On the Uniqueness of Equilibrium in Symmetric Two-Player Zero-Sum Games with Integer Payoffs*

Michel Le Breton
Université de Toulouse 1
Gremaq and Idei
manufacture F, Bat F
21, Allée de Brienne
31000, Toulouse (France)

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Abstract

Consider a symmetric two-player zero-sum game with integer payoffs. We prove that if there exists an integer such that all upper-diagonal payoff entries have the same non-zero reminder when divided by this integer, then the game has a unique equilibrium in mixed strategies.

1 Introduction

Laffond, Laslier and Le Breton (1997) proved that if the off-diagonal payoff entries of a finite symmetric two-player zero-sum game are odd integers, then the game has a unique equilibrium in mixed strategies. This result was generalizing itself a former result demonstrated by Fisher and Ryan (1992) and Laffond, Laslier and Le Breton (1993) for the special case where the off-diagonal entries are either equal to 1 or to -1.

The purpose of this note is to offer one more curiosity along the same lines. Precisely we prove that if the upper-diagonal payoff entries of a symmetric two-player zero-sum game are integers satisfying a certain congruence property, then the game has a unique equilibrium in mixed strategies. The congruence property is a generalization of the oddness condition and therefore the result in Laffond, Laslier and Le Breton (1997) follows from this result. The proof uses only elementary linear algebra and number theory. We also argue that

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the plurality game in Political Science may have payoff entries satisfying this congruence property.

2 The Result

A two-player zero-sum game is a triple (X_1, X_2, m) where X_1(resp. X_2) denotes the set of pure strategies of player 1 (resp. 2) and $m : X_{1\times}X_2 \to \Re$ denotes the payoff function of player 1 (The payoff function of player 2 is $-m$). It is finite if X_1 and X_2 are finite sets. It is symmetric if $X_1 = X_2 \equiv X$ and $m(x_1, x_2) + m(x_2, x_1) = 0$ for all $(x_2, x_1) \in X \times X$.

Consider a finite two-player zero-sum game (X_1, X_2, m). A mixed strategy for player 1 (resp. 2) is a probability distribution over X_1(resp. X_2). Denote by $P(X_1)$ (resp. $P(X_2)$) the set of probability distributions over X_1(resp. X_2). A pair (p_1, p_2) in $P(X_1) \times P(X_2)$ is an equilibrium (in mixed strategies) if it satisfies the following inequalities :

$$\begin{aligned} & \sum_{x_1 \in X_1} \sum_{x_2 \in X_2} p_1(x_1) p_2(x_2) m(x_1, x_2) \\ \geq & \sum_{x_1 \in X_1} \sum_{x_2 \in X_2} q_1(x_1) p_2(x_2) m(x_1, x_2) \text{ for all } q_1 \in P(X_1) \end{aligned}$$

and

$$\begin{aligned} & \sum_{x_1 \in X_1} \sum_{x_2 \in X_2} p_1(x_1) p_2(x_2) m(x_1, x_2) \\ \leq & \sum_{x_1 \in X_1} \sum_{x_2 \in X_2} p_1(x_1) q_2(x_2) m(x_1, x_2) \text{ for all } q_2 \in P(X_1) \end{aligned}$$

Since X_1 and X_2 are finite, there exists equilibria in mixed strategies and since it is zero-sum the set of equilibria is a product space $P_1 \times P_2 \subseteq P(X_1) \times P(X_2)$. Hereafter a strategy in P_1 (resp. P_2) will be referred to as an optimal play for player 1 (resp. 2). We denote by $Supp(p)$ the support of a probability distribution p. An equilibrium in mixed strategies (p_1, p_2) is quasi-strict (Harsanyi (1973)) if $Supp(p_1)$ (resp. $Supp(p_2)$ is the set of pure-stategy best responses to p_2 (resp. p_1). When the game is symmetric $P_1 = P_2$ and we will drop the reference to the player number. Our main result is the following:

Theorem 1 *Let (X, m) be a finite symmetric two-player zero-sum game with integer payoffs. If there exist an ordering $\prec$ of X and integers d and r such that $r \neq 0$ and $m(x, y) = r \pmod d$ for all $(x, y) \in X \times X$ such that $x \prec y$, then (X, m) has a unique equilibrium in mixed strategies.*

Remark 2 *We cannot replace the condition in the theorem by the weaker (when $d \geq 3$) condition : there exist an ordering $\prec$ of X and an integer d such* *that $m(x,y) \neq 0 \pmod d$ for all $(x,y) \in X \times X$ such that $x \prec y$. For instance all the off-diagonal payoff entries of the symmetric two-player zero-sum game represented by the matrix below are relatively prime with 3 and however it is easy to verify that there several optimal plays. The game has 4 strategies. It is tedious but elementary to show that if the game has only 3 strategies then whenever the off-diagonal payoff entries are different from 0, the game has a unique equilibrium which is either pure or completely mixed.*

$$\begin{pmatrix} 0 & 1 & 1 & -2 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & 1 \\ 2 & 1 & -1 & 0 \end{pmatrix}$$

Remark 3 *The condition of the theorem is verified when all the off-diagonal payoff entries are odd integers. In that case the ordering $\prec$ does not matter. Otherwise the ordering of the strategies may matter as illustrated by the symmetric two-player zero-sum game represented by the matrix below.*

$$\begin{pmatrix} 0 & -3 & 7 \\ 3 & 0 & -8 \\ -7 & 8 & 0 \end{pmatrix}$$

Remark 4 *The equilibrium is quasi-strict. This follows easily from the Equalizer Theorem (Rahgavan (1994)).*

Remark 5 *The number of strategies in the support of the optimal play is odd. This follows from Gale, Kuhn and Tucker (1950) or Kaplansky (1945).*

Remark 6 *A number of "artificial" generalizations of theorem 1 can be provided. Let (X_1, X_2, m) be a finite square ($\#X_1 = \#X_2$) two-player zero-sum game with integer payoffs. The proof technique of theorem 1 shows that if all diagonal entries are multiples of a nonzero integer d and all upper diagonal (or lower diagonal) entries have the same nonzero reminder r when divided by d, then (X_1, X_2, m) has at most one completely mixed equilibrium. The same assertion even holds for finite square two-player games (not necessarily zero-sum) with a unique vector of equilibrium payoffs, a property satisfied by two-player zero-sum games but by other games as well e.g. almost strictly competitive games (Aumann (1961)).*

Theorem 1 is can be used for the analysis of some games arising in Political Science. Consider the situation where two political parties compete for the votes of an electorate through their political platforms chosen in X. Denote by V the electorate and for all $v \in V$, let P_v be the preference of voter v over X. We assume that P_v is a linear order and denote by L the set of linear orders over X. Finally we assume that each party is interested in maximizing its electoral support[1]. To summarize, we are considering a symmetric two-player constant-sum game where X is the set of pure strategies of both parties and the payoff of party 1 when it plays x and party 2 plays y is $V_P(x,y)$ where $V_P(x,y) \equiv \#\{v \in V : xP_v y\}$[2]. It is straightforward to see that this game is strategically equivalent to the symmetric two-player zero-sum game (X, m) where $m(x,y) \equiv V_P(x,y) - V_P(y,x)$.

Since for all $x \neq y$, $m(x,y) = 2V_P(x,y) - \#V$, we observe that all the off-diagonal payoff entries inherit the parity of $\#V$. Debord (1987) has proved[3] that any skew-symmetric matrix whose off-diagonal entries are either all odd or all even can be obtained through this construction.

This result justifies the interest of our theorem for the analysis of the plurality game. Indeed take a skew-symmetric matrix satisfying the congruence property in the theorem and multiply by 2 all the entries. From Debord‘s result such matrix describes a plurality game for a particular electorate and preferences within the electorate. We are done since the game obtained by multiplying all the entries by 2 is strategically equivalent to the original game.

3 Proof of the Theorem.

Let $n = \#X$. For notational simplicity the elements of X are labelled from 1 to n and without loss of generality we assume that the order $\prec$ is the natural order. Further $m(i,j)$ is denoted simply by m_{ij}.

Claim 1: Let p be an optimal play. We have :

$$\sum_{i \in Supp(p)} m_{ij} p_i = 0 \text{ for all } j \in Supp(p)$$

Claim 2: If there are several optimal plays, there are several optimal plays with the same support.

The two claims follow from standard results on two-player zero-sum games (see e.g. Rahgavan (1994))[4]

By assumption, for all i and j such that $i \prec j$ there exists an integer d_{ij} such that : $m_{ij} = d_{ij} d + r$. Without loss of generality we may assume that d and r are relatively prime since if they were not, we would obtain two new integers relatively prime and satisfying the conditions of the theorem by dividing d and r by their common divisors.

From claim 2, it is enough to prove that there does not exist two optimal plays with the same support. Suppose on the contrary that there exist two optimal plays with the same support, say without loss of generality $\{1, 2, ..., k\}$. Then we deduce from claim 1 that the homogenous system of linear equations :

[1] This is the so-called plurality game (see e.g. Laffond, Laslier and Le Breton (1994) or Ordeshook (1988)).

[2] $V_P(x,x) \equiv \frac{\#N}{2}$ for all $x \in X$.

[3] For the sake of completeness, a short alternative proof of Debord's theorem is provided at the end of the manuscript.

[4] The value of a symmetric two-player zero-sum game is equal to 0.

$$\begin{aligned}\sum_{i=1}^{k} m_{ij} q_i &= 0 \text{ for all } i = 1, ..., k. \\ &\text{and} \\ \sum_{i=1}^{k} q_i &= 0\end{aligned}$$

has a non-zero solution i.e. that the nullspace of the linear operator from $\Re^k$ to $\Re^{k+1}$ described by the matrix A below has a dimension at least equal to 1.

$$A \equiv \begin{pmatrix} 0 & m_{12} & . & . & m_{1k} \\ -m_{12} & 0 & . & . & m_{2k} \\ . & . & . & . & . \\ . & . & . & . & . \\ -m_{1k} & -m_{2k} & . & . & 0 \\ 1 & 1 & . & . & 1 \end{pmatrix}$$

This implies that Rank$A \leq k-1$ i.e. that all the $k \times k$ determinants extracted from A are equal to 0. We show now that this is not possible. Consider the determinant D of the matrix extracted from A by deleting the k^{th} line of A i.e. :

$$D \equiv \begin{vmatrix} 0 & m_{12} & . & . & . & m_{1k-1} & m_{1k} \\ -m_{12} & 0 & . & . & . & m_{2k-1} & m_{2k} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ -m_{1k-1} & -m_{2k-1} & . & . & . & 0 & m_{k-1k} \\ 1 & 1 & . & . & . & 1 & 1 \end{vmatrix}$$

By using the linearity of D with respect to the first column, we obtain :

$$D = d \begin{vmatrix} 0 & m_{12} & . & . & . & m_{1k-1} & m_{1k} \\ -d_{12} & 0 & . & . & . & m_{2k-1} & m_{2k} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ -d_{1k-1} & -m_{2k-1} & . & . & . & 0 & m_{k-1k} \\ 0 & 1 & . & . & . & 1 & 1 \end{vmatrix} +$$

$$\begin{vmatrix} 0 & m_{12} & . & . & . & m_{1k-1} & m_{1k} \\ -r & 0 & . & . & . & m_{2k-1} & m_{2k} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ -r & -m_{2k-1} & . & . & . & 0 & m_{k-1k} \\ 1 & 1 & . & . & . & 1 & 1 \end{vmatrix}$$

The first term in the above sum is a multiple of d. Now consider the second determinant in the sum. By using the linearity of this determinant with respect to its second column, we obtain that it is the sum of a determinant which is a multiple of d and another determinant. After a repeated application of this argument, we obtain :

$$D = \text{ multiple of } d + r^{k-1} \begin{vmatrix} 0 & 1 & 1 & . & . & 1 & 1 \\ -1 & 0 & 1 & . & . & 1 & 1 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ -1 & -1 & -1 & . & . & 0 & 1 \\ 1 & 1 & 1 & . & . & 1 & 1 \end{vmatrix}$$

By using a straightforward induction argument, it is easy to show that :

$$\begin{vmatrix} 0 & 1 & 1 & . & . & 1 & 1 \\ -1 & 0 & 1 & . & . & 1 & 1 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ -1 & -1 & -1 & . & . & 0 & 1 \\ 1 & 1 & 1 & . & . & 1 & 1 \end{vmatrix} = (-1)^{k-1}$$

To summarize : $D =$ multiple of $d + (-r)^{k-1}$. Since d and r are relatively prime, d and $(-r)^{k-1}$ are relatively prime too. This implies, as desired that $D \neq 0$ ■

4 Proof of Debord ‘s Theorem.

Let M be a skew-symmetric matrix of order n such that either m_{ij} is even for all $i \neq j$ or m_{ij} is odd for all $i \neq j$. We prove[5] that there exists a finite set V and a function P from V into L such that : $m_{ij} \equiv V_P(i,j) - V_P(j,i)$ for all $i \neq j$.

Case 1 : m_{ij} is even for all $i \neq j$.

To every pair (i,j) be such that $i \neq j$ and $m_{ij} > 0$[6] we associate the following two linear orders denoted respectively by P^1_{ij} and P^2_{ij} ,where $\widetilde{P}_{ij}$ is an arbitrary linear order over $X \setminus \{i,j\}$.

$$iP^1_{ij}j, iP^1_{ij}k \text{ and } jP^1_{ij}k \text{ for all } k \neq i,j \text{ and } kP^1_{ij}k\text{‘} \text{ iff } k\widetilde{P}_{ij}k\text{‘} \text{ for all } k,k\text{‘} \neq i,j.$$

$$iP^2_{ij}j, kP^2_{ij}i \text{ and } kP^1_{ij}j \text{ for all } k \neq i,j \text{ and } kP^1_{ij}k\text{‘} \text{ iff } k\text{‘}\widetilde{P}_{ij}k \text{ for all } k,k\text{‘} \neq i,j.$$

[5] The proof is inspired by the proof of Mc Garvey (1953) for the case of digraphs.

[6] If all the entries m_{ij} are equal to 0, then just take an electorate with two voters having opposite preferences.

. Now consider an electorate $V = \bigcup_{\{i \neq j : m_{ij} > 0\}} V_{ij}$ where the sets V_{ij} are pairwise disjoints and $\# V_{ij} = m_{ij}$. Partition V_{ij} in two sets V_{ij}^1 and V_{ij}^2 of equal size and for each $v \in V_{ij}$, let $P_v = P_{ij}^1$ if $v \in V_{ij}^1$ and $P_v = P_{ij}^2$ if $v \in V_{ij}^2$. It is straightforward to verify that : $m_{ij} \equiv V_P(i,j) - V_P(j,i)$ for all $i \neq j$.

Case 2 : m_{ij} is odd for all $i \neq j$.

Let $M`$ be the matrix $M + N$ where :

$$N \equiv \begin{pmatrix} 0 & 1 & . & . & 1 \\ -1 & 0 & . & . & 1 \\ . & . & . & . & . \\ . & . & . & . & . \\ -1 & -1 & . & . & 0 \end{pmatrix}$$

From case 1, we deduce that there exists $V`$ and $P`$ from $V`$ into $L$ such that : $m`_{ij} \equiv V_{P`}(i,j) - V_{P`}(j,i)$ for all $i \neq j$. Now add to $V`$ one individual v with preference P_v defined by : iP_vj iff $i > j$. It is easy to verify that : $m_{ij} \equiv V_P(i,j) - V_P(j,i)$ for all $i \neq j$ ■

5 References.

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