Solving Asset Pricing Models with Habit Persistence

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Abstract

This paper provides a closed–form solution for the price–dividend ratio in a standard asset pricing model with habit formation when the growth rate of endowment is a first–order Gaussian autoregression. It determines conditions that guarantee the existence of a stationary bounded equilibrium and positivity of prices.

Key Words: Asset prices, Price–dividend ratio, Habit persistence

JEL Class.: C61, C62, G12

Introduction

A large part of the literature studying the behavior of asset prices assumes *iid* growth rate of endowment and/or separable utility over time to get closed–form solutions for the price to dividend ratio. However, Burnside (1998) proposed an exact solution for a standard asset pricing model under the assumption that the growth rate of endowment follows a first order autoregressive process with Gaussian shocks.[1] In this paper, we extend these results to non time–separable utility functions, when the non–separability stems from

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[1]Tsionas (2003) extends these results to a wider class of distributions.

habit formation. Following Abel (1990, 1999), we assume that the utility function of the representative agent can be written as a power function of the ratio of current to previous period consumption. We determine a closed–form solution for the price–dividend ratio and conditions that guarantee the existence of a stationary bounded equilibrium. Beside, we also provide some restrictions on the parameters that guarantee the positivity of prices. The paper is organized as follows. The next section presents the model. Section 2 gives the analytical form of the solution. In section 3 conditions for a bounded solution are discussed. A last section offers some concluding remarks.

1 An Asset Pricing Model with Habit Persistence

We consider the problem of an infinitely–lived representative agent who derives utility from consuming a single consumption good. The agent has preferences over both her current and her own past consumption, therefore reflecting the existence of some habit persistence phenomenon. She determines her consumption, asset holdings plans so as to maximize the expected sum of discounted future utility

$$\max E_t \sum_{s=0}^{\infty} \beta^s \frac{\mathscr{C}_{t+s}^{1-\theta}-1}{1-\theta} \qquad \text{where } \mathscr{C}_t \equiv \frac{C_t}{C_{t-1}^{\varphi}} \tag{1}$$

where $\beta > 0$ is a subjective discount factor, $\theta > 0$ denotes the curvature parameter. $\varphi \in [0,1]$ is the habit persistence parameter. For the moment, no further restrictions will be placed on these parameters.[2] When determining her consumption/asset holdings plans, the agent faces the budget constraint

$$P_t S_{t+1} + C_t \leq (P_t + D_t) S_t$$

where S_t denotes the share of the asset owned by the agent. P_t is the price of a share in period t. D_t denotes dividends, which should be thought of as the stochastic endowments paid to the owner of each unit of the asset held from period $t-1$ to t. The first order condition is given by

$$\frac{C_t^{-\theta}}{C_{t-1}^{\varphi(1-\theta)}} - \beta\varphi E_t\left[\frac{C_{t+1}^{1-\theta}}{C_t^{\varphi(1-\theta)+1}}\right] = \beta E_t\left[\left(\frac{P_{t+1}+D_{t+1}}{P_t}\right)\left(\frac{C_{t+1}^{-\theta}}{C_t^{\varphi(1-\theta)}} - \beta\varphi\frac{C_{t+2}^{1-\theta}}{C_{t+1}^{\varphi(1-\theta)+1}}\right)\right] \tag{2}$$

Since there is a single agent in this economy, market clearing imposes $S_t = 1$ for all t so that $C_t = D_t$ in equilibrium.

[2]However, we will discuss some restrictions in order to insure a bounded solution and positive price–dividend ratio in section 3.

2 A Closed–Form Solution

Up to now, no restrictions have been placed on the stochastic process of dividends. Most of the literature assumes the growth rate of dividend is *iid* and normally distributed (see e.g., Abel (1990) and (1999) among others). We will partially depart from this assumption, keeping with the normal distribution, but relaxing the *iid* assumption. The growth rate of the endowment $\gamma_t \equiv \log(D_t/D_{t-1})$ is indeed assumed to follow a Gaussian AR(1) process

$$\gamma_t = \rho\gamma_{t-1} + (1-\rho)\overline{\gamma} + \varepsilon_t$$

where $|\rho| < 1$ and $\varepsilon_t \rightsquigarrow \mathscr{N}(0, \sigma^2)$. Letting $v_t \equiv P_t/D_t$ denote the price–dividend ratio and defining $z_t \equiv \exp((1-\theta)\gamma_t - \varphi(1-\theta)\gamma_{t-1})$ and $y_t \equiv v_t[1 - \beta\varphi E_t z_{t+1}]$, equation (2) rewrites

$$y_t = \beta E_t \left(1 - \beta\varphi z_{t+2} + y_{t+1}\right) z_{t+1} \tag{3}$$

and has to be solved for y_t. This forward looking stochastic difference equation admits the closed form solution reported in the next proposition (see Appendix A).

Proposition 1 *The equilibrium price–dividend ratio is given by*

$$v_t = \frac{\beta\varphi\exp(a_0 + b_0(\gamma_t - \overline{\gamma})) + (1-\varphi)\sum_{i=1}^{\infty}\beta^i \exp(a_i + b_i(\gamma_t - \overline{\gamma}))}{1 - \beta\varphi\exp(a_0 + b_0(\gamma_t - \overline{\gamma}))} \tag{4}$$

where

$$a_0 = (1-\theta)(1-\varphi)\overline{\gamma} + (1-\theta)^2\frac{\sigma^2}{2}, \qquad b_0 = (1-\theta)(\rho-\varphi)$$

and

$$a_i = (1-\theta)(1-\varphi)\overline{\gamma}i + \left(\frac{1-\theta}{1-\rho}\right)^2\frac{\sigma^2}{2}\left[(1-\varphi)^2 i - 2\frac{(1-\varphi)(\rho-\varphi)}{1-\rho}(1-\rho^i) + \frac{(\rho-\varphi)^2}{1-\rho^2}(1-\rho^{2i})\right]$$

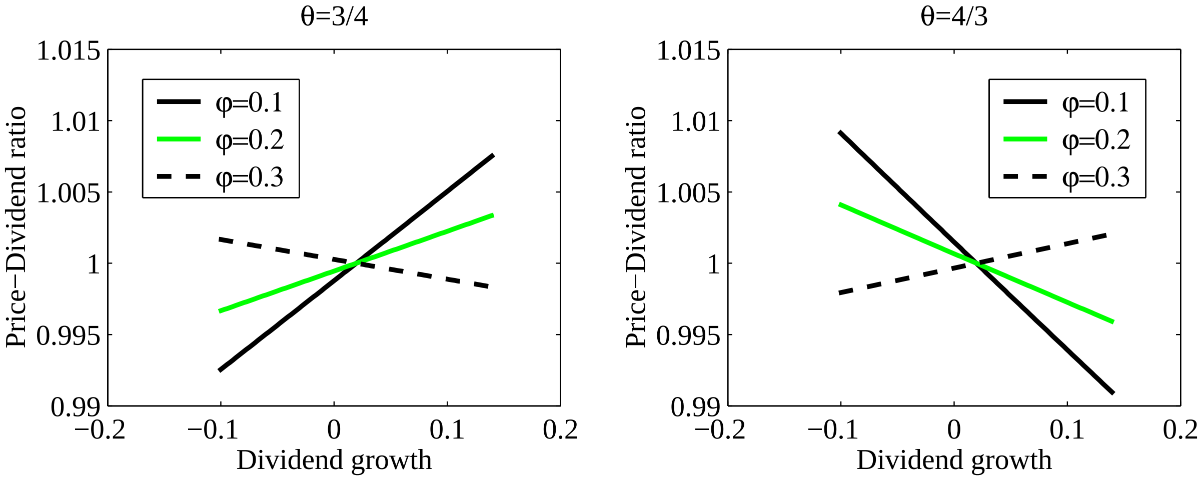
$$b_i = \frac{(1-\theta)(\rho-\varphi)}{1-\rho}(1-\rho^i) \qquad \textit{for} \quad i \geqslant 1$$

Equation (4) nests previous asset pricing formula. First, setting $\varphi = 0$ — *i.e.* imposing time separability in preferences — we recover Burnside's (1998) solution. Second, when the rate of growth of endowments is *iid* over time ($\gamma_t = \overline{\gamma} + \varepsilon_t$) and φ is set to 1, we recover the solution used by Abel (1990) to compute unconditional expected returns (see Table 1, panel C, p 41)

$$v_t = \frac{z_t}{1 - z_t} \quad \text{with} \quad z_t = \beta\exp\left((1-\theta)^2\frac{\sigma^2}{2} + (\theta - 1)(\gamma_t - \overline{\gamma})\right) \tag{5}$$

In this latter case, as equation (5) makes it clear, the price–dividend ratio is an increasing (resp. decreasing) and convex function of consumption growth if $\theta > 1$ (resp. $\theta < 1$). In other words, only the position of the curvature parameter around unity matters.[3] Things are actually more complicated when we consider the more general model. Indeed, both the position of the curvature parameter, θ, around 1 and the position of the habit persistence parameter, φ, around ρ matter. This is illustrated in Figure 1 that reports the price–dividend ratio as a function of dividend growth for different values for φ. For illustrative purposes, in each case we consider, the level of the decision rule was normalized to 1 when dividend growth equals its mean. As can be seen from the figure, when $\theta > 1$ (resp.

Figure 1: Price–Dividend Ratio v_t



Note : This figure is drawn for a calibration based on Chapman's (2002) estimation of the AR(1) process of consumption growth over the period 1949–1978. We therefore set $\rho = 0.269$, $\sigma = 0.01169$,and $\overline{\gamma} = 0.0196$. Following Abel (1990), we set $\beta = 0.99$. For illustrative purposes, the level of the decision rule was normalized to 1 when dividend growth equals its mean.

$\theta < 1$), the decision rule is increasing (resp. decreasing) with dividend growth when $\varphi > \rho$ (resp. $\varphi < \rho$). The economic intuition lying behind this result is clear. Let us, for example, consider a high curvature parameter (θ >1) which is associated with stronger wealth effects, and let us take φ as given. The behavior of an agent in face a positive shock on dividends essentially depends on the persistence of the process of endowments. When $\rho < \varphi$, dividend growth exhibits low persistence (relative to habit persistence). The rise in current consumption that follows the shock on the endowment triggers an increase in future consumption by force of habits. However, since the shock on the dividend is not persistent, the agent has to rely on assets in order to sustain next period consumption. This puts upward pressure on asset prices, which together with the fact that dividend are not persistent, implies that the price–dividend ratio increases. Consider now that $\rho > \varphi$, such that dividend growth is persistent with respect to habits. A positive shock on

[3]This can actually be seen straightforwardly in equation (5), where convexity holds if the growth rate of dividends satisfies $\exp((1-\theta)^2\sigma^2/2 + (\theta-1)(\gamma_t - \overline{\gamma})) < 1/\beta$ for all t.

endowments leads the agent to raise current and, by force of habits, future consumption. However, contrary to the previous case, the high persistence in dividend growth implies that future consumption can be sustained by future increases in endowments. Therefore, the upward pressure on asset prices — following the increase in the demand for securities — is not high enough to more than offset the increase in dividend. The price–dividend ratio decreases.

Note that the solution for the price–dividend ratio involves a series, which convergence properties have not been yet discussed. This is the object of the next section.

3 Conditions for a bounded and positive solution

The following proposition determines conditions for the existence of a stationary bounded equilibrium.[4]

Proposition 2 *The series in (4) converges if and only if*

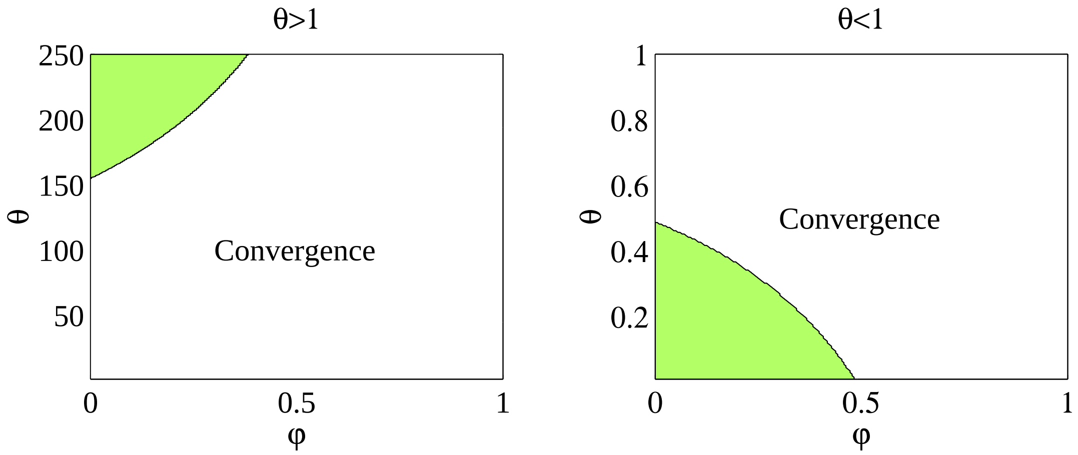
$$r \equiv \beta \exp\left[(1-\theta)(1-\varphi)\overline{\gamma} + \frac{\sigma^2}{2}\left(\frac{(1-\theta)(1-\varphi)}{1-\rho}\right)^2\right] < 1 \tag{6}$$

Figure 2 illustrates Proposition 2 and reports the zone of convergence and divergence of the series in (4) with respect to (φ, θ) when $\varphi \in (0,1)$. The left hand side panel of the figure considers the case where $\theta > 1$ and the right hand side panel depicts the zone for $\theta < 1$. Note that although we chose to focus (for our purpose) on zones determined by φ and θ, other representations could have been considered. In particular, as may be noticed from (6), $\beta < 1$ is neither necessary nor sufficient to insure finite asset prices, such that we might have, following Burnside (1998), considered a (β, θ) representation. In order to grasp some economic intuition for these results, let us focus on the case $\theta > 1$ and consider the time separable case (φ=0). If the future path of endowment is uncertain, risk adverse consumers (θ very large) are willing to purchase today a huge amount of the asset and insure themselves against future bad outcomes — *i.e.* the series would explode. Note that this effect would disappear in a deterministic setting[5] and any value for $\theta > 1$ would guarantee convergence. Conversely, when habit persistence is strong enough (large φ), the solution is bounded as the effect of uncertainty may be lowered by the smoother consumption path, even for large value of θ. In the limiting case where $\varphi = 1$, the price–dividend ratio is given by (5) and therefore the series drops out as the forecasting horizon reduces to 1 period ahead. Otherwise stated, discounted future risk would be inconsequential.

[4]The reader is left to refer to appendix B for a proof.

[5]In the deterministic case, the condition becomes: $r \equiv \beta \exp[(1-\theta)(1-\varphi)\overline{\gamma}] < 1$.

Figure 2: Convergence of the infinite series representation of v_t



Note : The region of divergence corresponds to the shaded area. This figure is drawn for the same calibration of β and the dividend growth process as Figure 1

As previously noted by Abel (1990), the asset pricing model with habit persistence does not preclude the existence of negative asset prices. The possibility of negative prices comes from *(i)* the log–normal assumption and *(ii)* the marginal utility of consumption which can be negative when the habit persistence and/or relative risk aversion are too large. Nevertheless, from a practical point of view, the law of motion of γ_t can be arbitrarily well represented by a Markov chain without loss of generality. In such a case it is possible to determine the upper and lower bounds on the process that guarantee positive prices. This can be achieved analyzing equation (4), from which we easily see that the price–dividend ratio is positive if and only if the denominator in the decision rule satisfies

$$1-\beta\varphi\exp(a_0+b_0(\gamma_t-\overline{\gamma}))>0 \Longleftrightarrow |a_0+b_0(\gamma_t-\overline{\gamma})|>-\log(\beta\varphi)$$

Using the triangular inequality, a sufficient condition for the positivity of asset prices is

$$\widehat{\gamma}_t<\Gamma\equiv-\frac{\log\beta\varphi+|(1-\theta)(1-\varphi)\overline{\gamma}+(1/2)\sigma^2(1-\theta)^2|}{|(1-\theta)(\rho-\varphi)|}$$

where $\widehat{\gamma}_t$ is the absolute deviation of γ_t from its mean, $|\gamma_t-\overline{\gamma}|$. This condition then provides upper and lower bounds for the support of γ in a Markov chain representation of the dividend growth process.

Note that $\lim_{\theta\uparrow 1}\Gamma=+\infty$ and $\lim_{\theta\downarrow 1}\Gamma=-\infty$, such that the bounds tends to infinity. A direct implication of that result is that price remains positive when $\overline{\gamma}$, ρ and σ are set to match the observed consumption growth provided the curvature parameter does not depart too much from unity. When $\varphi=0$, the price is always positive since the model just reduces to a time separable model and the denominator is equal to 1. When habit persistence is brought back, $\varphi\in[0,1]$, the condition is more stringent as φ increases.

4 Concluding remarks

This paper offers a closed–form solution for the price–dividend ratio in a standard asset pricing model with *(i)* a Gaussian autoregressive process for the endowments and *(ii)* habit formation — therefore extending Burnside's (1998) results. We establish conditions under which the solution is bounded and give some restrictions on the parameters to guarantee positive asset prices. These findings are useful because they allow to evaluate the accuracy of various approximation methods to non–linear rational expectation models. Furthermore, they can be used to perform simulation experiments to study the finite sample properties of various estimation methods.

References

Abel, A., 1990, "Asset prices under habit formation and catching up with the Joneses", *American Economic Review, Papers and Proceedings*, 80 (2), pp. 38-42.

Abel, A., 1999, "Risk premia and term premia in general equilibrium", *Journal of Monetary Economics*, 43, pp. 3–33.

Chapman, D.A., 2002, "Does Intrinsic Habit Formation Actually Resolve the Equity Premium Puzzle", Review of Economic Dynamics, 5, pp. 618–645.

Burnside, C., 1998, "Solving asset pricing models with Gaussian shocks", *Journal of Economic Dynamics and Control*, 22, pp. 329–340.

Tsionas, E., 2003, "Exact solution of asset pricing models with arbitrary shock distributions", *Journal of Economic Dynamics and Control*, 27, pp. 843–851.

A Proof of Proposition 1

Iterating forward, and imposing the transversality condition, a solution to this forward looking stochastic difference equation (3) is given by

$$y_t = \beta\varphi E_t z_{t+1} + (1-\varphi) E_t \sum_{i=1}^{\infty} \beta^i \prod_{j=1}^{i} z_{t+j}$$

Note that, from the definition of z_t, we have

$$\prod_{j=1}^{i} z_{t+j} = \exp\left((1-\theta) \sum_{j=1}^{i} \gamma_{t+j} - \varphi(1-\theta) \sum_{j=0}^{i-1} \gamma_{t+j} \right)$$

Since γ_t follows an AR(1) process, we have

$$\gamma_{t+j} = \overline{\gamma} + \rho^j(\gamma_t - \overline{\gamma}) + \sum_{k=0}^{j-1} \rho^k \varepsilon_{t+j-k}$$

which implies that

$$\sum_{j=1}^{i} \gamma_{t+j} = \frac{\rho}{1-\rho}(1-\rho^i)(\gamma_t - \overline{\gamma}) + i\overline{\gamma} + \sum_{k=0}^{i-1} \frac{1-\rho^{i-k}}{1-\rho}\varepsilon_{t+k+1} \tag{A.1}$$

and

$$\sum_{j=0}^{i-1} \gamma_{t+j} = \frac{1-\rho^i}{1-\rho}(\gamma_t - \overline{\gamma}) + i\overline{\gamma} + \sum_{k=0}^{i-2} \frac{1-\rho^{i-k-1}}{1-\rho}\varepsilon_{t+k+1} \tag{A.2}$$

Furthermore, since we assumed that dividend growth is normally distributed, we can make use of standard results on log–normal distributions, to compute $E_t(\prod_{j=1}^{i} z_{t+j}) = \exp(\mathscr{E} + \mathscr{V}/2)$, where

$$\mathscr{E} = E_t\left((1-\theta)\sum_{j=1}^{i}\gamma_{t+j} - \varphi(1-\theta)\sum_{j=0}^{i-1}\gamma_{t+j}\right)$$

and

$$\mathscr{V} = \mathrm{Var}_t\left((1-\theta)\sum_{j=1}^{i}\gamma_{t+j} - \varphi(1-\theta)\sum_{j=0}^{i-1}\gamma_{t+j}\right)$$

Using (A.1) and (A.2), the first term is simply given by

$$\mathscr{E} = \frac{(1-\theta)(\rho-\varphi)}{1-\rho}(1-\rho^i)(\gamma_t - \overline{\gamma}) + (1-\theta)(1-\varphi)\overline{\gamma}i$$

The calculation of $\mathscr{V}$ requires more algebra

$$\begin{aligned}
\mathscr{V} &= \mathrm{Var}_t\left[(1-\theta)\sum_{k=0}^{i-1}\frac{1-\rho^{i-k}}{1-\rho}\varepsilon_{t+k+1} - \varphi(1-\theta)\sum_{k=0}^{i-2}\frac{1-\rho^{i-k-1}}{1-\rho}\varepsilon_{t+k+1}\right]\\
&= \mathrm{Var}_t\left[\frac{1-\theta}{1-\rho}\sum_{j=1}^{i}\left(1-\varphi-(\rho-\varphi)\rho^{i-j}\right)\varepsilon_{t+j}\right]\\
&= \left(\frac{1-\theta}{1-\rho}\right)^2\sigma^2\left[(1-\varphi)^2 i - 2\frac{(1-\varphi)(\rho-\varphi)}{1-\rho}(1-\rho^i) + \frac{(\rho-\varphi)^2}{1-\rho^2}(1-\rho^{2i})\right]
\end{aligned}$$

Likewise,

$$E_t z_{t+1} = \exp\left((1-\theta)(\rho-\varphi)(\gamma_t - \overline{\gamma}) + (1-\theta)(1-\varphi)\overline{\gamma} + (1-\theta)^2\frac{\sigma^2}{2}\right)$$

Therefore, the solution to (3) is given by

$$y_t = \beta\varphi \exp(a_0 + b_0(\gamma_t - \overline{\gamma})) + (1-\varphi)\sum_{i=1}^{\infty} \beta^i \exp(a_i + b_i(\gamma_t - \overline{\gamma}))$$

where

$$a_0 = (1-\theta)(1-\varphi)\overline{\gamma} + (1-\theta)^2\frac{\sigma^2}{2} \quad \text{and} \quad b_0 = (1-\theta)(\rho-\varphi)$$

$$a_i = (1-\theta)(1-\varphi)\overline{\gamma}i + \left(\frac{1-\theta}{1-\rho}\right)^2 \frac{\sigma^2}{2}\left[(1-\varphi)^2 i - 2\frac{(1-\varphi)(\rho-\varphi)}{1-\rho}(1-\rho^i) + \frac{(\rho-\varphi)^2}{1-\rho^2}(1-\rho^{2i})\right]$$

$$b_i = \frac{(1-\theta)(\rho-\varphi)}{1-\rho}(1-\rho^i)$$

Recalling that $y_t = v_t[1-\beta\varphi E_t z_{t+1}]$ and making use of the calculation of $E_t z_{t+1}$, we finally get the price to dividend ratio. This completes the proof. $\square$

B Proof of Proposition 2

Let us define

$$w_i = \beta^i \exp(a_i + b_i(\gamma_t - \overline{\gamma}))$$

where a_i and b_i are defined in the main text. Then the series in v_t may be written as

$$y_t = \sum_{i=1}^{\infty} w_i$$

It follows that

$$\left|\frac{w_{i+1}}{w_i}\right| = \beta \exp(\Delta a_{i+1} + \Delta b_{i+1}(\gamma_t - \overline{\gamma}))$$

where

$$\Delta a_{i+1} = (1-\theta)(1-\varphi)\overline{\gamma} + \left(\frac{1-\theta}{1-\rho}\right)^2 \frac{\sigma^2}{2}\left\{(1-\varphi)^2 - 2(1-\varphi)(\rho-\varphi)\rho^i + (\rho-\varphi)\rho^{2i}\right\}$$

$$\Delta b_{i+1} = (1-\theta)(1-\varphi)\rho^i$$

Then, provided $|\rho| < 1$, we have

$$\lim_{i\to\infty} \Delta a_{i+1} = (1-\theta)(1-\varphi)\overline{\gamma} + \left(\frac{1-\theta}{1-\rho}\right)^2 \frac{\sigma^2}{2}(1-\varphi)^2$$

$$\lim_{i\to\infty} \Delta b_{i+1}(\gamma_t - \overline{\gamma}) = 0$$

Therefore

$$\lim_{i\to\infty}\left|\frac{w_{i+1}}{w_i}\right| = \beta \exp\left((1-\theta)(1-\varphi)\overline{\gamma} + \left(\frac{1-\theta}{1-\rho}\right)^2 \frac{\sigma^2}{2}(1-\varphi)^2\right) \equiv r$$

Then, the ratio test for convergence of a series implies that $\sum_{i=1}^{\infty} w_i$ converges iff $r < 1$. $\square$