Regulation of a Monopoly Generating Externalities

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July 2007

Abstract

We consider a monopoly that generates externalities. These depend on either the volume of services provided, or the number of clients, or both. We study the socially optimal prices and propose a regulatory mechanism to decentralize the optimum allocation. The mechanism, which is an generalised version of the price-cap scheme, only requires standard accounting data and straightforward estimates of the social marginal costs/benefits of externalities.

J.E.L. Classification Numbers: D42, D62, H21, H23, L51.
Keywords: Monopoly, Externalities, Regulation, Price-Cap.

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1 Introduction

Many industries, and in particular services of general interest, are plagued by two market imperfections: externalities and imperfect competition. The traditional remedy for externalities is taxation. As to imperfect competition, in the past typical responses were state ownership and/or administered prices. Over the last years, we have seen a general shift toward private property and the emergence of independent agencies caring for competition and prices. This observed trend also correspond to the rise of the so-called "new theory of regulation". A simultaneous treatment of both market imperfections can be found in a well established strand of the literature on taxation. The regulatory literature appears to be much poorer in that respect. In particular, to the best of our knowledge, the regulation of a monopoly generating externalities has never been considered. This is the purpose of our paper.

The proposed framework is designed to encompass a large set of environments. In particular, externalities depend on either the volume of services provided, or the number of clients, or both. Congestion (or, conversely, positive crowding) may occur in many instances. A case where greater use of the network may lower the quality of service is telecommunication. This is also an instance where the number of clients is considered to enhance the value of services, a phenomenon often referred to as "network externality". However, similar effects can be found in many different industries and contexts, *e.g.* energy markets and local public goods.

Consumption externalities already gave rise to a consistent stream of literature in the 70's. In particular, Diamond (1973) already considers corrective pricing to improve resource allocation. However, externalities may result in counter-intuitive effects. E.g. Diamond and Mirrlees (1973) identify the limited circumstances under which "a commodity that generates external economies should increase in quantity as one moves from a competitive equilibrium to a Pareto optimum". Soon, the complexity of the phenomenon brought pessimism about the very possibility of policy interventions. Indeed Littlechild (1975) concludes:

> "An attempt has been made, however, to characterize the optimal tariffs in terms of operational parameters, such as demand elasticities with respect to price, income, and the number of other subscribers in the system (...) In practice, things are much more complex than assumed here (...) It seems to me doubtful whether the present methods can be extended to give useful insights."

Building on the "new theory of regulation", we evidence that, despite the complexity of optimal allocations, the latter may be decentralised by means of a simple regulatory scheme, namely, an extended price-cap. In other words, we exhibit a general yet simple mechanism that allows to implement optimal regulation in economic environments with consumption externalities.

Along Littlechild (1975), we adopt a setup where producers price both access to the service (or connection to the network) and intensity of use. This leads us to consider non-linear pricing and in particular two-part tariffs.[1] Over the last years, several contributions have been looking at the working of two-parts tariffs in economic contexts with externalities. See, among others, Kanemoto (2000), Mitomo (2001) and Blonski (2002). However, most contributions are usually dedicated to specific markets and/or adopt quite specific setups. Moreover, a positive approach is generally adopted: the working of specific market mechanisms is considered. There is no attempt at looking for mechanisms able to implement what is to be considered as optimum. By contrast, in the present paper, we develop a unifying framework for the study of markets with con-

[1] We nevertheless also discuss the case where pricing is restricted to linear tariffs.

sumption externalities. The setup is fairly general. Furthermore, we provide an optimal (regulatory) mechanism.

Over the years, the literature on regulation is slowly but consistently bridging the gap between public economists' (theoretical) considerations and policy makers' (pragmatic) concerns. Central in that respect is the introduction of price-cap mechanisms and in particular the regulatory adjustment process proposed by Vogelsang and Finsinger (1979). Recent contributions look beyond the sole issue of affordability. An already consistent part of the literature looks at regulation and investment incentives. For a recent contribution, see Roques and Savva (2006). Concern for quality has also given rise to numerous contributions, recently surveyed by Sappington (2005). More elaborate objectives incorporating "social preferences" have also been given to regulators. See Poritz and Valentini (2002). The present work can be considered as a further development along these lines. More precisely, there is an interesting parallel to be drawn between quality and externality regulation. Quality is indeed an attribute of the consumed good that is set by the producer. Similarily, a consumption externality can be seen as an attribute of the consumption good. However, it is jointly and endogenously determined by the whole set of consumers. Moreover, in our set-up externalities may spill over to non-consumers. In that sense, the regulatory objectives are not limited to concerns streaming from the sole industry under scrutiny. They may also include concerns for other segments of society.

The analysis is drawn as follows. We first study consumer behaviour and decompose the effect of price changes into a direct effect and an indirect effect, that follows from the presence of the externality. This decomposition allows to identify the impact of externalities on market demand properties. Then we turn to a rather standard normative analysis and characterize in turn the first-best situation, the profit-maximising prices and the second-best allocation.

First-best prices are equal to the marginal costs of production plus/minus the marginal social costs/benefits of the externalities. The study of profit-maximisation and of the second-best allocation leads us to introduce the concept of virtual connection costs and virtual marginal costs. These "virtual costs" are those faced by the firm when all the effects of externalities are accounted for. We provide explicit formulae for these costs and analyze how they depend on the various externalities at work. At this point we are able to show that profit-maximising prices essentially do not differ from those provided by the Lerner formula. Similarly, the second-best allocation is characterised by optimal prices that have a Ramsey flavour. However a corrective term has to be added to account for the (direct) impact of externalities on social welfare. As already mentioned, we find that the second-best allocation can be implemented through a decentralised regulatory mechanism. The latter does only require standard accounting data and an estimate of the marginal impact of both commodities on social welfare.

The paper is organised as follows. Section 2 introduces the model and studies market demand. We then turn to the study of the First-best (Section 3), the profit-maximising price structure (Section 4) and the socially optimum prices, when the producer is required to break-even (Section 5). We then propose a regulatory mechanism that allows to decentralize the second-best allocation (Section 6). A short concluding section completes the paper.

2 The model

A monopolist delivers to N consumers a service produced in quantity X at a cost $C(X, N)$. The service is sold at a unit price b. In addition each consumer is charged a fee equal to a for having access to the service.

We adopt a quasi-linear framework for the representation of preferences. Let

$S_\theta(x_\theta, X, N)$ be the (gross) surplus of a consumer of type θ, where x_θ denotes his consumption. Net individual surplus is obtained by subtracting the individual expenditure for the service. In this setup, revenue effects are ruled out and the demand of infra-marginal consumers does not depend on a. We can write

$$x_\theta(b, X, N) = \arg\max_{x \geq 0} \{S_\theta(x, X, N) - (a + bx)\}, \tag{1}$$

whenever the corresponding net surplus

$$V_\theta(a, b) \equiv S_\theta[x_\theta(b, X, N), X, N] - [a + bx_\theta(b, X, N)] \tag{2a}$$

is not less than $S_\theta(0, X, N)$, *i.e.* when the consumer finds it beneficial to patronize the firm.

Let the population of types be distributed over $[0, +\infty]$ according to the density function $g(\theta)$ and the cumulative distribution function $G(\theta)$. We assume that the (gross) surplus S_θ is (weakly) increasing with θ,that is $(\partial S_\theta / \partial \theta) \geq 0$. As a result of the envelope theorem, the net surplus V_θ is also increasing with θ. Let θ_m be the type of the marginal consumer, who is indifferent between consuming the service or not, *i.e.*:

$$\max_x \{S_{\theta_m}(x, X, N) - (a + bx)\} = S_{\theta_m}(0, X, N). \tag{3}$$

Consumers with $\theta < \theta_m$ do not consume, while those with $\theta \geq \theta_m$ find it profitable to get access to the services. Aggregate demand is

$$X(a, b) = \int_{\theta_m}^{+\infty} x_\theta(a, b)\, g(\theta)\, d\theta, \tag{4}$$

while the number of consumers is

$$N(a, b) = \int_{\theta_m}^{+\infty} g(\theta)\, d\theta. \tag{5}$$

Given the assumed externalities, a preliminary step is a careful study of the impact of price changes on demand.

2.1 Consumer Behaviour

2.1.1 Impact of changes in the access fee a :

A change in a induces a shift in the marginal type θ_m, hence a change in the number of consumers N. More precisely, we know from equation (5) that

$$\frac{dN}{da} = -g\left(\theta_m\right)\frac{d\theta_m}{da}, \tag{6}$$

where, from the monotonicity of V_θ, the derivative $(d\theta_m/da)$ is certainly positive.

As already pointed out, the access fee a has no direct impact on the individual demand of inframarginal consumers. However it does impact $x_\theta\left(b, X, N\right)$ indirectly as a consequence of the externalities. To assess this change we consider the first-order condition that follows from the consumer program (1) that writes:

$$b = \frac{\partial S_\theta\left(x, X, N\right)}{\partial x}. \tag{7}$$

Differentiating with respect to a gives

$$\frac{dx_\theta}{da} = \left(\frac{-\partial^2 S_\theta}{\partial x^2}\right)^{-1}\left[\frac{\partial^2 S_\theta}{\partial x \partial X}\frac{dX}{da} + \frac{\partial^2 S_\theta}{\partial x \partial N}\frac{dN}{da}\right]. \tag{8}$$

We now turn to aggregate demand. By definition,

$$\frac{dX}{da} = \int_{\theta_m}^{+\infty}\frac{dx_\theta}{da}g\left(\theta\right)d\theta - g\left(\theta_m\right)x_{\theta_m}\frac{d\theta_m}{da}.$$

By using equation (8) and (6), the latter equation can be rewritten[2] as

$$\frac{dX}{da} = \frac{x_{\theta_m} + E_{xN}}{1 - E_{xX}}\frac{dN}{da}, \tag{9}$$

where

$$E_{xX} = \int_{\theta_m}^{+\infty}\left(\frac{-\partial^2 S_\theta}{\partial x^2}\right)^{-1}\frac{\partial^2 S_\theta}{\partial x \partial X}g\left(\theta\right)d\theta,$$
$$E_{xN} = \int_{\theta_m}^{+\infty}\left(\frac{-\partial^2 S_\theta}{\partial x^2}\right)^{-1}\frac{\partial^2 S_\theta}{\partial x \partial N}g\left(\theta\right)d\theta.$$

[2] Note that, in order to do so, we need to assume that externalities are "sufficiently small" so that $E_{xX} < 1$. This appears to be quite reasonable, since it amounts to suppose that cross-effects are smaller (in absolute value) than direct effects.

2.1.2 Impact of changes in the price b :

The effect of a change in the price b on the consumption of X can be decomposed into a marginal effect and an infra-marginal effect:

$$\frac{dX}{db} = -x_{\theta_m} g\left(\theta_m\right) \frac{d\theta_m}{db} + \int_{\theta_m}^{+\infty} \frac{dx_\theta}{db} g\left(\theta\right) d\theta, \tag{10}$$

where, from equation (5) ,we know that the marginal effect can be rewritten as:

$$-x_{\theta_m} g\left(\theta_m\right) \frac{d\theta_m}{db} = x_{\theta_m} \frac{dN}{db}. \tag{11}$$

The effect of a change in the price b on the consumption of an infra-marginal consumer can be decomposed in turn as a direct effect and as an indirect effect, the latter resulting from the presence of externalities. Indeed, differentiating with respect to b equation (7) that defines individual consumption gives

$$1 = \left(\frac{\partial^2 S_\theta}{\partial x^2}\right) \frac{dx_\theta}{db} + \left(\frac{\partial^2 S_\theta}{\partial x \partial X}\right) \frac{dX}{db} + \left(\frac{\partial^2 S_\theta}{\partial x \partial N}\right) \frac{dN}{db}.$$

It follows that

$$\frac{dx_\theta}{db} = \left(\frac{-\partial^2 S_\theta}{\partial x^2}\right)^{-1} \left[\left(\frac{\partial^2 S_\theta}{\partial x \partial X}\right) \frac{dX}{db} + \left(\frac{\partial^2 S_\theta}{\partial x \partial N}\right) \frac{dN}{db} - 1\right]$$

so that

$$\int_{\theta_m}^{+\infty} \frac{dx_\theta}{db} g\left(\theta\right) d\theta = \frac{\partial \widehat{X}}{\partial b} + E_{xX} \frac{dX}{db} + E_{xN} \frac{dN}{db} \tag{12}$$

where $\left(\partial \widehat{X} / \partial b\right)$ is nothing but the direct effect of b on the demand of infra-marginal consumers, given by

$$\frac{\partial \widehat{X}}{\partial b} = \int_{\theta_m}^{+\infty} \left(\frac{\partial^2 S_\theta}{\partial x^2}\right)^{-1} g\left(\theta\right) d\theta = \int_{\theta_m}^{+\infty} \frac{\partial x_\theta}{\partial b} g\left(\theta\right) d\theta,$$

while the two other terms on the right-hand side of (12) reflect the indirect effect of a change in b on the demand of infra-marginal consumers.

Plugging (11) and (12) into (10) yields:

$$\frac{dX}{db} = \frac{1}{1 - E_{xX}} \frac{\partial \widehat{X}}{\partial b} + \left(\frac{x_{\theta_m} + E_{xN}}{1 - E_{xX}}\right) \frac{dN}{db}. \tag{13}$$

3 First-best optimum

Let us start by characterizing the first-best allocation, that is the allocation that maximizes total surplus (consumer surplus plus profits). At this point the provider is not required to break even. We are thus implicitly assuming that fixed costs can be covered at no efficiency cost through a subsidy financed from the general budget. Such a solution is usually not feasible in practice. Nevertheless it provides us with an interesting benchmark.

We consider two formulations of the problem. The first one is direct and intuitive: we optimize with respect to individual decisions (access and consumption) and directly derive the optimal allocation. It appears that the latter can be decentralised by the means a two-part tariff. This allows us to propose a second approach that uses prices as decision variables. It is more complicated in a first-best setting but it will simplify the second-best problem significantly.

3.1 Direct approach

Social welfare writes as the difference between aggregate gross consumer surplus and total (production) costs:

$$W_1 = \int_0^{+\infty} S_\theta \left(x_\theta, X, N\right) g\left(\theta\right) d\theta - C\left(X, N\right), \tag{14}$$

where the aggregates X and N are defined by

$$\begin{aligned} X &= \int_0^{+\infty} x_\theta g\left(\theta\right) d\theta, \\ N &= \int_0^{+\infty} 1_{x_\theta > 0} g\left(\theta\right) d\theta. \end{aligned}$$

Consider first the determination of individual consumption. Differentiating (14) with respect to x_θ yields the following first-order condition:

$$\frac{\partial S_\theta}{\partial x_\theta} = \frac{\partial C}{\partial X} - \int_0^{+\infty} \frac{\partial S_\theta}{\partial X} g\left(\theta\right) d\theta. \tag{15}$$

Let E_X denote the marginal impact of X on aggregate gross consumer surplus:

$$E_X = \int_0^{+\infty} \frac{\partial S_\theta}{\partial X} g(\theta)\, d\theta. \tag{16}$$

By definition E_X does not depend on the specific individual considered. As a result, the optimal consumption pattern, can be decentralised by the means of a simple linear pricing scheme, *i.e.* by setting a unit price b for x to

$$b = \frac{\partial C}{\partial X} - E_X. \tag{17a}$$

We now shift to the access decision and assume that individual gross surplus is increasing with types in the first-best.[3] It follows that, at social optimum, if a type gets access to the service, all higher types are also given access to the service. In other words, the optimal access policy is uniquely defined by the lowest type to which access should be granted, $\theta_m \geq 0$.

A point we would like to make is that our framework does *not* entail any particular property for individual demand x_θ. We only require that the ranking of individual gross surplus corresponds to the ranking of types, at the first-best. But this is compatible with individual consumption decreasing (rather than increasing) with θ; and more complex patterns are not excluded.

Differentiating (14) with respect to θ_m and making use of the two relations $(dN/d\theta_m) = -g(\theta_m)$ and $(dX/d\theta_m) = -g(\theta_m)\, x_{\theta_m}$, the marginal type θ_m appears to be characterised by the implicit equation

$$S_{\theta_m}(0, X, N) = S(x_{\theta_m}, X, N) - \left[\left(\frac{\partial C}{\partial X} - E_X\right) x_{\theta_m} + \left(\frac{\partial C}{\partial N} - E_N\right)\right].$$

[3] The monotonicity of $S_\theta(x_\theta)$ may appear at first glance as a strong and *ad hoc* assumption. In fact, it is not. It may come out as a result in a variety of settings, *e.g.* under the assumption that marginal gross surplus from consumption is (weakly) increasing with types (together with the standard concavity of the gross surplus function). In fact, if $\left(\partial^2 S_\theta / \partial\theta \partial x_\theta\right) \geq 0$ and $\left(\partial^2 S_\theta / \partial x_\theta^2\right) < 0$, by differentiating $(17a)$ with respect to θ, one gets:
$\frac{\partial^2 S_\theta}{\partial\theta\partial x_\theta} + \frac{\partial^2 S_\theta}{\partial x_\theta^2} \frac{dx_\theta}{d\theta} = 0.$
This delivers demand monotonicity $(dx_\theta/d\theta) \geq 0$ and as a by-product optimal gross surplus monotonicity:
$\frac{d}{d\theta}\left[S_\theta(x_\theta)\right] = \frac{\partial S_\theta}{\partial\theta} + \frac{\partial S_\theta}{\partial x_\theta} \frac{dx_\theta}{d\theta} > 0.$

In fact, it is possible to show (See Appendix 9.1) that the first-best access policy can be decentralised by adopting for x the linear pricing scheme ($17a$) just obtained and by imposing in addition an access fee :

$$a = \frac{\partial C}{\partial N} - E_N, \tag{18}$$

where E_N denote the marginal impact of N on aggregate gross consumer surplus:

$$E_N = \int_0^{+\infty} \frac{\partial S_\theta}{\partial N} g(\theta)\, d\theta. \tag{19}$$

Observe that the first-best access pricing rule (18) that defines a depends on the very fact that equation ($17a$) holds (i.e. that the consumption price b is set correctly). By contrast, the first-best pricing rule ($17a$) that defines b does not require equation (18) to hold. It follows that, if the access price is exogenously determined, the (consumption) pricing rule ($17a$) can still be used. In particular, it continues to hold true if prices are restricted to be linear *i.e.* $a = 0$.

Expressions (18) and ($17a$) do not come as a surprise. They show that the first-best allocation can be decentralized through prices and have a number of interesting implications. First, they establish that the two-part tariff scheme proposed above do not introduce any inefficiency. Despite complex interactions accross heterogeneous consumers, this simple scheme is sufficient to decentralize the optimum allocation. Second, despite the externalities, the service should be sold at the same price whatever the quantity an individual consumes. Third, both prices a and b do not depend on consumer characteristics. Even if we allowed for perfect discrimination, it would not be desirable (on efficiency grounds) to charge different prices to different types; this is because social marginal costs do not depend on type. Fourth, both prices generally differ from marginal costs. In particular, if externalities are negative, the prices just obtained are higher than the corresponding marginal costs; as a result, the efficient allocation requires the monopolist to make strictly positive margins. Were margins reduced to zero, *i.e.* were prices set to marginal costs, over-consumption would obtain. Conversely, if the externality terms E_N and/or E_X are positive, the first-best allocation requires access and/or consumption to be subzidized. And marginal cost pricing would result in sub-optimal consumption.

Clearly, this pricing policy does not necessarily allow the provider to break even. For instance, in the presence of constant marginal costs, the provider may be unable to cover the fixed cost. Consequently, the first-best solution may not be feasible if the provider faces a break-even constraint. One then has to adopt a second-best solution where prices are set above (social) marginal cost in order to recover the fixed cost. This is studied in Section **??**. However, to facilitate the transition to the second-best setting, it is interesting to consider an alternative specification of the first-best problem.

3.2 Indirect approach

Alternatively we express total surplus as a function, not of quantities, but of prices, which then also become our decision variables. The objective function is then given by:

$$\begin{aligned} W_2 &= \int_0^{+\infty} V_\theta\left(a,b\right) g\left(\theta\right) d\theta \\ &\quad + aN\left(a,b\right) + bX\left(a,b\right) - C\left[X\left(a,b\right), N\left(a,b\right)\right]. \end{aligned}$$

The impact of prices on net surplus (indirect utility function) write

$$\begin{aligned} \frac{dV_\theta}{da} &= \frac{\partial S_\theta}{\partial X}\frac{dX}{da} + \frac{\partial S_\theta}{\partial N}\frac{dN}{da} - 1_{\theta\geq\theta_m}, \\ \frac{dV_\theta}{db} &= \frac{\partial S_\theta}{\partial X}\frac{dX}{db} + \frac{\partial S_\theta}{\partial N}\frac{dN}{db} - 1_{\theta\geq\theta_m} x_\theta\left(a,b\right). \end{aligned}$$

Thus, differentiating W_2 with respect to a and b and rearranging yields the FOCs:

$$\begin{aligned} \frac{dW_2}{da} &= \left(a - \frac{\partial C}{\partial N} + E_N\right)\frac{dN}{da} + \left(E_X + b - \frac{\partial C}{\partial X}\right)\frac{dX}{da} \\ \frac{dW_2}{db} &= \left(a - \frac{\partial C}{\partial N} + E_N\right)\frac{dN}{db} + \left(E_X + b - \frac{\partial C}{\partial X}\right)\frac{dX}{db} \end{aligned}$$

that clearly lead to the very same marginal cost pricing conditions we obtained above: (18) and (17a). Not surprisingly, both approaches are equivalent, thus yield the same results. However, while the direct approach is more convenient in a first-best setting, it is difficult to handle when a budget constraint is introduced.

4 Profit-maximising price structure

The firm may be subject to a taxation scheme (τ_X, τ_N), so that the general expression for its profits is given by:

$$\Pi = (a - \tau_N) N + (b - \tau_X) X - C(X, N). \tag{20}$$

Profit maximization gives thus rise to the following system of F.O.Cs:

$$\frac{d\Pi}{da} = N + \left(a - \frac{\partial C}{\partial N} - \tau_N\right)\frac{dN}{da} + \left(b - \frac{\partial C}{\partial X} - \tau_X\right)\frac{dX}{da} = 0, \tag{21}$$

$$\frac{d\Pi}{db} = X + \left(a - \frac{\partial C}{\partial N} - \tau_N\right)\frac{dN}{db} + \left(b - \frac{\partial C}{\partial X} - \tau_X\right)\frac{dX}{db} = 0. \tag{22}$$

4.0.1 Profit-maximising price a

By using (9), equation (21) can be rewritten to characterize the optimal access price a as

$$a - \frac{\partial C}{\partial N} - \tau_N + \left(b - \frac{\partial C}{\partial X} - \tau_X\right)\frac{x_{\theta_m} + E_{xN}}{1 - E_{xX}} = \frac{a}{\epsilon_N}, \tag{23}$$

where ϵ_N is the "price-elasticity" of the number of users N with respect to the access price a (i.e. the price-elasticity of the demand for access)

$$\epsilon_N = \left(-\frac{a}{N}\frac{dN}{da}\right). \tag{24}$$

Observe that (23) is nothing but a "standard" Lerner formula.

To see this, it may be useful to consider first the case without taxes and externalities. Then the equation (23) that characterizes the profit maximizing price a simplifies to

$$\frac{a - \frac{\partial C}{\partial N} + \left(b - \frac{\partial C}{\partial X}\right) x_{\theta_m}}{a} = \frac{1}{\epsilon_N}. \tag{25}$$

Giving access to an additional consumer, besides entailing the marginal cost $\partial C/\partial N$, also entails for the provider the profits from the sales resulting from his consumption. It is "as if" the firm were contemplating a "virtual cost" of connection

$$\frac{\partial C}{\partial N} - \left(b - \frac{\partial C}{\partial X}\right) x_{\theta_m}, \tag{26}$$

which in this simplified case is certainly lower than the marginal cost of connection $(\partial C/\partial N)$.Thus, absent taxes and externalities, the optimal access price a is lower than what a *naïve* application of the Lerner principle would suggest.

In the presence of taxes, the firm bases its decisions on prices net of taxes; furthermore, in the presence of externalities the connection of an additional consumer also impacts on the behaviour of other consumers, so the "virtual cost" contemplated by the firm is somewhat more complex than (26).

In fact, rearranging (23) gives

$$\frac{a - \widetilde{C}_N}{a} = \frac{1}{\epsilon_N} \tag{27}$$

where the expression of the "virtual cost" of connection $\widetilde{C}_N$ is

$$\widetilde{C}_N = \frac{\partial C}{\partial N} + \tau_N - \left(b - \frac{\partial C}{\partial X} - \tau_X\right) \frac{x_{\theta_m} + E_{xN}}{1 - E_{xX}}. \tag{28}$$

Observe that the ratio $(x_{\theta_m} + E_{xN}) / (1 - E_{xX})$ is nothing but the marginal change in demand that result from extending access to an additional consumer, as equation (9) makes it clear:

$$\frac{x_{\theta_m} + E_{xN}}{1 - E_{xX}} \equiv \frac{dX}{da} / \frac{dN}{da}.$$

The virtual cost $\widetilde{C}_N$ can thus be decomposed along the same line as (26): a (tax included) cost of connection minus the associated (net of taxes) revenue increase.

4.0.2 Profit-maximising price b

In the same manner as a change in a impacts on the (aggregate) quantity of services X sold by the firm, a variation in b impacts on the desirability of access, hence on the number of consumers N who actually get access to the service. A simple rewriting of (22) leads to a modified or "virtual" marginal cost of services that writes

$$\widetilde{C}_X = \frac{\partial C}{\partial X} + \tau_X - \left[a - \left(\frac{\partial C}{\partial N} + \tau_N\right)\right] \left(\frac{dN}{db} / \frac{dX}{db}\right) \tag{29}$$

that enters into a "standard" Lerner formula

$$\frac{b - \widetilde{C}_X}{b} = \frac{1}{\epsilon_X} \tag{30}$$

where ϵ_X is the standard price elasticity:

$$\epsilon_X = \left(-\frac{b}{X}\frac{dX}{db}\right). \tag{31}$$

The interpretation of 30 strictly parallels the interpretation given above with reference to connections. However, while the impact of an additional connection on total consumption is relativily easy to determine, the impact of an increase in total consumption on the number of consumers is less easy to evaluate. Indeed, while $\left(\frac{dN}{da} / \frac{dX}{da}\right)$ admits an explicit formulation, there is no closed form for $\left(\frac{dN}{db} / \frac{dX}{db}\right)$. So the formulae (29) and (30) are provided to illustrate the mechanisms at hand (and enhance their similarity), rather than for actual use. Thus, we now turn to a more convenient formulation of the optimal pricing rule, which holds true when the price a is set to its profit-maximizing level (23), *i.e.* when both prices a and b can be used as instruments by the monopolist.

Building on the analysis of consumer behaviour conducted above, in particular on equation (13), we rewrite now the FOC (22) as

$$\begin{aligned} 0 \quad = \quad & X + \left(b - \frac{\partial C}{\partial X} - \tau_X\right)\left(\frac{1}{1 - E_{xX}}\frac{\partial \widehat{X}}{\partial b}\right) \\ & + \left(a - \frac{\partial C}{\partial N} - \tau_N + \left(b - \frac{\partial C}{\partial X} - \tau_X\right)\frac{x_{\theta_m} + E_{xN}}{1 - E_{xX}}\right)\frac{dN}{db} \end{aligned} \tag{32}$$

If the price a is set to its profit-maximizing level (23), the latter equation boils down to:

$$0 = X + \left(b - \tau_X - \frac{\partial C}{\partial X}\right)\left(\frac{1}{1 - E_{xX}}\frac{\partial \widehat{X}}{\partial b}\right) - N\left(\frac{dN}{db}/\frac{dN}{da}\right).$$

Plugging into this equation the expression (52) derived in Appendix 9.2 one gets:

$$\frac{b - \widehat{C}_X}{b} = \frac{1 - x_{\theta_m}/\overline{x}}{\widehat{\epsilon_X}}. \tag{33}$$

Here, $\overline{x} = X/N$ denotes average consumption (which is assumed[4] to be greater than marginal consumption),

$$\widehat{C}_X = \frac{\partial C}{\partial X} + \tau_X - N\left(\frac{\partial S^+_{\theta_m}}{\partial X} - \frac{\partial S^-_{\theta_m}}{\partial X}\right) \tag{34}$$

is an alternative definition of virtual cost and

$$\widehat{\epsilon_X} = -\frac{b}{X}\frac{\partial \widehat{X}/\partial b}{1 - E_{xX}} \tag{35}$$

is the (absolute value of the) price elasticity of the demand of infra-marginal consumers. More precisely $\widehat{\epsilon_X}$ is the price elasticity that obtains when the number of consumers N is kept constant.

The Lerner formula (33) merits a few comments. First, everything happens as if the firm where contemplating a virtual marginal cost $\widehat{C}_X$. However, if there are no taxes and in addition either *(i)* there are no externalities or *(ii)* agents are affected by the externalities that derive from X whether they are connected or not, this virtual marginal cost $\widehat{C}_X$ is exactly identical to plain marginal cost $(\partial C/\partial X)$. Second, the "corrected elasticity" that is considered, namely $\widehat{\epsilon_X}/\left(1 - x_{\theta_m}/\overline{x}\right)$, is always larger[5] than $\widehat{\epsilon_X}$, whatever the nature of the

[4] Again, this would *follow* immediately from the previously mentioned assumption $\left(\partial^2 S_\theta/\partial\theta\partial x_\theta\right) \geq 0$. We prefer not to make this assumption so as to allow for more complex consumption patterns.

[5] In a very heterogeneous population, the consumption of the marginal consumer as compared to average consumption is usually negligible. In particular, if x_θ increases in θ (which would follow from the assumption $\left(\partial^2 S_\theta/\partial\theta\partial x_\theta\right) \geq 0$), the more heterogeneous the population, the smaller the correction.

externalities. However, $\widehat{\epsilon_X}$ and ϵ_X are difficult to compare. Absent externalities $\widehat{\epsilon_X} \leq \epsilon_X$ but there is no way to rank $\widehat{\epsilon_X}/\left(1 - x_{\theta_m}/\overline{x}\right)$ and ϵ_X in a systematic manner. As a result, even in the case where $\widehat{C}_X = C_X$, it is not possible to compare the price b, as defined by (33), with the price level that a *naïve* application of the standard Lerner formula would suggest.

Interestingly, this contrasts with the (profit maximising) access price a, as defined by (27): in the absence of externalities, $\widetilde{C}_N$ is always strictly smaller than $(\partial C/\partial N)$. As a result, absent externalities, the access price a should always be lower than what a *naïve* application of the standard Lerner formula would suggest.

5 Second-best prices

We now turn to the second-best solution which consists in maximizing W_2 subject to the producer break even constraint $\Pi \geq 0$. Let $\mathcal{L}$ be the Lagrangean expression associated with this problem while λ is the multiplier of the break-even constraint. We obtain the following first-order conditions:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial a} &= \int_0^{+\infty} \left[\frac{\partial S_\theta}{\partial X}\frac{dX}{da} + \frac{\partial S_\theta}{\partial N}\frac{dN}{da} - 1_{\theta \geq \theta_m}\right] g(\theta)\, d\theta \\
&\quad + (1+\lambda)\left[N + \left(a - \tau_N - \frac{\partial C}{\partial N}\right)\frac{dN}{da} + \left(b - \tau_X - \frac{\partial C}{\partial X}\right)\frac{dX}{da}\right] \quad (36) \\
\frac{\partial \mathcal{L}}{\partial b} &= \int_0^{+\infty} \left[\frac{\partial S_\theta}{\partial X}\frac{dX}{db} + \frac{\partial S_\theta}{\partial N}\frac{dN}{db} - x_\theta(a,b)\, 1_{\theta \geq \theta_m}\right] g(\theta)\, d\theta \\
&\quad + (1+\lambda)\left[X + \left(a - \tau_N - \frac{\partial C}{\partial N}\right)\frac{dN}{db} + \left(b - \tau_X - \frac{\partial C}{\partial X}\right)\frac{dX}{db}\right] \quad (37)
\end{aligned}$$

To rearrange and interpret these conditions, we make use of the notations introduced in (16), (19), (24), (29), (31), as well as in (35), and build on previous results, in particular equations (9) and (13). This allows us to obtain for the socially optimal price a the following characterization (see Appendix 9.3):

$$a-\left(\frac{\partial C}{\partial N}+\tau_N\right)+\left(b-\tau_X-\frac{\partial C}{\partial X}\right)\frac{x_{\theta_m}+E_{xN}}{1-E_{xX}}=\frac{\lambda}{1+\lambda}\frac{a}{\epsilon_N}-\frac{1}{1+\lambda}\left[E_N+E_X\left(\frac{x_{\theta_m}+E_{xN}}{1-E_{xX}}\right)\right] \tag{38}$$

or

$$a-\widetilde{C}_N=\frac{\lambda}{1+\lambda}\frac{a}{\epsilon_N}-\frac{1}{1+\lambda}\left[E_N+E_X\left(\frac{x_{\theta_m}+E_{xN}}{1-E_{xX}}\right)\right] \tag{39}$$

Similarily, the optimal price b should obey the equation

$$b-\widetilde{C}_X=\frac{\lambda}{1+\lambda}\frac{b}{\epsilon_X}-\frac{1}{1+\lambda}\left[E_X+E_N\left(\frac{dN}{db}/\frac{dX}{db}\right)\right]. \tag{40}$$

As for the equation (30) that defines the profit-maximising price b, this formulation, which is useful to highlight the mechanisms at hand, has the drawback that it rests on the ratio $\left(\frac{dN}{db}/\frac{dX}{db}\right)$ which is a priori difficult to estimate. However, if a can indeed be chosen by the monopolist, so equation (38) holds true, the latter expression can be rewritten as (See Appendix 9.3)

$$b-\left(\frac{\partial C}{\partial X}+\tau_X\right)+\frac{1}{1+\lambda}E_X+\frac{\lambda}{1+\lambda}N\left(\frac{\partial S_\theta^+}{\partial X}-\frac{\partial S_\theta^-}{\partial X}\right)=\frac{\lambda}{1+\lambda}\left(1-\frac{x_{\theta_m}}{X/N}\right)\frac{b}{\widehat{\epsilon_X}} \tag{41}$$

or

$$b-\widehat{C}_X=\frac{\lambda}{1+\lambda}\left(1-\frac{x_{\theta_m}}{X/N}\right)\frac{b}{\widehat{\epsilon_X}}-\frac{1}{1+\lambda}\left[E_X-N\left(\frac{\partial S_\theta^+}{\partial X}-\frac{\partial S_\theta^-}{\partial X}\right)\right]. \tag{42}$$

The second-best allocation is thus defined by means of equations (39) and either (40) or (42) . Obviously, besides providing the precise characterisation of the allocation, those formulae are of no practical use. Sure, they can interpreted along the lines proposed in previous sections. However, it is unrealistic to assume that the regulator may actually build on reliable estimates to compute these second-best prices. Hence the relevance of the mechanism proposed in next section.

In what follows we will use a star to denote the second-best solution derived in this section: (a^*, b^*).

6 Decentralization and global price-cap

So far we have concentrated on the pricing policy that would be chosen by a welfare maximizing (and well-informed) regulator. Let us now examine how this solution can be decentralized through a regulatory policy when the regulator faces a profit-maximizing provider. In other words, we study how the socially optimal prices (a^*, b^*) as defined by (38) and (40) can be achieved as a solution to the provider's profit maximization problem. It is plain that in the absence of regulation, the monopolist would not generally choose the socially optimal policy.[6] Some regulatory intervention is thus necessary to achieve the optimal outcome. The question is then, how "tight" has this regulation to be. More precisely, is it necessary to regulate every single price, or is some more "global" regulation sufficient?

To address this question, we consider a global price cap scheme - *i.e.* a constraint imposing an upper limit on the weighted average of the two prices - that bears some similarity with that proposed in Billette de Villemeur (2004).

Let the provider maximize its profits as defined in (20) subject to the global price-cap constraint given by

$$\alpha a + \beta b \leq \bar{p} + \varphi N + \psi X, \tag{43}$$

where α and β are the weights of goods access and service, respectively. Observe that theses weights are supposed to be exogenously given to the provider.

Let $\mathcal{L}$ be the Lagrangean of the provider's maximisation problem, while μ is the multiplier of the constraint (43). The first-order conditions are given by:

$$\frac{\partial \mathcal{L}}{\partial a} = N + \left(a - \frac{\partial C}{\partial N} - \tau_N + \mu\varphi\right)\frac{dN}{da} + \left(b - \frac{\partial C}{\partial X} - \tau_X + \mu\psi\right)\frac{dX}{da} - \mu\alpha, \tag{44}$$

$$\frac{\partial \mathcal{L}}{\partial b} = X + \left(a - \frac{\partial C}{\partial N} - \tau_N + \mu\varphi\right)\frac{dN}{db} + \left(b - \frac{\partial C}{\partial X} - \tau_X + \mu\psi\right)\frac{dX}{db} - \mu\beta. \tag{45}$$

[6] Except of course when the maximum achievable profit is equal to zero. In that case, the budget constraint can only be met if profit is maximized. Profit maximization and welfare maximization *subject to a break even constraint* then yield the same result.

Equation (44) rewrites simply as (see appendix 9.4):

$$a - \widetilde{C}_N = \left(1 - \mu\frac{\alpha}{N}\right)\frac{a}{\epsilon_N} - \mu\left(\varphi + \psi\frac{x_{\theta_m} + E_{xN}}{1 - E_{xX}}\right). \tag{46}$$

Again, if a is endogenous (can be chosen by the firm) so that it obeys (46), equation (45) rewrites (See appendix 9.4)

$$\begin{aligned} & b - \left(\frac{\partial C}{\partial X} + \tau_X\right) + \mu\psi + N\left(1 - \mu\frac{\alpha}{N}\right)\left(\frac{\partial S_\theta^+}{\partial X} - \frac{\partial S_\theta^-}{\partial X}\right) \\ = & \left(1 - \mu\frac{\beta}{X}\right)\left[1 - \left(\frac{1 - \mu\alpha/N}{1 - \mu\beta/X}\right)\frac{x_{\theta_m}}{X/N}\right]\frac{b}{\widehat{\epsilon_X}} \end{aligned} \tag{47}$$

The decentralization of the second-best solution requires that the solution (a, b) defined by (46) and (47) solves (36)–(37) for the appropriate value of μ. Comparing (39) and (42), the expressions determining the second-best solution to (46) and (47), we show that this is the case when

$$\mu = \frac{1}{1 + \lambda^*} \tag{48}$$

and

$$\begin{aligned} \alpha &= N(a^*, b^*), \quad \beta = X(a^*, b^*), \\ \varphi &= E_N^*, \quad \psi = E_X^*. \end{aligned} \tag{49}$$

Once these weights are determined, one can set $\bar{p}$ in such a way that profits go to zero. One readily verifies that (48) is then automatically also satisfied.

Of particular interest is the very nature of the extended price-cap coefficients, as defined in (49). The weights attached to prices (α and β) are nothing but the market demand for the corresponding goods. This does not differ from the "standard" price-cap results. The weights attached to the externality generating variables, that is $\varphi = E_N$ and $\psi = E_X$, are exactly equal to the social (or aggregate) marginal costs/benefits of the corresponding externalities. So, in the telecom example introduced above, φ is the (marginal) benefit that streams from having an additional user connected to the network; the weight ψ is the congestion cost that an additional call imposes on others. Clearly, if externalities are to be taken into account by the regulator, those are the simplest weights one can think of (and also the easiest to estimate). We already stressed that the presence of externalities makes the exact computation of the optimal allocation almost out of reach. By contrast, the proposed scheme appears to make optimal regulation surprisingly easy to implement.

7 Concluding comments

In this paper we investigate optimal pricing in presence of externalities. More precisely, we consider a monopoly providing goods or services. We assume that both access and intensity of use can be priced. Externalities may depend on aggregate output, number of subscribers or both. The setup is general enough to allow non-consumers to be affected by the externality.

Firstly, we thoroughly characterize various allocations of interest, including the second-best, defined as the social optimum when the monopolist is subject to the break-even constraint .

Secondly and more importantly, we propose an original regulatory policy of the price-cap category in order to decentralise the second best solution. In contrast to the complexity of the latter, the proposed mechanism is of surprising simplicity despite its generality. Moreover, it only relies on standard accounting data and two straightforward estimates of the social costs/benefits of the externalities.

Markets with externalities are often subject to taxation. We account for this fact by proposing a regulatory mechanism that considers explicitly the presence of taxes. However, taxation is taken as given. It would be of interest to consider the optimal simultaneous design of both taxation and regulation. This is left for further research.

8 References

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9 Appendix

9.1 First-Best Allocation

The welfare function writes

$$W_1 = \int_0^{\theta_m} S_\theta (0, X, N)\, g(\theta)\, d\theta + \int_{\theta_m}^{+\infty} S_\theta (x_\theta, X, N)\, g(\theta)\, d\theta - C(X, N)\,.$$

Differentiate with respect to θ_m :

$$\begin{aligned}
\frac{dW_1}{d\theta_m} &= \left[S_{\theta_m}(0, X, N) - S_{\theta_m}(x_{\theta_m}, X, N)\right] g(\theta_m) \\
&\quad + \int_0^{+\infty} \left[\frac{\partial S_\theta}{\partial X}\frac{dX}{d\theta_m} + \frac{\partial S_\theta}{\partial N}\frac{dN}{d\theta_m}\right] g(\theta)\, d\theta - \frac{\partial C}{\partial X}\frac{dX}{d\theta_m} - \frac{\partial C}{\partial N}\frac{dN}{d\theta_m} \\
&= \left[S_{\theta_m}(0, X, N) - S_{\theta_m}(x_{\theta_m}, X, N)\right] g(\theta_m) \\
&\quad + \left(E_X - \frac{\partial C}{\partial X}\right)\frac{dX}{d\theta_m} + \left(E_N - \frac{\partial C}{\partial N}\right)\frac{dN}{d\theta_m} \\
&= g(\theta_m)\left\{\left[S_{\theta_m}(0, X, N) - S(x_{\theta_m}, X, N)\right] - \left[\left(E_X - \frac{\partial C}{\partial X}\right) x_{\theta_m} + \left(E_N - \frac{\partial C}{\partial N}\right)\right]\right\}.
\end{aligned}$$

Clearly, assuming $g(\theta_m)$ to be strictly positive, $(dW_1/d\theta_m) = 0$ if and only if

$$S_{\theta_m}(0, X, N) = S(x_{\theta_m}, X, N) - \left[\left(\frac{\partial C}{\partial X} - E_X\right) x_{\theta_m} + \left(\frac{\partial C}{\partial N} - E_N\right)\right]$$

This says that, at first-best, the marginal consumer θ_m is indifferent between, on the one hand, getting no access and not being able to consume the good, and, on the other hand, being able to access the product, consume the optimal (first-best) quantity x_{θ_m} and pay for this the price

$$\frac{\partial C}{\partial N} - E_N + \left(\frac{\partial C}{\partial X} - E_X\right) x_{\theta_m}.$$

From the first FOC $(17a)$, we know that $\partial S_\theta/\partial x_\theta = \partial C/\partial X - E_X$, any θ, so that

$$\frac{d}{d\theta}\left[S_\theta(x_\theta, X, N) - \left(\frac{\partial C}{\partial X} - E_X\right) x_\theta\right] = \frac{\partial S_\theta}{\partial \theta} \geq 0.$$

As a result, if one type find it worth to pay an access fee a in order to be able to consume x at unit price $b = \partial C/\partial X - E_X$, all higher types will also find worth to do so. This says that the optimal access policy can be decentralised by setting the price b of x to $\partial C/\partial X - E_X$ and by imposing an access fee a that makes the marginal consumer indifferent, *i.e.*

$$a = \frac{\partial C}{\partial N} - E_N.$$

9.2 Computation of $\frac{dN}{db} / \frac{dN}{da}$.

By differentiating with respect to a equation (3) that defines the marginal type θ_m, it follows that

$$\begin{aligned} & \frac{\partial S_\theta^+}{\partial x}\frac{dx_{\theta_m}}{da} + \frac{\partial S_\theta^+}{\partial X}\frac{dX}{da} + \frac{\partial S_\theta^+}{\partial N}\frac{dN}{da} + \frac{\partial S_\theta^+}{\partial \theta}\frac{d\theta_m}{da} - \left(1 + b\frac{dx_{\theta_m}}{da}\right) \\ = \ & \frac{\partial S_\theta^-}{\partial X}\frac{dX}{da} + \frac{\partial S_\theta^-}{\partial N}\frac{dN}{da} + \frac{\partial S_\theta^-}{\partial \theta}\frac{d\theta_m}{da} \end{aligned}$$

where S_θ^+ stands for $S_{\theta_m}(x_{\theta_m}, X, N)$, i.e. the surplus function of the marginal consumer who actually gets access to the service, while S_θ^- stands for $S_{\theta_m}(0, X, N)$, i.e. the surplus of the marginal consumer who actually opts for not accessing the service.[7] By the envelope theorem, this boils down to:

$$\left(\frac{\partial S_\theta^+}{\partial X} - \frac{\partial S_\theta^-}{\partial X}\right)\frac{dX}{da} + \left(\frac{\partial S_\theta^+}{\partial N} - \frac{\partial S_\theta^-}{\partial N}\right)\frac{dN}{da} + \left(\frac{\partial S_\theta^+}{\partial \theta} - \frac{\partial S_\theta^-}{\partial \theta}\right)\frac{d\theta_m}{da} = 1. \quad (50)$$

Similarily, differentiating with respect to b equation (3) gives

$$\begin{aligned} & \frac{\partial S_\theta^+}{\partial x}\frac{dx_{\theta_m}}{db} + \frac{\partial S_\theta^+}{\partial X}\frac{dX}{db} + \frac{\partial S_\theta^+}{\partial N}\frac{dN}{db} + \frac{\partial S_\theta^+}{\partial \theta}\frac{d\theta_m}{db} - \left(x_{\theta_m} + b\frac{dx_{\theta_m}}{db}\right) \\ = \ & \frac{\partial S_\theta^-}{\partial X}\frac{dX}{db} + \frac{\partial S_\theta^-}{\partial N}\frac{dN}{db} + \frac{\partial S_\theta^-}{\partial \theta}\frac{d\theta_m}{db} \end{aligned}$$

hence

$$\left(\frac{\partial S_\theta^+}{\partial X} - \frac{\partial S_\theta^-}{\partial X}\right)\frac{dX}{db} + \left(\frac{\partial S_\theta^+}{\partial N} - \frac{\partial S_\theta^-}{\partial N}\right)\frac{dN}{db} + \left(\frac{\partial S_\theta^+}{\partial \theta} - \frac{\partial S_\theta^-}{\partial \theta}\right)\frac{d\theta_m}{db} = x_{\theta_m}. \quad (51)$$

[7] Remind that the marginal consumer is precisely indifferent between accessing or not so that the values of both functions are equals. Of course, the derivatives generally differ.

From (6) and (9), equation (50) rewrites:

$$\left[\frac{x_{\theta_m}+E_{xN}}{1-E_{xX}}\left(\frac{\partial S_\theta^+}{\partial X}-\frac{\partial S_\theta^-}{\partial X}\right)+\left(\frac{\partial S_\theta^+}{\partial N}-\frac{\partial S_\theta^-}{\partial N}\right)-\frac{1}{g\left(\theta_m\right)}\left(\frac{\partial S_\theta^+}{\partial \theta}-\frac{\partial S_\theta^-}{\partial \theta}\right)\right]\frac{dN}{da}=1.$$

From (11) and (13),equation (50) rewrites:

$$\begin{aligned} x_{\theta_m} &= \left(\frac{\partial S_\theta^+}{\partial X}-\frac{\partial S_\theta^-}{\partial X}\right)\frac{1}{1-E_{xX}}\frac{\partial \widehat{X}}{\partial b} \\ &\quad +\left[\frac{x_{\theta_m}+E_{xN}}{1-E_{xX}}\left(\frac{\partial S_\theta^+}{\partial X}-\frac{\partial S_\theta^-}{\partial X}\right)+\left(\frac{\partial S_\theta^+}{\partial N}-\frac{\partial S_\theta^-}{\partial N}\right)-\frac{1}{g\left(\theta_m\right)}\left(\frac{\partial S_\theta^+}{\partial \theta}-\frac{\partial S_\theta^-}{\partial \theta}\right)\right]\frac{dN}{db}\end{aligned}$$

hence

$$\frac{dN}{db}/\frac{dN}{da}=x_{\theta_m}-\left(\frac{\partial S_\theta^+}{\partial X}-\frac{\partial S_\theta^-}{\partial X}\right)\left[\frac{1}{1-E_{xX}}\frac{\partial \widehat{X}}{\partial b}\right] \tag{52}$$

9.3 Second_best

9.3.1 Computation of equations (38) − (39)

Making use of equation (9) and of notations (16) and (19), the FOC condition (36) yields directly

$$-\lambda N=\left(E_N+E_X\frac{x_{\theta_m}+E_{xN}}{1-E_{xX}}\right)\frac{dN}{da}+(1+\lambda)\left[\left(a-\tau_N-\frac{\partial C}{\partial N}\right)+\left(b-\tau_X-\frac{\partial C}{\partial X}\right)\frac{x_{\theta_m}+E_{xN}}{1-E_{xX}}\right]\frac{dN}{da}$$

With the elasticity ϵ_N defined in (24),this rewrites (38). Equation (39) is obtained by plugging in the definition of the virtual connection cost introduced in (28).

9.3.2 Computation of equations (41) − (42)

Making use of equation (13) and of notations (16) and (19), the FOC condition (37) writes directly

$$\begin{aligned} -\lambda X &= (1+\lambda)\left[\left(b-\tau_X-\frac{\partial C}{\partial X}\right)+\frac{1}{1+\lambda}E_X\right]\frac{1}{1-E_{xX}}\frac{\partial \widehat{X}}{\partial b} \\ &\quad +\left(E_N+E_X\frac{x_{\theta_m}+E_{xN}}{1-E_{xX}}\right)\frac{dN}{db} \\ &\quad +(1+\lambda)\left[\left(a-\tau_N-\frac{\partial C}{\partial N}\right)+\left(b-\tau_X-\frac{\partial C}{\partial X}\right)\left(\frac{x_{\theta_m}+E_{xN}}{1-E_{xX}}\right)\right]\frac{dN}{db}\end{aligned}$$

From (36) we know that:

$$-\lambda N = \left(E_N + E_X \frac{x_{\theta_m} + E_{xN}}{1 - E_{xX}}\right) \frac{dN}{da} + (1+\lambda)\left[\left(a - \tau_N - \frac{\partial C}{\partial N}\right) + \left(b - \tau_X - \frac{\partial C}{\partial X}\right) \frac{x_{\theta_m} + E_{xN}}{1 - E_{xX}}\right] \frac{dN}{da}.$$

hence condition (37) rewrites

$$-\lambda X = (1+\lambda)\left[\left(b - \tau_X - \frac{\partial C}{\partial X}\right) + \frac{1}{1+\lambda} E_X\right] \frac{1}{1 - E_{xX}} \frac{\partial \widetilde{X}}{\partial b} - \lambda N \left(\frac{dN}{db} / \frac{dN}{da}\right).$$

By using (52) ,one gets

$$\begin{aligned} -\lambda X &= (1+\lambda)\left[\left(b - \tau_X - \frac{\partial C}{\partial X}\right) + \frac{1}{1+\lambda} E_X\right] \frac{1}{1 - E_{xX}} \frac{\partial \widetilde{X}}{\partial b} \\ & \quad -\lambda N \left(x_{\theta_m} - \left(\frac{\partial S_\theta^+}{\partial X} - \frac{\partial S_\theta^-}{\partial X}\right)\left[\frac{1}{1 - E_{xX}} \frac{\partial \widetilde{X}}{\partial b}\right]\right). \end{aligned}$$

With the price elasticity of infra-marginal consumers $\widehat{\epsilon_X}$ defined in (35) this rewrites (41) . Equation (42) is obtained by plugging in the definition of the virtual marginal cost introduced in (34) .

9.4 Regulation and Global Price Cap

9.4.1 Computation of equation (46)

By using (9) , equation (44) rewrites directly as:

$$-N + \mu\left[\alpha - \left(\varphi + \psi \frac{x_{\theta_m} + E_{xN}}{1 - E_{xX}}\right) \frac{dN}{da}\right] = \left[a - \frac{\partial C}{\partial N} - \tau_N + \left(b - \frac{\partial C}{\partial X} - \tau_X\right) \frac{x_{\theta_m} + E_{xN}}{1 - E_{xX}}\right] \frac{dN}{da}. \tag{53}$$

It follows that

$$a - \frac{\partial C}{\partial N} - \tau_N + \left(b - \frac{\partial C}{\partial X} - \tau_X\right) \frac{x_{\theta_m} + E_{xN}}{1 - E_{xX}} = \left(1 - \mu \frac{\alpha}{N}\right) \frac{a}{\epsilon_N} - \mu \left(\varphi + \psi \frac{x_{\theta_m} + E_{xN}}{1 - E_{xX}}\right).$$

With the definition (28) of the virtual connection cost $\widetilde{C}_N$, one gets directly (46) .

9.4.2 Computation of equation (47)

Plugging (13) and (52) into (45) gives

$$\begin{aligned} 0 &= X - \mu\beta + \left(b - \frac{\partial C}{\partial X} - \tau_X + \mu\psi\right)\left(\frac{1}{1-E_{xX}}\frac{\partial \widehat{X}}{\partial b}\right) \\ &\quad + \left[a - \frac{\partial C}{\partial N} - \tau_N + \mu\varphi + \left(b - \frac{\partial C}{\partial X} - \tau_X + \mu\psi\right)\frac{x_{\theta_m} + E_{xN}}{1-E_{xX}}\right]\frac{dN}{db} \\ &= X - \mu\beta + \left(b - \frac{\partial C}{\partial X} - \tau_X + \mu\psi\right)\left(\frac{1}{1-E_{xX}}\frac{\partial \widehat{X}}{\partial b}\right) \\ &\quad + \left[a - \frac{\partial C}{\partial N} - \tau_N + \mu\varphi + \left(b - \frac{\partial C}{\partial X} - \tau_X + \mu\psi\right)\frac{x_{\theta_m} + E_{xN}}{1-E_{xX}}\right] \\ &\quad \times \left(x_{\theta_m} - \left(\frac{\partial S_\theta^+}{\partial X} - \frac{\partial S_\theta^-}{\partial X}\right)\left[\frac{1}{1-E_{xX}}\frac{\partial \widehat{X}}{\partial b}\right]\right)\frac{dN}{da}. \end{aligned}$$

Now from (53) , this expression rewrites

$$\begin{aligned} 0 &= X\left[\left(1 - \mu\frac{\beta}{X}\right) - \left(1 - \mu\frac{\alpha}{N}\right)\frac{x_{\theta_m}}{X/N}\right] \\ &\quad + \left(b - \frac{\partial C}{\partial X} - \tau_X + \mu\psi + N\left(1 - \mu\frac{\alpha}{N}\right)\left(\frac{\partial S_\theta^+}{\partial X} - \frac{\partial S_\theta^-}{\partial X}\right)\right)\left(\frac{1}{1-E_{xX}}\frac{\partial \widehat{X}}{\partial b}\right) \end{aligned}$$

that gives directly (47) .