambda\right),$$

where θ fully characterizes F_θ, and $\lambda = (\lambda_1, ..., \lambda_I)$.

We can notice that, statistically, a game model is characterized in the following way: the observable actions depend on unobservable variables (private information) and their distribution. This corresponds to the strategic aspect of the game, and causes a specific identification issue. Indeed, even if we consider that the strategic function φ is bijective and that θ would be identified if we could observe the ξ_i's, we may not be able to identify θ from the observable actions. We can illustrate this with two simple examples. First, let us assume that the private information ξ_i follows a parametric Normal distribution, $\xi_i \sim N\left(\mu, \sigma^2\right)$, and the observable actions are function of ξ_i taken in the following way: $x_i = \xi_i - \lambda$. We then have $x_i \sim N(\mu - \lambda, \sigma^2)$ and μ is not identified from the observations x_i's. Second, we can think about a nonparametric model, with the private information following a nonparametric distribution F, $\xi_i \sim F$, which would be identified if we would observe the ξ_i's, and $x_i = F(\xi_i)$, with F bijective. In this case, we have $x_i \sim U[0,1]$ and F is not identified from the

[2] We could define $i = (j, l)$. If we have the same number L of players from one game to the next, we can write $I = J \times L$.

observations.

In Florens, Protopopescu and Richard (2001), two criteria are found to identify F_θ, assuming symmetric risk neutral players, in both the parametric and nonparametric case. This was the first attempt to give a general identification condition that can be applied to any kind of game of incomplete information.

Our paper improves on previous work in several respects. First, we provide an identification condition for any parameters of the transformation φ_i (e.g., a parameter of risk aversion) and not only the parameters of the distribution function. This condition is valid in the parametric case as well as in the nonparametric case. Second, we allow for asymmetry among bidders. Third, we study different extensions and difficulties one can face in practice. We treat the case of partial observability of the information used by the players (exogenous variables). In this case we partially observe the information players use to decide their strategy: the vector of information $\xi = (\eta, z)$,is composed of the strictly private component η and an observable component z (e.g., the total number of players). The case of randomized strategies, such as mixed strategies, is also considered. Instead of assuming that action x is the result of a transformation φ, we assume that x follows a distribution H, $x \sim H$. Actually, $x = \varphi(.)$ corresponds to the special case $x \sim \delta_\varphi$, with δ_φ the Dirac measure which puts all the weight on $x = \varphi(.)$. We may have no analytical solutions for the first order conditions in the game model considered. Often, game theoretical models are complex and one cannot find a general analytical form for the strategic function φ. There are then non-closed form solutions for the first order conditions. We propose some solutions to adapt our local identification criteria to this problem. In most of the following sections, we will use the well-known First-Price Value auction model to illustrate the use of our identification approach.[3]

We concentrate on identification and not estimation. However, these issues are strongly related. Indeed, we will see in Section 2 that the model presented above generates a nonlinear

[3]More examples are available on Erwann Sbai web site: http://www.homes.eco.auckland.ac.nz/esba001/

inverse problem. This type of problem is usually considered locally by the analysis of a linear operator tangent to the original one at the true value. The (local) ill-posedness properties are analysed through this linear operator. In particular, the spectrum of this operator declines at a slope that determines the speed of convergence of the estimator. In this work the identification is considered as a one to one property of this operator, which is a first step in the analysis of ill-posedness.

After presenting the general framework in the next section, we derive in section 3 a general theorem which states our first identification condition. Section 4 extends the general theorem to the case of exogenous variables, when there is partial observability of players' information. In Section 5 we generalise our identification procedure to the case of randomized strategies, such as mixed strategies. Section 6 corresponds to the case of non-closed form solutions for the first order conditions of the game model considered. We conclude in Section 7.

2 General Framework and Specifications

We consider a game with I players. Player i draws private information $\xi_i \in \Xi_i \subset \mathbb{R}^p$, $i = 1, ..., I$. The joint distribution of $\xi = (\xi_1, ..., \xi_I) \in \Xi = \prod_i \Xi_i \subset \mathbb{R}^{pI}$ is denoted F_θ, completely determined by $\theta \in \Theta$ which is a (possibly functional) parameter. This situation covers the case where some elements of θ are specific to the distribution of ξ_i. We denote f_θ the associated joint density where this density is assumed to exist. Within each game, the ξ_i's are not necessarily independently and identically distributed (hereafter i.i.d.). We may for example consider some Common Value auction models. Nevertheless, we assume as usual that from one game to another the ξ_i's are i.i.d.. As we concentrate our attention on the identification issue, we consider a single observation model only. The distribution F_θ (or the value θ) is common knowledge to all players of the games, but θ is unknown to the econometrician.

The unobservable ξ_i are then transformed into observable actions $x_i \in \mathbb{R}^p$, by means of a transformation φ_i, say $x_i = \varphi_i(\xi_i, F_\theta, \lambda) \equiv \varphi_i(\xi_i, \theta, \lambda)$. In our framework, we assume that φ_i exists. We consider $\lambda \in \Lambda$ as a (possibly functional) parameter of the strategy function. Here also this presentation covers the case where some elements of λ are specific to the φ_i function. To simplify notation, we could also use the vector of the parameters of interest γ, where $\gamma = (\theta, \lambda)$.

Assumption 1 The parameters θ and λ are variation free, i.e. the domain of $\gamma = (\theta, \lambda)$ is $\Theta \times \Lambda$. For the (possibly functional) parameter γ we have: $\gamma \in \Gamma \subset \Gamma_0$ a normed vector space associated with the norm $\|\gamma\|$. We denote $\gamma_0 = (\theta_0, \lambda_0)$ as the true unknown values.

This assumption eliminates common elements between θ and λ. In particular this implies that derivatives of functions of θ with respect to (hereafter w.r.t.) λ are zero (and conversely).

Assumption 2 For all $i = 1, .., I$ we denote equivalently $\varphi_i \equiv \varphi_{i,\theta,\lambda} \equiv \varphi_{i,\gamma}$ where

$$\begin{aligned} \varphi_i &: \xi_i \longmapsto x_i \\ &: \Xi_i \longrightarrow \mathbb{X}_{i\gamma} \end{aligned}$$

Note that the range of φ_i usually depends on γ. We assume that φ_i is a one to one and strictly increasing function, i.e.

$$\xi_i < t_i \Longleftrightarrow \varphi_i(\xi_i) < \varphi_i(t_i)$$

We can denote φ as the vector of φ_i's, $\varphi = (\varphi_1, ..., \varphi_I)$. Also, we define the inverse function φ^{-1}. If we take a vector u, decomposed into I components of dimension p, we can write

$$\varphi^{-1}(u) = \left[\varphi_1^{-1}(u_1), ..., \varphi_I^{-1}(u_I)\right]$$

The structural model may be summarized by

$$\begin{cases} x = \varphi(\xi, \theta, \lambda) \\ \xi \sim F_\theta \end{cases} \tag{1}$$

Particularly interesting cases are symmetric games defined by the following assumption:

Assumption 3 We will refer to *symmetric* game models if two conditions hold. First, the ξ_i are i.i.d.:

$$\forall i = 1, ..., I,\ \Xi_i = \bar{\Xi} \text{ and } (\xi_1, ..., \xi_I) \sim \prod_{i=1}^{I} \overline{F}_\theta$$

where $\overline{F}_\theta$ is a distribution on $\bar{\Xi} \subset \mathbb{R}^p$. Second,

$$\begin{aligned} \forall i &= 1, ..., I,\ \varphi_i(\xi_i, \gamma) = \overline{\varphi}(\xi_i, \gamma) = \overline{\varphi}_\gamma(\xi_i) \\ \text{where } \overline{\varphi}_\gamma &: \bar{\Xi} \rightarrow \mathbb{R}^p \end{aligned}$$

Now, let us consider the reduced form of the model described by the distribution of the observable x. Through the different games the x are also i.i.d. and we only analyze a single observation. The distribution of x is denoted by G as follows:

$$\begin{aligned} G(u) &= P(x \leq u) \\ &= P(x_1 \leq u_1, ..., x_I \leq u_I) \\ &= P\left(\xi_1 \leq \varphi_{1,\gamma}^{-1}(u_1), ..., \xi_I \leq \varphi_{I,\gamma}^{-1}(u_I)\right) \\ &= F_\theta \circ \varphi_\gamma^{-1}(u) \end{aligned} \tag{2}$$

Equivalently the relation between the structural form parameter γ and reduced form parameter G takes the implicit form:

$$A(\gamma, G) = F_\theta - G \circ \varphi_\gamma = 0 \tag{3}$$

The function G is identified (and may be estimated) by the data generating process and the goal of the structural analysis of the model is to determine the γ parameters from this equation. This paper is devoted to identification, i.e. the uniqueness of the solution in γ given G. Generally, the resolution of (3), given an estimation of G, is actually a specific case of (nonlinear) inverse problem (see e.g. Carrasco et al. (2007)). The present paper is not concerned with estimation, but we may remark that in most of the cases this inverse problem is mildly ill-posed (see Florens, Protopopescu and Richard (2001) where it is proven in the Private Value auction model that the degree of ill-posedness is equal to one).

Let us now introduce some smoothness hypotheses.

Assumption 4 $\forall \gamma = (\theta, \lambda)$, $F_\theta \in C^q(\Xi)$ (is continuously differentiable up to order $q \geq 1$) and $\varphi_{i,\gamma}(.) \in C^q(\Xi)$. The set $C^q(\Xi)$ is endowed with a suitable topology defined by a norm $\|.\|_\Xi$.

For example if Ξ is bounded this norm may be an L^p norm or a Sobolev norm, with respect to the uniform distribution.

Assumption 5 The $p \times p$ matrix

$$\Delta\varphi_{i,\gamma_0} = \frac{\partial \varphi_{i,\gamma_0}}{\partial \xi_i'}$$

of partial derivatives of φ_{i,γ_0} w.r.t. the arguments of ξ_i is non-singular for all $\xi_i \in \Xi_i$.

This hypothesis may not be satisfied for some elements of Ξ_i in particular cases. A weaker version of assumption 5 is obtained by introducing an appropriate measure π on Ξ, where the marginal on Ξ_i is μ_i and by assuming regularity on the support of μ_i only (see Example 1 below).

Finally we may consider the function φ_i as an operator which associates to any γ a function $\varphi_i(., \gamma)$. We denote by Φ_i this nonlinear operator defined on Γ and taking values in $C^q(\Xi)$. We also consider F_θ as an operator, defined on Θ and with values $C^q(\Xi)$, which associates the function $F_\theta(.)$ to θ.

Assumption 6 $\forall i = 1, ..., I$, Φ_i is Fréchet differentiable, i.e. $\exists\ d_\gamma \Phi_{i,\gamma_0}(.)$ continuous linear operator from Γ_0 to $C^q(\Xi)$ such that

$$\Phi_i(\gamma) - \Phi_i(\gamma_0) = d_\gamma \Phi_{i,\gamma_0}(\gamma - \gamma_0) + \|\gamma - \gamma_0\| \varepsilon(\gamma - \gamma_0)$$

where $\varepsilon(\widetilde{\gamma}) \to 0$ if $\|\widetilde{\gamma}\| \to 0$.

Moreover F_θ is also assumed to be Fréchet differentiable and its derivative is denoted $d_\theta^F F_{\theta_0}\left(\tilde{\theta}\right)$.

In practice Fréchet differentials can be computed as Gâteaux differentials.

We say that, $\forall i = 1, ..., I$, Φ_i is Gâteaux differentiable in any direction $\widetilde{\gamma}$ at γ_0 if $\exists$ $\delta_\gamma \Phi_{i,\gamma_0}(\widetilde{\gamma})$ such that:

$$\delta_\gamma \Phi_{i,\gamma_0}(\widetilde{\gamma}) = \lim_{a \longrightarrow 0} \frac{d}{da} \mathbf{\Phi}_i(\gamma_0 + a\widetilde{\gamma}), \qquad a \in \mathbb{R}$$

Moreover F_θ considered as an operator between Θ_0 and $C^q(\Xi)$ may also have Gâteaux derivatives denoted $\delta_\theta F_{\theta_0}\left(\tilde{\theta}\right)$.

Relations between Gâteaux and Fréchet derivatives are standard topics in functional analysis (see Serfling (1980), Rieder (1994) or Nashed (1971) for example). We summarize these results by the following remarks:

- A Gâteaux derivative only uses a topology on $C^q(\Xi)$ and a Fréchet derivative uses the norms on Γ_0 and $C^q(\Xi)$ spaces.
- A Fréchet derivative is a linear continuous (bounded) operator but a Gâteaux derivative may be nonlinear or noncontinuous.
- If a Fréchet derivative exists, it is unique and equal to the Gâteaux derivative.
- The main interesting result is the following: if a Gâteaux derivative exists in a neighborhood of γ_0, is a linear continuous operator (as a function of $\widetilde{\gamma}$) and is continuous in

γ_0 (as an operator from Γ to the set of continuous linear function from Γ_0 to $C^q(\Xi)$), then this Gâteaux derivative is also a Fréchet derivative. There exist other sets of conditions which guarantee that a Gâteaux derivative is also a Fréchet derivative (see Nashed (1971)).

- Fréchet differentiability is required in order to apply the implicit function theorem in functional spaces (see Rieder (1994), theorem 1.4.7).

3 Local Identification Principle

3.1 A General Result

The parameters of interest are $(\theta, \lambda) = \gamma \in \Theta \times \Lambda = \Gamma \subset \Gamma_0$, while the observations $(x_i)_{i=1,\ldots,I}$, follow a distribution $G = F_\theta \circ \varphi_{\theta,\lambda}^{-1}$ which is identified.

Definition 1 *The parameters $\gamma = (\theta, \lambda)$ and $\gamma^* = (\theta^*, \lambda^*)$ are observationally equivalent ($(\theta, \lambda) \sim (\theta^*, \lambda^*)$ or $\gamma \sim \gamma^*$) if and only if (hereafter iff) $G = G^*$. Then, obviously γ^* is observationally equivalent to γ iff*

$$F_{\theta^*} - G \circ \varphi_{F_{\gamma^*}} = 0$$

Definition 2 *The true parameter $\gamma_0 = (\theta_0, \lambda_0) \in \Theta \times \Lambda$ is globally identified iff*

$$\forall\ \gamma \in \Gamma,\ \gamma_0 \sim \gamma \Longrightarrow \gamma_0 = \gamma$$

Definition 3 *The true parameter $\gamma_0 \in \Gamma \subset \Gamma_0$ is locally identified iff there exists a neighborhood $\mathcal{V}(\gamma_0)$ of γ_0 in Γ such that*

$$\forall\ \gamma \in \mathcal{V}(\gamma_0),\ \gamma \sim \gamma_0 \Longrightarrow \gamma = \gamma_0$$

Theorem 4 *Under Assumptions 1, 2, 4, 5 and 6, the game model (1) is locally identified if the bounded linear operator*

$$T_{\gamma_0}(\tilde{\gamma}) = d_\theta F_{\theta_0}\left(\tilde{\theta}\right) - \sum_{i=1}^{I}\left(\frac{\partial F_{\theta_0}(\xi)}{\partial \xi_i}\right)' [\Delta\varphi_{i,\gamma_0}]^{-1}\left(d_\theta \Phi_{i,\gamma_0}\left(\tilde{\theta}\right) + d_\lambda \Phi_{i,\gamma_0}\left(\tilde{\lambda}\right)\right)$$

is one to one.

Proof. Let us consider the operator $A(\gamma, G) = F_\theta - G \circ \varphi_\gamma$ from Γ to $C^q(\Xi)$. By definition of γ_0, $F_{\theta_0} - G_0 \circ \varphi_{\gamma_0} = 0$. Then, local identification is obtained through the implicit function theorem for general spaces (see Rieder (1994)). If the Fréchet derivative $d_\gamma A(\gamma_0, G_0)(\tilde{\gamma})$ is a one to one operator, the solution γ_0 is unique in a neighborhood of γ_0 and the model is locally identified.[4]

We now compute this derivative. First remark that $G_0 = F_{\theta_0} \circ \varphi_{\gamma_0}^{-1}$ is an element of $C^q(\Xi)$ where $q \geq 1$ because F_{θ_0}, and φ_{γ_0} (and then $\varphi_{\gamma_0}^{-1}$) are in $C^q(\Xi)$. Then using the chain rule for differentiation (valid for Fréchet derivative) we get:

$$d_\gamma\left(F_{\theta_0} - G_0 \circ \varphi_{\gamma_0}\right) = d_\theta F_{\theta_0} - \sum_{i=1}^{I}\left(\frac{\partial G}{\partial u_i'} \circ \varphi_{\gamma_0}\right)' d_\gamma \Phi_{i,\gamma_0}$$

In our notation, we consider G as a function of its arguments $(u_1, ..., u_I)$ where each $u_i \in \mathbb{R}^p$.

Moreover from $F_{\theta_0} - G_0 \circ \varphi_{\gamma_0} = 0$ we get, by derivation w.r.t. ξ_i:

$$\frac{\partial F_{\theta_0}}{\partial \xi_i} = \frac{\partial \varphi_{i,\gamma_0}'}{\partial \xi_i} \cdot \frac{\partial G}{\partial u_i} \circ \varphi_{\gamma_0}$$

and the result follows using assumption 5 and the following property:

$$d_\gamma \Phi_{i,\gamma_0}(\tilde{\gamma}) = \left(d_\theta \Phi_{i,\gamma_0}\left(\tilde{\theta}\right) + d_\lambda \Phi_{i,\gamma_0}\left(\tilde{\lambda}\right)\right)$$

[4] Actually the implicit function theorem says more: for any G in a neighborhood of G_0 we can solve the equation $A(\gamma, G) = 0$ into $\gamma = B(G)$ and characterize the derivative of B. This issue is fundamental for estimation but not for our identification problem.

where $\tilde{\gamma} = \left(\tilde{\theta}, \tilde{\lambda}\right)$. ■

We can mention three interesting features of this theorem.

First, it does not necessitate the use of the reduced form model, so we do not need to compute G.

Second, we can see that in the parametric case the condition for identification we find here, is just the usual necessary condition used to apply the Generalized Method of Moments (hereafter GMM). In GMM we have the moment equation $E\left[h\left(x,\theta\right)\right] = 0$. A necessary condition for identification is $rank\left\{E\left[\frac{\partial}{\partial\theta'}h\left(x,\theta\right)\right]\right\} = \dim\left(\theta\right)$, which corresponds to the one to one property. In our way of modeling, the moment equation is $A\left(\left(\theta_0,\lambda_0\right);G_0\right) = 0$ and in the parametric case the GMM rank condition corresponds exactly to our one to one condition on T_{γ_0}. In other words, our identification condition can be thought of as a functional version of GMM type condition.

Third, this formula may be interpreted in the following way. The operator is decomposed into three elements. The first one, $(d_\theta F_{\theta_0})$, describes the identification of the unobservable distribution through θ. The second one (including $d_\theta \Phi_{i,\gamma_0}\left(\tilde{\theta}\right)$) introduces the main element, i.e. the correction of identification coming from the dependence of φ_i on θ. If the actions of the players have no strategic component (φ does not depend on θ), then this term cancels. The last term, involving the derivatives w.r.t. λ comes from the introduction of unknown elements in the strategic function.

Additionally, in our general identification theorem, we do not need φ_i explicitly, but its (Fréchet) derivatives w.r.t. the private signal ξ and the vector of parameters of interest (θ,λ) (which can be nonparametric). In section 7 we will explain how we can find $d_{\theta,\lambda}\varphi_{i,\theta,\lambda}$ and how to treat the more complicated case of $\frac{\partial \varphi'_{i,\gamma_0}}{\partial \xi_i}$, when we consider non-closed form solutions for the strategic function φ.

The result of Theorem 4 depends on the choice of the norm $\|\|$ on the parameter space. To clarify this point let us consider another norm $\|\|_*$ on Γ such that $\|\gamma\|_* \geq \|\gamma\|$. This property implies that if assumption 6 is verified for $\|\|$, it is also verified for$\|\|_*$. However, local identification may be satisfied for $\|\|$ and not for $\|\|_*$. In terms of Theorem 4, if the operator T depends on $\|\|_*$, its one to one property may not be verified even if it was the case with $\|\|$. A question could arise concerning the choice of $\|\|$. If we think about estimation, the local identification condition is also a condition for consistency. In consequence, we can choose the norm on Γ used to obtain consistency.

As already discussed in Section 2, equation (3) defines an inverse problem: G may be replaced by an estimator $\widehat{G}$ (e.g. the empirical counterpart) and γ may be estimated by minimising $\|F_\theta - \widehat{G} \circ \varphi_\gamma\|^2 + \alpha\|\gamma\|^2$, where α is a regularisation parameter and $\alpha\|\gamma\|^2$ is a penalisation term. The degree of ill-posedness of this nonlinear problem is defined locally and, in our notation, is a measure of the shape of the spectrum of $T_{\gamma_0}(\widehat{\gamma})$. The model is linearly identified if T_{γ_0} has no null singular values (see Carrasco et al. 2007); it is not excessively ill-posed if there is no fast decline of these singular values.

3.2 Symmetric Independent Private Value Case

For the ease of presentation, in sections 4, 5 and 6 we present our results in the symmetric independent private value case.

In the symmetric independent private value case the ξ_i are i.i.d. and the φ_i are identical. This implies obviously that the x_i are i.i.d. and $G = \prod_{i=1}^{I} \bar{G}$. In order to study identification in that case, we may substitute G, F_θ and φ by their expressions in the previous result. However it is easier to notice that in the symmetric case observations are i.i.d. across players and games. So, it is sufficient to consider a single observation for one player and one game. In that case, the parameters $\gamma = (\theta, \lambda)$ are solution of the equation

$$\bar{F}_\theta - \bar{G} \circ \bar{\varphi}_\gamma = 0 \tag{4}$$

The previous results may be reproduced in this framework. Using the same proof as to Theorem 4 we get the following theorem.

Theorem 5 *Under Assumptions 1 to 6, the symmetric game model (4) is locally identified if the bounded linear operator:*

$$T_{\gamma_0}(\tilde{\gamma}) = d_\theta \bar{F}_{\theta_0}\left(\tilde{\theta}\right) - \left(\frac{\partial \bar{F}_{\theta_0}(\xi_i)}{\partial \xi_i}\right)' [\Delta \bar{\varphi}_{\gamma_0}]' \left(d_\theta \bar{\Phi}_{\gamma_0}\left(\tilde{\theta}\right) + d_\lambda \bar{\Phi}_{\gamma_0}\left(\tilde{\lambda}\right)\right)$$

is one to one.

Note that if $\xi_i \in \mathbb{R}$, i.e. $p = 1$, then $\frac{\partial \bar{F}_{\theta_0}(\xi_i)}{\partial \xi_i}$ is the density $\bar{f}_{\theta_0}$ of $\bar{F}_{\theta_0}$.

3.3 Example: First Price Private Value Auction Model

In order to understand the use of our identification theorem, we will provide an illustration based on a well-known model. We consider a model of independent private values auction, with symmetric players and i.i.d. private information, which is the same for Dutch and First-Price sealed-bid auctions. This same example will be presented below in the cases of risk neutrality and risk aversion.

3.3.1 Risk Neutrality

To introduce this example, we discuss first the case of risk neutrality. It has been extensively studied and global identification was established by Laffont and Vuong (1996). Local identification is considered by Florens et al. (2001) and we recall here that argument as a particular case of our general results.

The game is symmetric and for simplification we assume that $\xi_i \in \bar{\Xi} = [0, 1]$ and $\xi_i \sim \bar{F}$.[5] We treat the model nonparametrically in the sense that $\theta = \bar{F}$.

Remark: for the ease of notation in this part, we will write ξ instead of ξ_i. Since we assume symmetry among players, this has no consequences.

In the Florens et al. (2001) model, the bidding function $\bar{\varphi}$, derived from Nash equilibrium

[5] We do not exclude 0 as a possible value for the private information ξ_i, in order to avoid unnecessary restrictive assumptions. This should not make any difference for the implications of the model.

conditions, is perfectly known and equal to

$$\bar{\varphi}\left(\xi,\bar{F}\right)=\xi-\frac{\int_0^\xi \bar{F}^m\left(u\right)du}{\bar{F}^m\left(\xi\right)}$$

if $(m+1)$ players participate in the auction. Following Florens et al. (2001) we compute the following Gâteaux derivatives:

$$d_{F_\theta}\mathbf{\Phi}_{\bar{F}_0}\left(H\right)\left(\xi\right)=\frac{m}{\bar{F}^{m+1}\left(\xi\right)}\left[H\left(\xi\right)\int_0^\xi \bar{F}_0^m\left(u\right)du-\bar{F}_0\left(\xi\right)\int_0^\xi \bar{F}_0^{m-1}\left(u\right)H\left(u\right)du\right]$$

where H is an element of $C^q\left(\Xi\right)$ (replacing in this context the element $\tilde{\gamma}$), and

$$T_{\bar{F}}\left(H\right)\left(\xi\right)=\frac{\bar{F}_0\left(\xi\right)}{\int_0^\xi \bar{F}_0^m\left(u\right)du}\int_0^\xi \bar{F}_0^{m-1}\left(u\right)H\left(u\right)du.$$

This function is not defined when $\xi=0$. We then assume that the set Γ_0 is made of L^2 integrable functions of $C^1\left(\Xi\right)$, relative to a uniform measure on the set $[\varepsilon,1]$, where $\varepsilon>0$ is arbitrarily chosen. Using this measure one can easily verify that $T_{\bar{F}_0}\left(H\right)$ is a linear bounded operator (namely a Volterra type I integral operator). Indeed the Hilbert Schmidt norm of this operator is

$$\int_\varepsilon^1\int_\varepsilon^1\left(\frac{\bar{F}_0\left(\xi\right)}{\int_0^\xi \bar{F}_0^m\left(u\right)du}\bar{F}_0^{m-1}\left(u\right)1_{\left(u<\varepsilon\right)}\right)^2du\,d\xi$$

which is finite, because it is the integral of a continuous function on a compact set.

Moreover the application which associates to $\bar{F}$ the operator $T_{\bar{F}}$ is also continuous. Let $\bar{F}_n$ denote a sequence which converges to $\bar{F}_0$. Then the Hilbert Schmidt norm of $T_{\bar{F}_n}-T_{\bar{F}_0}$ (which is greater than the uniform norm) converges to zero. Using the Nashed (1971) result, it follows that T_F is the Fréchet derivative of $F-G\circ\varphi_F$.

The operator $T_{\bar{F}_0}$ is one to one because it is invertible:

$$T_{\bar{F}_0}\left(H\right)\left(\xi\right)=A\left(\xi\right)\int_0^\xi B\left(u\right)H\left(u\right)du=L\left(\xi\right)$$

implies

$$H=\frac{1}{B}\frac{\partial}{\partial\xi}\left(\frac{L}{A}\right)$$

This shows that $\bar{F}_0$ restricted to $[\varepsilon,1]$ is locally identified for any ε.[6]

Then $T_{\bar{F}_0}$ has no null singular values. It may be proved that the spectrum of $T_{\bar{F}_0}$ declines geometrically (i.e. singular values are proportional to $\frac{1}{j}, j=1,2,...$, see Florens (2007)), which characterises a mildly ill-posed inverse problem.

3.3.2 Risk Aversion

The case with risk aversion allow us to show more in details how we can use our identification theorem. Following Donald and Paarsch (1996), we have the following strategic function:

$$\bar{\varphi}\left(\xi,\bar{F},\lambda\right)=\xi-\frac{\int_0^\xi \bar{F}^{m\lambda}\left(v\right)dv}{\bar{F}^{m\lambda}\left(\xi\right)}$$

where $m+1$ is the number of bidders and $\lambda\subset[1,+\infty[\subset\mathbb{R}$. Now, $(F,\lambda)\in C^q\left(\Xi\right)\times\mathbb{R}$ and we actually face a semiparametric identification problem. We can now apply our identification Theorem. After some computations (see Appendix) we find that the operator of interest is

$$\begin{aligned}&T_{\bar{F},\lambda}\left(H,\beta\right)\left(\xi\right)=\\&\frac{\bar{F}\left(\xi\right)}{\int_0^\xi\left[\bar{F}\left(v\right)\right]^{m\lambda}dv}\\&\times\left\{\int_0^\xi H\left(v\right)\bar{F}^{m\lambda-1}\left(v\right)dv-\frac{\beta}{\lambda}\left[\ln\bar{F}\left(\xi\right)\int_0^\xi\left[\bar{F}\left(v\right)\right]^{m\lambda}dv-\int_0^\xi\ln\bar{F}\left(v\right)\left[\bar{F}\left(v\right)\right]^{m\lambda}dv\right]\right\}\end{aligned}$$

where H (respectively β) is the direction in $\bar{F}$ (respectively λ).

In order to determine if the model is identified in the semiparametric case, we can study

[6] Note that the neighborhood of $\bar{F}_0$ may depend on ε and then this result does not immediately imply that $\bar{F}_0$ is locally identified even in the case of continuous distribution functions.

$T_{\bar{F},\lambda}(H,\beta)(\xi)=0$. We can note that

$$T_{\bar{F},\lambda}(H,\beta)(\xi) = 0 \Leftrightarrow \beta C(F,\xi)=\lambda\int_0^{\xi} H(v)\,\bar{F}^{m\lambda-1}(v)\,dv \tag{5}$$

where $C\left(\bar{F},\xi\right)=\ln\bar{F}(\xi)\int_0^{\xi}\left[\bar{F}(v)\right]^{m\lambda}dv-\int_0^{\xi}\ln\bar{F}(v)\left[\bar{F}(v)\right]^{m\lambda}dv$.

Any H and β verifying (5) must also verify $\dfrac{\partial}{\partial\xi}T_{\bar{F},\lambda}(H,\beta)(\xi)=0$.

This implies $\beta\dfrac{\partial}{\partial\xi}C(F,\xi)=\lambda H(\xi)\,\bar{F}^{m\lambda-1}(\xi)$.

We can rewrite condition (5) and obtain

$$H(\xi)=\frac{\beta}{\lambda\bar{F}^{m\lambda-1}(\xi)}\frac{\partial}{\partial\xi}C\left(\bar{F},\xi\right) \tag{6}$$

with $\dfrac{\partial}{\partial\xi}C\left(\bar{F},\xi\right)=\int_0^{\xi}\left[\bar{F}(v)\right]^{m\lambda}dv\dfrac{\bar{f}(\xi)}{\bar{F}(\xi)}+\ln\bar{F}(\xi)\left\{\left[\bar{F}(\xi)\right]^{m\lambda}-\left[\bar{F}(\xi)\right]^{m\lambda}\right\}$
$=\int_0^{\xi}\left[\bar{F}(v)\right]^{m\lambda}dv\dfrac{\bar{f}(\xi)}{\bar{F}(\xi)}$

It follows from (6) that $\forall\beta$, $\exists H(\xi)$ such that $(H,\beta)\in\mathcal{N}\left[T_{\bar{F},\lambda}(H,\beta)\right]$. This implies that $\dim\left\{\mathcal{N}\left[T_{\bar{F},\lambda}(H,\beta)\right]\right\}\geq\dim\{\beta\}=1$ and thus that $T_{\bar{F},\lambda}(H,\beta)$ is not one to one.[7]

So we have shown the following result: *In an IPV Dutch or First-Price Sealed Bid model of auction, with the distribution of private information $\bar{F}$ and the parameter of risk aversion λ as parameters of interest, there is no semiparametric identification result.*[8]

Obviously, it is possible to find an identification result if we impose some parametric restriction on the distribution of the private information. For pedagogical purpose, we simplify our model by considering an exponential distribution for the private information and a procurement model, $\xi\sim E(\theta)$.

[7]Actually the null space of $T_{\bar{F},\lambda}$ is a one dimensional linear manifold.

[8]This result is consistent with Campo et al. (2002) result. Note that they find some restrictions in order to obtain global identification.

In this case the strategic function is the following[9]

$$\varphi\left(\xi, \bar{F}_{\theta}, \lambda\right) = \xi + \frac{1}{\theta m \lambda} \ .$$

After computation we find:

$$T_{\theta,\lambda}\left(\widetilde{\theta}, \tilde{\lambda}\right)(\xi) = -m\lambda\xi \exp\left(-\theta m \lambda \xi\right) \widetilde{\theta} + \left[\frac{\widetilde{\theta}}{\theta^2 m \lambda} + \frac{\tilde{\lambda}}{\theta m \lambda^2}\right] \theta \exp\left(-\theta\xi\right)$$

and we can also find that $\mathcal{N}\left[T_{\theta,\lambda}\left(\hat{\theta}, \tilde{\lambda}\right)\right] = 0$, which implies that the operator $T_{\theta,\lambda}$ is one to one.

We have the following result: *in the IPV procurement model with a cost ξ following an exponential distribution, $\xi \sim E(\theta)$, θ and the parameter of risk aversion λ are locally identified.*

4 Exogeneity

4.1 Game Model with Exogenous Variable

As we can see in our example with risk aversion, it is usually difficult to non parametrically identify the distribution of private information and other parameters of the strategy function in the i.i.d. symmetric case, because all the components are mixed in the distribution of the observable. The usual econometric intuition is naturally to consider exogenous variables which shift separately the distribution of private information and the strategy function in order to get identification. In order to formalize this more general framework we first consider the definition of exogeneity in game models and its implications for identification.

For simplicity we only consider a symmetric model with a single observation (one player in one game) and we drop the index i to clarify the notation. Let us decompose $\xi \in \mathbb{R}^p$ into

[9] See Paarsch (1992) for a more general discussion in the risk neutral case.

$(\eta, \zeta) \in \mathbb{R}^{r+s}$ (where $r + s = p$) and x into $(y, z) \in \mathbb{R}^{r+s}$.

Definition 6 *We define z as exogenous if:*

1. *the strategic function takes the form*

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \bar{\varphi}(\eta, z, \theta, \lambda) \\ \zeta \end{pmatrix}$$

   i.e., part of the private information vector is directly observable

2. *there exists a decomposition of θ into (ρ, μ) such that:*

   - *the marginal distribution of ζ only depends on ρ*
   - *the conditional distribution of η given ζ only depends on μ*
   - *ρ and μ are variation free*

   i.e., ρ, μ and z realize a cut in the model of unobservable variables

3. *$\bar{\varphi}$ only depends on θ through the element μ*

All the Assumptions 1 to 6 apply. Then:

Theorem 7 *Under Assumption 1, $\rho, (\mu, \lambda)$, and z realize a cut in the observable model (i.e., the marginal distribution of z only depends on ρ, the conditional distribution of y given z only depends on (μ, λ), and ρ and (μ, λ) are variation free.*

Proof. As θ and λ are variation free and μ and ρ variation free, we immediately conclude that ρ and (μ, λ) are variation free. The marginal distribution of z is identical to the marginal distribution of ζ and depends on ρ only. Let us consider the conditional distribution of y

given z.

$$\begin{aligned} P(y \le t|z) &= P(\bar{\varphi}(\eta, z, \mu, \lambda) \le t|z) \\ &= P\left(\eta \le \bar{\varphi}_{\mu,\lambda}^{-1}(t)\,|z\right) \\ &= \left(\bar{F}_{\mu}^{z} \circ \bar{\varphi}_{\mu,\lambda}^{-1}\right)(t) \end{aligned}$$

where $\bar{F}_{\mu}^{z}$ is the conditional distribution of η given z, and thus only depends on λ and μ. ■

We may now discuss identification.

- Identification of ρ does not raise specific problems and if μ and λ are our only parameters of interest, this question is not relevant.
- If we denote by G^{z} the conditional distribution function of y given z, the conditional game model is now fully characterized by the following (functional) nonlinear equation:

$$A(\mu, \lambda, G^{z}) = \bar{F}_{\mu}^{z} - \bar{G}^{z} \circ \bar{\varphi}_{\mu,\lambda} = 0 \tag{7}$$

  which is identical to (4) except that conditional distributions replace marginal ones. [10]

The same argument given in Theorem 1 applies here. Intuitively if the operator

$$d_{\mu}\bar{F}_{\mu_0}^{z} - \frac{\partial \bar{F}_{\mu_0}^{z}}{\partial \xi'}\left[\Delta\varphi_{\mu_0,\lambda_0}\right]^{-1}\left(d_{\mu}\bar{\Phi}_{\mu_0,\lambda_0} + d_{\lambda}\bar{\Phi}_{\mu_0,\lambda_0}\right)$$

is one to one, then μ and λ are locally identified. More precisely all assumptions 1 to 6 should be extended to this conditional model. These extensions can easily be seen by "fixing" the z at a particular value on the support of this variable.

[10] The notation $\bar{F}_{\mu}^{z}$ can also be understood as the conditional distribution $\bar{F}_{Y|Z}(.|.)$ so that one may have e.g. $\mu = \bar{F}_{Y|Z}(.|.)$ in the nonparametric case.

4.2 Exogeneity and Exclusion

Let us decompose z into z_1 and z_2 (or equivalently as $z = \zeta$, ζ_1, and ζ_2) and let us assume the following:

Assumption 7 z is an exogenous vector of variables, and the sampling model of unobservable verifies the following conditional independence :

$$\eta \perp\!\!\!\perp z_1 | z_2$$

where z_1 and z_2 are variation free.

In a game model exogenous variables may have many interpretations:

- some of the variables influence the distribution of types and are different throughout the games but are identical to all players (for example in an auction model the private values of the good may be dependent on some characteristics of the good).
- some of the variables characterize the players and are typically identified through different games. These variables are particularly interesting because they may be a way to construct an asymmetric game (by conditioning) from a symmetric game (in the joint model).
- finally some variables may describe the rules of the game. They are present in the φ function but not in the distribution of the unobservables.

In our presentation z_1 represents the last two categories of variables.

Our main result formalizes an identification strategy which is easy to implement in practice: first, check if the model is identified given λ (if the strategy is perfectly known as a function of η, z and μ); second, check a supplementary condition which guarantees the identification.

Theorem 8 *We assume assumptions 1 to 7 and η and $y \in \mathbb{R}$. If*

i) μ is identified in a model where λ is known (and equals to its true value)

ii)

$$\frac{\partial \bar{\varphi}_{\gamma_0}}{\partial z_1 \partial \eta} d_\gamma \bar{\Phi}_{\gamma_0}(\tilde{\gamma}) + \frac{\partial \bar{\varphi}_{\gamma_0}}{\partial \eta} \frac{\partial}{\partial z_1} d_\gamma \bar{\Phi}_{\gamma_0}(\tilde{\gamma}) = 0$$

implies $\tilde{\lambda} = 0$

then (μ, λ), elements of γ, are locally identified.

Proof. We have

$$\begin{aligned}
& d_\mu \bar{F}^{z_2}_{\mu_0} - \frac{\partial \bar{F}^{z_2}_{\mu_0}}{\partial \xi'} \left[\Delta \varphi_{\mu_0,\lambda_0}\right]^{-1} \left(d_\mu \bar{\Phi}_{\mu_0,\lambda_0} + d_\lambda \bar{\Phi}_{\mu_0,\lambda_0}\right) \\
&= d_\mu \bar{F}^{z_2}_{\mu_0}(\tilde{\mu}) - \tilde{f}^{z_2}_{\mu_0} \left(\frac{d_\mu \bar{\Phi}_{\mu_0,\lambda_0}(\tilde{\mu}) + d_\lambda \bar{\Phi}_{\mu_0,\lambda_0}\left(\tilde{\lambda}\right)}{\partial \varphi_{\mu_0,\lambda_0} / \partial \xi} \right) \\
&= d_\mu \bar{F}^{z_2}_{\mu_0}(\tilde{\mu}) - \tilde{f}^{z_2}_{\mu_0} \frac{d_\gamma \bar{\Phi}_{\gamma_0}(\tilde{\gamma})}{\partial \varphi_{\gamma_0} / \partial \eta}
\end{aligned}$$

From the previous results $\gamma = (\mu, \lambda)$ is locally identified if

$$d_\mu \bar{F}^{z_2}_{\mu_0}(\tilde{\mu}) - \tilde{f}^{z_2}_{\mu_0} \frac{d_\gamma \bar{\Phi}_{\gamma_0}(\tilde{\gamma})}{\partial \varphi_{\gamma_0} / \partial \eta} = 0 \tag{8}$$

implies $\tilde{\mu}$ and $\tilde{\lambda} = 0$.

Let us take the derivatives w.r.t. z_1 (or in the case of discrete variables take the discrete difference). Then the first term disappears and we get:

$$\frac{\partial \bar{\varphi}_{\gamma_0}}{\partial z_1 \partial \eta} \left(\frac{d_\gamma \bar{\Phi}_{\gamma_0}(\tilde{\gamma})}{\partial \varphi_{\gamma_0} / \partial \xi} \right) = 0$$

or equivalently

$$\frac{\partial \bar{\varphi}_{\gamma_0}}{\partial z_1 \partial \eta} d_\gamma \bar{\Phi}_{\gamma_0}(\tilde{\gamma}) + \frac{\partial \bar{\varphi}_{\gamma_0}}{\partial \eta} \frac{\partial}{\partial z_1} d_\gamma \bar{\Phi}_{\gamma_0}(\tilde{\gamma}) = 0 \ .$$

Let us now assume condition ii), which implies that $\tilde{\lambda} = 0$. We now come back to condition (8) where $\tilde{\lambda} = 0$. As we have noticed generally, verifying (8) with $\tilde{\lambda} = 0$ implies

$\tilde{\mu} = 0$ is equivalent to verifying the local identification of μ if λ is given. ■

This result separates the identification procedure into two relatively simple verifications. Moreover the second condition only depends on φ (and not on $\bar{F}_{\mu_0}^{z_2}$) and must be verified independently of any hypotheses on the distribution of unobservable variables.

4.3 Example (cont'd)

We consider again the First Price Private Value auction example with risk aversion. In the following, we simply discuss the use of exogenous variables characterized by the following assumptions (relative to one observation i.e. one player in one game).

$$\begin{aligned} \xi &\sim F, \quad \xi \in \mathbb{R} \ , \\ z &\sim F_z, \quad z \in \mathbb{R}^k \ , \\ & \xi \perp\!\!\!\perp z \ , \end{aligned} \tag{9}$$

$$x = \xi - \frac{\int_0^\xi \left[F\left(v\right)\right]^{\lambda(z)} dv}{\left[F\left(\xi\right)\right]^{\lambda(z)}} \in \mathbb{R} \ .$$

This model could have different interpretations. For example $\lambda(z) = \lambda z$ where $z+1$ is the number of participants (which varies in that model) or $\lambda(z) = m\bar{\lambda}(z)$ where $\bar{\lambda}(z)$ is the coefficient of risk aversion dependent on the characteristics and $m+1$ is the number of participants (fixed).[11]

We consider as parameters of the conditional model the two functions F and λ (with true values F_0 and λ_0). We assume F_z, F, and λ are variation free in order to treat z as an exogenous variable (actually z plays the role of z_1 and there is no z_2 in our example). The

[11] In order to preserve the interpretation of our strategic function as a Nash equilibrium, z should be identical for all the participants in each game. With this assumption we actually maintain symmetry between players.

linear operator associated with this game is

$$T_{F,\lambda}\left(\tilde{F},\tilde{\lambda}\right)(\xi)=$$
$$\frac{F_0(\xi)}{\int_0^\xi [F_0(t)]^{\lambda_0(z)}\,dt}$$
$$\times\left\{\int_0^\xi [F_0(t)]^{\lambda_0(z)}\tilde{F}(t)\,dt-\frac{\tilde{\lambda}(z)}{\lambda_0(z)}\left(\ln F_0(\xi)\int_0^\xi [F_0(t)]^{\lambda_0(z)}\,dt-\int_0^\xi \ln F_0(\xi)\,[F_0(t)]^{\lambda_0(z)}\,dt\right)\right\}$$

In order to show local identification, we have to show that $\mathcal{N}\left[T_{F,\lambda}\left(\tilde{F},\tilde{\lambda}\right)\right]=0$, i.e. show that $T_{F,\lambda}\left(\tilde{F},\tilde{\lambda}\right)=0$ implies $\tilde{F}$ and $\tilde{\lambda}=0$, where $\tilde{F}$ is a function of ξ and $\tilde{\lambda}$ a function of z.[12]

It is sufficient to consider the case where the sum between the brackets equals to 0. After some manipulation and a derivation w.r.t. ξ this equality implies

$$\tilde{\lambda}(z)\,a(z,\xi)=\tilde{F}(\xi)$$

where

$$a(z,\xi)=\frac{f_0(\xi)\int_0^\xi [F_0(t)]^{\lambda_0(z)}\,dt}{\lambda_0(z)\,[F_0(\xi)]^{\lambda_0(z)}}\ .$$

This then implies $\tilde{\lambda}(z)=0$ and $\tilde{F}(\xi)=0$ except if $\frac{\partial^2 \ln a}{\partial z\partial\xi}=0$. [13]

For simplicity we consider a single z. If many z appear, we can take the derivative w.r.t. any element of z.

To conclude the analysis we compute

$$\frac{\partial^2 \ln a}{\partial z\partial\xi}=\frac{\partial\lambda_0}{\partial z}\left\{\frac{\ln F_0\,[F_0]^{\lambda_0}\int_0^\xi [F_0]^{\lambda_0}-F_0\int_0^\xi \ln F_0\,[F_0]^{\lambda_0}}{\left(\int_0^\xi [F_0]^{\lambda_0}\right)^2}-\frac{f_0}{F_0}\right\}\ .$$

The sum between brackets is generically non zero (e.g. take $\xi=1$ and assume $f_0=0$,

[12]As in the case without exogenous variables, problems regarding integration around 0 also arise here. For simplicity this question is neglected here but may be treated as in Section 3.

[13]Indeed, if $\tilde{\lambda}(z)\neq 0$, a is the product of a function of z and a function of ξ and satisfies $\frac{\partial^2 \ln a}{\partial z\partial\xi}=0$.

then this sum should be strictly positive). We have then the following result.

Theorem 9 *In the model (9) the condition* $\frac{\partial \lambda_0(z)}{\partial z} \neq 0$ *is sufficient to establish local identification of* λ *and* F.

If many z *are present, the above condition for any one component of* z *is sufficient.*

This Theorem may be illustrated in the following way.

Let $z = (z_1, z_2)$,

$$\begin{aligned} \xi | z &\sim \xi | z_2 \sim F(.|z_2) \qquad \text{(i.e. } \xi \perp\!\!\!\perp z_1 | z_2 \text{ : conditional independence)} \\ &\text{and} \\ x &= \xi - \frac{\int_0^\xi [F(t|z_2)]^{\lambda(z)} dt}{[F(\xi|z_2)]^{\lambda(z)}} \end{aligned}$$

where the parameters are the functions $F(\xi|z_2)$ and $\lambda(z_1, z_2)$. Then all the previous computations may be extended and the sufficient condition for identification becomes

$$\frac{\partial \lambda_0(z)}{\partial z_1} \neq 0$$

This result proves that the introduction of exogenous variables in the distribution of ξ is not sufficient to reach identification and that specific exogenous variables should influence the λ parameter. [14]

5 Randomised Strategies

In our previous analysis we have always assumed that the relationship between the type or the signal ξ and the action x, played by the participants of the game, was deterministic. A natural extension of this framework is to consider the case when the strategic function is

[14] Guerre, Perrigne and Vuong (2009) consider a similar case for global identification. There are some clear differences, though. In particular, we do not specify endogenous participation. Also, we can directly apply our general approach, and after short derivations we can show the main result: in order to obtain identification, specific exogenous variables should influence the risk aversion parameter.

replaced by a conditional probability distribution of x given ξ (and dependent on the distribution generating ξ and probably other parameters). Here also we simplify the presentation by only considering symmetric games where a single observation (one player in one game) is used and here also we drop the index i. The original model is now replaced by

$$\begin{aligned} \xi &\sim \bar{F}_\theta \\ x|\xi &\sim H(.|\xi, \theta, \lambda) \end{aligned}$$

where $H(u|\xi, \theta, \lambda) = P(x \leq u|\xi, \theta, \lambda)$ is a conditional distribution on ξ depending on θ and λ. [15]

Exogenous variables may be also introduced in the framework of section 3. We do not explicitly consider this case, even if it is particularly relevant for applications.

The model is now constructed in a standard probabilistic way: given the marginal $\bar{F}_\theta$ on ξ and the conditional distribution H, we may construct a joint distribution on (ξ, x) from which the marginal

$$\bar{G}(u) = P(u \leq x|\theta, \lambda) = \int H(u|\xi, \theta, \lambda) \bar{F}_\theta(d\xi) \tag{10}$$

is the analog of equation (3) in the deterministic case.

Equation (10) defines another inversion problem which will be considered locally in order to construct a condition for local uniqueness of the solution.

Remark 10 *Note that properties such as monotonicity of $\bar{\varphi}$ are no longer required. The inversion of $\bar{\varphi}$ is replaced by the Bayes theorem which may be used to construct the conditional distribution of ξ given x (and θ and λ).*

We may extend our previous theorems to the case considered here, but the study of the null space of the linear operator may be a difficult task.

[15] The form of H is assumed to be known as the non randomized strategy φ was assumed to be known in previous sections.

To illustrate this point, let us take a model where $\theta = \bar{F}$. Indeed, in the deconvolution problem $x = \xi + \varepsilon$ there is no way to identify $\bar{F}$ and some components of ε. The distribution of ε should be completely known in order to identify $\bar{F}$. However a game theoretic model $x = \eta + \varepsilon$, where $\eta = \bar{\varphi}\left(\xi, \bar{F}, \lambda\right)$, may introduce some constraints in the distribution of η which may be used to identify some aspects of the distribution of ε. For example, let us consider the case where $\eta = \bar{\varphi}\left(\xi, \bar{F}, \lambda\right) = \bar{F}\left(\xi\right)$. In this case η is uniformly distributed for any $\bar{F}$ and the distribution of ε is fully identified by the model $x = \bar{F}\left(\xi\right) + \varepsilon$, even if $\bar{F}$ is non identified. The general identification question of the additive model where $\bar{\varphi}$ is more general, is a non trivial problem.

In the parametric case we may develop a rank argument based on a linear approximation of Equation (10). Indeed, the parameters are locally identified if the family of functions

$$\left\{ \begin{array}{c} \int \frac{\partial}{\partial\theta} H_{\theta,\lambda} \bar{F}_{\theta}\left(d\xi\right) + \int H_{\theta,\lambda} \frac{\partial}{\partial\theta} \bar{F}_{\theta}\left(d\xi\right) \\ \int \frac{\partial}{\partial\lambda} H_{\theta,\lambda} \bar{F}_{\theta}\left(d\xi\right) \end{array} \right\}$$

is linearly independent.

In general, the first order conditions of local identification, expressed without technicalities, may be written as

$$\begin{aligned} & \int \left[d_{\theta} H_{\theta_0,\lambda_0}\left(\tilde{\theta}\right) + d_{\lambda} H_{\theta_0,\lambda_0}\left(\tilde{\lambda}\right) \right] \bar{F}_{\theta_0}\left(d\xi\right) + \int H_{\theta_0,\lambda_0} d_{\theta} \bar{F}_{\theta}\left(\tilde{\theta}\right)\left(d\xi\right) = 0 \\ \Longrightarrow \quad & \tilde{\theta} = 0 \text{ and } \tilde{\lambda} = 0 \end{aligned}$$

and solving this linear equation may be a difficult task.

6 Non-Closed Form Solutions for First-Order Conditions

In numerous complex games the vector of strategies $\bar{\varphi} = (\varphi_1, ..., \varphi_I)$ is not explicit but is the solution of a functional set of equations summarized by:

$$A(\bar{\varphi}, \gamma) = 0 \ . \tag{11}$$

Let us take for example the case of Nash equilibrium. For a player i the value of playing $x_i = \varphi_i(\xi_i)$ if other players play $\varphi_j(\xi_j) \ \ (j \neq i)$, is in general:

$$U_i(\varphi_1(\xi_1), ..., \varphi_I(\xi_I), \xi_1, ..., \xi_I, \lambda) = U_i(\varphi_i(\xi_i), \varphi_{-i}(\xi_{-i}), \xi_i, \xi_{-i}, \lambda) \ .$$

Each player maximizes expected utility given ξ_i:

$$E_\theta [U_i(\varphi_i(\xi_i), \varphi_{-i}(\xi_{-i}), \xi_i, \xi_{-i}, \lambda) | \xi_i] \ ,$$

and the set of first order conditions creates the set of conditions:

$$\frac{\partial}{\partial x_i} E_\theta [U_i(x_i, \varphi_{-i}(\xi_{-i}), \xi_i, \xi_{-i}, \lambda) | \xi_i] |_{x_i = \varphi_i(\xi_i)} = 0 \qquad i = 1, ..., I \ .$$

We may summarize this set of equations by (11).

The next step would be to solve this problem and to get the set of $\varphi_{i\gamma}(\xi_i)$ from which $d_\gamma \Phi_{i\gamma_0}$ and $\partial \varphi_{i\gamma_0}$ may be described and used in the local identification condition.

We may however remark that those elements may be, in some cases, directly obtained without computation of $\bar{\varphi}$, but just using the implicit function $A(\bar{\varphi}, \gamma) = 0$.

For instance, the implicit function theorem may be used to compute $d_\gamma \Phi_{i\gamma_0}$. Indeed

$$d_\gamma \bar{\Phi}_{i\gamma_0}(\tilde{\gamma}) = -(d_{\bar{\Phi}} A_{\bar{\Phi}_0})^{-1} \circ d_\gamma A_{\gamma_0}(\tilde{\gamma}) \tag{12}$$

This result has a particular interest if A is a nonlinear operator, because equation (12) implies the resolution of a linear functional equation, namely:

$$d_{\bar{\varphi}} A_{\varphi_0}(y) = x$$

where x and y are in suitable spaces.

However, we also need to compute $\left(\frac{\partial}{\partial \xi_i} \varphi_{i\gamma}\right)_{i=1,..,I}$, which cannot be derived from (11) by implicit function theorem argument.

However equation $A(\bar{\varphi}, \gamma) = 0$ may be transformed into equation $B\left(\bar{\psi}, \gamma\right) = 0$ where $\bar{\psi}$ is the vector of inverse functions $\left(\varphi_{1\gamma}^{-1}, ..., \varphi_{I\gamma}^{-1}\right)$. As we see in the example below, in numerous cases the equation is naturally of the form $B\left(\bar{\psi}, \gamma\right) = 0$. We may apply in that case the implicit function theorem to obtain the derivative of $\bar{\Psi}_\gamma$ (the operator which associates to γ the set of functions $\left(\varphi_{1\gamma}^{-1}, ..., \varphi_{I\gamma}^{-1}\right)$. Indeed

$$d_\gamma \bar{\Psi}_{\gamma_0}(\tilde{\gamma}) = -\left(d_{\bar{\psi}} B_{\bar{\psi}_0}\right)^{-1} \circ d_\gamma A_{\gamma_0}(\tilde{\gamma}) \tag{13}$$

Here also this computation actually needs the inversion of a linear operator. Finally, by derivation of the identities $\varphi_{i\gamma}\left(\varphi_{i\gamma}^{-1}(x)\right) = x$ we find

$$\left[d_\gamma \bar{\Psi}_{i\gamma_0}(\tilde{\gamma})\right]\left(\varphi_{i\gamma_0}(\xi)\right) = \frac{d_\gamma \bar{\Phi}_{i\gamma_0}(\tilde{\gamma})}{\frac{\partial}{\partial \xi_i}\varphi_{i\gamma_0}}(\xi) \ .$$

The right hand side of this equation is precisely the element depending on $\bar{\varphi}_\gamma$ we need for the computation of the operator T and it may be obtained by solving (13) and by a composition with $\varphi_{i\gamma_0}$. [16]

Note that the methodology we develop in this section should be adapted to specific case and some computation may be simplified using the specific features of these particular cases.

[16] In general $\varphi_{i\gamma_0}$ does not need to be computed in order to check the one to one property. In many cases this property may be verified only thanks to the increasing property of the $\varphi_{i\gamma_0}$.

Interestingly, Armantier and Sbaï (2006) apply our local identification procedure. They estimate a structural model with a sample of French Treasury auctions to test whether participants are symmetric, and to determine which auction format is preferable in this context. More precisely, their model is a parametric Common Value auction of shares with asymmetry and risk aversion. In this case, first order conditions cannot be solved analytically, and the concept of Constrained Strategic Equilibrium, proposed by Armantier, Florens and Richard (2008), is used. To the best of our knowledge, no global identification result can be found for their empirically pertinent but complex model. This illustrates the importance to develop new identification tools such as the ones given here.

7 Conclusion

This paper has established a new general and flexible procedure to study local identification for a broad class of games. Parameters of interest are not necessarily only those that characterize the distribution of players' private value, but may also be dissociated structural parameters, such as a parameter of risk aversion. We also allow for asymmetric players with affiliated information by using a general form of the implicit function theorem to present a local identification principle.

A clear advantage of our local identification approach is that it is much more general than the identification results already existing for some specific models. The approach is flexible in that it is applicable to games with partially or fully observable exogenous variables, randomized (mixed) strategies, and no closed formed solution for the first order conditions. In the last case, an application has been already successfully implemented in Armantier and Sbaï (2006). A clear benefit from considering specific models is the ability to establish identifiability assumptions or restrictions, in a way similar to Athey and Haile (2002). This is a possible extension to our work, as well as the possibility to establish a class of distributions for which we can obtain identification in games that were a priori unidentified.

Sbaï (2007) follows this direction and provides more illustrations using our identification approach. Incentive regulation models, as studied in Perrigne and Vuong (2004), may receive more specific attention.

To conclude, we contribute to the growing literature on the structural econometric analysis of game theoretic models (overviewed in Paarsch and Hong (2006)). A general approach to identification is provided that opens up the econometric study of a broad class of models, and more particularly expanding the limits of structural analysis in empirical games of incomplete information.

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8 Appendix

8.1 First Price Private Value Auction Model with Risk Aversion

We present here in detail the necessary calculus to obtain the different elements of our operator of interest $T_{\bar{F},\lambda}$.

8.1.1 Computation of $d_{\bar{F}}\mathbf{\Phi}_{\bar{F},\lambda}(H)$

We recall that

$$d_{\bar{F}}\mathbf{\Phi}_{\bar{F},\lambda}(H) = \lim_{a\to 0^+}\frac{d}{da}\mathbf{\Phi}\left(\bar{F}+aH,\lambda\right)$$

We expand the right-hand side term:

$$\frac{d}{da}\mathbf{\Phi}\left(\bar{F}+aH,\lambda\right)(\xi) = \frac{d}{da}\varphi\left(\xi,\bar{F}+aH,\lambda\right)$$

$$= \frac{d}{da}\left[\xi - \frac{\int_{\xi_0}^{\xi}\left[\bar{F}(v)+aH(v)\right]^{m\lambda}dv}{\left[\bar{F}(\xi)+aH(\xi)\right]^{m\lambda}}\right]$$

$$= -\frac{\int_{\xi_0}^{\xi}\left[\bar{F}(v)+\lambda H(v)\right]^{m\lambda}dv\,(-m\lambda H(\xi))}{\left[\bar{F}(\xi)+aH(\xi)\right]^{m\lambda+1}} - \frac{\int_{\xi_0}^{\xi}m\lambda H(v)\left[\bar{F}(v)+aH(v)\right]^{m\lambda-1}dv}{\left[\bar{F}(\xi)+aH(\xi)\right]^{m\lambda}}$$

Thus we find

$$d_{\bar{F}}\mathbf{\Phi}_{\bar{F},\lambda}(H)(\xi) = \frac{m\lambda}{\left[\bar{F}(\xi)\right]^{m\lambda+1}}\times\left\{H(\xi)\int_{\xi_0}^{\xi}\bar{F}^{m\lambda}(v)\,dv - \bar{F}(\xi)\int_{\xi_0}^{\xi}H(v)\,\bar{F}^{m\lambda-1}(v)\,dv\right\}$$

8.1.2 Computation of $d_{\lambda}\mathbf{\Phi}_{\bar{F},\lambda}$

We have

$$d_{\lambda}\mathbf{\Phi}_{\bar{F},\lambda}(\beta) = \lim_{a\to 0^+}\frac{d}{da}\mathbf{\Phi}\left(\bar{F},\lambda+a\beta\right)$$

We expand the right-hand side term:

$$\tfrac{d}{da}\mathbf{\Phi}\left(\bar{F},\lambda+a\beta\right)(\xi)=\tfrac{d}{da}\varphi\left(\xi,\bar{F},\lambda+a\beta\right)$$

$$=\tfrac{d}{da}\left[\xi-\frac{\int_{\xi_0}^{\xi}\left[\bar{F}(v)\right]^{m(\lambda+a\beta)}dv}{\left[\bar{F}(\xi)\right]^{m(\lambda+a\beta)}}\right]$$

$$=\frac{m\beta\ln\bar{F}(\xi)\int_{\xi_0}^{\xi}\left[\bar{F}(v)\right]^{m(\lambda+a\beta)}dv}{\left[\bar{F}(\xi)\right]^{m(\lambda+a\beta)}}-\frac{\int_{\xi_0}^{\xi}m\beta\ln\bar{F}(v)\left[\bar{F}(v)\right]^{m(\lambda+a\beta)}dv}{\left[\bar{F}(\xi)\right]^{m(\lambda+a\beta)}}$$

Thus we find

$$d_\lambda\mathbf{\Phi}_{\bar{F},\lambda}(\beta)(\xi)\quad=\frac{m\beta}{\left[\bar{F}(\xi)\right]^{m\lambda}}\left\{\ln\bar{F}(\xi)\int_{\xi_0}^{\xi}\left[\bar{F}(v)\right]^{m\lambda}dv-\int_{\xi_0}^{\xi}\ln\bar{F}(v)\left[\bar{F}(v)\right]^{m\lambda}dv\right\}$$

8.1.3 Computation of $T_{\bar{F},\lambda}(H,\beta)(\xi)$

We can compute

$$\begin{aligned}\partial\varphi_{\bar{F},\lambda}(\xi)&=\frac{\partial}{\partial\xi}\varphi_{\bar{F},\lambda}(\xi)=1+\frac{m\lambda\bar{F}(\xi)}{\bar{F}^{m\lambda+1}(\xi)}\int_{\xi_0}^{\xi}\bar{F}^{m\lambda}(u)\,du-\frac{\bar{F}^{m\lambda}(\xi)}{\bar{F}^{m\lambda}(\xi)}\\&=\frac{m\lambda\bar{F}(\xi)}{\bar{F}^{m\lambda+1}(\xi)}\int_{\xi_0}^{\xi}\bar{F}^{m\lambda}(u)\,du\end{aligned}$$

and then deduce

1. $\dfrac{\bar{F}}{\partial\varphi_{\bar{F},\lambda}}d_{\bar{F}}\mathbf{\Phi}_{\bar{F},\lambda}(H)(\xi)=H(\xi)-\dfrac{\bar{F}(\xi)}{\int_{\xi_0}^{\xi}\left[\bar{F}(v)\right]^{m\lambda}dv}\int_{\xi_0}^{\xi}H(v)\,\bar{F}^{m\lambda-1}(v)\,dv$
2. $\dfrac{\bar{F}}{\partial\varphi_{\bar{F},\lambda}}d_{\lambda}\mathbf{\Phi}_{\bar{F},\lambda}\left(\tilde{\lambda}\right)(\xi)=\dfrac{\bar{F}(\xi)\,\beta}{\lambda}\left\{\ln\bar{F}(\xi)-\dfrac{\int_{\xi_0}^{\xi}\ln\bar{F}(v)\left[\bar{F}(v)\right]^{m\lambda}dv}{\int_{\xi_0}^{\xi}\left[\bar{F}(v)\right]^{m\lambda}dv}\right\}$

Thus we can write

$$T_{\bar{F},\lambda}(H,\beta)(\xi) =$$
$$H(\xi) - H(\xi)$$
$$+ \frac{\bar{F}(\xi)}{\int_{\xi_0}^{\xi}\left[\bar{F}(v)\right]^{m\lambda}dv}\int_{\xi_0}^{\xi} H(v)\,\bar{F}^{m\lambda-1}(v)\,dv - \frac{\bar{F}(\xi)\,\beta}{\lambda}\left\{\ln\bar{F}(\xi) - \frac{\int_{\xi_0}^{\xi}\ln\bar{F}(v)\left[\bar{F}(v)\right]^{m\lambda}dv}{\int_{\xi_0}^{\xi}\left[\bar{F}(v)\right]^{m\lambda}dv}\right\}$$

Finally we find :

$$T_{\bar{F},\lambda}(H,\beta)(\xi) =$$
$$\frac{\bar{F}(\xi)}{\int_{\xi_0}^{\xi}\left[\bar{F}(v)\right]^{m\lambda}dv}$$
$$\times\left\{\int_{\xi_0}^{\xi} H(v)\,\bar{F}^{m\lambda-1}(v)\,dv - \frac{\beta}{\lambda}\left[\ln\bar{F}(\xi)\int_{\xi_0}^{\xi}\left[\bar{F}(v)\right]^{m\lambda}dv - \int_{\xi_0}^{\xi}\ln\bar{F}(v)\left[\bar{F}(v)\right]^{m\lambda}dv\right]\right\}$$