Smooth Inequality Measurement : Approximation Theorems*

Michel le Breton†

IDEI-GREMAQ, University of Toulouse.

Eugenio Peluso‡

Department of Economics, University of Verona.

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Abstract

The main purpose of this paper is to prove that smoothness is not a restrictive assumption in inequality measurement as any inequality measure (preorder) can be approximated in a well defined sense by a a smooth inequality measure (preorder).

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†Manufacture des Tabacs, Aile Jean-Jacques Laffont 21, allée de Brienne, F-31000 Toulouse. E-mail: lebreton@cict.fr.

‡Via dell'Università 4, 37129 Verona. E-mail: eugenio.peluso@univr.it.

1 Introduction

Inequality measurement is certainly one of the most popular area in applied welfare economics. It aims to provide numerical or ordinal inequality measures to evaluate the evolution of inequality in the distribution of some personal characteristic such as income or wealth. Inequality may vary across space and time and under the impulse of economic and social policies like for instance income taxation and social expenditures. It is important to determine the contribution of each factor to the observed changes in the distribution.

Unfortunately, there is not a single universally accepted inequality measure that would impose itself as the canonical tool to deal with such questions. The axiomatic approach aims to select a family of measures (sometimes a single one) on the basis of a set of properties that may be considered appealing, desirable or expected for an inequality measure. The choice of these axioms is, of course, itself controversial but the merit of this approach is to offer a transparent description of the respective qualities and shortcomings of the measures and to lay the foundations of a comparative analysis. Further, while compatible with a multiplicity of inequality measures, some important axioms impose significant limitations on the ways in which inequality comparisons should be done. Sometimes, the axiom even allows unambiguous inequality comparisons.

Among these axioms, the most celebrated one is the Pigou-Dalton principle of transfers. From the theorem of Hardy-Littlewood and Polya (1934), we know that a measure satisfies the Pigou-Dalton principle of transfers if and only if it is strictly-Schur convex. This theorem also states that this property is equivalent to monotonicity with respect to the Lorenz criterion. Of course some other properties like decomposability may also be included in the list but their inclusion in the list is mostly motivated by practical considerations. Schur-convexity is truly a property which aims to offer a quantitative content to the vague concepts of inequality/equality. This paper is an investigation of the all class of measures which satisfy this property i.e. the all class of Schur-convex functions.

Our contribution examines one particular feature of this family of measures. Precisely, we investigate whether the technical property of smoothness can be considered (or not) as being innocuous. Can we, "without loss of generality", limit our attention to smooth Schur-convex measures and benefit, therefore from the practical advantages attached to differentiability? After all, one of the most famous index, the Gini measure, is not differentiable everywhere! The greatest advantage offered by smoothness is the easy necessary and sufficient differential test of Schur-convexity (the so-called Schur-Ostrowski's test presented as Theorem 1 hereafter and called "rectifiance" by Kolm (1968, 1976 a,b)) which can be considered under this property. Sometimes, it is quite difficult to check Schur-convexity through a direct application of the definition and this alternative route which require to compare two partial derivatives turns to be very useful.

The answer to the above question(s) will depend obviously upon the exact meaning given to the expressions "innocuous" or "without loss of generality". Fortunately for us, a somewhat similar question has been formulated in traditional microeconomics for the family of numerical and ordinal conventional utility measures, where instead of Schur-convexity, quasi-concavity and increasingness are the key properties imposed on preferences. This question addressed by Kannai (1974) and Mas-Colell (1974) is formulated in terms of approximation theorems: Is it true that any measure in the original set can be approximated (in a well-defined topological sense), as close as desired, by a smooth measure? Their papers answer affirmatively this question. The main purpose of our paper is to prove that the same conclusion holds true in our setting of inequality measures. We prove that the answer to this question is: yes, in the sense that any inequality index can be approximated arbitrarily close by a smooth one (all these terms will be carefully defined later one). The proper formal formulation of this property is the statement of a density theorem in a suitable topological framework. We prove a numerical and an ordinal version of this approximation theorem and present some side complements.

The paper is organized as follows. In Section 2, we give the notations and basic definitions that are used in the paper. Then, in Section 3, we state and prove our main approximation theorem for inequality measures and discusses various versions of the result. Finally, in Section 4 we state and prove the ordinal versions of the result.

2 Inequality Measurement: Schur-Convexity, Rectifiance and Smoothness

The main purpose of this paragraph is to introduce some of the main properties encountered in the area of economic inequality measurement and the celebrated Ostrowski-Schur's differential characterization of Schur convexity. In this paper, we limit our attention to income distributions described by discrete probability distributions, i.e. to probability distribution P of the following type:[1]

$$P = \sum_{i=1}^{n} p_i \delta_{x_i}, \text{ where } x_1 \leq x_2 \leq .... \leq x_n,\ p_i \geq 0\ \forall i = 1, ...n \text{ and } \sum_{i=1}^{n} p_i = 1,$$

P describes an income distribution in a society divided into n groups from the poorest denoted by 1 to the richest denoted by n; x_i and p_i denotes respectively the mean outcome and the the population size (in percentage) of group i. Since any discrete probability distribution can be approximated by a distribution where the probabilities p_i are all equal, we limit hereafter our attention to those distributions whose support is contained in $\Re_+$ and consists of at most n points. This set is in a one to one relationship with the cone K_n defined as follows.

$$K_n = \left\{ x \in \Re_+^n : x_1 \leq x_2 \leq .... \leq x_n \right\}.$$

This point of view postulates from the very beginning that the identities of the groups are irrelevant from the perspective of inequality measurement. While we will maintain this assumption through the all paper, it is useful to consider that the set of income distributions is $\Re_+^n$ in order to prepare further generalizations. Hereafter, we will denote respectively by $\mu(x)$ and $\sigma^2(x)$ the mean income and the variance of incomes attached to the distribution x i.e.

$$\mu(x) = \frac{1}{n} \sum_{i=1}^{n} x_i \text{ and } \sigma^2(x) = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu(x))^2$$

A square matrix $B = (b_{ij})_{1 \leq i,j \leq n}$ of order n is *doubly stochastic* if $\sum_{i=1}^{n} b_{ij} = 1$ for all $j = 1, ..., n$ and $\sum_{j=1}^{n} b_{ij} = 1$ for all $i = 1, ..., n$. A square matrix P of order n is a *permutation*

[1] For all $s \in \Re$, the abstract but useful symbol δ_s is used to denote the degenerate probability where all the mass is concentrated on s.

matrix if it is a doubly stochastic matrix with exactly one positive entry in each row and each column. We denote respectively by $\mathcal{D}_n$ and $\pm_n$ the set of doubly stochastic and permutation matrices of order n.

A real valued function f defined over $D \subseteq \Re^n_+$ is *Schur-convex*[2] if:

$$f(Bx) \leq f(x) \text{ for all } x \in D \text{ and } B \in \mathcal{D}_n \text{ such that } Bx \in D$$

f is strictly Schur-convex if:

$$f(Bx) < f(x) \text{ for all } x \in D \text{ and } B \in \mathcal{D}_n \backslash \pm_n \text{ such that } Bx \in D$$

Finally, f is *symmetric* if:

$$f(Bx) = f(x) \text{ for all } x \in D \text{ and } P \in \pm_n \text{ such that } Px \in D$$

Similarly, a set $D \subseteq \Re^n$ is Schur-convex if all $x \in D$ and $B \in \mathcal{D}_n : Bx \in D$. D is symmetric if all $x \in D$ and $P \in \pm_n : Px \in D$. A set A is Schur-convex (symmetric) if the indicator function 1_A is Schur-convex (symmetric). Alternatively, a function f is Schur-convex if, for all $x \in D$, the lower contour set $\{y \in D : f(y) \leq f(x)\}$ is a Schur-convex set. Typically, inequality measurement refers to comparison of income distributions x and y such that $\mu(x) = \mu(y)$. The properties of Schur-convexity and symmetry are essential. When $D = S_n$, the unitary simplex in $\Re^n$ i.e. $S_n = \left\{x \in \Re^n_+ : \sum_{i=1}^n x_i = 1\right\}$, an inequality measure is a real valued function which is continuous and strictly Schur-convex.[3] We will denote by $\mathcal{I}_1$ the set of inequality measures on S_n.

Practitioners are often confronted to the necessity of comparing income distributions x and y which differ according to the mean. For instance, we may have to compare x and y such that $\mu(x) > \mu(y)$ and $\sigma^2(x) > \sigma^2(y)$. In such situation, the per capita income has increased when we move from x to y but the dispersion of incomes has also increased. To conclude,

[2] After the seminal pionnering work of Schur who was the first to introduce formally this class of functions.

[3] We can demonstrate that continuity and strict Schur-convexity implies Schur-convexity and then symmetry.

we need a welfare measure which combines inequality and "growth" considerations. A real valued function defined over $\Re^n_+$ is a *welfare induced inequality measure* if it is continuous, strictly Schur-convex and strictly decreasing. We will denote by $\mathcal{I}_2$ the set of welfare induced inequality measures on $D = \Re^n_+$. Finally, we may decide to focus on inequality and to adopt a principle to compare income distributions belonging to different simplices. One such principle is invariance with respect to a proportional growth of all individual incomes i.e. homogeneity of degree 0. A real valued function defined over $\Re^n_+ \setminus \{0\}$ is an *invariance induced inequality measure* if it is continuous, strictly Schur-convex and homogeneous of degree 0. We will denote by $\mathcal{I}_3$ the set of invariant[4] inequality measures on $D = \Re^n_+ \setminus \{0\}$.

It is interesting to remark that Schur-convexity is truly a monotonicity property with respect to a partial preorder[5]. Precisely, if we define the preorder $\succeq$ on D as follows:

$$x \succeq y \text{ iff there exists a doubly stochastic matrix } B \text{ such that } y = Bx$$

then a function f over D is Schur-convex if f is increasing with respect to $\succeq$ i.e. if $x \succeq y \Rightarrow f(x) \geq f(y)$. The celebrated Hardy, Littlewood and Polya's theorem[6] asserts that this preorder $\succeq$ is equivalent to three other preorders : $x \succeq y$ iff

$$y \text{ is in the convex hull of the set of vectors } \{Px\}_{P \in \pm_n}$$

$$\sum_{i=1}^{n} v(x_i) \geq \sum_{i=1}^{n} v(y_i) \text{ for every convex function } v : \Re \rightarrow \Re$$

$$\sum_{i=1}^{k} y_i^* \geq \sum_{i=1}^{k} x_i^* \text{ for all } k = 1, ..., n-1$$

where for any $z \in \Re^n_+$, z^* denotes the vector where the coordinates of z have been rearranged in increasing order. The importance of this theorem in economics was first pointed out by

[4] Some other principles of invariance could be considered. Kolm (1976b) discusses several alternative axioms of invariance.

[5] This observation is also formulated by Marshall and Olkin (1979).

[6] See Hardy, Littlewood and Polya (1934).

Kolm (1968).[7] Several variants of that theorem[8] can be found in applied mathematics under the heading "theory of majorization" (Marshall and Olkin (1979) and alternative presentations and extensions of this result are also analyzed in the area of stochastic dominance (Atkinson (1970), Le Breton (1987)).

From that perspective, checking whether a function f is Schur-convex or not amounts to verify the behavior of f with respect to the partial preorder $\succeq$. In some occasions, the task may be tricky i.e. it may be cumbersome to verify if f is increasing with respect to $\succeq$. Some general sufficient conditions on f to be Schur-convex are well known. For instance if f is quasi-convex (in particular if f is convex or log-convex) and symmetric then f is Schur-convex. Note however that Schur-convexity is much less demanding than quasi-convexity. A function f is quasi convex if for all $x \in D$, the lower contour set $\{y \in D : f(y) \leq f(x)\}$ is a convex subset. Convexity is not preserved by union while in contrast the union of two Schur-convex sets is a Schur convex set. The indicator function of the set $A \cup B$ where A and B are the two symmetric convex sets depicted on figure 1 is Schur-convex but is not quasi-convex. The class of Schur-convex functions is much larger that the class of quasi-convex functions.

[7] The importance of this theorem has been stressed by many authors (see e.g. Dasgupta, Sen and Starrett (1973) and Sen (1973)).

[8] In particular, to handle the sets of functions $\mathcal{I}_2$ and $\mathcal{I}_3$.

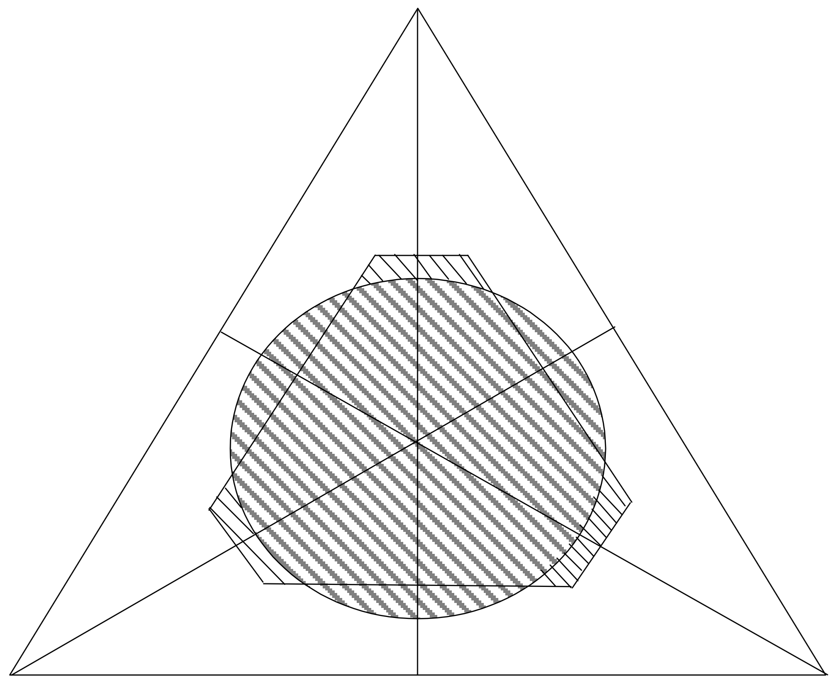

Figure 1

When f is differentiable, the task to verify if it is Schur-convex or not is much more easy as it amounts to check the sign of some derivatives. The following key result which formulates a two-coordinate characterization of Schur-convexity is due to Schur (1923) and Ostrowski (1952).[9]

Theorem 1 *Let D be an open and convex subset of $\Re^n$ and f be a differentiable real valued function defined on D. Then:*

(i) If for all $x \in D$ with $x_i \neq x_j$, $(x_i - x_j)\ (x_i - x_j) \left(\frac{\partial f}{\partial x_i}(x) - \frac{\partial f}{\partial x_j}(x)\right) > 0$, f is strictly Schur-convex.

(ii) f is Schur-convex iff for all $x \in D$, $(x_i - x_j) \left(\frac{\partial f}{\partial x_i}(x) - \frac{\partial f}{\partial x_j}(x)\right) \geq 0$

This theorem needs several comments. Note that the conditions (i) and (ii) constitute differential versions of the Pigou-Dalton principles of transfers. Kolm (1976) calls respectively

[9] Berge (1965) reproduces up to some simplifications the very elegant proof of Ostrowski. Notice that, due to symmetry, the rectifiance condition can be limited to the first two variables.

strict and weak rectifiance the properties (i) and (ii). It is important to observe that (i) is sufficient but not necessary for the strict Schur-convexity of f. It can be demonstrated however that if f is strictly Schur-convex, then the property of strict rectifiance is verified almost everywhere. Note also that some technical adjustments of the definition of differentiability are required if D is not an open and convex subset of $\Re^n$ as it is the case for instance when $D = S_n$ and $D = \Re^n_+$.

The importance of Theorem 1 lies in its operational character as it provides a handy way to test (strict) Schur-convexity. Besides those considered by Ostrowski, functions which are rectifiant (but not always quasi-convex) appears[10] in Elezovic and Pecaric (2000), Guan (2006), Karlin and Rinott (1981), Li, Zhao and Chen (2006), Sandor (2007), Shi (2007), Stepniak (2007), Zhang (1998 a,b). Karlin and Rinott uses rectifiance to prove the Schur-convexity of a class of generalized entropy functions. Sandor uses rectifiance to prove the Schur-convexity of the Stolarsky and Gini means. Elezovic and Pecaric and Shi uses it to prove the preservation of Schur-convexity through an averaging operation. Guan and Ostrowski proves the Schur-convexity of the complete elementary function:

$$c_r(x) \equiv \sum_{(i_1,i_2,\ldots,i_n)\in\mathbb{N}^n:\sum_{1\leq k\leq n} i_k=r} (x_1)^{i_1}(x_2)^{i_2}\ldots..(x_n)^{i_n}$$

where r is a fixed integer and subsequently the Schur-convexity of the function $\psi_r(x) \equiv \frac{c_r(x)}{c_{r-1}(x)}$. Li, Zhao and Chen proves the rectifiance of the function ϕ_m defined over $\left(-\frac{1}{m}, +\infty\right)^n$ as follow:

$$\phi_m(x) \equiv \prod_{i=1}^{n} \frac{\Gamma(mx_i+1)}{\Gamma^m(x_i+1)}$$

where $m \geq 2$ is a fixed integer and Γ denotes the celebrated gamma function defined over $\Re_+$ as follows:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

[10] Of course, this is just a sample. The theory of majorization is in fact mostly a systematic investigation of the class of Schur-convex functions to derive inequalities on pair of probability distributions. We refer the reader to chapter 3 in Marshall and Olkin (1979) which is entirely dedicated to this topic.

In this paper, we assume that inequality measures are defined on sets of (deterministic) income allocations but we could consider instead consider stochastic income allocations where the income ultimately received by an individual proceeds from a random device. The inequality measure is then defined upon the set of parameters describing this random device and it is natural to examine whether the measure is Schur-convex with respect to this vector of parameters. For instance, we could consider the case where the realized income distribution $x = (x_1, x_2, ..., x_n)$ is integer valued and such that $\sum_{1 \leq i \leq n} x_i = N$ where N is an exogenous integer and assume that it is drawn according to the multinomial distribution X:

$$P(X = x) = \left(\begin{array}{c} N \\ \\ x_1, ., x_n \end{array} \right) \prod_{i=1}^{n} \theta_i^{x_i}$$

where $\theta = (\theta_1, \theta_2, ..., \theta_n) \in S_n$. Let ϕ be a Schur-convex function over $\Re_+^n$. It can be demonstrated (Rinott (1973)) that the function $\Psi(\theta)$ defined as the expectation

$$\sum_{\left\{x : \sum_{1 \leq i \leq n} x_i = N\right\}} \phi(x) \left(\begin{array}{c} N \\ \\ x_1, ., x_n \end{array} \right) \prod_{i=1}^{n} \theta_i^{x_i}$$

is a Schur-convex function.

This example is an illustration of what Marshall and Olkin define as a family of distributions functions parametrized to preserve Schur-convexity. In their chapter 11 on stochastic majorization, they offer some general results and show how they apply to particular popular probability distributions. The differential test is often used to demonstrate Schur-convexity.

Hereafter, we will be interested in the class of strictly rectifiant inequality measures which are continuously differentiable at any order. An inequality measure f over D will be called *smooth* if it $f \in C^{\infty}(D, \Re)$ and is strictly rectifiant and we will denote by $\mathcal{I}_l^s$ the subset of smooth inequality measures in $\mathcal{I}_l$ for $l = 1, 2, 3$.

3 Numerical Approximation

The main purpose of this section is to demonstrate that the subset of smooth inequality measures if dense in the set of inequality measures. This result holds true for three alternative subsets of inequality measures introduced in the preceding section. We state the result in the case where $D = S_n$ i.e. the subset $\mathcal{I}_1$.

Theorem 2 *Let f be an inequality measure in I_1. Then there exists a sequence $(f_k)_{k\geq 1}$ of inequality measures in I_1^s converging uniformly to f over S_n.*

The proof of Theorem 2 will proceed from the combination of the following sequence of lemmas.

Lemma 1 *There exists a sequence of functions $(\varepsilon_k)_{k\geq 1}$ from $\Re^n$ into $\Re_+$ such that for all k:*

(ii) $\varepsilon_k \in C^\infty(\Re^n, \Re_+)$

(i) ε_k *is Schur-concave*

(iii) Supp[11]$(\varepsilon_k) \subseteq B(0, \frac{1}{k}) \cap \Re_-^n$

(iv) $\int_{\Re^n} \varepsilon_k(x)dx = 1$

Proof : Let h, $\Psi : \Re \to \Re$ be the functions defined as follows (the graph of Ψ is depicted on figure 2):

$$h(x) = \mu(x)^2 - n\sigma^2(x)$$

$$\Psi(t) = \begin{cases} e - ^{\frac{1}{t}} & \text{if } t > 0 \\ 0 \text{ if } t \leq 0 \end{cases}$$

[11] Given a real valued function g, $Supp(g)$ denotes its support i.e. the closure of the set $\{x \in D : g(x) \neq 0\}$; $B(x, \delta)$ denotes the ball centered on x with radius δ.

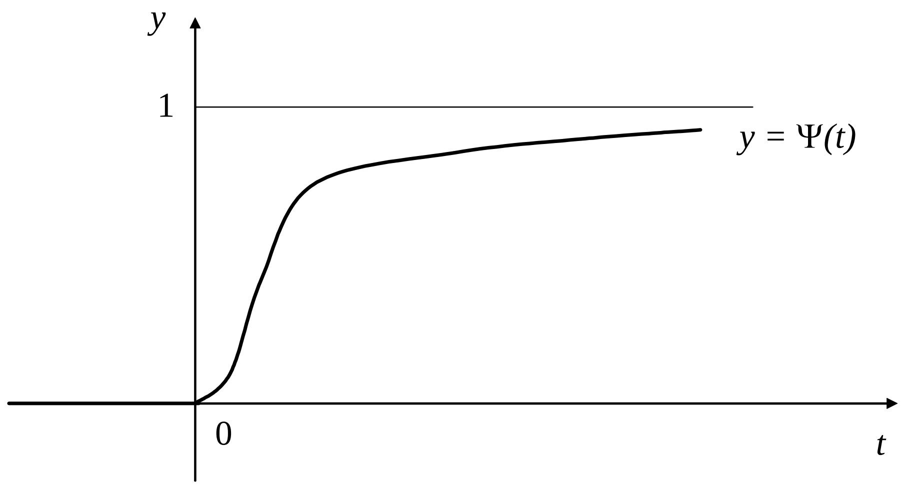

Figure 2

It is easy to verify that $\Psi \in C^{\infty}(\Re, \Re)$ and $h \in C^{\infty}(\Re^n, \Re)$. Further, h is Schur-concave. Therefore, $\Psi \circ h \in C^{\infty}(\Re^n, \Re)$ and is Schur-concave. Define $\widetilde{\varepsilon}_k : \Re^n \to \Re_+$ as follows:

$$\widetilde{\varepsilon}_k(x) \equiv \Psi\left(\frac{1}{k} + \mu(x)\right)\Psi(-n\mu(x))\,\Psi \circ h(x)$$

It is easy to check that the sequence of functions $(\varepsilon_k)_{k \geq 1}$ where:

$$\varepsilon_k(x) \equiv \frac{\widetilde{\varepsilon}_k(x)}{\int_{\Re^n} \widetilde{\varepsilon}_k(x)\,dx}$$

satisfies the four properties of the lemma. The support of is ε_k depicted on Figure 3 in the case where $n = 2$ $\square$

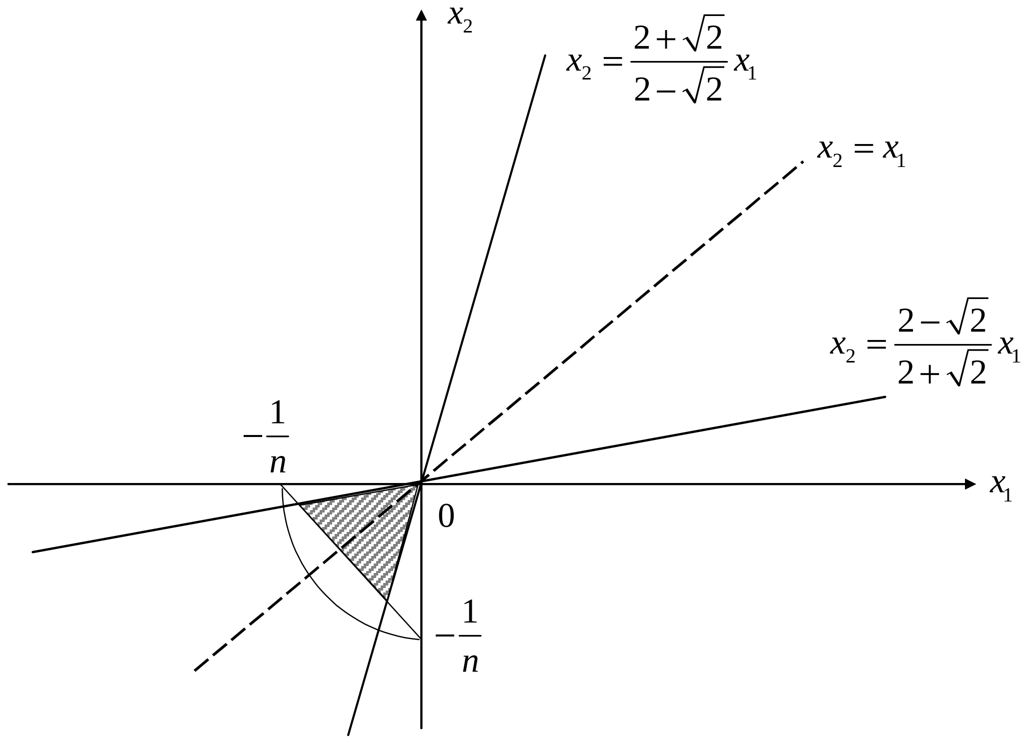

Figure 3

Lemma 2[12] *Let $f \in C_c^\infty(\Re^n, \Re)$ and $g \in L_{loc}^1(\Re^n, \Re)$. Then the convolution product $f * g$ defined as follows*

$$(f * g)(x) \equiv \int_{\Re^n} f(x-y) g(y) dy$$

is well defined and $f * g \in C^\infty(\Re^n, \Re)$.

The following key step is due to Marshall and Olkin (1974).

Lemma 3 *Let f and g be Schur-concave functions on $\Re^n$. Then $f * g$ (whenever it is defined) is Schur-concave. Moreover, if f is increasing (decreasing) and g is non-negative, then $f * g$ is increasing (decreasing).*

Proof of Theorem 2. Let f be an inequality measure in $\mathcal{I}_1$ and let $g = -f$. We extend g on $\Re_+^n$ as follows

[12] The proof of this assertion can be found in Yosida (1965).

$$\hat{g}(x) = \begin{cases} n\mu(x)\, g\left(\frac{x}{n\mu(x)}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

By construction, this extension of g is continuous and Schur-concave on $\Re^n_+$. Finally, we extend g on $\Re^n$ as follows.

$$\tilde{g}(x) = \begin{cases} \min\limits_{y \in S(x)} \hat{g}(y) \text{ if } \mu(x) \geq 0 \\ 0 \text{ if } \mu(x) < 0 \end{cases}$$

where $S(x) \equiv \left\{y \in \Re^n_+ : \mu(y) = \mu(x)\right\}$. It is easy to check that this extension of g is Schur-concave and belongs to $L^1_{loc}(\Re^n, \Re)$. We show that when k tends to ∞, $\tilde{g}* \, \varepsilon_k$ converges uniformly to $\hat{g}$ on any compact subset of $\Re^n_+$. Let K be a compact subset of $\Re^n_+$. From property (iii) and (iv) in Lemma 1, we deduce that for all $x \in \Re^n_+$

$$\begin{aligned} (\tilde{g} * \varepsilon_k)(x) - \hat{g}(x) &= \int_{\Re^n} (\tilde{g}(x-y) - \hat{g}(x))\, \varepsilon_k(y) dy \\ &= \int_{\left\{y \in B(0,1) \cap \Re^n_-\right\}} (\tilde{g}(x-y) - \hat{g}(x))\, \varepsilon_k(y) dy \end{aligned}$$

Since $\widehat{g}$ is uniformly continuous on $K + B(0,1)$, for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that:

$$\text{For all } x, y \in K + B(0,1) : \| x - y \| \leq \delta(\varepsilon) \Rightarrow | \widehat{g}(x) - \widehat{g}(y) | \leq \frac{\varepsilon}{2}$$

Therefore, if $k \geq \frac{1}{\delta(\varepsilon)}$, we obtain

$$\underset{x \in K}{Sup} \mid (\tilde{g} * \varepsilon_k)(x) - \hat{g}(x) \mid \leq \varepsilon \int_{\left\{y \in B(0,1) \cap \Re^n_-\right\}} \varepsilon_k(y) dy = \frac{\varepsilon}{2}$$

From property (i) in Lemma 1 and Lemma 2, $\widetilde{g} * \varepsilon_k \in C^\infty(\Re^n, \Re)$ and from property (ii) in Lemma 1 and Lemma 3, $\widetilde{g} * \varepsilon_k$ is Schur-concave. Further, from the above construction, we deduce that:

$$\underset{x \in S_n}{Sup} \mid -(\widetilde{g} * \varepsilon_k)(x) - f(x) \mid \leq \frac{\varepsilon}{2}$$

Let f_k be defined on S_n as follows

$$f_k(x) = -\left(\widetilde{g} * \varepsilon_k\right)(x) + \frac{\sigma^2(x)}{k} \text{ for all } k \geq 1$$

It is immediate to verify that f_k is a smooth inequality measure in $\mathcal{I}_1$ and that:

$$\underset{x \in S_n}{Sup} \mid \frac{\sigma^2(x)}{k} \mid \leq \frac{\varepsilon}{2}$$

if $k \geq \frac{2(n-1)(n-2)}{\varepsilon n^3}$. This completes the proof of Theorem 2 $\square$

An analogous result can be established for the sets $\mathcal{I}_2$ and $\mathcal{I}_3$. Indeed, any careful reader will notice that, up some minor adjustments, the same argument works for the space of inequality measures $\mathcal{I}_2$ and $\mathcal{I}_3$. For the set $\mathcal{I}_2$, we only need to extend g from $\Re^n_+$ to $\Re^n$ and use the second part of Lemma 3. For the space $\mathcal{I}_3$, we only needs to consider the (unique) zero homogeneous extension to $\Re^n_+ \setminus \{0\}$ of the approximating sequence defined in the proof of Theorem 2.

One key argument in the proof of Theorem 2 is the preservation of Schur-concavity by the convolution operator. This property and many of its important extensions have been analyzed in the mathematical literature (Nevius, Proschan and Sethuraman (1977), Proschan and Sethuraman (1977)) where Schur-concavity is shown to be preserved under the action of broader classes of operators.

In contrast, it is not immediate to adjust the proof in order to deal with the subsets of quasi-convex and log-convex inequality measures. The convolution argument does not work for quasi-concave functions (Dubuc (1978)) and while it works for log-concave functions (Ibragimov (1956), Davidovic, Korenbljum and Hacet (1969), Prékopa (1973)), log-concavity is not preserved by some monotonic transformations used in the proof. Under the presumption that the approximation property holds for these two subsets, a new proof is needed.

In addition to the above three sets of inequality measures, we could consider the subset of those which are decomposable (satisfying the property of "independence" according to Kolm (1968). An inequality measure f over D is decomposable if there exists a real valued convex function v from the projection of D over $\Re$ such that:

$$f(x) = \xi\left(\sum_{i=1}^{n} v(x_i)\right)$$

where ζ is a strictly increasing numerical function. It is interesting to point out that the approximation property holds true in restriction to the subset of decomposable inequality measures. This is an immediate consequence of the fact that any convex real valued function can be approximated by a smooth concave real valued function.[13]

Lemma 4 Let v be a convex function on $[0,1]$. Then there exists a sequence $(v_k)_{k\geq 1}$ of convex functions in $C^{\infty}([0,1],\Re)$ converging uniformly to v on $[0,1]$.

Proof. Let v be an arbitrary convex function on $[0,1]$. For any k in $\mathbb{N}\backslash\{0\}$, let v_k be the polynomial defined as follows

$$v_k(x) = \sum_{j=1}^{k} C_k^j v\left(\frac{j}{k}\right) x^j (1-x)^{k-j}.$$

It is well know that the sequence $(v_k)_{k\geq 1}$ converges uniformly[14] to v when k tends to ∞. We now show that for all k, v_k is convex. Since

$$v_k'(x) = \sum_{j=1}^{k} C_k^j a_j \left[j x^{j-1}(1-x)^{k-j} - (k-j)\, x^j (1-x)^{k-j-1}\right]$$

where $a_j \equiv v\left(\frac{j}{k}\right)$, we obtain:

$$\begin{aligned} v_k'(x) &= k\left(\sum_{j=0}^{k} \frac{(k-1)!}{j!\,(k-j)!} a_j x^{j-1}(1-x)^{k-j} - \sum_{j=0}^{k} \frac{(k-1)!}{j!\,(k-j)!} a_j x^j (1-x)^{k-j-1}\right) \\ &= k\left(\sum_{j=0}^{k} \frac{(k-1)!}{(j-1)!\,(k-j)!} a_j x^{j-1}(1-x)^{k-j} - \sum_{j=0}^{k} \frac{(k-1)!}{j!\,(k-j-1)!} a_j x^j (1-x)^{k-j-1}\right) \end{aligned}$$

i.e. after a change of variable

$$v_k'(x) = k\sum_{j=0}^{k-1} C_{k-1}^j (a_{j+1} - a_j)\, x^j (1-x)^{k-1-j}$$

[13] This result is probably known in the mathematical literature. For the sake of completeness, we include its simple proof here. We consider here the case where $D = S_n$ for which the projection of D over $\Re$ is $[0,1]$ but the result holds more generally.

[14] The function v_k is called the Bernstein's polynomial of order k attached to v on $[0,1]$.

Repeating this procedure for $v_k''(x)$, we obtain:

$$v_k''(x) = k\,(k-1)\sum_{j=0}^{k-2} C_{k-2}^{j}\,(a_{j+2} - 2a_{j+1} + a_j)\,x^j\,(1-x)^{k-2-j}$$

Since v is convex, $(a_{j+2} - 2a_{j+1} + a_j) \geq 0$ and therefore $v_k''(x) \geq 0$. The proof is complete

$\square$

4 Ordinal Approximation

Many practitioners in the area of inequality measurement disregard the numerical content of inequality measures and only pay attention to the (pre)ordering of income distributions induced by the measure. Any preorder derived from an inequality measure will be called hereafter an inequality preorder. Given an inequality measure f on domain D, we will denote by I_f the inequality preorder on D induced by f i.e.

$$\text{For all } x, y \in D : xI_f y \text{ iff } f(x) \geq f(y)$$

The properties introduced in the previous section have an immediate transposition into this ordinal setting. For instance, an arbitrary preorder I on D is Schur convex if for all $x \in D$, the upper contour sets $\{y \in D : yIx\}$ are Schur-convex sets. It is symmetric if the upper contour sets are symmetric. It is strictly Schur-convex if the upper contour sets are strictly Schur convex. An inequality preorder is continuous and Schur-convex. Given a continuous and strictly-Schur convex preorder I on a domain D , it is easy to deduce from Debreu's theorem (1964) that there exists an inequality measure f on D such that $I = I_f$; such a measure f constitutes a numerical representation of I. Hereafter, we denote respectively by $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ the set of inequality preorders induced by the sets $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ of inequality measures. An inequality preorder is smooth if it is induced by a smooth inequality measure. We denote by $\mathcal{P}_1^s$, $\mathcal{P}_2^s$ and $\mathcal{P}_3^s$ the sets of smooth inequality preorders induced by the sets $\mathcal{I}_1^s$, $\mathcal{I}_2^s$ and $\mathcal{I}_3^s$ of smooth inequality measures.

In this section, we provide an ordinal version of Theorem 2. Here, an inequality preorder I is defined by its graph G_I in $S_n \times S_n$: $(x, y) \in G_I$ iff xIy and the distance between two inequality preorders I and I' is defined as the Hausdorff's distance $\Delta(G_I, G_{I'})$ between their graphs. Given two non empty subsets A and B of D,

$$\Delta(A,B) \equiv Inf\{\varepsilon > 0 : A \subseteq B(B,\varepsilon) \text{ and } B \subseteq B(A,\varepsilon)\}$$

where, for any nonempty subset C of D, $B(C,\varepsilon)$ denotes the ε-neighborhood of C i.e. $B(C,\varepsilon) \equiv \{x \in D : \| x - y \| \leq \varepsilon \text{ for some } y \in D\}$.

When the set D is compact, the Hausdorff's topology coincides with the topology of closed convergence which is the standard topology employed in economics (Hildenbrand (1974)) to define proximity between preferences[15]. When D is non compact, some straightforward adjustments are needed. In the case of $\mathcal{P}_2$, we can use the standard Kannai's metric (1970) to proceed and in the case of $\mathcal{P}_3$, there is an immediate reduction to the simplex. Hereafter, we will concentrate our attention on the set $\mathcal{P}_1$.

Theorem 3 *Let I be an inequality preorder on S_n. Then there exists a sequence $(I_k)_{k\geq 1}$ of smooth inequality preorders on S_n such that $\Delta(G_I, G_{I_k})$ tends to 0 when k tends to ∞.*

The proof of Theorem 3 combines Theorem 1 and the following key lemma which constitutes a generalization of Lemma 1 in Mas-Colell (1974).

Lemma 5. *Let K be a compact subset of $\Re^n$ such that $\overset{\circ}{K} \neq \varnothing$ and $f : \Re^n \to \Re$ be continuous on K. Suppose that $f(Max(f,K)) \cap f(Min(f,K)) = \varnothing$. Then for all $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that $\Delta\left(G_{I_g} \cap (K \times K), G_{I_h} \cap (K \times K)\right) \leq \varepsilon$ for all g and h continuous on K and such that $\underset{x\in K}{Sup} \mid g(x) - f(x) \mid \leq \delta(\varepsilon)$ and $\underset{x\in K}{Sup} \mid h(x) - f(x) \mid \leq \eta(\varepsilon)$.*

Proof. Let $\varepsilon > 0$ and denote respectively by $Max_\varepsilon(f,K)$ and $Min_\varepsilon(f,K)$ the sets defined respectively as follows

$$Max_\varepsilon(f,K) \equiv \{x \in K : f(x) \geq f(y) \ \forall y \in B(x,\varepsilon) \cap K\}$$

$$Min_\varepsilon(f,K) \equiv \{x \in K : f(x) \leq f(y) \ \forall y \in B(x,\varepsilon) \cap K\}$$

[15] Approximation (in that sense) of arbitrary economic preferences by smooth economic preferences have been established by Kannai (1974) and Mas-Colell (1974).

The sets $Max_\varepsilon(f,K)$ and $Min_\varepsilon(f,K)$ are non-empty closed subsets of K. From our assumption on f, we deduce that $\mid f(x)-f(y)\mid>0$ for all $x\in Max_\varepsilon(f,K)$ and all $y\in Min_\varepsilon(f,K)$ and consequently $Max_\varepsilon(f,K)\cap Min_\varepsilon(f,K)=\varnothing$. Let $A(\varepsilon)$ and $B(\varepsilon)$ be open neighborhoods of $Max_\varepsilon(f,K)$ and $Min_\varepsilon(f,K)$ such that $\overline{A(\varepsilon)}\cap\overline{B(\varepsilon)}=\varnothing$ and $\mid f(x)-f(y)\mid>0$ for all $x\in\overline{A(\varepsilon)}$ and $y\in\overline{B(\varepsilon)}$. Let:

$$\overline{t}_\varepsilon(x)\equiv Sup_{y\in B(x,\varepsilon)\cap K}\ f(y)-f(x)$$

and

$$\underline{t}_\varepsilon(x)\equiv Inf_{y\in B(x,\varepsilon)\cap K}\ f(y)-f(x)$$

It is straightforward to check that the functions $\overline{t}_\varepsilon$ and $\underline{t}_\varepsilon$ are well defined and continuous on K. Since the sets $K\backslash A(\varepsilon)$, $K\backslash B(\varepsilon)$, $\overline{A(\varepsilon)}$ and $\overline{B(\varepsilon)}$ are compact, we deduce that:

$$\exists\overline{\gamma}_\varepsilon>0 \text{ such that } \overline{t}_\varepsilon(x)>\overline{\gamma}_\varepsilon \text{ for all } x\in K\backslash A(\varepsilon)$$

$$\exists\underline{\gamma}_\varepsilon<0 \text{ such that } \underline{t}_\varepsilon(x)>\underline{\gamma}_\varepsilon \text{ for all } x\in K\backslash B(\varepsilon)$$

and

$$\exists\lambda>0 \text{ such that } \mid f(x)-f(y)\mid>\lambda \text{ for all } x\in A(\varepsilon) \text{ for all } y\in B(\varepsilon)$$

Let $\eta(\epsilon)\equiv\frac{1}{8}Min\left(\overline{\gamma}_\varepsilon,-\underline{\gamma}_\varepsilon,\lambda\right)$ and consider g and h such that $\underset{x\in K}{Sup}\mid g(x)-f(x)\mid\leq\eta(\epsilon)$ and $\underset{x\in K}{Sup}\mid h(x)-f(x)\mid\leq\eta(\epsilon)$. Let $(x,y)\in G_{I_g}\cap(K\times K)$. We prove the existence of $(x',y')\in(B(x,\varepsilon)\times B(y,\varepsilon))\times(K\times K)$ such that $(x',y')\in G_{I_h}\cap(K\times K)$. Consider three distinct cases.

Case 1. $x\in A(\varepsilon)$ and $y\in B(\varepsilon)$.

In such case, $f(x)-f(y)>0$. Indeed, suppose on the contrary that $f(x)-f(y)<0$. Then we deduce then that $f(x)-f(y)<-\lambda\leq-8\eta(\epsilon)$ and therefore $g(x)-g(y)<-6\eta(\epsilon)<0$ which contradicts our assumption that $(x,y) \in G_{I_g} \cap (K \times K)$. Since $f(x) - f(y) > 0$, we deduce from the construction of $8\eta(\epsilon)$ that $f(x) - f(y) > 8\eta(\epsilon)$. Since:

$$\begin{aligned} g(x) - g(y) &= (g(x) - f(x)) + (f(x) - f(y)) + (f(y) - g(y)) \\ &\geq 6\eta(\epsilon) \end{aligned}$$

we deduce

$$h(x) - h(y) \geq 4\eta(\epsilon) \geq 0$$

i.e. $(x', y') = (x, y) \in G_{I_h} \cap (K \times K)$.

Case 2 $x \in K \backslash A(\varepsilon)$

In such case, consider $x' \in B(x, \varepsilon) \cap K$ such that $f(x') - f(x) \geq \overline{\gamma}_{\varepsilon} \geq 8\eta(\epsilon)$. Since:

$$\begin{aligned} g(x') - g(y) &= \big(g(x') - f(x')\big) + \big(f(x') - f(x)\big) + (f(x) - f(y)) + (f(y) - g(y)) \\ &\geq 6\eta(\epsilon) + (f(x) - f(y)) \end{aligned}$$

and since

$$\begin{aligned} f(x) - f(y) &= (f(x) - g(x)) + (g(x) - g(y)) + (g(y) - f(y)) \\ &\geq -2\eta(\epsilon) \end{aligned}$$

we obtain:

$$g(x') - g(y) \geq 4\eta(\epsilon)$$

Therefore, since:

$$\begin{aligned} h(x') - h(y) &= \big(h(x') - g(x')\big) + \big(g(x') - g(y)\big) + (g(y) - h(y)) \\ &\geq -2\eta(\epsilon) + 4\eta(\epsilon) - 2\eta(\epsilon) \geq 0 \end{aligned}$$

we obtain that $(x', y') = (x', y) \in G_{I_h} \cap (K \times K)$.

Case 3 $y \in K \backslash B(\varepsilon)$

The proof parallels the proof of case 2 by considering $(x', y') = (x, y')$ where $y' \in B(y, \varepsilon) \cap K$ such that $f(y) - f(y') \geq -\underline{\gamma}_{\varepsilon} \geq 8\eta(\epsilon)$. The conclusion follows by a symmetry argument and the definition of Δ □

Proof of Theorem 3. Let f be a numerical representation of I. >From theorem 1, there exists a sequence $(f_k)_{k \geq 1}$ converging uniformly to f on S_n. From strict Schur-convexity, we deduce that for all $\varepsilon > 0$, $Max_{\varepsilon}(f, K) = \{(1, 0, ..., 0), (0, 1, ..., 0), ..., (0, 0, ..., 1)\}$ and $Min_{\varepsilon}(f, K) = \left\{\left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)\right\}$ which implies $Max_{\varepsilon}(f, K) \cap Min_{\varepsilon}(f, K) = \varnothing$. The conclusion follows from Lemma 5 □

As already mentioned, an analogous result can be established for the sets $\mathcal{P}_2$ and $\mathcal{P}_3$.

5 Concluding Remarks

The results of this paper can be completed and/or generalized in several directions. We outline three of them that seem particularly promising.

First, we could explore whether the approximation results established in this paper for income distributions with finite support extend to continuous income distributions. The nice functional extension of Schur-Ostrowski obtained by Chan, Proschan and Sethuraman (1977) would be a first step in that direction.

Second, we could consider multivariate generalizations i.e. situations where each individual i is described by a vector $x^i = \left(x_1^i, x_2^i, ..., x_m^i\right)$ in the $m-$dimensional Euclidean space, instead of a single real number: each coordinate $j = 1, ..., m$ refers to a specific individual attribute (income, health status,...) A distribution is now a collection of n vectors $x^1, x^2, ..., x^n$ in $\Re^m$ which can be arranged into a matrix $x = \left(x_j^i\right)_{1 \leq i \leq n, 1 \leq j \leq m}$. Rinott (1973) extends the notions of Schur-convexity and symmetry to this multivariate setting. He derives a differential characterization of Schur-convex functions which extends the Ostrowski-Schur characterization in the univariate case. It would be worthwhile to investigate whether our approximation results hold in this multivariate setting.

Third, we have assumed through the paper that inequality measures were symmetric. In some cases, we may want to depart from this postulate. These will be the case when some observable characteristics of the groups suggest that they don't have the needs due (for instance) to differences in the demographic characteristics of the households. Any extension in that direction calls for an asymmetric generalization of Schur-convexity. Such extension has been developed notably by Hwang and Rothblum (1993, 1996). Hwang and Rothblum (1993) generalizes the classical concept of majorization defined earlier as one of the equivalent form of the partial preorder $\succeq$ and define the corresponding notion of Schur-convexity for which Schur-Ostrowski type characterizations are obtained. Hwang and Rothblum (1996) use quasi-directional convexity to extend the scope of Schur-convexity to functions which are not symmetric. An important relaxation of symmetry has also been explored by Eaton and Perlman (1977). Their approach consists in considering an arbitrary group G of orthonormal matrices of order n. Given $x \in \Re^n$, we denote by $C(x)$ the convex hull of the $G-$orbit of x i.e; the set of points $\{gx : g \in G\}$ and define the preorder $\succeq$ on $\Re^n$ as follows:

$$x \succeq y \text{ iff } y \in C(x)$$

They analyze the class of real valued functions over $\Re^n$ which are increasing with respect to $\succeq$. They call $G-$increasing any such function. When $G = \pm_n$, the partial order is the partial order of majorization introduced in Section 2 and the class of $G-$increasing functions is then the class of Schur-convex functions. They focus mostly on the case where the group G is a reflection group and demonstrate (among other things) that the class of $G-$increasing functions is preserved under convolution. This generalization of Lemma 2 would constitute an important step towards a generalization of our approximation technique in an asymmetric setting. They also obtain differential characterizations a la Schur-Ostrowski of the class of $G-$monotonicity. This question has been investigated further[16] by many authors among whom Niezgoda (1998a, b) and Tan (2002).

[16] Eaton (1982) is also a nice overview of this area of research.

6 References

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