Optimal Collusion in Oligopoly Supergames:

Marginal Costs Matter

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Abstract

In a standard oligopoly supergame with identical firms, a necessary condition on the level of marginal costs is derived for optimal collusion to be sustainable, either in prices or in quantities, for any degree of product differentiation, and any number of firms. Only in the Cournot version of the model, and with at most three firms, the level of marginal costs plays no role in the sustainability of collusion. Otherwise, for any degree of differentiation and any size of industry, the marginal cost must be higher than a threshold value for an optimal single-period punishment to be obtained. When parameter values are such that no optimal single-period punishment can be implemented, a multi-period punishment can always be constructed that leads firms to sustain collusion at the profit-maximizing price or quantity level. In that case, punishments last longer when the level of marginal costs decreases. By contrast, the length of the punishment phase does not impact the minimum discount factor for which collusion can be implementable. More differentiation and/or concentration unambiguously facilitate collusion. When firms use prices, the optimal punishment phase lasts longer than when they use quantities, and the boundary condition on the discount factor is more stringent.

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1 Introduction

Antitrust authorities are interested in relying on simple guidelines in order to investigate the behavior of firms. This motivates the understanding of the impact of product differentiation and industry concentration on the sustainability of collusive agreements in oligopolistic industries. In a legal context in which collusive agreements cannot be contractually enforced, it is well known that each firm may find it profitable to deviate from a tacitly collusive price or quantity that maximizes industry profits. This may not be the case however if a deviation can be sufficiently and credibly “punished” via lower industry prices or larger quantities in a subsequent period of time. In order to identify structural conditions for the sustainability of collusive strategies, the theoretical literature has constructed a class of dynamic models usually referred to as supergames. These models feature a repeated market game in which firms maximize a flow of discounted individual profits by non-cooperatively choosing a price or a quantity over an infinite number of periods.

One stream of that literature follows Friedman (1971) by considering trigger strategies (commonly referred to as “grim” strategies) which call for reversion to the one-shot stage game Nash equilibrium forever when a deviation is detected in a previous period. A general result is that collusion can be sustained if the discount factor is above a threshold value. In a duopoly model with horizontal differentiation, Deneckere (1983, 1984) finds that tacitly collusive agreements in prices are less easily sustained than in quantities unless goods are highly substitutable. This is sensitive to the two-firm assumption, since in an extension of the same model Majerus (1988) demonstrates that, with three firms or more, a collusion in prices is less easily sustained than in quantities for all degrees of substitutability. Other studies examine the relationship between alternative formalizations of product differentiation and the sustainability of collusion. They include papers by Chang (1991) and Ross (1992), who confirm in two spatial competition models
that collusion is more easily sustained when products are increasingly differentiated.¹

A weakness that is common to all models of collusion with trigger strategies (i.e. punishments of infinite duration) is that they rule out the possibility of modulating the level of punishments. More precisely, by assuming that firms revert to the Nash equilibrium of the one-shot stage game in all periods that follow a deviation, they put an upper bound on the severity of punishments.

This problem is tackled by a second stream of the literature, which reconsiders the impact of various model specifications on the sustainability of collusion with a stick-and-carrot penal code. In the latter mechanism, if a firm deviates from the collusive strategy, all firms play a punishment strategy before returning to the collusive price or quantity. In a repeated Cournot (i.e., quantity-setting) oligopoly model, with identical sellers of a homogenous good and constant marginal costs, Abreu (1986) characterizes the most severe punishments as “optimal” by proving that they lead firms to sustain the most profitable collusive strategy. A dual interpretation is to consider the minimum discount factor that sustains a given collusive outcome. Also in a repeated Cournot model, Wernerfelt (1989) uses a stick-and-carrot framework to find that more product differentiation may render collusion less sustainable when the number of firms is relatively large.² In a repeated Bertrand (i.e., price-setting) duopoly model, with spatial differentiation, and constant marginal costs normalized to zero, Häckner (1996) demonstrates that a symmetric stick-and-carrot structure is an optimal price path, and confirms that differentiation tends to facilitate collusive agreements. It is also demonstrated that, when the punishment price is constrained to be non-negative, a prolonged price war can sustain the same collusive

¹With vertical differentiation, Häckner (1994) finds that more differentiation unambiguously implies more sustainable collusion in a Bertrand setup, with a trigger penal code. In the present paper we concentrate on horizontal differentiation exclusively.
²Although of interest, this ambiguous result is derived from demand assumptions (adapted from Deneckere, 1983) which are not standard (on this see Osterdal, 2003, pp. 54-55).
strategy as in the unrestricted case. With two firms and constant marginal costs again, but with another specification of the horizontal differentiation assumption, Lambertini and Sasaki (2002) find a qualitatively similar relationship between product substitutability and collusion sustainability.\(^3\) In addition, for all degrees of product differentiation, they show that collusion is less easily sustainable in Bertrand than in Cournot. This is obtained in a setup where quantities are constrained to be non-negative but prices may fall below zero.\(^4\)

In this paper, we show that the level of constant marginal costs matters for collusion sustainability. Our starting point is the intuitive proposition by Lambertini and Sasaki (2001) that collusion in a repeated oligopoly game with high marginal costs may be more easily sustainable than with lower marginal costs, all other things remaining equal. The underlying idea is that, if prices are constrained to be positive, the ability to punish a deviation by charging prices below marginal costs is limited when these costs are close to zero. We unveil a condition on the level of marginal costs that is necessary for collusion to be sustainable, either in prices or in quantities, in a stick-and-carrot setup. This is done with standard demand specifications, for any degree of product differentiation, and any number of firms.

The introduction of positive marginal costs, together with a non-negativity constraint on prices as in Hӓckner (1996), leads to several new results. We find that only in the Cournot version of the model, and with at most three firms, the level of marginal costs plays no role in the

\(^3\)Lambertini and Sasaki (2002) also consider complements. Under non-negative quantity constraints, they find that longer (i.e., multi-period) punishments are required only when products are close complements and the quantity is the strategic variable.

\(^4\)Beyond the degree of product differentiation and the number of firms, other studies focus on the role of various cost and demand specifications. For example, Rothschild (1999) demonstrates that collusion sustainability depends on firms’ relative efficiencies. Other examples include Lambertini (1996), Tyagi (1999), and Collie (2004), who investigate the relationship between assumptions on the form of the demand function and firms’ ability to collude in a repeated games with trigger strategies.
sustainability of collusion. Otherwise, for any degree of differentiation and any number of firms, the marginal cost must be higher than a threshold value. This comes in addition to the usual lower bound condition on the discount factor. High marginal costs can thus be included in the list of factors that facilitate collusion. When parameter values are such that no optimal single-period punishment can be implemented, we find that a longer – i.e., multi-period – punishment can always be constructed that leads firms to sustain collusion. More precisely, the length of the punishment phase is demonstrated to be a substitute for the level of marginal costs. By contrast, it does not impact on the critical threshold value of the discount factor that allow to sustain collusion in multi-period punishment scheme. A comparative statics analysis establishes that more differentiation and/or concentration in the industry relax the sustainability conditions on the marginal cost and the discount factor, and thereby facilitate collusion. Eventually, when firms use prices as opposed to quantities, we find that the optimal punishment phase lasts longer, and the boundary condition on the discount factor is more stringent. In that sense, collusion is less easily sustainable in Bertrand than in Cournot.

The remainder of the paper is organized as follows. Section 2 describes the model. In Section 3, we investigate the role of the level of marginal costs on the existence of an optimal single-period punishment, either in prices or in quantities. In Section 4, we consider situations in which collusion is not sustainable with a single-period punishment. Conditions of sustainability of optimal collusion for the Bertrand and Cournot cases are compared in Section 5. Final remarks appear in Section 6. Computations are in the appendix.
2 The Model

Identical firms in $N = \{1, \ldots, n\}$ maximize intertemporal profits by simultaneously and non-cooperatively choosing either quantities (Cournot supergame) or prices (Bertrand supergame) in an infinitely repeated game over $t = 1, 2, \ldots, \infty$. The discount factor $\delta = 1/(1 + r)$, where $r$ is the single period interest rate, is common to all firms. Each firm $i$ incurs a constant marginal (unit) cost $c \in [0, 1]$ to supply a non-negative quantity $q_i$. Over all periods, each consumer has the same identical utility function, adapted from Häckner (2000), of the form

$$U(q, I) = \sum_{i=1}^{n} q_i - \frac{1}{2} \left( \sum_{i=1}^{n} q_i^2 + 2\gamma \sum_{i \neq j} q_i q_j \right) + I,$$  \hspace{1cm} (1)

which is quadratic in the consumption of $q$-products and linear in the consumption of the composite $I$-good (i.e., the numeraire).\(^5\) The parameter $\gamma \in (0, 1)$ measures product substitutability as perceived by consumers. If $\gamma \to 0$, the demand for the different product varieties are independent and each firm has monopolistic market power, while if $\gamma \to 1$, the products are perfect substitutes. Consumers maximize utility subject to the budget constraint $\sum p_i q_i + I \leq m$, where $m$ denotes income, $p_i$ is the non-negative price of product $i$, and the price of the composite good is normalized to one. By symmetry, we note $\sum_{j \neq i} q_j = (n-1)q_j$.

**Bertrand** In the price-setting version of the model, from (1) the demand function for firm $i$ writes

$$q_i(p_i, p_j) = \begin{cases} 1 - p_i & \text{if } p_i \leq \tilde{p}_i(p_j) \\ \frac{1}{1+\gamma(n-1)} \left( 1 - \frac{1+\gamma(n-2)}{1-\gamma} p_i + \frac{\gamma(n-1)}{1-\gamma} p_j \right) & \text{if } \tilde{p}_i(p_j) \leq p_i \leq \tilde{p}_i(p_j) \\ 0 & \text{if } p_i \geq \tilde{p}_i(p_j) \end{cases}, \hspace{1cm} (2)$$

\(^5\)In Häckner (2000), quantities $q_i$ are multiplied by a parameter $a_i$, that is a measure the distinctive quality of each variety $i$. Here we exclude vertical product differentiation by assuming that $a_i = 1$, all $i \in N$. 


and the demand for each other symmetric firm $j$ writes

$$ q_j(p_i, p_j) = \begin{cases} 0 & \text{if } p_i \leq \tilde{p}_i(p_j) \\ \frac{1}{1+\gamma(n-1)} \left( 1 - \frac{1}{1+\gamma} p_j + \frac{1}{1+\gamma} p_i \right) & \text{if } \tilde{p}_i(p_j) \leq p_i \leq \hat{p}_i(p_j) \\ \frac{1-p_j}{1+\gamma(n-2)} & \text{if } p_i \geq \hat{p}_i(p_j) \end{cases} $$

(3)

with $\tilde{p}_i(p_j) = \frac{1}{\gamma} \left[ p_j - (1 - \gamma) \right]$ and $\hat{p}_i(p_j) = \frac{1}{1+\gamma(n-2)} (1 - \gamma + \gamma (n-1) p_j)$. The piecewise definition of demand functions is a consequence of the non-negativity constraint we impose on quantities. Remark however that the two functions (24) and (25) are continuous in their respective arguments. This implies that profit functions are also continuous. One can check that they are also concave in the relevant decision variable.

**Cournot** In the quantity-setting version of the model, from (1) firm $i$'s inverse demand function in each period is

$$ p_i(q_i, q_j) = \max \{0, 1 - q_i - \gamma(n-1)q_j\}, $$(4)

all $q_j \geq 0, j \neq i$, and the inverse demand for each other symmetric firm $j$ is defined by

$$ p_j(q_i, q_j) = \max \{0, 1 - \gamma q_i - (1 + \gamma(n-2))q_j\}. $$

(5)

Here again, it is straightforward to check that a firm’s profit function is continuous and the associated maximization problem is convex.

**Penal Codes** Both in the Bertrand and Cournot versions of the model, a strategy profile is a set of available actions $a$, including a collusive price (quantity), which yields joint profit maximization, and a punishment price (quantity), which leads to low per-firm profits. In the stage game, firms’ strategies in prices (quantities) may differ from period to period. An action path $\{a_t\}_{t=1}^{\infty}$ is defined as an infinite sequence of $n$-dimensional vector prices (quantities), as
chosen by each firm in each period. We assume all firms initially follow a collusive path by choosing the collusive action $a_M$. The collusive price and quantity are given by $p_M = (1 + c) / 2$ and $q_M = (1 - c) / [2(1 + \gamma(n - 1))]$ respectively. If the collusive strategies are played by all firms in all periods, each firm earns a sum of profits $\Pi^M = \pi^M / (1 - \delta)$, where $\pi^M$ refers to symmetric single-period collusive profits. All firms have a short-run incentive to deviate, that is to lower (increase) its own price (quantity) in order to increase individual profits at every other firm’s expense. If such a deviation by one firm $i$ in $N$ is detected in period $t$, all firms switch to the punishment action $a_P$ in period $t + 1$.

As standard in the literature, we consider two alternative penal codes, namely the “trigger” penal code and the “stick-and-carrot” penal code. We are interested in investigating the conditions on marginal costs, product differentiation, and size of the industry, in which collusion is sustainable in a stick-and-carrot setup. We compare them with conditions on the parameter values for which firms collude in equilibrium in the trigger model, we thereby use as a benchmark. Both in the trigger and stick-and-carrot cases, we have a collusive equilibrium of the supergame if all firms find it profitable to adopt the collusive action whenever the action path calls for them to do so, and to implement the penal code whenever the action path calls for them to do so.

Following Friedman (1971), in the trigger penal code, all firms revert forever to the stage game Nash equilibrium in the period that follows any deviation from the collusive strategy. In this case, each firm earns a discounted flow of profits $\Pi^d_i = \pi^d_i + \delta \pi^N / (1 - \delta)$, all $i$, where $\pi^d_i$ refers to a firm $i$’s single-period deviation profits, and $\pi^N$ refers to symmetric single period Nash equilibrium profits. For the collusive strategy to be sustainable, the discounted flow of profits from adhering to the collusive strategy forever must exceed the discounted flow of
profits obtained in the case of deviation, i.e. $\Pi^M \geq \Pi^d_i$. It follows that collusion is sustainable with a trigger penal code if and only if the discount factor is higher than a threshold level

$$\delta_{\text{trigger}} = \frac{(\pi^d_i - \pi^M)}{(\pi^d_i - \pi^N)}.$$

Remark that this result holds true for all $(\gamma, c) \in [0, 1)^2$ and all $n \geq 2$.

In a single-period *stick-and-carrot* penal code à la Abreu (1986) also, all firms initially follow a collusive path by choosing the collusive action $a_M$. If a deviation by any firm occurs, all firms switch to the punishment action $a_P$ in the next period (the stick). After one period of punishment, if any deviation from $a_P$ is detected, the punishment phase restarts, otherwise all firms resume the collusive behavior by adopting $a_M$ forever (the carrot).

The need for a punishment is rooted in the fact that each individual firm, assuming that all other firms play the collusive action, has a short-run incentive to lower (increase) its own price (quantity) to increase individual profits at every other firm’s expense. Then the choice of a low (high) punishment price (quantity) $a_P$ in the next period renders free-riding less attractive. As standard in the penal code literature, we say punishments are optimal if, for a given discount parameter $\delta$, they lead firms to obtain the highest level of sustainable collusive profits. As established by Abreu (1986, p. 192), these optimal punishments are also the most severe. Following Lambertini and Sasaki (2002), we adopt an equivalent interpretation according to which a penal code is optimal if, given a collusive action $a_M$, it requires the lowest critical threshold in the discount factor in order to sustain the collusive equilibrium path. Intuitively, this is because the lower the discount factor, the smaller the present value of future collusive profits, and the higher the incentives to deviate.
3 Collusion Sustainability

In this section, we investigate the role of the level of marginal costs on the existence of an optimal single-period punishment, either in prices or in quantities, in the stick-and-carrot mechanism. As in the existing literature, we also examine the effects of the degree of product differentiation and of the number of firms.

For each player to have no incentive to deviate, a deviation must be followed by a punishment that leads the discounted flow of profits to be less than the actualized stream of collusive equilibrium profits. Moreover, for the punishment to be a credible threat, one should verify that firms do implement the punishment strategy. This occurs if individual gains to deviate from the punishment phase are smaller than the loss incurred by prolonging the punishment by one more period. The strategy profile \( \{a_M, a_P\} \) is a subgame perfect equilibrium of the supergame if and only if the two conditions are satisfied, as made formal in the system of incentive constraints

\[
\pi_i^d(a_M) - \pi(a_M) \leq \delta (\pi(a_M) - \pi(a_P)),
\]

\[
\pi_i^d(a_P) - \pi(a_P) \leq \delta (\pi(a_M) - \pi(a_P)),
\]

where \( \pi(a) \) denotes each firm’s stage profit when all firms play \( a \), and \( \pi_i^d(a) \) is firm \( i \)’s profit from a one-shot best deviation from the action \( a \) selected by all firms \( j \neq i \), all \( i \), with \( a = a_M, a_P \).

The first condition says that the incentive for the initial deviation must be smaller than what is lost due to the punishment phase. The second condition says that the incentive to deviate from the punishment phase must be smaller than the loss incurred by prolonging the punishment by one more period. We look for an optimal punishment action \( a_P^* \) and the threshold level of the discount factor \( \delta_K \) such that, if \( a_P^* \) is prescribed and \( \delta \geq \delta_K \), then firms can sustain collusion
at $a_M$. The subscript $K = B, C$ refers to the Bertrand or Cournot versions of the model, respectively.

**Characterization of an optimal punishment** The optimal punishment action $a^*_P$, and the threshold level of the discount factor $\hat{\delta}_K$, are such that (6) and (7) hold with equality. To see that, assume first that none of the two inequalities is binding for $a = a^*_P$ and $\hat{\delta} = \hat{\delta}_K$. Since both expressions on the right-hand side of the inequality sign are continuous in $\hat{\delta}$ and monotonically decreasing when the discount parameter is decreasing, there exists a value $\hat{\delta} < \hat{\delta}_K$ for which the system still holds true, a contradiction. Assume now that exactly one inequality is binding. Recalling that profit functions $\pi^d(\cdot)$ and $\pi(\cdot)$ are continuous in firms’ strategies, by changing slightly the punishment action from $a^*_P$ to $\tilde{a}_P$ one can relax the binding constraint and still let the other inequality hold. Now with $\delta = \hat{\delta}_K$ and $\tilde{a}_P$, both constraints (6) and (7) are slack, contradicting the previously established claim. As a result both constraints must be binding for $a = a^*_P$ and $\delta = \hat{\delta}_K$. It follows that

$$\hat{\delta}_K = \frac{\pi^d(a_M) - \pi(a_M)}{\pi(a_M) - \pi(a_P)}.$$  \hfill (8)

A final condition is that firms have incentives to participate in the game after deviation also. This occurs when the individual rationality constraint is satisfied (Lambson, 1987), that is

$$\pi(a^*_P) + \sum_{\tau=1}^\infty (\hat{\delta}_K)^\tau \pi^M \geq 0.$$  \hfill (9)

In words, the losses incurred by a firm in the punishment period must not overbalance the discounted stream of collusive profits earned in the following periods. We find that this constraint is always satisfied for the specifications of the model (on this see the Appendix, sections 7.1.1. and 7.1.2).

When firms detect a deviation from the collusive strategy $a^M$, they all switch to the pun-

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6This is a restatement of Abreu (1986, Theorem 15) in the new context of the present model specifications.
ishment phase before returning to collusive profits. As a change in parameters \((\gamma, n, c)\) impacts each firm’s deviation profits and collusive profits (i.e., \(\pi_i^d(a_M), \pi_i^d(a_F)\), and \(\pi(a_M)\)), from (6) and (7) it also impacts the optimal punishment action \(a^*_P\) and thereby the lowest discount factor \(\delta_K\). The following proposition gives conditions on collusion sustainability under an optimal single-period punishment.

**Proposition 1** Optimal collusion is implementable with a single-period punishment if and only if \(c \geq \max\{0, \xi_K\}\) and \(\delta \geq \delta_K\), where \(K = B, C\). If \(K = C\) and \(n \leq 3\) then \(\xi_K = 0\) for all \(\gamma, n\), otherwise \(\xi_K > 0\). In all cases \(\delta_K \leq \delta_{\text{trigger}}\).

**Proof** See Appendix 7.1. ■

There are several results in the proposition. As standard in the literature, we obtain that the single-period punishment profits must be sufficiently valued by a deviating firm. This means that the discount factor must be sufficiently close to 1, i.e. \(\delta \geq \delta_K\). Another result is that the level of marginal costs matters for collusion sustainability.

For an intuitive interpretation of this claim, recall first that, in all models of tacit collusion, a necessary condition for the sustainability of agreements is the existence of a sufficiently severe punishment mechanism, as formalized in (6). This holds when the discount factor is high enough and punishment profits are low enough. More precisely, for a given discount factor, the higher the deviation payoffs, and the lower the collusive profits, the tougher the punishment must be for collusion to remain sustainable. In a trigger penal code, a deviation implies that firms stop colluding and revert to the one-shot stage game Nash equilibrium forever. A stick-and-carrot setup authorizes a shorter and more severe punishment phase that may lead firms to earn negative profits for some time without violating the individual rationality constraint (9). The severity of punishment depends on the extend to which firms can charge a non-negative
price sufficiently below marginal costs. This leads to identify the following fact: the higher the level of $c$, the larger the range $[0, c]$ of below-cost non-negative prices that firms can possibly charge in a punishment phase.

Secondly, note that a more severe punishment is likely to render deviation from the punishment path more profitable. This is captured in the present stick-and-carrot penal code, by a second necessary condition displayed in (7), which requires that the punishment is enforced when a previous-period deviation is observed. When punishment profits are negative (which holds true with below marginal cost pricing), the deviator can be better off by interrupting production temporarily. This occurs when goods are relatively close substitutes and the marginal cost is sufficiently high.\footnote{When goods are differentiated enough and/or the marginal cost is very low, a firm can earn positive profits by deviating from the punishment strategy (on this see Appendices 7.1.1 and 7.1.2).} We thus have another effect that may potentially countervail the former one: the higher the marginal cost $c$, the more severe the maximal punishment, and the higher the incentive to deviate from a severe punishment path.

Proposition 1 establishes that the latter effect is always dominated by the former one. This leads to the conclusion that $c$ must lie above a threshold value $c_K$ for collusion to be sustainable. Remark that this constraint on $c$ is ineffective in the quantity-setting version of the model with two or three firms. This implies that results obtained in the literature with a duopoly and a constant marginal cost normalized to zero, all other specifications remaining equal, are robust to the introduction of a positive marginal cost. Otherwise, the constraint on $c$ must be satisfied for a single-period punishment to be optimal. This result contrasts sharply with trigger penal code models, in which one can easily check that the sustainability of collusion is not directly connected to the level of marginal costs (at least in the linear cost setup).
Comparative statics We now turn to the impact of a change in the differentiation parameter \( \gamma \) or in the number of firms \( n \) on the threshold values \( c_K \) and \( \delta_K \).

**Proposition 2** For \( K = B, C \) and for all \( \gamma, n \):

(i) \( \frac{\partial c_K}{\partial \gamma} \geq 0, \quad \frac{\partial \delta_K}{\partial \gamma} > 0; \)

(ii) \( \frac{\partial c_K}{\partial n} \geq 0, \quad \frac{\partial \delta_K}{\partial n} > 0. \)

**Proof** The sign of derivatives can be obtained by direct computation of expressions given in the Lemmas 1 and 2 of Appendix 7.1.

This proposition establishes that the threshold values \( c_K \) and \( \delta_K \) evolve in the same direction when parameter values change. We find that an increase in product differentiation (i.e., a reduction in \( \gamma \)) facilitates collusive agreements, as it relaxes the constraints on \( c \) and \( \delta \). This extends to the present model specifications a previous result unveiled by Chang (1991) with a trigger penal code, and by Häckner (1996) and Lambertini and Sasaki (2002) in a stick-and-carrot framework with two firms. The driving force behind this claim is that a rise in product substitutability renders a deviation from the collusive or punishment path individually more profitable. This effect dominates all other changes in the expression of \( \delta_K \) as displayed in (8). This contrasts with the fact that, with a trigger penal code, \( \delta_{\text{trigger}} \) is not monotonic in \( \gamma \). An interesting and novel result relates to the level of marginal costs. When product substitutability increases, individual deviation profits rise, but simultaneously punishment profits are lowered. We find that the magnitude of the former effect dominates the latter. In consequence, an increase in product substitutability makes a deviation from the collusive path more likely. This results in the strengthening of the constraint on \( c \), that is Proposition 2(i). An increase in the number of firms has similar effects, as claimed in Proposition 2(ii). Again, this is because an individual deviation becomes most profitable, both with price and quantity competition,

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whereas the reduction in punishment profits is relatively limited.

4 Multi-Period Punishments

In this section we focus on situations in which optimal collusion is not sustainable with a single-period penal code. This occurs when parameter values are such that one of the two constraints on $c$ and $\delta$ we described in Proposition 1 is violated. To do that, we first introduce a multiple-period extension to our model, before comparing optimal symmetric punishments with the trigger penal code equilibrium benchmark.

As in Lambertini and Sasaki (2002), consider a multi-period stick-and-carrot penal code in which, if any deviation from $a_M$ by any firm is detected, all firms switch to a $l$-period punishment phase (the stick) during which they play $a_{P,k}$, with $k = 1, \ldots, l$. In any period of punishment, if a deviation from the punishment action by a firm is detected, the punishment phase restarts for $l$ more periods, after which all firms revert to the initial collusive path $a_M$ forever (the carrot).

For firms not to be incentivized to deviate from the collusive path, a deviation should lead them to earn a discounted flow of profits which is less than their discounted flow of collusive equilibrium profits. This writes

$$
\pi^d_i(a_M) + \sum_{k=1}^{l-1} \delta^k \pi(a_{P,k}) + \delta^l \pi(a_{P,l}) + \sum_{\tau=l+1}^{\infty} \delta^\tau \pi(a_M) \leq \sum_{\tau=0}^{\infty} \delta^\tau \pi(a_M),
$$

where $\pi(a)$ denotes each firm’s stage symmetric profits when all firms charge $a$, and $\pi^d_i(a)$ is firm $i$’s profits from a one-shot best deviation from the action $a$, as selected by all firms $j \neq i$, all $i, j$. For some parameter values, which satisfy conditions described in the previous section, this inequality may hold with a single-period punishment, that is $l = 1$. However, a longer duration of punishment is needed when parameter values are such that the severity of a single-period
punishment is not sufficient to deter deviation, in which case \( l > 1 \). In the latter situation, we consider punishment phases in which the \( l - 1 \) first periods of punishment are the toughest possible. The most severe action, we denote by \( a_P \), is such that prices are driven down to zero. This occurs when, in the Bertrand version of the model, all firms charge \( p_P = 0 \), while in the Cournot version all firms expand their output so that \( p_i(q_P, q_{i}) = 0 \), all \( i \). Then in the \( l \)-th final period of punishment, firms play an optimal strategy \( a_{P,l} \). Remark that, for all \( l \geq 1 \), when marginal costs are positive firms may earn negative profits if they play the punishment action \( a_{P,l} \).

When a firm deviates from the collusive strategy, a second incentive constraint must be satisfied so that firms implement the penal code. In words, the individual gain to deviate from the punishment phase must be smaller than the loss incurred by prolonging the punishment phase of \( l \) more periods. Formally, this writes

\[
\sum_{k=1}^{s-1} \delta^k \pi(a_P) + \delta^s \pi^d(a_P^*) + \sum_{k=s+1}^{s+l-1} \delta^k \pi(a_P) + \delta^{s+l} \pi(a_{P,l}) + \sum_{\tau=s+l+1}^{\infty} \delta^\tau \pi^M \\
\leq \sum_{k=1}^{l-1} \delta^k \pi(a_P) + \delta^l \pi(a_{P,l}) + \sum_{\tau=l+1}^{\infty} \delta^\tau \pi^M,
\]

for any period \( s \) in which a firm deviates from the penal code, with \( 1 \leq s \leq l \) (we adopt the convention that \( \sum_{k=1}^{s-1} \delta^k \pi(a_P) = 0 \) if \( s = 1 \)).

The strategy profile \( \{a_M, a_P, a_{P,l}\} \) is a subgame perfect equilibrium of the supergame if and only if (10) and (11) are satisfied. There are \( l \) constraints in (11). The only one that will be binding is for \( s = 1 \), since the incentive to deviate from the penal code is higher in the first period of punishment than in any subsequent period. The system of inequalities can thus be rewritten as

\[
\pi^d(a_M) - \pi(a_M) \leq \delta^l \left( \pi(a_M) - \pi(a_{P,l}) \right) + \left( \delta(1 - \delta^{l-1})/(1 - \delta) \right) \left( \pi(a_M) - \pi(a_P) \right),
\]

\[
\pi^d(a_P) - \pi(a_P) \leq \delta^l \left( \pi(a_M) - \pi(a_{P,l}) \right) + \delta^{l-1} \left( \pi(a_{P,l}) - \pi(a_P) \right).
\]

(12)
The optimal punishment action $a_{P,l}^*$ and the threshold level of the discount factor $\delta_{K,l}$ are such that (12) holds with equality (the proof proceeds as in the single-period case, by observing that for any $l > 1$, both expressions on the right-hand side of the inequality sign are monotonically decreasing when $\delta$ decreases). A reorganization of terms leads to

$$\delta_{K,l} = \frac{\pi^d(a_M) - \pi(a_M)}{\pi^i(a_M) - \pi^i(a_P)}, \quad \text{all } l > 1,$$

(13)

In this multi-period penal code, the individual rationality constraint for a deviation not to end the game writes

$$\sum_{k=1}^{l-1} \delta^k \pi(a_P) + \delta^l \pi(a_{P,l}^*) + \sum_{\tau=l+1}^{\infty} \delta^\tau \pi^M \geq 0.$$ (14)

Remark that a shift to a longer punishment phase always results in lower intertemporal punishment profits. This is because $\pi(a_P) \geq \pi(a_P^*)$ and $\pi(a_M) \geq \pi(a_{P,l+1})$. In words, for a given period, the profits obtained in a short (i.e., $l$-period) punishment phase are bounded from below by the profits earned in a longer $(l+1$-period) phase. The following proposition gives conditions on collusion sustainability under an optimal multi-period punishment.

**Proposition 3** Optimal collusion is implementable with a $l$-period punishment if and only if $\max\{0, \zeta_{K,l}\} \leq c \leq \max\{0, \zeta_{K,l-1}\}$ and $\delta \geq \delta_{K,l}(c)$, where $K = B, C$ and $l > 1$. The threshold value $\zeta_{K,l}$ is strictly decreasing when $l$ increases, but $\delta_{K,l}(c)$ is constant in $l$. Moreover, for all $c$, there always exists a length $l$ of punishment phase such that optimal collusion is implementable if $\delta \geq \delta_{K,l}(c)$, with $\delta_{K,l}(c) \leq \delta_{\text{trigger}}$.

**Proof** Recall that tacit collusion is implementable if and only if (i) there exists a sufficiently severe punishment mechanism, (ii) there are no incentive to deviate from this punishment, and (iii) there are no incentives to drop out from the game, i.e. the individual rationality constraint (14) holds...
true. The first two conditions stated in (12) can be rewritten as

\[ X_l(\delta, c) \leq \pi(a_{P,l}, c) \leq Y_l(\delta, c), \tag{15} \]

where the argument \( c \) highlights the fact that all expressions depend on the value of the marginal cost while the lower-bound \( X_l(\delta, c) \) and the upper-bound \( Y_l(\delta, c) \) of the last-period punishment profits \( \pi(a_{P,l}, c) \) are given by:

\[ X_l(\delta, c) \equiv \frac{1}{\delta^{l+1}} \frac{1}{1-\delta} \left( \pi^d(a_P, c) - \pi(a_P, c) \right) - \frac{\delta}{1-\delta} \pi(a_M, c) + \frac{1}{1-\delta} \pi(a_P, c), \tag{16} \]

\[ Y_l(\delta, c) \equiv \pi(a_M, c) - \frac{1}{\delta^l} \left( \pi^d(a_M, c) - \pi(a_M, c) \right) + \frac{1}{\delta^{l-1}} \frac{1}{1-\delta} \left( \pi(a_M, c) - \pi(a_P, c) \right). \tag{17} \]

Condition (14) can also be rewritten as \( Z_l(\delta, c) \geq 0 \), where \( Z_l \) increases when \( \delta \) increases. Clearly, since an increase in \( \delta \) relax the three constraints, if collusion is implementable with a discount factor \( \delta_{K,l} \) it is implementable for any discount factor \( \delta \geq \delta_{K,l} \). It is also plain that the lower bound of the discount factor \( \delta_{K,l} \) depends a priori on the marginal cost \( c \) since the inequality (16) must be binding for \( a = a_{P,l}^* \) and \( \delta = \delta_{K,l} \) and both \( X_l(\delta, c) \) and \( Y_l(\delta, c) \) depends on \( c \). In contrast, the impact of \( c \) on collusion sustainability is less straightforward. Since \( Z_l(\delta, c) \) is decreasing in \( c \), an increase in the marginal costs makes the third constraint more stringent. Moreover, the last-period punishment profits \( \pi(a_{P,l}, c) \) also depend on \( c \).

Note that, while a longer length of punishment results in lower intertemporal profits, the profits in the last period of the punishment phase increase as the length \( l \) of the punishment phase increases. To see that, observe that \( \pi^d(a_P) > \pi(a_P) \) implies that the lower bound of the last period punishment profits \( X_l(\delta) \) strictly increases with \( l \). Since the inequality (16) must be binding for \( a = a_{P,l}^* \) and \( \delta = \delta_{K,l} \) and the critical discount factor \( \delta_{K,l} \) as defined by equation (13) does not change with
$l$, it follows that $\pi(a^*_{P,l+1}) > \pi(a^*_{P,l})$. Note also that obvious feasibility constraints impose that $\pi(a_P) \leq \pi(a^*_{P,l}) \leq \pi(a_M)$, all $l$.

The threshold level of marginal cost $\zeta_{K,l}$ is the lowest marginal cost that allow to implement collusion by the means of a punishment period of length $l$. It happens to be also the highest marginal cost that allow to implement collusion by the means of a punishment period of length $l + 1$. It corresponds indeed to the level of marginal costs such that both feasibility constraints $\pi(a_P) \leq \pi(a^*_{P,l})$ and $\pi(a^*_{P,l+1}) \leq \pi(a_M)$ are binding. Formally

$$\pi(a_P, \zeta_{K,l}) = -\zeta_{K,l} q(a_P) = X_l (\Delta_{K,l}, \zeta_{K,l}) = Y_l (\Delta_{K,l}, \zeta_{K,l}),$$

$$\pi(a_M, \zeta_{K,l}) = X_{l+1} (\Delta_{K,l}, \zeta_{K,l}) = X_{l+1} (\Delta_{K,l+1}, \zeta_{K,l}).$$

By definition, collusion implementation by the means of a $l$-period punishment with $a^*_{P,l} = a_P$ is tantamount to collusion implementation by the means of a $l + 1$-period punishment with $a^*_{P,l+1} = a_M$. The get the result that collusion can be implemented by the means of a $l$-period punishment for any $\zeta_{K,l} \leq c \leq \zeta_{K,l-1}$, it is thus sufficient to show that, if collusion can be implemented with last period punishment profits $\pi(a_P) < \pi(a^*_{P,l}) < \pi(a_M)$, then collusion can also be implemented by the means of a $l$-period punishment for a strictly smaller and a strictly larger marginal cost $c$. Both $X_l$ and $Y_l$ depends continuously on $\delta$ and $c$. Moreover the difference $Y_l - X_l$ is monotonic, strictly increasing in $\delta$. It follows that it is always possible to have $X_l (\delta, c) \leq Y_l (\delta, c)$, possibly by adjusting $\delta$, both after an increase or a decrease in $c$. The only limit to this changes in the marginal cost follow from the ability to match the value $X_l (\Delta_{K,l}, c) = Y_l (\Delta_{K,l}, c)$ with punishment profits $\pi(a_{P,l}, c)$. One can show however that the difference $\pi(a_{P,l}, c) - X_l (\delta, c)$ is continuous and increasing $a_{P,l}$ and continuous and decreasing in $c$. Indeed,

$$\frac{\partial [\pi(a_{P,l}, c) - X_l (\delta, c)]}{\partial c} = \frac{1}{\delta - 1} - \frac{1}{\delta} q(a_P) - q(a_{P,l}) - \frac{\delta q(a_M) + 1}{1 - \delta} q(a_P) - q(a_{P,l}) < 0.$$

To see that, observe that since $q(a_{P,l})$ is decreasing with $a_{P,l}$ it is sufficient to show that the inequality
holds for $a_{P,l} = a_M$. Since $q(q_{MP}) > q_{MP}^2$, it is also sufficient to focus on the case $l = 2$. Thus the monotonicity of $\pi(a_{P,l}, c) - X_l(\delta, c)$ in $c$ will follow from
\[
\delta < \frac{q(q_{MP}) - q_{MP}^2}{q(q_{MP}) - q(a_M)}.
\]
As a result $\pi(a_{P,l}, c) - X_l(\delta, c)$ is always decreasing in $c$ since $q_{MP}^2 \leq q(a_M)$. It follows that, if collusion can be implemented by the means of a $l$-period punishment for level $c$ of marginal costs, it can also be implemented for a strictly higher (resp. lower) $c$ as long as $a_{P,l}$ can be increased (resp. decreased). One also get that $c_{K,2} < c_{K,l-1}$.

To sum up, if collusion can be implemented by using a $l$-period punishment for some $(\delta, c)$, one can define $\delta_{K,l+1}(c) = \delta_{K,l}(c)$ and $c_{K,l+1} < c_{K,l}$ so that collusion can also be implemented for any $\delta \geq \delta_{K,l}(c)$ and $c_{K,l+1} \leq c \leq c_{K,l}$ by using a $l + 1$-period punishment. The proof proceeds by induction. Appendix 7.2 focus on the 2-period case and establishes that collusion can indeed be sustained for $\delta \geq \delta_{K,2}(c)$ and $c_{K,2} \leq c \leq c_{K}$. In particular, the threshold value $\delta_{K,l}(c) = \delta_{K,2}(c)$ is exhibited.

It remains to demonstrate that a trigger penal code can be replicated by a stick-and-carrot penal code. Observe that, for any particular set of values taken by each of the structural parameters $(c, n, \gamma)$, it is always possible to construct a multi-period stick-and-carrot penal code which associates to each $r_m$ a stream of profits which, evaluated in the first period of punishment, is exactly equal to the actualized sum of profits earned in the punishment phase of the trigger penal code. In our notation, this replication argument is equivalent to saying that there exists a pair $(l, a_{P,l})$ such that
\[
\sum_{k=1}^{l-1} \delta^k q_{MP}^2 + \delta^l q_{MP}^2 + \sum_{\tau=l+1}^{\infty} \delta^\tau M = \sum_{\tau=1}^{\infty} \delta^\tau N,
\]
where the subscript $S&C$ stands for “stick-and-carrot”, $q_{MP}$ is the most severe action (i.e., either a price equal to zero or a quantity which drives each firm’s price down to zero), and $l \geq 1$. To see that, it is
sufficient to find an integer $l$ such that $\Pi_{SKC} (l + 1, a_M) = \Pi_{SKC} (l, a_P) \leq \Pi_{trigger} \leq \Pi_{SKC} (l, a_M)$ holds true. By continuity of $\Pi_{SKC} (l, a_{P,l})$ in $a_{P,l}$, there exist a last period punishment value such that $\Pi_{SKC} (l, a_{P,l}) = \Pi_{trigger}$. Rewrite the expression on the left-hand side of the equality side in (18) for $a_{P,l} = a_M$:

$$\Pi_{SKC} (l, a_M) = \frac{\delta (1 - \delta^{l-1})}{1 - \delta} \pi(a_P) + \delta^l \pi^M + \frac{\delta^{l+1}}{1 - \delta} \pi^M.$$  (19)

Recalling that $\pi(a_P) \leq 0$ (by construction) together with $\pi^M > 0$ makes it clear that $\Pi_{SKC} (l, a^M)$ is decreasing when $l$ increases. Moreover, as $\pi^M > \pi^N$ (by definition), it follows that $\Pi_{SKC} (l = 1, a^M) = (\delta / (1 - \delta)) \pi^M > \Pi_{trigger}$ and $\lim_{l \to +\infty} \Pi_{SKC} (l, a^M) = \delta / (1 - \delta) \pi(a_P) < 0 < \Pi_{trigger}$. Therefore there exists a value for $l$ such that $\Pi_{SKC} (l + 1, a_M) \leq \Pi_{trigger} \leq \Pi_{SKC} (l, a_M)$ hence a pair $(l, a_{P,l})$ such that (18) holds true. ■

There are several results in this proposition. Firstly, the constraint on the marginal cost $c$ we identified in Proposition 1 for the single-period punishment case finds a counterpart in the multi-period setup. Secondly, the latter constraint on $c$ is relaxed by increasing the length of the punishment phase. In other words, a longer duration of punishment is a “substitute” to a high marginal cost. In contrast, the constraint on the discount factor cannot be relaxed by the increase in $l$ since $\delta_{K,l}$ depends on $c$ (as well as $n$ and $\gamma$) but not on the length of the punishment phase $l$. The last result of Proposition 3 consists in showing that the constraint on marginal costs can be fully relaxed by sufficiently lengthening the punishment phase. In other words, if $\delta \geq \delta_{K,l}$, there exists a punishment code that allow to implement collusion for any level of the marginal costs $c$. This echoes an existing result by Häckner (1996), who demonstrates that an optimal stick-and-carrot symmetric punishment $a^*_P$ always exists (Proposition 4, p. 624) with two firms, constant marginal costs normalized to zero, and in a spatial differentiation framework. Proposition 3 extends this result to the general case, that is with $n \geq 2$ firms, constant marginal costs $c$ in $[0,1)$, and for any value taken by the exogenous differentiation.
parameter $\gamma$ in $[0, 1)$.

**Corollary 4** $\hat{\delta}_{K,l} \geq \hat{\delta}_K$, all $l > 1$, $K = B, C$.

**Proof** Since $\hat{\delta}_{K,l}$ does not depend on $l$, it is sufficient to show that $\hat{\delta}_{K,l=2} \geq \hat{\delta}_K$. This is done by comparing values of $\hat{\delta}_{K,l=2}$ and $\hat{\delta}_K$ as given in Appendices 7.1 and 7.2. ■

To illustrate, we now consider the simplest possible case of multiple-period punishment, that is $l = 2$, to obtain the following result.

**Remark 1** For $K = B, C$ and for all $\gamma, n$:

If $K = C$ and $n \leq 4$ then $e_{K,l=2} = 0$ for all $\gamma, n$; otherwise $e_{K,l=2} > 0$. Moreover $\frac{\partial e_{K,l=2}}{\partial \gamma} > 0$, $\frac{\partial e_{K,l=2}}{\partial n} > 0$, and $\frac{\partial e_{K,l=2}}{\partial n} > 0$.

It remains to compare the role of the nature of competition on the sustainability of collusion. We tackle this next.

## 5 Price vs. Quantity

We now compare the threshold values $e_K$ and $\hat{e}_K$, as defined in Section 3, in the Bertrand ($K = B$) and Cournot ($K = C$) versions of the model.

**Proposition 5** For all $\gamma, n$: $e_B > e_C$ and $\hat{e}_B > \hat{e}_C$.

This proposition establishes that collusion is less easily sustainable in a price-setting oligopoly than in a quantity-setting one. This means first that, in a Bertrand setup, a longer punishment than in the Cournot case may be required. Indeed, since $e_B > e_C$, there exist parameters $(c, n, \gamma)$ which are such that collusion is not sustainable with a single-period punishment in Bertrand, while a single-period punishment is sufficient in Cournot. From Proposition 3, we know there
exists \( l \) such that \( \varphi_{\gamma,l} \leq c \), hence optimal collusion with be implementable in Bertrand with a multi-period punishment. Moreover, the constraint on the discount factor is more stringent when firms use prices as opposed to quantities. This claim holds for any degree of product substitutability, and for all numbers of firms.

This can be made intuitive by observing that the incentives to deviate in the Bertrand case are higher than in the Cournot setup. This is because a deviating firm can capture the whole demand in a price-setting oligopoly. This does not apply in a quantity-setting oligopoly, because a unilateral expansion in some firm’s output cannot eliminate its rivals’ demand. By contrast, the range of punishment possibilities \([-cq, \pi^M]\) is the same in the Bertrand and Cournot cases, since we use symmetric punishments. In both versions of the model, the severity of punishments thus depends only on the level of marginal costs.

To illustrate the multi-period case, we now focus on two-period punishment phases. For the sake of clarity, denote by \( \underline{c}_{K,l=1} \) and \( \hat{\delta}_{K,l=1} \) the threshold values defined in the single-period framework.

**Remark 2** For all \( \gamma, n : \underline{c}_{\gamma,l} > \varphi_{\gamma,l} \); and for all \( c, \gamma, n : \hat{\delta}_{\gamma,l} > \hat{\delta}_{\gamma,l'} \), \( l \geq l' \), with \( l, l' = 1, 2 \).

This remark establishes that the results offered in Proposition 4 are still valid in a two-period context. It can be easily extended to more than two periods. This is because the most severe punishments \( \pi(a_P) = -cq \) inflicted in the first \( l - 1 \) periods are the same in Bertrand and Cournot, with \( a_P = p_P = 0 \) in the price-setting case, and \( p_i(a_P, a_P) = p_i(q_P, q_P) = 0 \) in the quantity version. The difference between the Bertrand and Cournot cases is therefore fully captured by the difference in the unilateral deviation profits \( \pi^d(a_{P,l}) \) and the optimal punishment profits \( \pi(a_{P,l}) \) in the last period \( l \), as in the single-period framework.
6 Conclusion

We have presented an oligopoly supergame model with differentiated goods and positive marginal costs, under Bertrand or Cournot behavior. In this setup, we are able to derive quite general results which are policy relevant. They confirm that antitrust authorities should concentrate investigations on industries in which firms are few and sell highly differentiated goods. On top of that, we argue that the level of marginal costs should be added to the list of structural factors that facilitate collusion. Another interesting finding is that a longer punishment phase is a substitute for a low level of marginal costs. However, real-world circumstances may limit the duration of below cost price wars. In this case, a low level of marginal costs would constitute an impediment to tacitly collusive agreements.

7 Appendix

In this appendix, we determine conditions of collusion sustainability by computing the threshold values of $c$ and $\delta$ for all parameter values.

7.1 Single-period punishment

The optimal punishment action $a_P$ and the minimum threshold $\delta_K$ are defined by

$$\pi_i^d (a_P) - \pi (a_P) = \pi_i^d (a_M) - \pi (a_M)$$

and

$$\delta_K = \frac{\pi_i^d (a_M) - \pi (a_M)}{\pi (a_M) - \pi (a_P)}.$$ 

We check that the individual rationality constraint in the punishment phase is satisfied by verifying that the value of discounted profits from the punishment period onwards is strictly
positive, that is
\[ \pi(a_P) + \frac{\delta}{1-\delta} \pi(a_M) \geq 0. \] (22)

Since \( \frac{\partial}{\partial \delta} \left( \pi(a_P) + \frac{\delta}{1-\delta} \pi(a_M) \right) > 0 \), it will be sufficient to check in the following that
\[ \pi(a_P) + \frac{\delta \kappa}{1-\delta \kappa} \pi(a_M) \geq 0 \] (23)
to conclude that the individual rationality constraint is satisfied for all \( \delta > \delta \kappa \).

Both in the Bertrand and in the Cournot cases, the collusive profits, price and individual quantities are respectively given by
\[ \pi_i^M = \frac{(1-c)^2}{4(1+\gamma(n-1))}, \quad p_M = \frac{(1+c)}{2} \text{ and} \quad q_M = \frac{(1-c)}{2(1+\gamma(n-1))}. \]

7.1.1 Bertrand case

We start by recalling the Bertrand related content of Proposition 1, and by displaying threshold values of \( c \) and \( \delta \). Then we turn to the proof.

Lemma 6 Collusion can be sustained by a single-period punishment price if and only if \( c \geq c_B \) and \( \delta \geq \delta_B \), where:

\[ \delta_B = \begin{cases} 
\hat{c}_B^0 & \text{if } 0 \leq \gamma < \min \left\{ \frac{2(n-3)+(n-1)\sqrt{2}}{2n+n^2-\gamma}, \frac{n-3+\sqrt{n^2-1}}{3n-5} \right\} \\
\hat{c}_B^1 & \text{if } \min \left\{ \frac{2(n-3)+(n-1)\sqrt{2}}{2n+n^2-\gamma}, \frac{n-3+\sqrt{n^2-1}}{3n-5} \right\} \leq \gamma \\
\hat{c}_B^2 & \text{if } \gamma \geq \max \left\{ \frac{n-3+\sqrt{n^2-1}}{3n-5}, \frac{(n-3)(n^2-2n+5)+(n^2-1)\sqrt{n^2-2n+5}}{2(12n-7n^2+2n^3+1)} \right\} 
\end{cases} \]

with
\[ \hat{c}_B^1 = \begin{cases} 
\hat{c}_B^1 & \text{for } n = 2 \\
\hat{c}_B^2 & \text{for } n > 3 
\end{cases} \]
and
\[
\delta_B = \begin{cases} 
\frac{\gamma(\gamma(n-3)+2)^2}{16(1-\gamma)(1+\gamma(n-2)+1)} & \text{if } 0 \leq \gamma \leq \min\left\{\frac{n-3+\sqrt{n^2-1}}{3n-6}, \gamma_1\right\} \text{ and } \min\{0, \tilde{c}_B\} \leq c \leq 1 \text{ for all } n \\
\frac{\gamma^2(2n-3)+\gamma(3-n)-1}{(2\gamma-1)(1+\gamma(n-2)+\gamma)} & \text{if } \max\left\{\frac{n-3+\sqrt{n^2-1}}{3n-6}, \gamma_2\right\} \leq \gamma \leq 1 \text{ and } \tilde{c}_B^2 \leq c \leq 1 \text{ for all } n \\
\frac{(2\gamma-1)(1+\gamma)-\gamma^2}{(\gamma^2-1)(\gamma^2-4\gamma+2)+4\gamma^2(1-\gamma)/(\gamma^2+1)} & \text{if } \frac{n-3+\sqrt{n^2-1}}{3n-6} \leq \gamma \leq \gamma_2 \text{ and } \tilde{c}_B^1 \leq c \leq 1 \text{ for } n = 2 \\
\gamma^2 \frac{(n-1)^2}{(\gamma(n-3)+2)^2} & \text{if } \gamma_1 \leq \gamma \leq \frac{n-3+\sqrt{n^2-1}}{3n-6} \text{ and } \tilde{c}_B^1 \leq c \leq f(\gamma) \text{ for } n > 3 
\end{cases}
\]
where \(\gamma_1\) and \(\gamma_2\) are expressions of \(n\), and \(f(\gamma) = (1-\gamma)/(1+\gamma(n-2))\).

**Proof:** The demand function is given by
\[
q_i(p_i, p_j) = \begin{cases} 
1 - p_i & \text{if } p_i \leq \bar{p}_i(p_j) \\
\frac{1}{1+\gamma(n-1)} \left(1 - \frac{1+\gamma(n-2)}{1-\gamma} p_i + \frac{\gamma(n-1)}{1-\gamma} p_j\right) & \text{if } \bar{p}_i(p_j) \leq p_i \leq \tilde{p}_i(p_j) \\
0 & \text{if } p_i \geq \tilde{p}_i(p_j)
\end{cases}
\quad (24)
\]
and demand for each other symmetric firm \(j\) is defined by
\[
q_j(p_i, p_j) = \begin{cases} 
0 & \text{if } p_i \leq \bar{p}_i(p_j) \\
\frac{1}{1+\gamma(n-2)} \left(1 - \frac{1+\gamma(n-1)}{1-\gamma} p_j + \frac{\gamma(n-1)}{1-\gamma} p_i\right) & \text{if } \bar{p}_i(p_j) \leq p_i \leq \tilde{p}_i(p_j) \\
1-p_j & \text{if } p_i \geq \tilde{p}_i(p_j)
\end{cases}
\quad (25)
\]
with \(\bar{p}_i(p_j) = \frac{1}{\gamma} [p_j - (1-\gamma)]\) and \(\tilde{p}_i(p_j) = \frac{1}{1+\gamma(n-2)} (1-\gamma + \gamma(n-1)p_j)\).

Firm \(i\)'s profit writes \(\pi_i(p_i, p_j) = (p_i - c) q_i(p_i, p_j)\), with \(c < 1\). This profit function is locally
concave in each market structure. One can find that the best deviation profit for firm $i$ writes

$$
\pi^d_i(p) = \begin{cases} 
0 & \text{if } p \leq p_j \\
\frac{(1-\gamma)(1-c)+\gamma(n-1)(p-c)^2}{4(1+\gamma(n-1))(1+\gamma(n-2))(1-\gamma)} & \text{if } p \leq \overline{p}_j \\
\frac{(1-p_j)(p-c)-(1-\gamma)(1-c)}{(1-c)^2} & \text{if } \overline{p}_j \leq p \leq \overline{p}_j \\
(1-c)^2 & \text{if } p \geq \overline{p}_j 
\end{cases}
$$

(26)

where $p_j - c \equiv \frac{(1-\gamma)(1-c)}{\gamma(n-1)}$, $\overline{p}_j - c \equiv \frac{(1-\gamma)(1-c)(2+\gamma(2n-3))}{2(1-\gamma)+\gamma(n-1)(2-\gamma)}$, $\overline{p}_j - c \equiv \frac{(1-c)(2-\gamma)}{2}$. Note that

$$p_j > 0 \iff c > \frac{1-\gamma}{(1+\gamma(n-2))} \equiv f(\gamma).$$

We now compute the best deviation to the collusive price. Since $p_j < p_M < \overline{p}_j$ and $p_M \leq \overline{p}_j \iff \gamma \leq (n-3+\sqrt{n^2-1})/(3n-5) < 1$, when all other firms charge $p_M$, firm $i$'s maximum deviation profit is given by

$$
\pi^d_i(p_M) = \begin{cases} 
\frac{(1-c)^2(2+\gamma(n-3)^2)}{16(1+\gamma(n-1))(1+\gamma(n-2))(1-\gamma)} & \text{if } \gamma \leq \frac{n-3+\sqrt{n^2-1}}{3n-5}, \\
\frac{(1-c)^2}{(2\gamma-1)} & \text{if } \gamma \geq \frac{n-3+\sqrt{n^2-1}}{3n-5}. 
\end{cases}
$$

(27)

In order to solve (20), we distinguish several cases that depend on the form of $\pi^d_i(p_F)$ and $\pi^d_i(p_M)$.

- **Case 1.** $\pi^d_i(p_F) = 0$ if $0 \leq p_F \leq \underline{p}_j$ and $c > f(\gamma)$

The left-hand side term of (20) is a convex parabolic function of $p$ that takes value 0 if either $p = c$ or $p = 1$. Since $\underline{p}_j < c < 1$, we know that it is monotone decreasing for all $p \leq \underline{p}_j$.

**Case 1.1.** $\gamma \leq \frac{n-3+\sqrt{n^2-1}}{3n-5}$

We find:

(i) $\pi^d_i(0) - \pi(0) > (\equiv) (\pi^d_i(p_M) - \pi(p_M)) \iff c > (\equiv) \frac{1}{\gamma^2} \equiv \left(\frac{17-10n+n^2}{n^2-8(3-n)\gamma+8(2+\gamma(2n-3))\sqrt{(1-\gamma)(1+\gamma(n-2))}}\right).$

(28)
Remark that $\tilde{c}_B^1$ is a monotone increasing function of $\gamma$, with $\lim_{\gamma \to 1} \tilde{c}_B^1 = 1$, and that for $\gamma = \frac{n-3+\sqrt{n^2-1}}{3n-5}$ the difference $\tilde{c}_B^1 - f(\gamma)$ is negative if $n = 2$, nil if $n = 3$ and positive if $n > 3$.

$$(ii) \quad \pi_0^d\left(p_{\infty}\right) - \pi\left(p_{\infty}\right) \equiv \gamma > (\gamma_1)$$

with $\gamma_1 \equiv \frac{2((n-3)+(n-1)\sqrt{2})}{2n+n^2-7}$,

We conclude that (20) admits a unique solution $p_P \in (0, p_{\infty})$ if $n > 3$, $\gamma_1 \leq \gamma \leq \frac{n-3+\sqrt{n^2-1}}{3n-5}$ and $c \geq \tilde{c}_B^1$. The threshold value of the discount factor is given by

$$\delta_B^1 \equiv \frac{\gamma^2(n-1)^2}{(\gamma(n-3)+2)^2} \in (0, 1).$$

From (23), the individual rationality constraint is satisfied since one can easily check that

$$\pi(p_P) + \frac{\delta_B^1 \pi(p_M)}{1-\delta_B} = 0.$$

**Case 1.2. $\gamma \geq \frac{n-3+\sqrt{n^2-1}}{3n-5}$**

We find:

$$(i) \quad \pi_0^d(0) - \pi(0) > \pi_0^d(p_M) - \pi(p_M) \equiv c > \tilde{c}_B^2$$

with

$$\tilde{c}_B^2 \equiv \frac{(2n-1)\gamma^2 + (3-n)\gamma - 1 - 2\gamma(2\gamma - 1)(1 + \gamma(n-1))}{(2\gamma - 1)(1 + \gamma(n-1)) - \gamma^2},$$

which is a monotone increasing function of $\gamma$, with $\lim_{\gamma \to 1} \tilde{c}_B^2 = (\sqrt{n} - 1)/(\sqrt{n} + 1) < 1$. For $\gamma = \frac{n-3+\sqrt{n^2-1}}{3n-5}$ we check that $\tilde{c}_B^1 = \tilde{c}_B^2$ and $\tilde{c}_B^2 - f(\gamma)$ is negative if $n = 2$, nil if $n = 3$ and positive if $n > 3$.

$$(ii) \quad \pi_0^d\left(p_{\infty}\right) - \pi\left(p_{\infty}\right) \equiv \gamma > \tilde{\gamma}_2$$

with

$$\tilde{\gamma}_2 \equiv \frac{(n-3)\left(n^2 - 2n + 5\right) + (n^2 - 1)\sqrt{(n^2 - 2n + 5)}}{2(2n^3 - 7n^2 + 12n - 11)}$$

and $\tilde{\gamma}_2 \equiv \frac{n-3+\sqrt{n^2-1}}{3n-5} \equiv n < (\equiv) 3$ and $\tilde{c}_B^2 = f(\gamma)$ if $\gamma = \tilde{\gamma}_2$. 

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We conclude that (20) admits a unique solution \( p_P \in (0, p_j) \) if \( \gamma \geq \max\{\frac{n-3+\sqrt{n^2-1}}{3n-5}; \bar{\gamma}_2\} \) and \( c \geq \delta_B^2 \). The discount factor threshold is given by

\[
\delta_B^2 = \gamma^2 \frac{2n-3 + \gamma (3-n) - 1}{(2\gamma-1)(1+\gamma(n-1))} \in (0,1).
\]

From (23), the individual rationality constraint is satisfied since \( \pi(p_P) + \frac{\delta_B^2 \pi(p_M)}{1-2\gamma} = 0 \).

**Case 2 :** \( \gamma \leq \frac{n-3+\sqrt{n^2-1}}{3n-5} \)

The lower bound of \( p_P \) is \( \sup\{0, p_j\} \) since \( p_P \) must be non-negative. The left-hand side term of (20) is a convex parabolic function of \( p_P \) which takes value 0 for \( p_P = 1+c-\gamma(1-c)(n-2) \) \( \equiv \bar{\gamma}_1 \) only. As \( p_j < \bar{\gamma}_1 < p_j \), this function is monotone decreasing on \( [p_j, \bar{\gamma}_1] \) and monotone non-decreasing on \( [\bar{\gamma}_1, p_j] \).

**Case 2.1.** \( \gamma \leq \frac{n-3+\sqrt{n^2-1}}{3n-5} \)

The only admissible solution to (20) is \( p_P - c = \frac{1}{2} \frac{\gamma(3n-5)}{2+\gamma(3n-5)} \), with \( p_P < \bar{\gamma}_1 < p_j \) for all parameter values. We find \( p_P > p_j \Leftrightarrow \gamma < \bar{\gamma}_1 \) and \( p_P > 0 \Leftrightarrow c > \delta_B^0 \) with

\[
\delta_B^0 = \frac{-2+\gamma(n+1)}{2+\gamma(3n-5)}.
\]

We check that (i) \( \frac{2+\gamma(n+1)}{2+\gamma(3n-5)} > 0 \Leftrightarrow \gamma > \frac{2}{(n+1)} < \frac{n-3+\sqrt{n^2-1}}{3n-5} \) and (ii) \( \frac{2}{(n+1)} < \bar{\gamma}_1 \), all \( n \).

We conclude that there is a solution \( p_P \) such that \( p_j < p_P < \bar{p}_j \) for all \( \gamma < \min\{\bar{\gamma}_1, \frac{n-3+\sqrt{n^2-1}}{3n-5}\} \) and \( c \geq \delta_B^0 \). We also find

\[
\delta_B^3 = \frac{1}{16} \frac{(\gamma(n-3) + 2)^2}{(1-\gamma)(\gamma(n-2) + 1)} \in (0,1).
\]

From (23), the individual rationality constraint is satisfied since \( \pi(p_P) + \frac{\delta_B^3 \pi(p_M)}{1-2\gamma} > 0 \) for all \( \gamma < \min\{\bar{\gamma}_1, \frac{n-3+\sqrt{n^2-1}}{3n-5}\} \).
Case 2.2. \( \gamma \geq \frac{n-3+\sqrt{n^2-1}}{3n-5} \)

The only admissible solution \( p_P \) to (20) is such that \( p_P < p_j \) for all parameters, \( p_P \geq p_j \Leftrightarrow [\gamma \leq \bar{\gamma}_2 \text{ and } c \geq f(\gamma)] \) and \( p_P \geq 0 \Leftrightarrow c \geq \bar{\gamma}_j \), with

\[
\bar{\gamma}_j = \frac{\gamma(\gamma-1) + \sqrt{(1-\gamma)(\gamma+\gamma^2-1)}}{\gamma + \sqrt{(1-\gamma)(\gamma+\gamma^2-1)}}
\]

We check that \( \bar{\gamma}_j \leq f(\gamma) \) whenever \( \frac{n-3+\sqrt{n^2-1}}{3n-5} \leq \gamma \leq \bar{\gamma}_2 \), that \( \bar{\gamma}_j = f(\gamma) \) if \( \gamma = \bar{\gamma}_2 \) and that \( \bar{\gamma}_j = \bar{\gamma}_j^0 \) if \( \gamma = \frac{n-3+\sqrt{n^2-1}}{3n-5} \). We conclude that there exists a solution \( p_P \) such that \( p_P < p_j \) whenever \( n-3+\sqrt{n^2-1} \leq \gamma \leq \bar{\gamma}_2 \) and \( c \geq \bar{\gamma}_j \), which is a non empty case only for \( n = 2 \). Now for \( n = 2 \), we find

\[
\delta_B^1 = \frac{(2\gamma - 1)(\gamma + 1) - \gamma^2}{(\gamma^2 - 2)\gamma^2 + 4\gamma^2\sqrt{(1-\gamma)(\gamma+\gamma^2-1)}}(\gamma - 2)^2 \in (0, 1)
\]

From (23), the individual rationality constraint is satisfied since \( \pi(p_P) + \frac{\pi(p_M)p_M}{1-\delta_B^1} > 0 \).

- **Case 3**: \( \pi_i^d(p_P) = \frac{(1-p_j)}{\gamma} ((p_j - c) - (1-\gamma)(1-c)) \) if \( p_j \leq p_P \leq \bar{p}_j \)

The left-hand side of (20) is increasing when \( p \) increases, all \( p \in [p_j, \bar{p}_j] \).

Case 3.1. \( \gamma \leq \frac{n-3+\sqrt{n^2-1}}{3n-5} \)

We use \( \pi_i^d(p_P) - \pi(p_P) > \pi_i^d(p_M) - \pi(p_M) \Leftrightarrow \gamma \leq \frac{n-3+\sqrt{n^2-1}}{3n-5} \) to conclude that (20) has no solution in the range \( [p_j, \bar{p}_j] \) and for \( \gamma \leq \frac{n-3+\sqrt{n^2-1}}{3n-5} \).

Case 3.2. \( \gamma \geq \frac{n-3+\sqrt{n^2-1}}{3n-5} \)

There is no admissible solution to (20) in the range \( [p_j, \bar{p}_j] \) and for \( \gamma \geq \frac{n-3+\sqrt{n^2-1}}{3n-5} \).

### 7.1.2 Cournot case

Let us start by recalling the Cournot related content of Proposition 1 before expressing the threshold values of \( c \) and \( \delta \).
Lemma 7 An optimal single-period punishment quantity $q_p$ exists in any Cournot supergame if $\delta \geq \delta_C$ with $\gamma \in (0, 1)$ for $n = 2, 3$, or $c \geq c_C$ and $n > 3$, where

$$c_C = \begin{cases} \gamma(n-1) - 2 & \text{if } \frac{2}{n-1} \leq \gamma \leq \min \left\{ \frac{2 + \sqrt{2}}{n-1}, 1 \right\} \\ \frac{2 + \sqrt{2}}{(1+\gamma(n-1)) + (2+\gamma(n-1))\sqrt{(1+\gamma(n-1))}} & \text{if } \min \left\{ \frac{2 + \sqrt{2}}{n-1}, 1 \right\} \leq \gamma. \end{cases}$$

and

$$\delta_C = \begin{cases} \frac{1}{16} (2+\gamma(n-1))^2 & \text{if } n < 6 \text{ or if } n \geq 6 \text{ and } \frac{2}{n-1} \leq \gamma \leq \min \left\{ \frac{2 + \sqrt{2}}{n-1}, 1 \right\} \\ \frac{(2+\gamma(n-1)) \sqrt{(1+\gamma(n-1)) - 2(1+\gamma(n-1))}}{(2+\gamma(n-1)) + (2+\gamma(n-1))\sqrt{(1+\gamma(n-1))}} & \text{if } n \geq 6 \text{ and } \min \left\{ \frac{2 + \sqrt{2}}{n-1}, 1 \right\} \leq \gamma. \end{cases}$$

Proof: The inverse demand function for firm $i$ and all other symmetric firms $j$ are given by

$$p_i(q_i, q_j) = \max\{0, 1 - q_i - \gamma(n-1)q_j\}, \text{ all } q_j \geq 0, j \neq i,$$

$$p_j(q_i, q_j) = \max\{0, 1 - \gamma q_i - (1 + \gamma(n-2))q_j\}. \quad (28)$$

Symmetric profits write

$$\pi(q) = \begin{cases} (1 - q (1 + \gamma(n-1)) - c)q & \text{if } q \leq \frac{1}{1+\gamma(n-1)}, \\ -cq & \text{if } q \geq \frac{1}{1+\gamma(n-1)}. \end{cases} \quad (29)$$

One can easily check that the one-shot best deviation profit is

$$\pi_i^d(q) = \begin{cases} \frac{1}{4} [1 - c - \gamma (n-1) q]^2 & \text{if } q \leq \frac{1-c}{\gamma(n-1)}; \\ 0 & \text{otherwise}. \end{cases} \quad (30)$$

Since $q_M < \frac{1-c}{\gamma(n-1)}$ for all parameter values, $\pi_i^d(q_M) = \frac{(1-c)^2 (\gamma(n-1)+2)^2}{16 (1+\gamma(n-1))}$. Now, the optimal punishment strategy $q_p$ is a solution in $q$ of

$$\pi_i^d(q) - \pi(q) = \pi_i^d(q_M) - \pi(q_M). \quad (31)$$

There are two cases:
**Case 1:** \( \pi_i^d(q) = \frac{1}{4} (1 - c - \gamma(n - 1)) q_j^2 \) if \( q \leq \frac{1-c}{\gamma(n-1)} \).

Solution to (31) is \( q_P = \frac{1}{4} (1 - c) \frac{3\gamma(n-1)+2}{2+\gamma(n-1)} > 0 \) which is admissible if and only if \( q_P \leq \frac{1-c}{\gamma(n-1)} \Leftrightarrow \gamma \in \left[0, \min \left\{ 2 \left(1 + \sqrt{2}\right)/(n-1), 1 \right\} \right] \), where \( \min \left\{ 2 \left(1 + \sqrt{2}\right)/(n-1), 1 \right\} = 1 \Leftrightarrow n < 6 \). The price \( p_i(q_P, q) \geq 0 \) if \( c \geq c_0^C \) with

\[
c_0^C = \frac{\gamma(n-1) - 2}{3\gamma(n-1) + 2}.
\]

This frontier \( c_0^C \) is upward sloping and intersects \( c = 0 \) for \( \gamma = \frac{2}{n-1} \). Note that we need \( \frac{2}{n-1} < 1 \Leftrightarrow n > 3 \) for the frontier to be above zero. Now we compute the threshold value of \( \delta \) to find

\[
\delta_2^C = \frac{1}{16} (2 + \gamma(n-1))^2 < 1.
\]

From (23), the individual rationality constraint is satisfied since \( \pi(q_P) + \frac{\delta_2^C \pi(q_M)}{1-\delta_2^C} > 0 \) if \( \gamma \in \left( \frac{2(3-2\sqrt{3})}{(n-1)}, \frac{2(3+2\sqrt{3})}{(n-1)} \right) \) which is always true since \( \gamma \leq \frac{2\left(1 + \sqrt{2}\right)}{(n-1)} < \frac{2\left(3+2\sqrt{3}\right)}{(n-1)} \) for all \( n \).

**Case 2:** \( \pi_i^d(q) = 0 \) if \( q > \frac{1-c}{\gamma(n-1)} \).

The only admissible solution to (31) is \( q_P = \frac{(1-c)(2+\gamma(n-1)) \sqrt{2+\gamma(n-1)}}{4(1+\gamma(n-1))^2} \) if and only if \( \gamma \in \left[ \frac{2 + \sqrt{3}}{n-1}, 1 \right] \) and \( n \geq 6 \). We find that \( q_P \leq (1-c)/\gamma(n-1) \Leftrightarrow c > c_1^C \) with

\[
c_1^C = \frac{(2 + \gamma(n-1)) \sqrt{1+\gamma(n-1)} - 2(1 + \gamma(n-1))}{2(1 + \gamma(n-1)) + (2 + \gamma(n-1)) \sqrt{1+\gamma(n-1)}}.
\]

This frontier \( c_1^C \) is upward sloping and we check that \( c_1^C = c_0^C \) for \( \gamma = \frac{2 + \sqrt{3}}{n-1} \). Now we compute the threshold value of \( \delta \) to find:

\[
\delta_2^C = \left( \frac{\gamma(n-1)}{2 + \gamma(n-1)} \right)^2 < 1.
\]

From (23), the individual rationality constraint is satisfied since \( \pi(q_P) + \frac{\delta_2^C \pi(q_M)}{1-\delta_2^C} = 0 \).

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7.2 Two-period punishment

We solve the two-period punishment case which occurs when collusion cannot be sustained by using a single-period punishment, that is if \( c \) is below a threshold \( c_K \). We examine in turn the Bertrand and Cournot cases.

7.2.1 Bertrand

During the first period of punishment, firms inflict the most severe punishment, that is \( p_F = 0 \).

We solve in \( p_{P,2} \) and \( \delta \) the system

\[
\begin{align*}
\pi_d^i(p_M) - \pi^i(p_M) &= \delta^2 (\pi^i(p_M) - \pi(p_{P,2})) + \delta (\pi(p_M) - \pi(0)), \\
\pi_d^i(0) - \pi(0) &= \delta^2 (\pi^i(p_M) - \pi(p_{P,2})) + \delta (\pi(p_{P,2}) - \pi(0)),
\end{align*}
\]

with

\[
\pi_d^i(0) = \begin{cases} \frac{[(1-\gamma)(1-c)+\gamma(n-1)(-c)]^2}{4(1+\gamma(n-1))(1+\gamma(n-2))(1-\gamma)} & \text{if } 0 \leq c \leq f(\gamma), \\
0 & \text{if } c \geq f(\gamma),
\end{cases}
\]

\[
\pi_d^i(p_M) = \begin{cases} \frac{(1-c)^2(2+\gamma(n-3))}{16(1+\gamma(n-1))(1+\gamma(n-2))(1-\gamma)} & \text{if } \gamma \leq \frac{n-3+\sqrt{2\gamma^2-1}}{3n-5}, \\
\left(\frac{1-c}{2\gamma}\right)^2(2\gamma - 1) & \text{if } \gamma \geq \frac{n-3+\sqrt{2\gamma^2-1}}{3n-5}.
\end{cases}
\]

and

\[
\pi(p_M) = \frac{(1-c)^2}{4(1+\gamma(n-1))} \quad \text{and} \quad \pi(0) = \frac{-c}{1+\gamma(n-1)}.
\]

We start by recalling a partial result (that appears in Proposition 2) before displaying the thresholds values of \( c \) and \( \delta \).

**Lemma 8** For \( n \geq 2 \), an optimal two-period punishment penal code exists for \( c^B \geq c \geq \ldots \)

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$\mathcal{L}^{B,l=2}(\gamma)$ and $\delta > \mathcal{L}^{B,l=2}$ where:

$$\mathcal{L}_{B,l=2} = \begin{cases} 
0 & \text{if } \frac{2}{n+4} \leq \gamma \leq \tilde{\gamma}_4 \text{ and } n = 2 \\
\mathcal{L}^0_{B,l=2} & \text{if } \tilde{\gamma}_5 \leq \gamma \leq \min \left\{ (n-3 + \sqrt{n^2-1}) / (3n-5), \tilde{\gamma}_6 \right\} \text{ and } n \geq 3 \\
\mathcal{L}^1_{B,l=2} & \text{if } \tilde{\gamma}_6 \leq \gamma \leq (n-3 + \sqrt{n^2-1}) / (3n-5) \text{ and } n > 4 \\
\mathcal{L}^2_{B,l=2} & \text{if } \max \left\{ \tilde{\gamma}_4, (n-3 + \sqrt{n^2-1}) / (3n-5) \right\} \leq \gamma \leq \tilde{\gamma}_3 \\
\mathcal{L}^3_{B,l=2} & \text{if } \max \left\{ \tilde{\gamma}_3, (n-3 + \sqrt{n^2-1}) / (3n-5) \right\} \leq \gamma \leq 1 
\end{cases}$$

and for $n = 2$:

$$\delta_{B,l=2} = \begin{cases} 
\frac{\gamma(c-1)^2}{(c+1)(\gamma-c(3+4(1-\gamma)))} \text{ if } \frac{2}{n+4} \leq \gamma \leq \frac{n-3+\sqrt{n^2-1}}{3n-5} \\
\frac{(\gamma^2+c-1)(-1+\gamma)}{(c-1)^2+c(1-\gamma)+2c(1-c)\gamma(\gamma)} \text{ if } \frac{n-3+\sqrt{n^2-1}}{3n-5} \leq \gamma \leq \tilde{\gamma}_3 \\
\frac{\gamma+c-1}{(2c-1)(1+c)} \text{ if } \tilde{\gamma}_3 \leq \gamma \leq 1 
\end{cases}$$

and for $n \geq 3$:

$$\delta_{B,l=2} = \begin{cases} 
\frac{-(c-1)^2(n-1)}{(c+1)(\gamma+c(n-3)+4(1-c)\gamma(\gamma-6))} \text{ if } \frac{2}{n+4} \leq \gamma \leq \min \left\{ \frac{n-3+\sqrt{n^2-1}}{3n-5}, \tilde{\gamma}_6 \right\} \text{ and } \\
\min\{0, \mathcal{L}^0_{B,l=2}\} \leq c \leq \min\{\mathcal{L}^0_{B}, f(\gamma)\} \\
\frac{\gamma^2(n-1)^2}{(2c+3\gamma(n-1)+2c(1-c)\gamma(\gamma-1))} \text{ if } \frac{n-3+\sqrt{n^2-1}}{3n-5} \leq \gamma \leq 1 \text{ and } \max\{f(\gamma), \mathcal{L}^0_{B,l=2}\} \leq c \leq \mathcal{L}^0_{B} \\
\frac{\gamma n(2\gamma-1)-3\gamma(\gamma-1)-1}{(\gamma-1)^2+c(1-\gamma)(n-1))} \text{ if } \frac{n-3+\sqrt{n^2-1}}{3n-5} \leq \gamma \leq 1 \text{ and } \max\{f(\gamma), \mathcal{L}^0_{B,l=2}\} \leq c \leq \mathcal{L}^0_{B} \\
\frac{(c-1)^2(\gamma-1)(\gamma^2(2n-3)+\gamma(3-n)(1+\gamma(n-2)))}{D \delta} \text{ if } \frac{n-3+\sqrt{n^2-1}}{3n-5} \leq \gamma \leq \tilde{\gamma}_3 \text{ and } \\
\mathcal{L}^1_{B,l=2} \leq c \leq f(\gamma) \text{ and } n = 3, 4 
\end{cases}$$

with $D = 1 + \left( (2(1-2c) + 3c^2) n^2 - 2 (3 - 7c + 5c^2) n + 5 - 12c + 8c^2 \right) \gamma^4 + (-3 (c-1)^2 n^2 + (15c - 13) (c-1) n - 14 + 30c - 16c^2) \gamma^3 + \left( (1-c)^2 n^2 - 9(c-1)^2 n + 14(c-1)^2 \right) \gamma^2 + 2(c-1)^2 (n-
3) \( \gamma + c^2 - 2c \). We verify that \( \delta_{\beta, l=2} \) is continuous over the range of \( \gamma \).

**Proof:** There are two cases:

- **Case 1.** \( \pi^d_i(0) = \frac{(1-\gamma)(1-\gamma) + \gamma(n-1)(-\gamma)}{4(1+\gamma(n-1))(1+\gamma(n-2))(1-\gamma)} \) if \( 0 \leq c \leq f(\gamma) \)

We have to distinguish two subcases according to the optimal deviation to the collusive price.

**Case 1.1.** \( \pi^d_i(p_M) = \frac{\gamma(n-1)(\gamma^2 - (24 - 8n)\gamma + 8) - 4(2 + \gamma(n-3))^2\sqrt{(1-\gamma)(1+\gamma(n-2))}}{(-3\gamma + 13 \gamma - 59\gamma + 39\gamma^2 - 15\gamma + 15)} \) for \( n \geq 3 \), all \( \gamma \). We also obtain

\[
\delta_{\beta, l=2}^0 = \frac{-(c - 1)^2 (n - 1) \gamma}{(c + 1) ((n(3c - 1) - 7c + 5) \gamma - 4(1 - c))}.
\]

**Case 1.2.** \( \pi^d_i(p_M) = \left( \frac{1-c}{2\gamma} \right)^2 (2\gamma - 1) \) if \( \gamma \geq \frac{n-3+\sqrt{n^2-4}}{3n-5} \)

System (32) has only one admissible solution \( p_P \). Now for \( n = 2 \), \( p_P \) has four roots in \( c \). Two of them are negative and two are positive, we denote by \( c' \) and \( c'' \), with \( 0 < c' < c'' \). We show that \( c' = 0 \) for \( \gamma = \gamma_4 \). Since \( p_P < 0 \iff \gamma > \gamma_4 \) for \( c = 0 \), we conclude that \( p_P > 0 \iff c > \max \{c',0\} \) and we define \( c_{\beta, l=2}^1 \equiv c' \). For \( n = 3,4 \), since we have to consider only \( \gamma > \gamma_4 \), we know that \( p_P < 0 \) for \( c = 0 \). We also find that \( p_P > 0 \) for \( c = f(\gamma) \) and for \( \gamma \geq \frac{n-3+\sqrt{n^2-4}}{3n-5} \). Eventually, for \( n \geq 5 \), (32) has no solution in \( p_P \) since \( c_{\beta, l=2}^1 > f(\gamma) \) for all \( \gamma \geq \frac{n-3+\sqrt{n^2-4}}{3n-5} \). We also obtain

\[
\delta_{\beta, l=2} = \frac{(c - 1)^2 (\gamma - 1) \left( (2n - 3) \gamma^2 + (3 - n) \gamma - 1 \right) (1 + \gamma(n - 2))}{D},
\]
where $D$ is as expressed above.

- **Case 2.** $\pi_i^d(0) = 0$ if \( \frac{(1-\gamma)}{(1+\gamma(n-2))} \equiv f(\gamma) \leq c \leq \bar{c}_B \)

Again we distinguish two subcases.

**Case 2.1.**

$$\pi_i^d(p_M) = \frac{(1-c)^2[2+\gamma(n-3)]^2}{16(1+\gamma(n-1))(1+\gamma(n-2))(1-\gamma)}$$

if $\gamma \leq \frac{n-3+\sqrt{n^2-1}}{3n-6}$

Remark that this subcase cannot occur if $n = 2, 3$. Moreover, for $n = 4$, we have $\bar{c}_{B,l=2}^0 < f(\gamma)$ and collusion can always be sustained with a two-period punishment. For $n > 4$, only one solution to (32) is below the collusive price and is positive if $c > \bar{c}_B^0$ with

$$\bar{c}_B^0 \equiv \frac{A-4(2+\gamma(n-3))\sqrt{2(1-\gamma)(1+\gamma(n-2))(5-4n+n^2)\gamma^2+(2n-6)\gamma+2}}{(2+\gamma(n-3))^2}$$

where $A = (161 - 212n + 102n^2 - 20n^3 + n^4) \gamma^4 + 16(n - 3)^3 \gamma^3 + 48(n - 3)^2 \gamma^2 + 64(n - 3)\gamma + 32$.

Let $\bar{\gamma}_0$ be defined implicitly by $\bar{c}_{B,l=2}^0 = \bar{c}_{B,l=2}^0 = f(\bar{\gamma}_0)$. We also obtain

$$\delta_{B,l=2}^2 = \frac{\gamma^2(n-1)^2}{(2+\gamma(n-3))^2}$$

**Case 2.2.**

$$\pi_i^d(p_M) = \left(\frac{1-c}{2\gamma}\right)^2(2\gamma - 1)$$

if $\gamma \geq \frac{n-3+\sqrt{n^2-1}}{3n-6}$

Only one solution to (32) is below the collusive price and is positive if $c > \bar{c}_{B,l=2}^2$ with

$$\bar{c}_{B,l=2}^2 \equiv \frac{B-2\gamma(2\gamma-1)(\gamma n-\gamma+1)\sqrt{(4n-5)\gamma^2+(2n+6)\gamma-2}}{(2n-3)\gamma^2+(2n+3)\gamma-1}$$

where $B = (-1 - 4n + 4n^2) \gamma^4 - 2(2n - 1)(n - 3)\gamma^3 + (11 - 10n + n^2)\gamma^2 - (6 - 2n)\gamma + 1$. We also obtain

$$\delta^2_{B,l=2} = \frac{\gamma n(2\gamma - 1) - 3\gamma\gamma - 1}{(2\gamma - 1)(1+\gamma(n-1))}$$

We also show that $\bar{c}_B^2 > \bar{c}_{B,l=2}^2(\gamma)$ on the range $\left(\max\left\{\bar{\gamma}_2, \frac{n-3+\sqrt{n^2-1}}{3n-6}\right\}, 1\right]$ for all $n \geq 2$. 37
7.2.2 Cournot

We solve in $q_{P,2}$ and $\delta$ the following system

\[
\pi_i^d(q_{M}) - \pi(q_{M}) = \delta^2 (\pi(q_{M}) - \pi(q_{P,2})) + \delta \left( \pi(q_{M}) - \pi(q_{P}) \right), \\
\pi_i^d(q_{P}) - \pi(q_{P}) = \delta^2 (\pi(q_{M}) - \pi(q_{P,2})) + \delta \left( \pi(q_{P,2}) - \pi(q_{P}) \right),
\]

(33)

where $q_{P}$ is the first-period punishment quantity which is such that $p_i(q_p, q_{P}) = 0$ which yields to $\pi(q_{P}) = -c q_{P}$. We have

\[
\pi_i^d(q_{P}) = \begin{cases} 
\frac{1}{4} \left( 1 - c - \gamma (n-1) q_{P} \right)^2 & \text{if } q_{P} \leq \frac{1-c}{\gamma (n-1)} \Leftrightarrow c < \frac{1}{1+\gamma (n-1)}, \\
0 & \text{otherwise}.
\end{cases}
\]

The two-period case requires that $c < c^C(\gamma)$. For $n = 4, 5, \pi_i^d(q_{P}) = 0$ is impossible since $\frac{1}{1+\gamma (n-1)} > c^0$. For $n \geq 6$, since $\frac{1}{1+\gamma (n-1)} < c^1$ for $\gamma = 1$, there exists a $\hat{\gamma}$ such that $\frac{1}{1+\gamma (n-1)} < c^C \Rightarrow \gamma > \hat{\gamma}$.

There are two cases:

- **Case 1.** $\pi_i^d(q_{P}) = \frac{1}{4} \left( 1 - c - \gamma (n-1) q_{P} \right)^2$ if $0 < c < \min \left\{ \frac{1}{1+\gamma (n-1)}, c^C \right\}$

  Only one solution to (33) is above the collusive quantity and is below $q_{P}$ if $c > c_{1,2}^0$ with

  \[
  c_{1,2}^0 = \frac{\gamma (n-1) ((3(1-n)^2)/\gamma^2 + (8(n-1))\gamma + 8) - 4(2+\gamma(n-1))^2 \sqrt{1+\gamma (n-1)}}{2(n-1)^2 \gamma^3 + 2(1-\gamma(n-1)\gamma + 4(1-n)\gamma + 46}},
  \]

  which is below $c^C$ for all $\gamma$ and $n > 4$. Moreover $c_{1,2}^0$ intersects $\frac{1}{1+\gamma (n-1)}$ for $n > 9$. We find also

  \[
  c_{2,1}^1 = \frac{\gamma (1-c)^2 (n-1)}{(1+c)((n(1+3c) - 1 + 3c)\gamma + 4 - 4c)},
  \]

- **Case 2.** $\pi_i^d(q_{P}) = 0$ if $\frac{1}{1+\gamma (n-1)} \leq c \leq c^C$ which is possible for $n \geq 6$ only.
Here again, there is only one solution to (33) above the collusive quantity. This second-period punishment is below \( q_1 \) if \( c > q^1_{C,l=2} \) with

\[
q^1_{C,l=2} = \frac{E - 4(2 + \gamma(n-1))^2 \sqrt{2(1+\gamma(n-1))(1-n)^2 \gamma^2 + 2(n-1)\gamma + 2}}{\gamma^4(n-1)^4},
\]

where \( E = (1 - n)^4 \gamma^4 - 16 (1 - n)^3 \gamma^3 + 48 (1 - n)^2 \gamma^2 - 64(1 - n)\gamma + 32 \). We also obtain

\[
\delta^2_{C,l=2} = \frac{\gamma^2 (n - 1)^2}{(\gamma(n - 1) + 2)^2}.
\]

We check that \( q^1_{C,l=2} \leq q_1 \) for all \( \gamma \) and all \( n > 4 \), and that \( q^1_{C,l=2} > \frac{1}{1+\gamma(n-1)} \) only for \( n \geq 9 \).
8 References


