The Rawlsian Principle and Secession-Proofness in Large Heterogeneous Societies

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Abstract

This paper examines a model of multi-jurisdictional formation considered by Alesina and Spolaore (1997) and Le Breton and Weber (2003), where the distribution of individuals is given by Lebesgue measure over the given (finite or infinite) interval. Every jurisdiction chooses a location of a public good and shares the cost of production among its residents. Each individual is responsible for both *transportation cost* to the location of the public good and her contribution towards the production of the public good. We consider a notion of a *secession-proof allocation* where no group of individuals can make all its members better off by choosing a location of the public good and a cost-sharing mechanism among its members. We show that if the support of individuals' distribution is the real line $\Re$, the only secession-proof allocation is *Rawlsian* that equalizes the utilities of *all* individuals in the society. In the case of bounded support we show that there is a degree of approximation to the Rawlsian solution that reconciles the secession-proofness and the weakened Rawlsian principle.

Keywords: Optimal jurisdictions, Secession-proofness, Rawlsian allocations, Efficiency.

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1 Introduction

In this paper we consider a large heterogeneous society whose members make selections from the given policy space. The selection may consist of a unique policy in which case the society remains undivided. If more than one policy is chosen, it creates a partition of the entire society into smaller subsocieties, called hereafter *jurisdictions*, each assigned to one of the chosen policies. The basic reason for a possible partition of the society into smaller groups is the conflict between increasing returns to scale on one hand and heterogeneity of agents' preferences on the other. Indeed, in a large group the per capita contributions towards financing of a public project could be lower than in a smaller group. However, the policy choices made by a large group could be quite distant from the ideal choices of some of its members. Thus, the benefits of groups size are not unlimited and the increasing returns could be outweighed by costs of heterogeneity of agents' characteristics and tastes. It is possible, therefore, that a decentralized organization might be superior to the grand coalition that embraces the entire society.

The policy choices of the society and the composition of the formed jurisdictions do not provide, however, the complete description of the problem we consider. Since the policies are costly in our framework, one needs a mechanism of sharing the these costs within each jurisdiction. Thus, when a partition of individuals has been formed, a set of appropriate policies has been chosen, the allocation of the cost among the individuals should be determined. When all three elements of the collective choice problem, namely projects' choice, assignment to the projects, and cost allocation are in place, one can turn to a stability test of the proposed arrangement. There could be a single individual or a group of individuals (not necessarily from the same jurisdiction) who object the proposed arrangement. Their threat of "secession" is credible if there is a policy and the allocation of its costs among the members of the deviating group that would make each of them better off with respect to the original arrangement. The set of cost allocation allocations that do not allow credible secession threats would be called "secession-proof" and the analysis of the structure of the set of secession-proof cost allocations is the main focus of this paper.

We examine a society with the infinite population of individuals, who make policy choices from the unidimensional policy space. The cost of each policy is given by the positive parameter g. The preferences of each individual are single-peaked and symmetric with respect to her peak, which represents her favorite policy choice. The heterogeneity of individuals' preferences is described by the distribution of the peaks over the real line. Alesina and Spolaore (1997) study this problem in the case where the distribution λ is the Lebesgue measure over the unit interval. In their framework the only cost allocation available for each jurisdiction is the *equal-share*, namely, when all individuals within the same jurisdiction make identical contribution towards the policy costs. Le Breton and Weber (2003) examine a large class of absolutely continuous distributions and demonstrate the existence of secession-proof cost allocations for high values of policy costs g that make it optimal to form a single jurisdiction. They establish a principle of partial equalization asserting that, in general, secession-proofness entails some, but not full, compensation of citizens disadvantaged by the assigned public policy. In particular, neither equal-share allocation (no equalization) nor Rawlsian allocation (full equalization) are not, in general, secession-proof. Haimanko, Le Breton and Weber (2004) consider an arbitrary probability measure with bounded support and establish the existence of secession-proof cost allocations regardless of the value of policy costs g.

In this paper we consider distributions of individuals' ideal points by the Lebesgue measure on large (bounded or unbounded) intervals. Obviously, for a fixed value of g and large populations, it is efficient to partition the society into many jurisdictions. We demonstrate the crucial role played under these circumstances by the Rawlsian principle that maximizes the lowest utility level among all society members. It entails the full compensation to every individual for being assigned to the policy different from her favorable one. Our main result is that in the case where the population distribution is represented by the Lebesgue measure on the entire real line, the *unique* secession-proof allocation is Rawlsian. This is a surprising and sharp implication of secession-proofness that shows that the only way to achieving stability in large societies leads to the full equalization of utilities of all its members.

We show that in the case of finite support, the Rawlsian and secession-proofness requirements are consistent only for a non generic set of cost values of public projects. Moreover, even if we relax the fairness requirements imposed by the Rawlsian principle, the weak Rawlsian principle and secession-proofness are still, in general, incompatible. We then introduce the notion of approximate Rawlsian allocation, where the number of individuals receiving a fixed level of subsidy relative to the Rawlsian allocation, is sufficiently small and show that in this sense the approximate Rawlsian principle can be consistent with secession-proofness. We also examine the basic differences between the multi-jurisdictional and one-jurisdictional set-ups. In particular, we show that the result of Le Breton and Weber (2003) asserting the existence of a family of secession-proof cost allocations that satisfy the principle of partial equalization, does not, in general, extend from the one-jurisdictional to the multi-jurisdictional framework.

The paper is organized as follows. In the next section we present the model and analyze the efficient partitions of the entire society. in Section 3 we introduce the notion of a cost allocation, the principle of partial equalization and define symmetric, neutral and Rawlsian allocations. Section 4, where the notion of secession-proofness is introduced, contains our main result that in the unbounded case the Rawlsian allocation is the only one that meets the test of secession-proofness. We also show that in the finite framework the Rawlsian principle and secession-proofness are, in general, incompatible. In Section 5 we show that even a relaxation of the Rawlsian principle does not bring the reconciliation with secession-proofness. We demonstrate, however, that in a large finite society the secession-proofness yields cost allocations that satisfy the approximate Rawlsian principle. The proofs of all results are relegated to the Appendix.

2 The Model and Efficient Partitions

We consider a group of individuals with the preferences over the set of feasible policies represented by the real line $\Re$. Every individual has an ideal point in $\Re$ and her preferences are single-peaked and symmetric with respect to the ideal point, which allows to identify an individual with her ideal point. The distribution of all ideal points (and, thus, of individuals' themselves) is assumed to be given by the Lebesgue measure λ over the interval $R_\theta = [-\theta, \theta]$ where θ is either a positive real number or infinity. (In the latter case $\Re_\infty = \Re$.)

The group choice consists of three elements:

partition of R_θ into measurable subsets with positive measure, called *jurisdictions*;

one policy in R_θ for each jurisdiction;

cost-sharing scheme among individuals, where the sum of individual contributions covers the total cost of public projects.

For illustrative purposes, we may interpret this problem as follows: an urban population located on the all real line is faced with the choice of selecting a number and location of public projects (say, libraries), an assignment of each individual to one of the projects, and, finally, an allocation of projects costs to society members. We assume that the cost of every project is a positive number g.[1] We therefore adopt a spatial interpretation of R_θ, where a policy choice is represented by the location of the public project. That allows us to introduce the disutility or transportation cost incurred by individuals. An individual located in t (for simplicity, labelled t henceforth) faces the transportation cost $d(t, p)$ between t and the location p of the public project she is assigned to. We assume that transportation cost is simply represented by the distance,[2] i.e., $d(t, p) = |t - p|$.

[1] This assumption can be easily weakened. We can, for example, assume that the project costs in each jurisdiction depend on its size and be of the form $g(S) = g + \alpha\lambda(S)$, where g and α are positive constants.

[2] That also can be easily generalized to cover the case where the transportation cost is a continuous and convex function of the distance.

For any bounded measurable subset $S \subset \Re$, denote by

$$D(S) = \inf_{p \in I} \int_S d(t, p) dt,$$

the minimal aggregate transportation cost of the individuals in S.. It is useful to make the following observation:

Remark 2.1: It is easy to see that every jurisdiction S would minimize its total transportation cost by choosing the location of the public project at the ideal point of its "median voter". That is, the median of S, denoted $m(S)$, is the cost-minimizing location for S. Thus, if the group S is an interval of length s, its total transportation cost $D(S)$ is given by

$$\int_S \mid t - m(S) \mid dt = \frac{s^2}{4}. \tag{1}$$

At this point we do not examine the issue of a cost allocation and do not define how the monetary contribution of each individual t towards the cost of the public projects is determined. The only condition is the *balanced budget* that requires the society to finance all chosen public projects.

Since the total cost incurred by jurisdiction S consists of policy and transportation components, the average cost of an individual t in S

$$\frac{g + D(S)}{\lambda(S)}.$$

The following remark, which plays an important role in our analysis, determines the optimal size of a jurisdiction that minimizes the per capita cost of its members:

Remark 2.2: If a jurisdiction is represented by an interval of length s, by (1), the per capita total cost of its members is

$$f(s) \equiv \frac{s}{4} + \frac{g}{s}.$$

For a given value of $g > 0$, the function $\frac{s}{4} + \frac{g}{s}$ is convex and obtains its minimum at $s^* = 2\sqrt{g}$. Therefore, the value of s^* can be viewed as an *optimal size* of a jurisdiction, and an interval of length s^* as an *optimal jurisdiction.*

For every measurable subset S of R_θ denote by $\mathcal{P}_\mathcal{S}$ the set of all finite partitions of the set S into bounded measurable subsets with positive measure. If $S = R_\theta$ we will simply write $\mathcal{P}$. Let S be a measurable subset of R_θ and $P \in \mathcal{P}$ be a partition of R_θ. Denote $S'_k = S_k \bigcup S$ for all $S_k \in P$. Then the truncated partition P^S consists of all non-negligible intersections of the elements of P with the set S:

$$P^S = \{S'_k | \lambda(S'_k) > 0\}.$$

Definition 2.3: Two partitions $P, P' \in \mathcal{P}$ have a *bounded difference* if there exists a bounded interval $I \subset Re_\theta$ such that two truncated partitions $P_{R_\theta \setminus I}$ and $P'_{R_\theta \setminus I}$ coincide. Obviously, if θ is finite, every two partitions in $\mathcal{P}$ have a bounded difference.

Definition 2.4: The partition $P' \in \mathcal{P}$ is *efficient* if for every partition $P' \in \mathcal{P}$ that has a bounded difference with P, we have

$$\sum_{S \in P \setminus P'} [D(S + g] \leq \sum_{S' \in P' \setminus P} [D(S') + g].$$

That is, the total cost over all elements in P that do not belong to the partition P' is smaller than over those that belong to the partition P' but not the partition P. Obviously if θ is finite, an efficient partition simply minimizes the aggregate sum of transportation and policy costs over all partitions in $\mathcal{P}$.

The main result of this section guarantees that, in general, there is a unique efficient jurisdictional structure which consists of jurisdictions of equal size.

Proposition 2.5: For every $0 < \theta \leq \infty$, every efficient partition consists of intervals of equal size. Moreover,

(i) if $\theta = \infty$, every jurisdiction in an efficient partition is an interval of the optimal size s^*.

(ii) Let $\theta < \infty$. Denote by $\lfloor \frac{2\theta}{s^*} \rfloor$ the largest integer (called integer part) that does not exceed $\frac{2\theta}{s^*}$. Then the optimal number of jurisdictions $N(R_\theta)$ is given by[3]

$$N(R_\theta) = \begin{cases} \lfloor \frac{2\theta}{s^*} \rfloor & if \quad \frac{2\theta}{s^*} \leq \sqrt{\lfloor \frac{2\theta}{s^*} \rfloor (\lfloor \frac{2\theta}{s^*} \rfloor + 1)} \\ \lfloor \frac{2\theta}{s^*} \rfloor + 1 & if \quad \frac{2\theta}{s^*} \geq \sqrt{\lfloor \frac{2\theta}{s^*} \rfloor (\lfloor \frac{2\theta}{s^*} \rfloor + 1)}. \end{cases}$$

The size of each jurisdiction in an efficient partition is

$$s(R_\theta) = \frac{2\theta}{N(R_\theta)}.$$

Remark 2.6: It would be useful to expand the notion of optimal number of jurisdictions to an arbitrary interval of the length L and to consider an efficient partition of such an interval into $N(L)$ equal size intervals. As in Proposition 2.5 we have:

$$N(L) = \begin{cases} \lfloor \frac{L}{s^*} \rfloor & if \quad \frac{L}{s^*} \leq \sqrt{\lfloor \frac{L}{s^*} \rfloor (\lfloor \frac{L}{s^*} \rfloor + 1)} \\ \lfloor \frac{L}{s^*} \rfloor + 1 & if \quad \frac{L}{s^*} \geq \sqrt{\lfloor \frac{L}{s^*} \rfloor (\lfloor \frac{L}{s^*} \rfloor + 1)}. \end{cases}$$

The optimal number of jurisdictions $N(R_\theta)$ is determined through interplay of two opposite forces. A creation of a new jurisdiction reduces the aggregate transportation cost but adds an additional cost of public project g. Once we determine that all jurisdictions are intervals of the same length, we can formulate our problem in terms of minimization of the total aggregate cost with respect to the length of a typical jurisdiction s. In general, the optimal value s^* is such that $\frac{2\theta}{s^*}$ is not an integer. We solve this problem by taking advantage of convexity of the total aggregate cost as a function of s. After deriving the unconstrained minimum s^* of the total aggregate cost with respect to s, it just remains to consider the nearest value(s) of s on the right and on the left of s^* such that $\frac{2\theta}{s}$ is an integer. An evaluation of the total aggregate cost in these two points leads to the optimal number of jurisdictions. In the (non generic) case where these two values are equal, there are two optimal jurisdictional structures with a different number of jurisdictions.

The "connectedness" of all optimal jurisdictions is a general property that can be derived under more general assumptions on the individual transportation cost functions and on the

[3]Obviously, if $\lfloor \frac{2\theta}{s^*} \rfloor = 0$, then $N(R_\theta) = 1$.

distribution of individuals over R_θ. However, the fact that these intervals are of equal length, is driven by our assumption of the uniform distribution of the population over R_θ. In the next proposition we evaluate the optimal length of jurisdictions $s(L)$ for an interval of size L as well as the average contribution of individuals in an optimal partition:

Proposition 2.7: For all positive values L we have:

$$(i)\ 2\sqrt{g} - \frac{4g}{L+2\sqrt{g}} \leq s(L) \leq 2\sqrt{g} + \frac{4g}{L-2\sqrt{g}}.$$

$$(ii)\ f(s(L)) \leq f(2\sqrt{g}) + \frac{27\sqrt{g^3}}{L^2}.$$

3 Cost Allocations

Let us introduce the notion of a *cost allocation* that determines the monetary contribution of each individual t towards the cost of public projects in group S.

Definition 3.1: A measurable function x, defined on the bounded subset S is called a S-cost allocation[4] if it satisfies the budget constraint, i.e., the total contribution of all members of S, $x(S)$, is equal to the cost of the public project:

$$x(S) \equiv \int_S x(t)dt = g.$$

We allow for lump sum transfers and do not restrict the mechanism for reallocation of benefits within each potential jurisdiction S.

Definition 3.2: A measurable function x defined on Re_θ is called a P-*cost allocation* if it satisfies the budget constraint[5]:

$$\sum_{S \in P} (x(S) - g) = 0. \tag{2}$$

[4]We use the term cost allocation without qualification when $S = R_\theta$ represents the entire population.

[5]Note that in the case of $\theta = \infty$, the summation in (2) is expanded over a countable number of jurisdictions.

The results of the previous section show that efficiency leads to a well-defined jurisdictional structure, which generates a total aggregate cost to be shared among individuals. In this section, we introduce the sets of normative principles that will put some constraints on the choice of cost allocations. These principles formulate various fairness requirements that may be considered in the current cost sharing issue; a special attention will be paid to the Rawlsian principle and its various versions.

We now define *symmetry* and *the principle of partial equalization* of cost allocations introduced in Le Breton and Weber (2003), as well as the concept of *Rawlsian allocation*. Let S be an arbitrary jurisdiction and x a S-cost allocation.

Definition 3.3 - Symmetry: x is symmetric if $x(t) = x(s)$ whenever $d(t, m(S)) = d(s, m(S))$.

Definition 3.4 - Partial equalization: x satisfies the principle of partial equalization if

$$0 \leq x(t) - x(s) \leq t - s \text{ for all } t, s \in S \text{ such that } s < t \leq m(S)$$

and

$$0 \leq x(t) - x(s) \leq s - t \text{ for all } t, s \in S \text{ such that } s > t \geq m(S).$$

Since the project supported by group S is located at its median $m(S)$, the symmetry requires that individuals who are equidistant from the public project should make an equal contribution. The principle of partial equalization suggests some form of compensation across members of the same jurisdiction. Precisely, those closer to the project location contribute more than those who are further away. But these contribution differentials cannot exceed the transportation cost differentials and ultimately those closer to the project location would still have a (weakly) lower aggregate cost.

If the principle of partial equalization is replaced by full equalization, the appropriate cost allocation would equalize the total of transportation costs and contribution towards the cost of public project for all members of S. This allocation is called *Rawlsian* as it minimizes the highest total cost burden among all individuals in S:

Definition 3.5 - Full Equalization (Rawlsian allocation): An S-cost allocation x is called Rawlsian if it yields the *full equalization*, i.e.,

$$x(t) - x(s) = t - s \text{ for all } t, s \in S \text{ such that } s < t \leq m(S) \qquad \text{and}$$
$$x(t) - x(s) = s - t \text{ for all } t, s \in S \text{ such that } s > t \geq m(S).$$

The Rawlsian allocation will be denoted x_R^S.

Note that the efficiency requires the public project in S to be located at its median, $m(S)$. Thus, the total individual contribution of citizen t in S is given by the term $d(t, m(S)) + x(t)$. Since, moreover, the aggregate transportation and policy costs in S combine to $D(S) + g$, it follows that

$$x_R^S(t) = \frac{g + D(S)}{\lambda(S)} - d(t, m(S)).$$

Note that if $S \subset R_\theta$ is a connected interval $[a, b]$, the allocation x

is symmetric if $x(t) = x(b + a - t)$ for all $t \in S$,

satisfies the principle of partial equalization if two conditions are satisfied

$$0 \leq x(t) - x(s) \leq t - s \text{ for all pairs} s, t \text{ such that } a \leq s \leq t \leq \frac{a+b}{2}$$
$$0 \leq x(t) - x(s) \leq s - t \text{ for all pairs} s, t \text{ such that } b \geq s \geq t \geq \frac{a+b}{2}.$$

is Rawlsian if

$$x(t) = x_R^S(t) = \begin{cases} t + \frac{g}{b-a} - \frac{b-a}{4} & if \quad t \in \left[a, \frac{a+b}{2}\right] \\ -t + \frac{g}{b-a} + \frac{3(b-a)}{4} & if \quad t \in \left[\frac{a+b}{2}, b\right] \end{cases}.$$

It is natural to extend the notions of symmetry, partial equalization and Rawlsian allocation by applying these principles separately to each jurisdiction. Consider a pair (P, x), where P is a partition of R_θ and x is a P-cost allocation. x is symmetric (respectively, Rawlsian or satisfies the principle of partial equalization) if its truncation to S is symmetric (respectively, Rawlsian or satisfies the principle of partial equalization) for every $S \in P$. We shall denote by x_R^P the Rawlsian allocation associated with the partition P. It should be noted that these principles do not impose any link between cost allocations in different jurisdictions and do not restrict the sharing of the total public projects costs Ng across N jurisdictions. Since x is a P-cost allocation, by (2), we have

$$\sum_{S \in P} x(S) = gN,$$

and if $x(S) \neq g$ for some S, then a cross subsidization takes place. If $x(S) < g$, then the jurisdiction S receives a subsidy to cover the cost of its public project, and if $x(S) > g$, the jurisdiction S is a donor that subsidizes other jurisdictions.

To introduce the last principle of this section, we limit our attention to the case where, as for efficient jurisdictional structures, the partition P consists of intervals of equal length. Since all jurisdictions are of the same size, from a normative point of view it makes sense to rule out a subsidization across jurisdictions and to require that cost allocations should be the same in all jurisdictions. Indeed, an external impartial observer contemplating that problem could well argue that if we promote the use of x in a given jurisdiction, then x should be promoted in all other jurisdictions identical to that under consideration.

Definition 3.6: Let P be a partition of R_θ into n jurisdictions. A P-cost allocation x is *neutral* if the equality

$$x(t + \frac{2n\theta}{N}) = x(t)$$

holds for all $t \in R_\theta$ such that $t + \frac{2n\theta}{N} \in R_\theta$.

Symmetry, partial equalization and neutrality are general normative principles which are useful to guide a selection of the cost allocation. The set of cost allocations satisfying the three properties is nonempty, as it always contains the Rawlsian allocation. Given that the Rawlsian allocation meets a series of desirable fairness criteria, the natural question in our analysis of stability is whether the Rawlsian principle is compatible with the requirement of secession-proofness. The next section provides a negative answer to this question. We show, moreover, that even if the set of acceptable cost allocations is expanded to include all symmetric neutral cost allocations satisfying the principle of partial equalization, we still fall short of the reconciliation of our fairness requirements with secession-proofness.

4 Secession-Proofness and the Rawlsian Principle

We now turn to the examination of stable partitions that are immune against a threat of deviation or secession by a group of individuals. We study a collective choice problem whose solution must be accepted by all individuals and all coalitions of individuals. That is, we require a voluntary participation of the individuals in potentially seceding groups. This necessitates an examination of a mechanism that allocates benefits among all individuals in such a way that no coalition can generate a higher payoff to all its members. However, if a coalition S can make its members better off relative to the current arrangement, we will say that S is *prone to secession*. Formally,

Definition 4.1: Consider a pair (P, x), where P is a partition in $\mathcal{P}$ and x is a cost allocation.

The jurisdiction S is prone to secession (given (P, x)) if

$$\int_S (d(t, m(S)) + x(t))dt > D(S) + g.$$

If no jurisdiction is prone to secession, then the pair (P, x) is called *secession-proof*; if there is no ambiguity we drop the first argument of the pair and simply refer to secession-proof cost allocation.

The concept of secession-proofness introduced here is closely related to the notion of the core of a game with coalition structures (Aumann and Drèze (1974)), whose set of players is given by R_θ, and the set of feasible outcomes of a coalition S is determined by all S-cost allocations. Since we do not use the game-theoretical machinery here, we chose to formulate our results without relying on it.

We would like to point out two immediate but nevertheless useful implications of secession-proofness:

Remark 4.2: (i) In every jurisdiction the budget is balanced. Indeed, no jurisdiction can be a net donor. Otherwise it would secede and save the amount of the net transfer. It cannot be a net recipient either as this would imply from (2) that another jurisdiction is a net donor.

(ii) In every jurisdiction, all individuals make a nonnegative contribution to the financing of the public project; otherwise the coalition of other individuals within the jurisdiction who do make a positive contribution would be better off by breaking away from the jurisdiction.

As we argued above, the condition of secession-proofness is quite demanding as it requires that no coalition should be able to deviate by improving welfare of all its members. In this section, we explore to which extent this principle can coexist with the normative principles introduced in the previous section.

In the single-jurisdiction case (when the neutrality property is obviously vacuous), Le Breton and Weber (2003) show that secession-proofness is compatible with symmetry and the principle of partial equalization for a large class of population distributions containing the Lebesgue measure. In the multi-jurisdictional set-up, the situation becomes more complicated. Indeed, secession-proofness requires, in particular, that there is no threat of secession arising from a subset of existing jurisdiction. Even if the neutrality requirement is imposed, we have to face all additional constraints of preventing secession by groups containing individuals from different jurisdictions.

First we examine the limit case where the population is unbounded with $R_\theta = \Re$. Proposition 2.5 implies that an efficient partition must consist of optimal jurisdictions, i.e., intervals of length s^*. We denote such a partition[6] by P^*. We proceed by showing that the Rawlsian allocation associated with the partition, denoted by simply $x^*_R(t)$ instead of $x^{P^*}_R(t)$, is secession-proof. Moreover, we demonstrate that, in cost terms, the Rawlsian principle yields

[6] A partition of the real line into equal intervals of a given size is not unique. However, if any of endpoints of an interval in the partition is fixed, the uniqueness would be restored.

the unique cost allocation. In the infinite case there are no endpoints and, in some sense, it is impossible to discriminate ex ante among identical individuals.

Proposition 4.3 - Unbounded case: Let $R_\theta = \Re$. A pair (P, x), where $P \in \mathcal{P}$ is a partition and x is a P-cost allocation, is secession-proof if and only $P = P^*$ and $x = x_R^*(t)$ is the Rawlsian allocation.

The proposition ensures not only the compatibility of secession-proofness with the Rawlsian principle, but, even more importantly, it identifies the Rawlsian allocation as the unique cost sharing mechanism meeting the stability requirement. This result demonstrates that in absence of full equalization that compensates all individuals for their disadvantaged locations, there always be a group of individuals that would reject an assigned cost allocation.

Let us now examine the bounded case, where the population represented by the finite interval R_θ. We will show that the assertions of Proposition 4.3 do not hold in this case and it is, in general, impossible to reconcile the Rawlsian principle with the stability requirement of secession-proofness. Denote by $P(\theta)$ an efficient partition of R_θ into $N(\theta)$ jurisdictions of equal size and by x_R^θ the Rawlsian allocation associated with partition P_θ. Then

Proposition 4.4 - Bounded case: Let $\sqrt{g} \leq \theta < \infty$. The pair (P_θ, x_R^θ) is secession-proof if and only if the ratio $\frac{\theta}{\sqrt{g}}$ is an integer.[7] That is, the Rawlsian allocation is secession-proof only for a non generic subset of values of θ.

Given the society R_θ and the cost of public project g, we denote by $SP(\theta, g)$ the set of secession-proof cost allocations. The general existence result of Haimanko, Le Breton and Weber (2004) guarantees that this set is nonempty. Furthermore, denote by $NPES(\theta, g)$ the set of neutral and symmetric cost allocations that satisfy the principle of partial equalization. This set is nonempty as well as it contains the Rawlsian allocation. One may wonder whether

[7]This is reminiscent of the integer problem examined first by Pauly (1967), (1970) in the context of the theory of clubs.

the compatibility of symmetry, partial equalization and neutrality with secession-proofness, shown in Le Breton and Weber (2003) in the single jurisdiction case, can be extended to the multi-jurisdictional framework. The following result shows that, in general, it is not the case:

Proposition 4.5 - Bounded Case: If $N(R_\theta) > 1$, $SP(\theta, g) \cap NPES(\theta, g) \neq \emptyset$ if and only if θ is a multiple of $\sqrt{g}$.

In the next section we consider modified versions of the Rawlsian principle and examine their compatibility with the stability requirement of secession-proofness.

5 Modifications of the Rawlsian Principle and Secession-Proofness

In the previous section we have shown that in the bounded case it is in general impossible to reconcile the standard Rawlsian principle with secession-proofness principle. When the requirement of secession-proofness is maintained it is therefore important to investigate possible modifications of the Rawlsian principle.

Remark 2.2 shows that the lowest possible aggregate cost for the bounded society R_θ is $2g$, which can be obtained only if the society is partitioned into jurisdictions of optimal size s^*. Since this would imply that 2θ is a multiple of $2\sqrt{g}$, these circumstances are rather exceptional and, in general, the society will have to incur a per capita cost larger than $\sqrt{g}$. We would then require a "fair" cost allocation should not assign any individual $t \in R_\theta$ a cost contribution less than $\sqrt{g}- \mid t - m(S(t)) \mid$, which would imply that the total contribution, including the transportation cost, falls below $\sqrt{g}$. When this principle is applied to groups rather than individuals, it amounts to the following inequality:

$$\int_S (x(t) + d(t, m(S(t))))dt \geq \lambda((S)\sqrt{g} \quad (3)$$

for all $S \subseteq R_\theta$. In fact, we will consider a weaker version of that condition and require it to hold only for intervals of optimal size s^*:

Definition 5.1: A cost allocation x satisfies the *weak Rawlsian principle* or is a *weakly Rawlsian* if the inequalities (3) are satisfied for all groups $S \subseteq R_\theta$ such that $S = [t, t+s^*]$. Obviously every Rawlsian allocation is weakly Rawlsian.

The logic behind this principle is again that the reservation payoff $\sqrt{g}$ is the lowest average best payoff attainable for a group. It can be obtained only when the economies of scale generated by a large jurisdiction are perfectly balanced by the heterogeneity costs incurred by its size. Unless one wishes to provide an individual or a group with specific favors or privileges, it is hard to offer a priori reason, immune to an ethical or normative appraisal, to assign an individual or a group a cost contribution below the level given in the right hand side of inequalities (3).

We show, however, the weak Rawlsian principle is, in general, incompatible with secession-proofness in the multi-jurisdictional framework (or when $\sqrt{g} < \theta$) when we consider continuous cost allocations:

Proposition 5.2: Let $\sqrt{g} < \theta$. Then the set of continuous, secession-proof and weakly Rawlsian allocations on R_θ is nonempty if and only if θ is a multiple of $\sqrt{g}$.

Let $\theta \geq \sqrt{g}$. Every secession-proof allocation x satisfies

$$\int_z^{z+2\sqrt{g}} (x(t) + d(t, m(S(t))))dt \leq 2g$$

for all $z \in \left[-\theta, \theta - 2\sqrt{g}\right]$. Since every optimal group group yields the opposite inequality (3), we have

$$\int_S (x(t) + d(t, m(S(t))))dt = 2g \tag{4}$$

for all optimal jurisdictions S.[8] It immediately implies that the total cost $x(t)+d(t, m(S(t)))$ is a periodic function with the period of $2\sqrt{g}$.

[8]Note that there are no optimal groups when $\theta < \sqrt{g}$, and the quations (4) are vacuous.

In order to reconcile a version of the Rawlsian principle with secession-proofness, we examine an "approximate" Rawlsian principle and demonstrate that for large populations with a bounded support, secession-proof cost allocations are in some sense not "too far" from the Rawlsian solution. We define a notion of subsidy to a group of individuals when each of its members contributes less than the amount prescribed by the Rawlsian allocation reduced by some given value δ.

Definition 5.3: Let $0 < \theta < \infty$ and $\delta > 0$ be given. Let P be an efficient partition of R_θ. We say that a P-cost allocation provides a *preferential treatment of magnitude* δ (with respect to the Rawlsian allocation x_R^θ) to the group S if for all $t \in S$ the following inequality holds:

$$x(t) \leq x_R^\theta(t) - \delta.$$

We denote by $\mathcal{G}(x, \delta, \theta)$ the family of measurable sets S that receive a preferential treatment of δ via allocation x.

The main result of this section shows that if a society is large enough then the size of a coalition receiving preferential treatment of a given magnitude via secession-proof allocation is relatively small:

Proposition 5.4: Let $\delta > 0$ and the bounded set $S \subset \Re$ be given. Then there exists $\bar{\theta}$ with $S \subset R_\theta$ such that for every $\theta > \bar{\theta}$ and every secession-proof allocation $x \in SP(\theta, g)$,

$$S \notin \mathcal{G}(x, \delta, \theta).$$

Proposition 5.4 asserts that if x is secession-proof, then for any subsidy level, the size of a subsidized group must be small if the size of the entire population is large enough. In some sense, Proposition 5.4 is a formal statement of the claim that secession-proofness implies the approximate Rawlsian recommendation.

The assertion of Proposition 5.4 can actually be strengthened. Let us denote by $\mathcal{G}'(x, \delta, \theta)$ the collection of groups S such that:

$$\int_S (x(t)+ \mid t - m(S(t)) \mid) \, dt \leq \int_S (x^*_R(t)+ \mid t - m(S(t)) \mid) \, dt - \delta\lambda(S),$$

where the subsidy $\delta\lambda(S)$ is given to the entire set S. The average benefit of members of S is δ which is weaker requirement than insisting on the subsidy level of δ for every member of S and obviously,

$$\mathcal{G}(x, \delta) \subseteq \mathcal{G}'(x, \delta).$$

We have a stronger version of Proposition 5.4:

Proposition 5.5: Let $\delta > 0$ and the group $S \in \Re$ be given. Then there exists $\bar{\theta}$ with $S \subset R_\theta$ such that for every $\theta > \bar{\theta}$ and every secession-proof allocation $x \in SP(\theta, g)$,

$$S \notin \mathcal{G}'(x, \delta, \theta).$$

6 Appendix

We start by stating several claims used to prove our results. Unless stated otherwise, the claims cover both bounded and unbounded cases.

Claim A.1: Let P be an efficient partition. Then $S_P(t) \in \arg\min_{S \in P} \mid t - m(S) \mid$. That is, every individual is assigned to the public project closest to her location.

Follows immediately from efficiency of P.

Claim A.2: Let P be an efficient partition. Then every element $S \in P$ is an interval.

Proof: Consider $S \in P$ and $a, b \in S$ with $m(S) \leq a < b$. We claim that every point $c \in [a, b]$ belongs to S. Indeed, suppose that such an c is assigned to another jurisdiction, say, S'. First note that $m(S) \neq m(S')$. Otherwise, the merger of jurisdictions S and S' would violate the efficiency of P. Claim A.1 implies that $m(S') > m(S)$, but the inequality

$\mid c - m(S') \mid \leq \mid c - m(S) \mid$ yields $\mid b - m(S') \mid < \mid b - m(S) \mid$. Thus, b is closer to $m(S')$ than to $m(S)$, a contradiction to Claim A.1. $\square$

Claim A.3: Let P be an efficient partition. Then all intervals S in P have the same length.

Proof: Let S and S' be two adjacent intervals in P and let l and l' denote their respective lengths. Since P is efficient, $D(S) + D(S')$ is minimal among all possible partitions of $S \cup S'$ into two intervals. Thus, (1) implies that $l^2 + (l')^2 \leq x^2 + y^2$ for all nonnegative numbers x, y satisfying $x + y = l + l'$. Since under the constraint $x + y = l + l'$ the convex function $x^2 + y^2$ reaches its minimum at $x = y = \frac{l+l'}{2}$, it follows that l must be equal to l'. $\square$

Proof of Proposition 2.5: (i) Consider first the unbounded case. By Claims A.1-A.3 it suffices to show that every partition P' of $\Re$ into equal intervals of length s is inefficient when $s \neq s^*$. Consider the case where $s > s^*$ (the case $s < s^*$ is treated in a similar manner). Since $\frac{s}{s^*} > 1$, there exists a rational number $\frac{m}{n}$ such that

$$\frac{s}{s^*} > \frac{m}{n} > 1.$$

Thus, there exists $\tilde{s}$ with $s > \tilde{s} > s^*$ such that $\tilde{s}m = sn$. Take partition $\tilde{P}$, which is created by replacing n adjacent intervals of length s in P' by m adjacent intervals of length $\tilde{s}$. Partitions P' and $\tilde{P}$ have a bounded difference and to show that P' is inefficient, it suffices to demonstrate that:

$$mg + m\frac{\tilde{s}^2}{4} < ng + \frac{s^2}{4}.$$

Put $L = \tilde{s}m = sn$. Then the last inequality is equivalent to

$$Lf(\tilde{s}) < Lf(s),$$

where $f(t) = \frac{g}{t} + \frac{t}{4}$. Since $s > \tilde{s} > s^*$, and, by Remark 2.2, function $f(t)$ increases for $t \geq s^*$, it follows that only efficient partition consists of intervals of the size s^*.

(ii) We prove here a more general version of Proposition 2.5 by considering an interval S of the length L rather than the set R_θ. Let $N(L)$ be the efficient number of jurisdictions when the population is distributed over S. Let $L = 2m\sqrt{g} + r$, where $m = \lfloor \frac{L}{2\sqrt{g}} \rfloor$, is an integer and $0 \leq r < 2\sqrt{g}$. We consider the case where $m > 0$ (otherwise, $N(L)$ is trivially equal to 1).

When S is divided in N intervals of equal size, the total aggregate cost is

$$N\left(g + \frac{L^2}{4N^2}\right) = Ng + \frac{L^2}{4N}$$

The first order condition over the set of real numbers yields

$$g - \frac{L^2}{4N^2} = 0, or$$

$$N' = \frac{L}{2\sqrt{g}} = \frac{2m\sqrt{g}+r}{2\sqrt{g}} = m + \frac{r}{2\sqrt{g}}.$$

Since $Ng + \frac{L^2}{4N}$ is convex in N, and N' is not necessarily an integer, it follows that $N(L)$ is either m or $m+1$. The optimal choice is m if and only if

$$gm + \frac{L^2}{4m} \leq g(m+1) + \frac{L^2}{4(m+1)},$$

which, after simplifications, is equivalent to

$$\frac{L^2}{4g} \leq m(m+1), \quad \text{or} \quad \frac{L}{s^*} \leq \sqrt{m(m+1)}.$$

□

Proof of Proposition 2.7: (i) Let $L = 2m\sqrt{g} + r$, where $m = \lfloor \frac{L}{2\sqrt{g}} \rfloor$ is an integer and $0 < r < 2\sqrt{g}$. By Proposition 2.5, the optimal number of intervals is either m or $m+1$. This implies that $s(L)$ is either equal to $\frac{L}{m}$ or to $\frac{L}{m+1}$. Thus,

$$\frac{2m\sqrt{g}+r}{m+1} \leq s(L) \leq \frac{2m\sqrt{g}+r}{m},$$

and, therefore,

$$2\sqrt{g}\frac{r-2\sqrt{g}}{L-r+2\sqrt{g}} \leq \frac{r-2\sqrt{g}}{m+1} \leq s(L)-2\sqrt{g} \leq \frac{r}{m} = 2\sqrt{g}\frac{r}{L-r}.$$

Since $0<r<2\sqrt{g}$, this implies

$$2\sqrt{g}-\frac{4g}{L+2\sqrt{g}} \leq s(L) \leq 2\sqrt{g}+\frac{4g}{L-2\sqrt{g}}.$$

(ii) The per capita aggregate cost of an individual in an interval of length s:

$$f(s)=\frac{1}{s}(g+\frac{s^2}{4}).$$

We evaluate the values $f(s(L))$ when L tends to infinity. By assertion (i), the value of $s(L)$ for large L is close to $2\sqrt{g}$, and we examine the second order Taylor expansion of f at $2\sqrt{g}$. Assertion (i) implies that there is

$$z(L) \in [2\sqrt{g}-\frac{4g}{L+2\sqrt{g}}, 2\sqrt{g}+\frac{4g}{L-2\sqrt{g}}], suchthat$$

$$f(s(L))-f(2\sqrt{g}) = f''(z(L))\frac{(s(L)-2\sqrt{g})^2}{2} = \frac{g(s(L)-2\sqrt{g})^2}{z(L)^3}.$$

Since $f'''(s)=-\frac{6g}{s^4}<0$, the last inequality yields

$$f(s(L))-f(2\sqrt{g}) \leq \frac{16g^3(\frac{1}{L-2\sqrt{g}})^2}{8\sqrt{g^3}(\frac{L}{L+2\sqrt{g}})^3}. \tag{5}$$

If $L>4\sqrt{g}$ then $\frac{L}{L+2\sqrt{g}}>\frac{2}{3}$ and $\frac{1}{L-2\sqrt{g}}<\frac{2}{L}$, yielding

$$f(s(L))-f(2\sqrt{g}) \leq \frac{27\sqrt{g^3}}{L^2}.$$

If $L \leq 4\sqrt{g}$ then $s(L)$ is either 1 or 2, and $f(s(L)$ is either $\frac{g}{L}+\frac{L}{4}$ or $\frac{2g}{L}+\frac{L}{8}$. It is easy to verify that (5) holds in this case as well. □

Proof of Proposition 4.3 relies on the proof of Proposition 5.4, and will be presented following the proof of that proposition.

Proof of Proposition 4.4: Assume first that $\frac{\theta}{s^*}$ is an integer. Then the size of each jurisdiction in an optimal partition is s^* and the total cost incurred by an individual $t \in R_\theta$ under allocation x_R^* is $\sqrt{g}$.

Consider a jurisdiction S. By Remark 2.1, the total cost incurred by S is at least $g+\frac{(\lambda(S))^2}{4}$ and each $t \in S$ on average contributes at least

$$f(\lambda(S)) = \frac{g}{\lambda(S)} + \frac{\lambda(S)}{4}.$$

Since the minimum of the function $\frac{g}{s} + \frac{s}{4}$ is attained at $s = 2\sqrt{g}$, it follows that $\frac{g}{\lambda(S)} + \frac{\lambda(S)}{4} \geq \sqrt{g}$, and the coalition S is not prone to secession. Thus, the sufficiency part is completed.

Assume now that the value $\frac{\theta}{\sqrt{g}}$ exceeds 1 and is not an integer. SSuppose that the Rawlsian allocation x_R^θ is secession-proof. The efficiency implies that the total cost incurred by the entire society R_θ is

$$2\theta \left[\frac{gN(R_\theta)}{2\theta} + \frac{\theta}{2N(R_\theta)}\right].$$

Then

$$\lambda(S) \left[\frac{gN(R_\theta)}{2\theta} + \frac{\theta}{2N(R_\theta)}\right] \leq g + \frac{\lambda(S)^2}{4}$$

for all S. Since $f(\lambda(S)) = \frac{g}{\lambda(S)} + \frac{\lambda(S)}{4}$ is minimal for at $2\sqrt{g}$, which does not exceed 2θ, the secession-proofness of x_R^θ implies that

$$\frac{gN(R_\theta)}{2\theta} + \frac{\theta}{2N(R_\theta)} \leq \sqrt{g}.$$

But this inequality is violated whenever $N(R_\theta) \neq \frac{\theta}{\sqrt{g}}$, a contradiction. Thus, the Rawlsian allocation is not secession-proof.□

Proof of Proposition 4.5: Let $\frac{\theta}{\sqrt{g}}$ be an integer. By Proposition 4.4, the set $SP(\theta, g)$ contains the Rawlsian allocation, which is, as we observed earlier, is an element of $NPES(\theta, g)$. Thus, the intersection of the sets $SP(\theta, g)$ and $NPES(\theta, g)$ is non-empty.

Assume now that $\frac{\theta}{\sqrt{g}}$ is not an integer and let x be a cost allocation in $NPES(\theta, g)$. By Proposition 2.5, there are two possibilities.

If $N = \lfloor \frac{\theta}{\sqrt{g}} \rfloor$, then the length $s = s(R_\theta)$ of a jurisdiction in P is larger than $2\sqrt{g}$. Since there are at least two jurisdictions, consider the jurisdictions $S_1 \equiv [-\theta, -\theta+s]$, $S_2 \equiv [-\theta+s, -\theta+2s]$. Take an optimal group $S^* = [-\theta+s-\sqrt{g}, -\theta+s+\sqrt{g}]$. Since x satisfies neutrality and symmetry we have

$$\int_{S^*}(x(t)+d(t,m(S(t)))dt = 2\int_{-\theta+s-\sqrt{g}}^{-\theta+s}(x(t)+d(t,m(S(t)))dt. \tag{6}$$

Moreover, the principle of partial equalization and the fact that $s > 2\sqrt{g}$ imply

$$\frac{\int_{-\theta+s-\sqrt{g}}^{-\theta+s}(x(t)+d(t,m(S(t)))dt}{\sqrt{g}} \geq \frac{2}{s}\int_{-\theta+\frac{s}{2}}^{-\theta+s}(x(t)+\mid t-m(S_1)\mid)dt. \tag{7}$$

Since

$$\int_{-\theta+\frac{s}{2}}^{-\theta+s}(x(t)+\mid t-m(S_1)\mid)dt = \frac{g}{2}+\frac{s^2}{8} > \frac{s\sqrt{g}}{2}. \tag{8}$$

By combining (6)-(8), we obtain

$$\int_{S^*}(x(t)+d(t,m(S(t)))dt > 2g.$$

However, the jurisdiction S^* is optimal and, therefore,

$$g + D(S^*) = \int_{-\theta+s-\sqrt{g}}^{-\theta+s+\sqrt{g}}(\mid t+\theta-s\mid)dt = 2g.$$

Thus, x is not secession-proof.

Consider now the case where $N(R_\theta) = \lfloor \frac{\theta}{\sqrt{g}} \rfloor + 1$. The optimal size $s = s(R_\theta)$ is smaller than $2\sqrt{g}$. However, it is still bounded from below, namely,

$$s > \frac{4}{3}\sqrt{g}. \tag{9}$$

Indeed, since

$$N(R_\theta) < \frac{\theta}{\sqrt{g}} + 1,$$

it follows that

$$s > 2\sqrt{g}\frac{\theta}{\theta+\sqrt{g}}.$$

If $\theta \geq 2\sqrt{g}$, (9) follows immediately. If $\theta < 2\sqrt{g}$ then $N(R_\theta) = \lfloor \frac{\theta}{\sqrt{g}} \rfloor + 1 = 2$. Since a two-jurisdictional structure yields a lower aggregate cost than the grand coalition, by Remark 2.1, we have

$$2(g + \frac{\theta^2}{4}) \leq g + \frac{(2\theta)^2}{4},$$

or $\theta > \sqrt{2}g$. Since $s = \theta$ and $\sqrt{2} > \frac{4}{3}$ it follows that (9) holds in this case as well.

Consider the coalition $S' \equiv \left[-\theta, -\theta + 2\sqrt{g}\right]$. Since $s \neq 2\sqrt{g}$, it follows

$$\int_{-\theta}^{-\theta+s} (x(t) + \mid t - m(S') \mid) dt > s\sqrt{g}.$$

Note that (9) implies $2\sqrt{g} < \frac{3s}{2}$, that is, the point $-\theta + 2\sqrt{g}$ is located to the left of the median of the adjacent jurisdiction $-\theta + \frac{3}{2}s$. The principle of partial equalization implies

$$\int_{-\theta+s}^{-\theta+2\sqrt{g}} (x(t) + \mid t - m(S') \mid) dt > (2\sqrt{g} - s)\sqrt{g},$$

which leads to

$$\int_{-\theta}^{-\theta+2\sqrt{g}} (x(t) + \mid t - m(S) \mid) dt > 2g.$$

However, the equality $g + D(S') = 2g$ implies that x is not secession-proof. $\square$

Proof of Proposition 5.2: It remains only to show that if $N(R_\theta) > 1$ and θ is not a multiple of $\sqrt{g}$, then any continuous secession proof allocation is not weakly Rawlsian. Suppose, in negation, that there a continuous secession-proof weakly Rawlsian allocation x.

Since θ is not a multiple of $2\sqrt{g}$, Remark 2.2 implies that there exists $\bar{t} \in (-\theta, \theta)$ such that:

$$x(\bar{t}) + d(\bar{t}, m(S(\bar{t}))) > \sqrt{g}.$$

Consider the coalition $S(\delta) \equiv \left[\bar{t} - 2\sqrt{g}, \bar{t} + \delta\right]$ where $\delta > 0$. By (1) of Remark 2.1, we have

$$g + D(S(\delta)) = 2g + \delta\sqrt{g} + \frac{\delta^2}{4}, \tag{10}$$

where, to recall,

$$D(S) = \min_{p \in I} \int_S d(t,p)dt,$$

is the minimal transportation cost of the individuals of S.

Since x is continuous, there exists $\delta > 0$ and $\eta > 0$ small enough such that

$$x(t) + d(t, m(S(t))) > \sqrt{g} + \eta$$

for all $t \in [\bar{t}, \bar{t} + \delta]$. Furthermore, since x satisfies the weak Rawlsian principle, then

$$\int_{S(\delta)} (x(t) + d(t, m(S(t))))dt > 2g + \delta\sqrt{g} + \eta\delta, \tag{11}$$

a contradiction, as inequalities (10) and (11) are incompatible for δ small enough.$\square$

In the proof of our next propositions we use some results from the basic measure theory. The first two rely on the regularity of the Lebesgue measure (Billingsley (1995), Theorem 12.3):

Claim A.4: If S is a bounded and measurable subset of $\Re$ then for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq S$ with $\lambda(S \setminus K_\varepsilon) \leq \varepsilon$.

Claim A.5: If S is a bounded and measurable subset of $\Re$ then for every $\varepsilon > 0$ there exists an open set $O_\varepsilon \supseteq S$ with $\lambda(O_\varepsilon \setminus S) \leq \varepsilon$.

Another is a well-known result (Billingsley (1995), page 231):

Claim A.6 - Lusin's theorem: Let A be a bounded measurable subset of $\Re$ and h is a measurable function on A. Then for every $\varepsilon > 0$ there exists a compact set $C_\varepsilon \subseteq A$ with $\lambda(\tilde{K}_\varepsilon) \geq \lambda(A) - \varepsilon$ and h is continuous on C_ε.

Finally, we will utilize the property of *essential boundedness* of secession-proof allocations in a multi-jurisdictional framework:

Claim A.7: Let $N(R_\theta) > 1$ and $x \in SP(\theta, g)$. Then there exists a constant $q > 0$ such that $x(t) \leq C$ almost everywhere on R_θ, or $\lambda(\{t \in R_\theta : x(t) > C\}) = 0$. In fact, q can be any number exceeding $2\sqrt{g}$.

Proof: Let $N(R_\theta) > 1$ and $x \in SP(\theta, g)$. We shall show that

$$\lambda(\{t \in R_\theta : x(t) + \mid t - m(S(t)) \mid > 2s(R_\theta)\}) = 0,$$

where $s(R_\theta)$ is an optimal jurisdictional size in R_θ.

Suppose, to the contrary, that $\lambda(S) > 0$, where $S \equiv \{t \in \Re : x(t) + \mid t - m(S(t)) \mid > 2s(R_\theta)\}$. Let S_1, S_2 be two adjacent jurisdictions in an optimal partition, with $\lambda(S') > 0$, where $S' = S_2 \cap S$. Denote $T \equiv S_1 \cup S'$ and define the T-cost allocation y as follows:

$$y(t) = \begin{cases} x(t) & if \quad t \in S_1 \\ 0 & if \quad t \in S'. \end{cases}$$

We have

$$D(T) + g \leq \int_T (y(t) + \mid t - m(S_1) \mid) dt < \int_{S_1} x(t) dt \int_S \mid t - m((S_1) \mid dt.$$

Since $\mid t - m(S_1) \mid < x(t) + \mid t - m(S_2) \mid$ for all $t \in S'$, it follows that

$$D(T) + g < \int_T (x(t) + \mid t - m(S(t)) \mid) dt.$$

That is, T is prone to secession, contradicting our assumption that x is secession-proof.□

Proof of Proposition 5.4: Let $\delta > 0$. By Proposition 2.7, the Rawlsian allocation x_R^θ is close to $\sqrt{g}$ and the optimal jurisdictional size is close to $2\sqrt{g}$. Thus, we can choose $\tilde{\theta}$ such that $\mid x_R^\theta - \sqrt{g} \mid < \frac{\delta}{4}$ and $s(R_\theta) < \frac{5}{2}\sqrt{g}$ for every $\theta > \tilde{\theta}$.

Let $\theta > \tilde{\theta}$. Suppose now that a cost allocation $y \in SP(\theta, g)$ and a set $S \in R_\theta$ with a positive measure are such that S receives preferential treatment of magnitude δ via allocation y. Let $\lambda(S) = \rho > 0$. We will use the following claim:

Claim A.8: There exists a finite family of pairwise disjoint intervals $\tilde{\mathcal{I}} = \{I_1, \ldots, I_m\}$ such that

(i)

$$\lambda(I) \geq \frac{\rho}{2}, \quad \text{where } I \equiv \bigcup_{I_i \in \tilde{\mathcal{I}}} I_i,$$

(ii)

$$\int_I z(t)dt < \int_I \left(x_\theta^R(t) + \mid t - m(S(t)) \mid\right) dt - \frac{\delta}{2}\lambda(I).$$

Proof: Let $\eta > 0$ satisfy $\eta < \frac{\delta}{10\sqrt{g}}$. Claim A.4 implies that there exists a compact subset K of S such that $\lambda(K) > \frac{3}{4}\rho$ and an open set O_η for all $\eta > 0$ with $O_\eta \supseteq S$ such that

$$\lambda\left(O_\eta \backslash K\right) \leq \eta \lambda\left(K\right). \tag{12}$$

For every $t \in K$, let $I(t) \subset O_\eta$ be an interval that contains t. Since K is compact, the cover $\{I(t)\}_{t \in K}$ admits a subcover $\mathcal{I} = I_1, \ldots, I_N$. We may assume, without loss of generality, that all intervals in $\mathcal{I}$ are pairwise disjoint, and, moreover that O_η consists only of elements of $\mathcal{I}$.

Denote by $\tilde{I}$ the following subset of $\mathcal{I}$:

$$\tilde{I} = \{I_i \in \mathcal{I} | \frac{\lambda\left(K^c \cap I_i\right)}{\lambda\left(I_i\right)} \leq 3\eta\}.$$

By (12) we have

$$\sum_{I_i \in \mathcal{I}} \lambda\left(K^c \cap I_i\right) = \lambda\left(O_\eta \backslash K\right) \leq \eta \lambda\left(K\right) < \frac{3}{4}\eta\rho.$$

But

$$\sum_{I_i \in \mathcal{I}} \lambda\left(K^c \cap I_i\right) = \sum_{I_i \in \tilde{\mathcal{I}}} \lambda\left(K^c \cap I_i\right) + \sum_{I_i \notin \tilde{\mathcal{I}}} \lambda\left(K^c \cap I_i\right) > 3\eta \sum_{I_i \notin \tilde{\mathcal{I}}} \lambda\left(I_i\right).$$

Thus,

$$\lambda(\bigcup_{I_i \notin \overset{\heartsuit}{\mathcal{I}}} I_i) < \frac{1}{4}\lambda(K), \quad \text{and } \lambda(\tilde{I}) > \frac{3}{4}\lambda(K) > \frac{\rho}{2},$$

where $I = \bigcup_{I_i \in \overset{\heartsuit}{\mathcal{I}}} I_i$. Moreover,

$$\int_I z(t)dt = \int_{K \cap I} z(t)dt + \int_{K^c \cap I} z(t)dt.$$

We have

$$\int_{K\cap I} z(t)dt < \lambda(I)(\sqrt{g}-\delta),$$

By Claim A.7, $z(t) < 5\sqrt{g}$ almost everywhere on R_θ, yielding

$$\int_{K^c\cap I} z(t)dt \le 5\sqrt{g}\eta\lambda(I).$$

Since $\eta < \frac{\delta}{10\sqrt{g}}$, we have

$$\int_{\tilde{I}} z(t)dt < \lambda(I)(\sqrt{g}-\frac{\delta}{2}),$$

which completes the proof of the claim. □

Let $\mathcal{I}$ be a family of m intervals that satisfy conditions (i) and (ii) of Claim A.8. Then there exists an interval $I^* \in \mathcal{I}$ such that $\lambda(I^*) > \frac{\rho}{2m}$. Let $I^* = (a,b)$. We have

$$\int_I z(t)dt < \int_I (x_R^\theta(t)+ \mid t-m(S(t)) \mid)dt - \frac{\delta(b-a)}{2}$$

and

$$\int_{R_\theta\setminus I} z(t)dt > \int_{R_\theta\setminus I} (x_R^\theta(t)+ \mid t-m(S(t)) \mid)dt + \frac{\delta(b-a)}{2}. \tag{13}$$

Consider the coalition T, which is the union of two closed sets T_1 and T_2, where $T_1 = [-\theta, a]$ and $T_2 = [b, \theta]$.

By assertion (ii) of Proposition 2.7, the total contribution of individuals in T_1 and T_2 satisfies

$$f(s(T_j))\lambda(T_j) \le \lambda(T_j)\sqrt{g} + \frac{27\sqrt{g^3}}{\lambda(T_j)} \le \lambda(T_j)\sqrt{g} + \frac{54\sqrt{g^3}}{\theta}$$

for $j = 1, 2$. Thus, the total contribution of the individuals in T, $F(T) = f(s(T_1))\lambda(T_1) + f(s(T_2))\lambda(T_2)$ satisfies

$$F(T) \le \lambda(T)\sqrt{g} + \frac{108\sqrt{g^3}}{\theta} \le \int_T (x_R^\theta(t)+ \mid t-m(S(t)) \mid)dt + \frac{108\sqrt{g^3}}{\theta}.$$

But, by (13),

$$\int_T (y(t)+ \mid t-m(S(t)) \mid)dt > \int_T (x_R^\theta(t)+ \mid t-m(S(t)) \mid)dt + \frac{\delta(b-a)}{2}.$$

The secession-proofness of y implies

$$\int_T (y(t)+ \mid t - m(S(t)) \mid) dt \leq F(T)$$

and, therefore,

$$b - a \leq \frac{216\sqrt{g^3}}{\delta\theta}.$$

The inequality $b - a \geq \frac{\rho}{2m}$ immediately implies that

$$\rho \leq \frac{432m\sqrt{g^3}}{\delta\theta}.$$

Since m is independent of θ, it follows that the last inequality is violated when δ is large enough. $\square$

Proof of Proposition 4.3: First let us show that the pair (P^*, x_R^*) is secession-proof. The total cost for any coalition S, if it secedes, is no less than $g + \frac{(\lambda(S))^2}{4}$. However, given (P^*, x_R^*), the total contribution of members of S is $\lambda(S)\sqrt{g}$. Since, by Remark 2.2, the value $f(s) = \frac{g}{s} + \frac{s}{4}$ is minimal when $s = 2\sqrt{g}$, it follows that the inequality $\frac{g}{s} + \frac{s}{4} \geq \sqrt{g}$ holds for all s and the conclusion follows.

Let us demonstrate that the Rawlsian allocation x_R^* is the unique secession-proof cost allocation. Let y be a secession-proof cost allocation, associated with partition P^*, that differs from x_R^* on the set of positive measure. Since per capita total cost in R_θ is equal to $\sqrt{g}$, this means that there is a group S with $\lambda(S) = \rho > 0$, such that

$$y(t)+ \mid t - m(S(t)) \mid < \sqrt{g}$$

for all $t \in S$. By Claims A.5 and A.6, there exists a compact set $C \subseteq S$ with $\lambda(C) \geq \frac{\rho}{2}$, such that the function

$$z(t) = y(t)+ \mid t - m(S(t)) \mid - \sqrt{g}$$

is continuous on C. Since the set C is compact, there exists $\delta > 0$ such that $z(t) < -\delta$ for all $t \in C$, or

$$y(t)+ \mid t - m(S(t)) \mid < \sqrt{g} - \delta$$

for all $t \in C$.

Take an integer M and consider the truncation of the real line to M optimal intervals of the length $2\sqrt{g}$. Remark 4.2 implies that y is budget balanced on the interval $[-\theta, \theta]$ and represents a cost allocation for the truncated society $[-\theta, \theta]$. Thus,

$$\int_{-\theta}^{\theta} (y(t)+ \mid t - m(S(t)) \mid) dt = 4Mg.$$

The proof of Proposition 5.4 implies that if C receives preferential treatment of magnitude δ via secession-proof allocation, for every positive ε there exists $M(\varepsilon)$ such that $\frac{\rho}{2} \leq \lambda(C) < \varepsilon$ for all $M \geq K(\varepsilon)$. But this contradicts the fact that the value of ρ is independent of M. $\square$

7 References

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