Collective investment decision making with heterogeneous time preferences

Christian Gollier   Richard Zeckhauser*
University of Toulouse   Harvard University

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Abstract

We examine the investment decision problem of a group whose members have heterogeneous time preferences. In particular, they have different discount factors for utility, possibly not exponential. We characterize the properties of efficient allocations of resources, and of the shadow prices that would decentralize such allocations. We show in particular that the representative agent discounts future utility hyperbolically when all group's members discount their own future utility exponentially and have DARA preferences. We also exhibit conditions that lead the representative agent to have a rate of impatience that decreases with GDP per capita. We apply these findings to the determination of the term structure of interest rates.

Keywords: aggregation of preferences, hyperbolic discounting, impatience, time preference, investment and consumption.

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1 Introduction

Saving and investment decisions are among the most important choices of economic agents. They strongly affect the lifetime welfares of individuals and the prosperity of nations. Such decisions reflect time preferences. Most people prefer an immediate utility reward to the same reward experienced later. This pure preference for the present, or impatience, has long been recognized by economists and psychologists. The classical model introduced by Samuelson (1937) takes this into account by assuming that consumers maximize the discounted value of their flow of utility, utilizing an exponentially decreasing discount factor, i.e., with a constant rate of impatience. However, the intensity of impatience is a subject that is little understood and fiercely debated.

Frederick, Loewenstein and O'Donoghue (2002) survey several attempts to estimate individuals' discount rates. Two noteworthy findings emerge. First, high discount rates predominate. For example, Warner and Pleeter (2001) study actual financial decisions made by U.S. military servicemen, and find an estimated mean discount rate of 17.5% per year. Second, and more important, there is spectacular variation across studies and within studies across individuals, with no convergence toward an agreed-upon or unique rate of impatience. In their study, Warner and Pleeter (2001) found that individual discount rates vary between 0% and 70% per year. Barsky, Juster, Kimball and Shapiro (1997) estimated a negative mean rate of impatience by using survey responses in the Health and Retirement Study. In the literature more generally, rates range from −6% to 55700%.

These variations could stem in part from differences in the time horizon considered in the various experiments and field studies. There is no reason to believe that consumers use the same discount rate per period when discounting utility over different time horizons. Strotz (1956) was the first economist to discuss horizon-dependent discount rates. Empirical evidence suggests that agents discount future happiness at a rate that declines with the time at which the happiness will be experienced. Most typically, people use "hyperbolic discounting", i.e., declining discount rate with respect to time-horizon, rather than exponential discounting. This leads to a time-inconsistency problem that emerged recently as a "hot" topic in our profes-

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sion.[1]

This paper follows another path to explain the wide range of estimates for individual discount rates. There is no reason to believe that different consumers have identical time preferences for utility streams. Let us assume that sizeable disparities in discount rates arise because individuals strongly differ in their attitude towards time. Day-to-day evidence, say in pursuing education or bad habits, is compatible with heterogeneous time preferences. Such heterogeneity raises several questions that we explore in this paper.

When individuals use different rates of impatience to discount their future utilities, it is not clear a priori which discount rate should be used, for example, for public investments. This raises the more general question of the aggregation of preferences in a group. We consider a general model where each agent maximizes a time-additive lifetime utility. The discount rate is heterogenous in the population and it may depend upon either the time of receipt (hyperbolic discounting) or on current consumption. Agents may also have different instantaneous utility functions. We assume that the group is able to allocate consumption within the group in a Pareto-efficient way. We first show that the behavior of the group towards time can be duplicated by a representative agent whose lifetime utility functional is also time additive. Time additivity is essential to define a concept of impatience. Rubinstein (1974) also examined the question of aggregating heterogenous rates of impatience, but he derives a solution only for exponential and logarithmic utility function in a two-period model.

One of the key findings of this paper is that if individuals have heterogeneous constant rates of impatience, the representative agent will not in general use a constant rate to discount the future. More precisely, we show that if individuals have decreasing absolute risk aversion (DARA), as would seem reasonable, then the representative agent has a declining discount rate. In short, *heterogeneous individual exponential discounting yields collective hyperbolic discounting.* Under some realistic calibrations of the economy, the collective discount factor duplicates either the one discussed by Loewenstein and Prelec (1992), or its simplified "quasi-hyperbolic" version (Laibson (1997)).

To get to this result, we need to examine how agents should share resources intertemporally in an exchange economy, what might be labeled the

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[1]See for example Laibson (1997), Carrillo and Mariotti (2000), Benabou and Tirole (2000) and O'Donoghue and Rabin (2001).

multiple-cakes problem. Obviously, it is Pareto-efficient for the most impatient members to receive a larger share of the period's cake early in life; that share will be decreasing with time. This allocation is a best compromise between individual relative impatience and the agents' willingness to smooth consumption over time. This trait of individual preferences is measured by the concavity of their utility function. As shown by Wilson (1968), it is best to use the notion of (absolute) tolerance to consumption fluctuations over time. If $u(.)$ denotes the utility function of an agent, her tolerance to fluctuations is measured by $T(.) = -u'(.)/u''(.)$. It is Pareto-efficient to request that more tolerant agents bear a larger share of fluctuations of aggregate incomes. Extending a well-known result by Wilson, we show that the group's tolerance to fluctuations in the per-capita income is the unweighted mean of its members' tolerance.

Turning to pure time preferences, we show that the rate of impatience of the representative agent equals a weighted mean of individual rates of impatience. These weights are proportional to the individual tolerances to consumption fluctuations. Intuitively, agents with a very low tolerance want to smooth their consumption independent of their degree of impatience. The group will therefore not take account of these agents when determining their collective degree of impatience. Except for exponential utility functions, the weights in computing the weighted mean of individual discount rates will fluctuate over time. When T increases with wealth (DARA), those members with a smaller discount rate will see their weight increasing in tandem with time in parallel to their level of consumption. Therefore, we obtain that the rate of impatience of the representative agent decreases with time .

The fact that the representative agent uses hyperbolic discount factors in no way implies that the group faces a time-consistency problem. Suppose that each individual in the group discounts future utility in an exponential way. As is well-known, such individuals will not want to modify their portfolio of future credit/saving contracts as time moves forward. In short, the allocation of future consumption will still be Pareto-efficient tomorrow. This future allocation will correspond to another set of Pareto weights. A time-inconsistency problem arises only if some members in the group are themselves time inconsistent.

In the classical model of intertemporal choices, it is assumed that the rate of impatience is independent of wealth. We show by contrast that, even if individual rates of impatience are independent of consumption, the rate of impatience that should be used by the social planner is generally not indepen-

dent of the group's per capita consumption. We provide a sufficient condition that guarantees that more developed economies should use a smaller rate of impatience.

We apply our findings to the determination of equilibrium interest rates in the economy. We consider two equilibrium models, one with infinitely lived consumers, and another with overlapping generations. In the model with infinitely lived agents, we focus the analysis on the term structure of interest rates. Cox, Ingersoll and Ross (1985a,b) were the first to examine this question using a consumption-based approach. Whether we should use a decreasing rate to discount cash-flows occurring in the far distant future received intense debate. Significant long-term risks, such as global warming, are a new ingredient in the discussion, and beyond the scope of this analysis. In a risk-free economy with no growth, the competitive interest rate equals the rate of impatience of the representative agent. Thus, our results sustain the recommendation to use a decreasing rate of interest to discount cash-flows occurring in a more distant future. The theoretical basis for this recommendation strongly differs from those developed earlier by Weitzman (2001) and Gollier (2002a,b).

We also examine an overlapping generation growth (OLG) model, potentially with a production sector. In the classical OLG model with cohorts living only for two periods, the only possible transactions are within each cohorts. It implies that the effects of the heterogeneity of time preferences on the aggregate variables of the economy are completely characterized by the preferences of the representative agent of each cohort.

2   Assumptions on individual preferences

We consider a cohort or a group of heterogeneous agents indexed by θ in a type set Θ. They all live from date 0 to date N. Types are distributed according to cumulative distribution function $H : \Theta \to [0, 1]$. We assume that the lifetime utility $\mathcal{U}(\theta)$ of consumer θ is time-additive.[2] This excludes habit formation and anticipatory feelings. The lifetime utility of agent θ is

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[2]Following Koopmans (1960), time additivity can be derived from the independence axiom stating that if two intertemporal prospects share a common outcome at a given date, then preference between them is determined solely by the remaining outcomes that differ.

evaluated at date 0 by

$$\mathcal{U}(\theta) = \int_0^N u(c(t), t, \theta) dt, \tag{1}$$

where $c(.)$ is the consumption plan of the agent, and $u(c, t, \theta)$ is the discounted utility extracted by agent of type θ consuming c at time t. We assume that

$$u'(c, t, \theta) = \frac{\partial u}{\partial c}(c, t, \theta)$$

is continuously differentiable in (c, t), and is nonincreasing in c. If u' is decreasing in t, consumers are impatient, i.e., at any given consumption level they value future marginal utility less than current marginal utility. We hereafter redefine the two well-known indexes of sensitivity of marginal utility either with respect to t and with respect to c.

The instantaneous rate of pure time preference of agent θ consuming c at time t equals

$$\delta(c, t, \theta) = -\frac{\partial \ln u'(c, t, \theta)}{\partial t} = -\frac{\frac{\partial u'}{\partial t}(c, t, \theta)}{u'(c, t, \theta)}. \tag{2}$$

It measures the rate at which marginal utility decreases with time with consumption held constant. This definition can be rewritten as

$$u'(c, t', \theta) = u'(c, t, \theta) \exp\left[-\int_t^{t'} \delta(c, \tau, \theta) d\tau \right].$$

Impatient people have a positive δ. In the special case of a multiplicatively separable utility function $u(c, t, \theta) = \beta(t, \theta) h(c, \theta)$, δ is independent of c. If we add the assumption of exponential discounting $(\beta(t, \theta) = \exp(-\delta(\theta)t))$, δ is also independent of t. In the case of hyperbolic discounting, δ is independent of c, but is decreasing in t. In the face of the consistency problem that a non constant δ raises, we assume at this stage that agents can commit on their future consumption plan at date $t = 0$.

In a parallel way, one can define the absolute aversion to consumption fluctuations over time through the following equation:

$$A(c, t, \theta) = -\frac{\partial \ln u'(c, t, \theta)}{\partial c} = -\frac{u''(c, t, \theta)}{u'(c, t, \theta)}. \tag{3}$$

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Thus A measures the rate at which marginal utility decreases with consumption at a given time. In the risk context, it corresponds to the Arrow-Pratt index of the concavity of u with respect to c. As stated by Pratt (1964), in a static framework it satisfies:

$$u'(c',t,\theta) = u'(c,t,\theta) \exp\left[-\int_c^{c'} A(y,t,\theta)dy\right].$$

Under our assumptions, A is nonnegative. Again, under the condition of a multiplicatively separable utility function, A would be independent of t. If we assume as well exponential utility ($h(c,\theta) = -\exp(-a(\theta)c)$), A would also be independent of c. But it is usually assumed that A is decreasing in c, i.e., DARA applies. In the following, we will most often use the inverse of A, which is the index of absolute tolerance to consumption fluctuations. It is denoted $T(c,t,\theta) = 1/A(c,t,\theta)$.

3 Efficient cake sharing with heterogeneous preferences

Characterizing the optimal investment decision of a cohort of heterogeneous agents requires understanding how the cohort will share the cash flows generated by any such investment, namely how it divides the multiple cakes that become available, one per period. Suppose that agents of type θ are endowed with a flow $z(.,\theta) : [0,N] \to R$ of the single consumption good. We assume that endowments are risk free. An allocation is characterized by a set of consumption profiles $C(.,.) : R^+ \times \Theta \to R$, where $c(t,\theta)$ is the consumption of agent θ at time t. Such an allocation is feasible if at each instant of time average consumption equals average income:

$$EC(t,\widetilde{\theta}) = z(t) =_{def} Ez(t,\widetilde{\theta}) \qquad \forall t \in [0,N], \tag{4}$$

where $Ef(\widetilde{\theta}) = \int_\Theta f(\theta)dH(\theta)$ is the mean of $f(\widetilde{\theta})$ with respect to the type distribution H in the cohort.

The only restriction that we impose on the cohort's sharing of the cakes is that it be Pareto-efficient. An allocation $C(.,.)$ is Pareto-efficient if it is feasible and there is no other feasible allocation that raises the lifetime utility of at least one type without reducing the lifetime utility of the other types.

To any such efficient allocation, there exists a weight function $\lambda(.) : \Theta \to R^+$ such that it is the solution of the following social planner's program:

$$SWF(\lambda) = \max_{C(.,.)} \quad E\left[\lambda(\widetilde{\theta}) \int_0^N u(C(t,\widetilde{\theta}),t,\widetilde{\theta})dt\right] \quad s.t. \quad (4). \qquad (5)$$

In his classic analysis of the static syndicate problem, Wilson (1968) considered a decision problem that is similar to (5). He examined a decision under uncertainty for expected-utility (EU) maximizers with heterogeneous utility functions, but homogeneous beliefs. Except for this restriction, the additivity property made in the EU model under static uncertainty and in the time-additive model (1) under dynamic certainty makes these two problems equivalent. Wilson (1968) proved that the optimal collective decision policy is isomorphic to the optimal decision policy of a representative agent who also maximizes the expected value of a function of consumption per capita in the cohort. The EU functional of the representative agent is additively separable with respect to the states of nature. Wilson's result can be extended to a dynamic framework, as Constantinides (1982) has shown. The existence of a representative agent with such simple aggregative properties has become a cornerstone of theories in finance and macroeconomics. In Wilson's model, the probability weights that are used to measure individual expected utilities are the same for all individuals given that beliefs are homogeneous.[3] The parallel assumption for intertemporal choices under certainty is that agents use the same discounting function to measure their lifetime utility.

The following Proposition shows that the existence of a representative agent with a time separable lifetime utility does not require any restriction on time preference beyond the time separability of $\mathcal{U}(\theta)$ for all θ.

Proposition 1 *(Representative Agent) To any positive weight function $\lambda(.)$, there exists a representative agent with a time-additive utility function v such that $SWF(\lambda) = \int_0^N v(z(t),t)dt$. Function v is defined by*

$$v(z,t) = \max_{c(z,t,.)} E\left[\lambda(\widetilde{\theta})u(c(z,t,\widetilde{\theta}),t,\widetilde{\theta})\right] \quad s.t. \quad Ec(z,t,\widetilde{\theta}) = z. \qquad (6)$$

The associated efficient allocation is characterized by $C(t,\theta) = c(z(t),t,\theta)$ for all t and θ.

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[3]Leland (1980) reconsidered Wilson's model when agents have heterogeneous beliefs. Gollier (2003) uses techniques similar to those presented in this paper to determine how to aggregate heterogeneous beliefs in a group that can share risk in a Pareto-efficient way.

Proof: This is a direct consequence of the additivity of $SWF(\lambda)$ with respect to types and time, which implies that

$$E\left[\lambda(\widetilde{\theta})\int_0^N u(C(t,\widetilde{\theta}),t,\widetilde{\theta})dt\right] = \int_0^N E\left[\lambda(\widetilde{\theta})u(C(t,\widetilde{\theta}),t,\widetilde{\theta})\right]dt.\blacksquare$$

This Proposition enables us to decompose the multiperiod maximization program (5) into a sequence of static maximization programs (6). By imposing at each time t the feasible allocation of cake $z(t)$ that maximizes the weighted sum of discounted utilities, the social planner obtains an ex ante allocation plan that maximizes the weighted sum of the members' lifetime utilities. The time additivity of individual preference functionals is, of course, essential to get this result.

Proposition 1 disentangles the two impacts that time has on the efficient sharing of the cake and on the utility of the representative agent. First, it has a direct effect coming from the dependence of individual members' utilities on time. Second, it plays a role because income per capita, $z(t)$, is a function of time. It is useful to separate these two effects by defining $C(t,\theta) = c(z(t),t,\theta)$. In the following, we examine the properties of functions $v(.,.)$ and $c(.,.,\theta)$.

By the concavity of u with respect to its first argument, the solution to program (6) is unique. Its first-order condition is written as

$$\lambda(\theta)u'(c(z,t,\theta),t,\theta) = \psi(z,t), \tag{7}$$

for all (z,t) and θ, where ψ is the Lagrange multiplier of the feasibility constraint associated with time t and average endowment z. By the envelope theorem, the marginal value of wealth at time t is the Lagrange multiplier associated with time t. Thus we have that

$$\frac{\partial v}{\partial z}(z,t) = \psi(z,t), \tag{8}$$

for all (z,t).

4 The group's tolerance to aggregate fluctuations

We now characterize the cohort's tolerance to aggregate fluctuations in earnings. To do so, we need to determine how these fluctuations will be allocated

among the different agents in the cohort. Consider a marginal increase in the per capita income z. At time t with average consumption z, agent $\theta's$ sensitivity of consumption to such an increase is given by $\frac{\partial c}{\partial z}(z,t,\theta)$. This is referred to as the marginal propensity to consume (MPC). It tells us how fluctuations in z get transferred to fluctuations in individual consumption levels. Given the feasibility constraint $Ec(z,t,\widetilde{\theta}) = z$, it must be that, for all (z,t),

$$E\frac{\partial c}{\partial z}(z,t,\widetilde{\theta}) = 1. \tag{9}$$

The fluctuation of average consumption must equal the fluctuation of average earnings in the cohort. The following Proposition characterizes the MPC.

Proposition 2 *(Cake-Sharing) The marginal propensity to consume of agent θ at time t when the average endowment is z, is proportional to this agent's tolerance to consumption fluctuations evaluated at $c(z,t,\theta)$:*

$$\frac{\partial c}{\partial z}(z,t,\theta) = \frac{T(c(z,t,\theta),t,\theta)}{ET(c(z,t,\widetilde{\theta}),t,\widetilde{\theta})}. \tag{10}$$

Proof: Fully differentiating first-order condition (7) with respect to z yields

$$\lambda(\theta)u''(c(z,t,\theta),t,\theta)\frac{\partial c}{\partial z}(z,t,\theta) = \frac{\partial \psi}{\partial z}(z,t).$$

Eliminating $\lambda(\theta)$ by using (7) again, we rewrite the above condition as

$$\frac{\partial c}{\partial z}(z,t,\theta) = -\frac{\frac{\partial \psi}{\partial z}(z,t)}{\psi(z,t)}T(c(z,t,\theta),t,\theta). \tag{11}$$

Moreover, combining this result with condition (9) implies that

$$-\frac{\frac{\partial \psi}{\partial z}(z,t)}{\psi(z,t)}ET(c(z,t,\widetilde{\theta}),t,\widetilde{\theta}) = 1. \tag{12}$$

Eliminating the ratio in (11) and (12) yields the result.■

The Cake-Sharing Proposition states that more tolerant agents have larger marginal propensities to consume. It is intuitively appealing that people who are more tolerant to consumption fluctuations should be allocated a larger share of aggregate fluctuations. By contrast, agents who strongly dislike fluctuations, i.e., those with a small T, enjoy an efficient consumption plan that

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is relatively insensitive to aggregate fluctuations. Proposition 2 also shows that all agents have a nonnegative marginal propensity to consume out of aggregate incomes. All consumption levels are procyclical, but some are more procyclical than others.

Knowing how the cohort allocates fluctuations in aggregate earnings to different individuals determines the cohort's attitude towards fluctuations in the size of the cake. The cohort's tolerance to fluctuations in average earnings is given by

$$T_v(z,t) =_{def} -\frac{\frac{\partial v}{\partial z}(z,t)}{\frac{\partial^2 v}{\partial z^2}(z,t)}. \tag{13}$$

When the per capita endowment $z(.)$ is increasing in t, an increase in T_v will increase the cohort's propensity to invest in a normal project, i.e., a project that yields an increasing cash-flow over time. Equation (12) yields the following result.

Proposition 3 *(Tolerance to Consumption Fluctuations) The cohort's absolute tolerance to consumption fluctuations is the mean of its members' tolerances:*

$$T_v(z,t) = ET(\widetilde{c}, t, \widetilde{\theta}). \tag{14}$$

Proposition 3 has several consequences. For example, because equation (14) implies that

$$\frac{\partial T_v}{\partial z}(z,t) = E\left[T'(\widetilde{c},t,\widetilde{\theta})\frac{\partial c}{\partial z}(z,t,\widetilde{\theta})\right] = \frac{E\left[T'(\widetilde{c},t,\widetilde{\theta})T(\widetilde{c},t,\widetilde{\theta})\right]}{ET(\widetilde{c},t,\widetilde{\theta})},$$

we conclude that DARA is inherited by v from u. In other words, if all members have a tolerance that increases in c, then the cohort has a tolerance that increases with z. This extends a result obtained by Carroll and Kimball (1996) to the group context. Notice that v being DARA implies that the group as a whole is ready to pay more to smooth consumption when it is poor than when it is wealthy.

5 The group's rate of impatience

In the classic case with homogenous exponential discount factors, individuals' consumption levels vary only with fluctuations in the aggregate endowment

$z(.)$. When discount rates are heterogenous, by contrast, time enters as an additional factor. We examine the partial derivative of individual consumption levels with respect to time: $\partial c/\partial t$. When the average income z remains constant over time, it is intuitive that impatient people will trade later consumption for earlier consumption with those who are more patient. The impatient ones will have a decreasing consumption path, and vice versa.

Again, given the feasibility constraint, it must be that

$$E\frac{\partial c}{\partial t}(z, t, \widetilde{\theta}) = 0. \tag{15}$$

When the average income remains constant over time, increases in consumption by some members of the cohort must be compensated by equivalent reductions to others. Fully differentiating the first-order condition (7) yields

$$\lambda(\theta)\frac{\partial u'}{\partial t}(c, t, \theta) + \lambda(\theta)u''(c, t, \theta)\frac{\partial c}{\partial t} = \frac{\partial \psi}{\partial t}.$$

Using (2), (3) and (7) again to eliminate the Lagrange multiplier λ, we can rewrite the above equality as

$$-\delta(c, t, \theta) - A(c, t, \theta)\frac{\partial c}{\partial t} = \frac{\frac{\partial \psi}{\partial t}}{\psi},$$

or equivalently,

$$\frac{\partial c}{\partial t} = -T(c, t, \theta)\left[\frac{\frac{\partial \psi}{\partial t}}{\psi} + \delta(c, t, \theta)\right]. \tag{16}$$

Replacing $\partial c/\partial t$ in (15) by its expression from (16) yields

$$\frac{\frac{\partial \psi}{\partial t}}{\psi} = -\frac{E\left[\delta(\widetilde{c}, t, \widetilde{\theta})T(\widetilde{c}, t, \widetilde{\theta})\right]}{E\left[T(\widetilde{c}, t, \widetilde{\theta})\right]}, \tag{17}$$

where $\widetilde{c} = c(z, t, \widetilde{\theta})$. Proposition 4 characterizes the time profile of individual consumption plans when people have heterogenous discount rates. It flows from properties (16) and (17).

Proposition 4 *(Individual Consumption Path) The increase in consumption through time of agent θ is decreasing in the agent's discount rate $\delta(\theta)$:*

$$\frac{\partial c}{\partial t}(z, t, \theta) = T(c(z, t, \theta), t, \theta)\left[\overline{\delta}(z, t) - \delta(c(z, t, \theta), t, \theta)\right], \tag{18}$$

with

$$\overline{\delta}(z,t) = \frac{E\left[\delta(\widetilde{c},t,\widetilde{\theta})T(\widetilde{c},t,\widetilde{\theta})\right]}{ET(\widetilde{c},t,\widetilde{\theta})}. \tag{19}$$

The Individual-Consumption-Path Proposition determines how more patient people should substitute current consumption for future consumption. Notice that the consumption path of agent θ increases locally in t if and only if her rate of impatience is smaller than the weighted mean $\overline{\delta}$ of individual rates of impatience. More patient members postpone their consumption to the future in exchange for a positive return on their savings. Because both δ and $\overline{\delta}$ are a function of z and t, efficient consumption profiles need not to be monotone. In technical terms, the above Proposition requires that $\partial c/\partial t$ be increasing in $\delta(\theta)$ when agents have the same tolerance to consumption fluctuations. The following Corollary exhibits the weaker property of single-crossing.

Corollary 1 *Suppose that agents have the same tolerance to consumption fluctuations: $\partial T(c,t,\theta)/\partial\theta \equiv 0$. Then, individual consumption profiles satisfy the single-crossing property: $\forall(\theta,\theta') \in \Theta^2 : \delta(c,t,\theta) > \delta(c,t,\theta') \quad \forall(c,t)$ implies that*

$$c(z,t,\theta) = c(z,t,\theta') \Longrightarrow \frac{\partial c}{\partial t}(z,t,\theta) \leq \frac{\partial c}{\partial t}(z,t,\theta').$$

Proof: This is a direct consequence of equation (18).■

We can now turn to the central aim of this paper, which is to characterize the aggregation of individual discount rates. Impatience comes from the fact that, seen from $t = 0$, the marginal value of an increase in consumption decreases with the time at which it takes place. One can make impatience more explicit in the definition of the cohort's preferences by defining the cohort's instantaneous rate of impatience as

$$\delta_v(z,t) =_{def} -\frac{\partial \ln \frac{\partial v}{\partial z}(z,t)}{\partial t} = -\frac{\frac{\partial^2 v}{\partial z \partial t}(z,t)}{\frac{\partial v}{\partial z}(z,t)} = -\frac{\frac{\partial \psi}{\partial t}(z,t)}{\psi(z,t)}. \tag{20}$$

Combining conditions (20) and (17) yields the following result:

Proposition 5 *(Collective Impatience) The instantaneous rate of pure preference for the present of the representative agent defined by (20) is a weighted*

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mean of individual members' instantaneous rates:

$$\delta_v(z,t) = \frac{E\left[\delta(\widetilde{c},t,\widetilde{\theta})T(\widetilde{c},t,\widetilde{\theta})\right]}{E\left[T(\widetilde{c},t,\widetilde{\theta})\right]}. \tag{21}$$

Not surprisingly, the (implicit) psychological discount rate of the representative agent is a weighted mean of the individual rates of impatience in the cohort: $\delta_v(z,t) = \overline{\delta}(z,t)$. In such an environment, in the absence of growth, a marginal investment incurring a cost at t and yielding a benefit at $t + \Delta t$ would be socially desirable if and only if its net return would exceed $\delta_v(z,t)$. The cohort's rate of impatience can be rewritten as

$$\delta_v(z,t) = \widehat{E}_{z,t}\delta(c(z,t,\widetilde{\theta}),t,\widetilde{\theta}),$$

where $\widehat{E}_{z,t}$ is a "risk-neutral" expectation operator defined as

$$\widehat{E}_{z,t}f(\widetilde{\theta}) = \frac{\int f(\theta)T(c(z,t,\theta),t,\theta)dH(\theta)}{\int T(c(z,t,\theta),t,\theta)dH(\theta)}. \tag{22}$$

The mean of individual rates of impatience is weighted by the corresponding individual tolerances to consumption fluctuations. This weighting of the mean of δ is intuitive. Patient agents will be willing to save strongly early in life only if they are sufficiently tolerant to the consumption fluctuations they will face. To illustrate, consider a cohort with two agents. Agent 1 has a high discount rate δ_h and is somewhat tolerant to consumption fluctuations. Agent 2, by contrast, has a lower discount rate δ_l, but has a zero tolerance to consumption fluctuation. Despite his patience, agent 2 will prefer to smooth his consumption completely. Therefore, agent 1 will bear the entire burden of aggregate fluctuations. The cohort's attitude towards time is therefore determined entirely by agent 1's preferences. In particular, the cohort's degree of impatience will be the larger δ_h.

6 The term structure of the group's rate of impatience

As a direct consequence of the fact that δ_v is a weighted mean, it is bounded below and above by the smallest and largest individual rates of impatience:

$$\min_{\theta \in \Theta} \delta(c(z,t,\theta),t,\theta) \leq \delta_v(z,t) \leq \max_{\theta \in \Theta} \delta(c(z,t,\theta),t,\theta).$$

It is important to notice that the weighting function T is a function of both z and t. This is made explicit in the notation by indexing the expectation operator $\widehat{E}$ by (z,t). Thus, even if δ is independent of z and t, it is generally not true that δ_v is independent of these variables. We now examine the term structure of the cohort's rate of impatience.

Fully differentiating condition (21) with respect to t and using condition (18) yields

$$\frac{\partial \delta_v}{\partial t}(z,t) = 2(\widehat{E}\delta T')(\widehat{E}\delta) - \widehat{E}\delta^2 T' - (\widehat{E}\delta)^2(\widehat{E}T') \tag{23}$$
$$+\widehat{E}\frac{\partial \delta}{\partial t} + \widehat{E}\frac{\partial \delta}{\partial c}T(\delta_v - \delta) - \widehat{E}\frac{1}{T}\frac{\partial T}{\partial t}(\delta_v - \delta),$$

where T, δ and their derivatives are evaluated at $(c(z,t,\widetilde{\theta}),t,\widetilde{\theta})$. To simplify notation, we dropped the index to operator $\widehat{E}_{z,t}$. To examine this property, let us first focus on the traditional model of a cohort whose members are exponential discounters. In this benchmark case, individual utility functions are multiplicatively separable: $u'(c,t,\theta) = \beta(t,\theta)h'(c,\theta)$. This means that δ is independent of c, and that T is independent of t. The fact that individual members have exponential discount functions implies furthermore that $\beta(t,\theta) = \exp(-d(\theta)t)$, or that $\delta \equiv d$ is also independent of t. For this situation the second line in property (23) vanishes.

The problem here is to determine whether the cohort as a whole should use exponential discounting when all of its members use exponential discounting. It trivially should when all members have the same discount rate. A more interesting case arises when individual preferences satisfy the ISHARA property. A utility function exhibits Harmonic Absolute Risk Aversion (HARA) if its absolute risk tolerance is linear in consumption. Quadratic, exponential, power and logarithmic functions are HARA. A set of utility functions satisfies the Identically Sloped HARA (ISHARA) property if its members' absolute risk tolerances are linear in consumption with the same slope: $T'(c,\theta) = 1/\gamma$. The set of utility functions that satisfies this differential equation are represented:

$$u(c,t,\theta) = k \exp(-\delta(\theta)t)\left(\frac{c - a(\theta)}{\gamma}\right)^{1-\gamma}. \tag{24}$$

These utility functions are defined over the consumption domain such that $\gamma^{-1}(c - a(\theta)) > 0$. When $\gamma > 0$, parameter $a(\theta)$ is often referred to as the minimum level of subsistence. This preference set includes preferences with

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heterogeneous exponential utility functions $u(c,t,\theta) = -\beta(t,\theta)\exp(-A(\theta)c)$ when γ tends to $+T$, and $a(\theta)/\gamma$ tends to $-1/A(\theta)$. Taking $a(\theta) = 0$ for all θ, it also includes the set of power (and logarithmic) utility functions with the same relative concavity index γ for all θ. Under this set of conditions, equation (23) simplifies to

$$\frac{\partial \delta_v}{\partial t}(z,t) = -T'\left[\widehat{E}\delta^2 - (\widehat{E}\delta)^2\right].$$

Using Jensen's inequality, we conclude that the term structure of the social discount rate is decreasing if T' is a positive constant, and that it is increasing when T' is a negative constant. T' positive is a standard assumption in economics; it corresponds to decreasing absolute risk aversion (DARA) in the context of uncertainty. DARA means that agents have a tolerance to consumption fluctuations that increases with their wealth. Proposition 6 shows that the constancy of T' can be relaxed at no additional cost.

Proposition 6 *(Hyperbolic Collective Impatience) Suppose that agents have multiplicatively separable utility functions with exponential discount: $u(c,t,\theta)$ $= k\exp(-\delta(\theta)t)\nu(c,\theta)$. The term structure of the social rate of impatience δ_v is decreasing (increasing) if all utility functions $\nu(.,\theta)$, $\theta \in \Theta$, exhibit decreasing (increasing) absolute risk aversion.*

Proof: We consider the case of DARA (T' positive). Dividing equation (23) by $\widehat{E}T'$, the discount rate δ_v is decreasing with respect to time if

$$2\left(\widehat{E}\delta(\widetilde{\theta})\right)\left(\overline{E}\delta(\widetilde{\theta})\right) \leq \left(\widehat{E}\delta(\widetilde{\theta})\right)^2 + \overline{E}(\delta(\widetilde{\theta}))^2, \tag{25}$$

where $\overline{E}\delta(\widetilde{\theta}) = \int \delta(\theta)dF(\theta)$, and

$$dF(\theta) = \frac{T(c(\theta),\theta)T'(c(\theta),\theta)}{ET(c(\widetilde{\theta}),\widetilde{\theta})T'(c(\widetilde{\theta}),\widetilde{\theta})}dH(\theta).$$

Because T' is uniformly positive, F can be interpreted as a cumulative probability function. Observe that

$$2\left(\widehat{E}\delta(\widetilde{\theta})\right)\left(\overline{E}\delta(\widetilde{\theta})\right) \leq \left(\widehat{E}\delta(\widetilde{\theta})\right)^2 + \left(\overline{E}\delta(\widetilde{\theta})\right)^2. \tag{26}$$

Moreover, we know from Jensen's inequality that

$$\left(\overline{E}\delta(\widetilde{\theta})\right)^2 \leq \overline{E}(\delta(\widetilde{\theta}))^2. \tag{27}$$

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Obviously, combining (26) and (27) yields (25), which concludes the proof.■

Notice that we neither restrict T to be linear, nor assume any correlation between rates of impatience and degrees of tolerance to fluctuations. The monotonicity of these degrees of tolerance is the only thing that matters for the slope of the term structure of δ_v. Simple intuition supports this important result. From equation (18), we know that more patient consumers have an increasing consumption profile. Under DARA, their tolerance to consumption fluctuations increases through time. This implies that when time goes forward, consumers with a low δ see their weight growing in the mean $\delta_v(z,t) = \hat{E}_{z,t}\delta(\widetilde{\theta})$. This implies that the social rate of impatience decreases with time.

Example 1 *We illustrate this result with a simple example. There are two groups of agents, respectively with constant rates of impatience δ_l and $\delta_h > \delta_l$. All agents have the same felicity function $\nu(c,\theta) = \min(b(c-a), d(c-a))$, with $0 < d < b$. This function is piecewise linear with a kink at $c = a$. We consider the case where b tends to infinity, which means that the left branch of the curve becomes vertical. Parameter a is the minimum level of subsistence. On the relevant domain $[a, +\infty[$ of this limit function, agents have a nondecreasing tolerance (DARA), with a zero tolerance at $c = a$, and an infinite tolerance for all $c > a$. In this economy, any Pareto-efficient sharing of the cake produces a flip-flop consumption pattern. Prior to some identified date τ, the patient group functions at subsistence, and the impatient group consumes any surplus. After time τ, the impatient group falls to subsistence, and the patient group enjoys any surplus. As a consequence, the social rate of impatience $\delta_v(z,t)$ equals δ_h prior to τ, and δ_l thereafter. The term structure is a simple downward step function in this case. Rates of impatience that are decreasing with time horizon are often refered to as "hyperbolic" discounting. Phelps and Pollak (1968), then followed by Laibson (1997) and many others afterwards, introduced this stepwise functional form to describe observed psychological discount rates. This special case is often refered to the "beta-delta" model.*

Example 2 *This discounting functional would emerge as the socially efficient rule for less extreme examples. Let us replace the piecewise-linear felicity function by a power felicity function. The two equally weighted groups have the same constant relative risk aversion γ. Under this "fair" efficient*
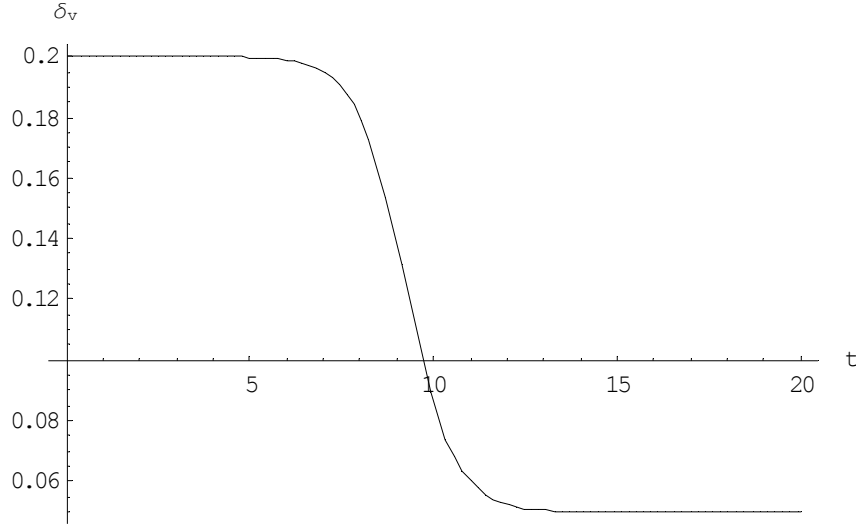
Figure 1: The discount rate as a function of time horizon for two-agent group with $\delta_h = 20\%$ and $\delta_l = 5\%$, when $\nu(c, \theta) = c^{0.9}$.

allocation of resources, the group's discount rate as a whole can be written as

$$\delta_v(z, t) = \frac{\delta_l^{\frac{1+\gamma}{\gamma}} e^{-\frac{\delta_l t}{\gamma}} + \delta_h^{\frac{1+\gamma}{\gamma}} e^{-\frac{\delta_h t}{\gamma}}}{\delta_l^{\frac{1}{\gamma}} e^{-\frac{\delta_l t}{\gamma}} + \delta_h^{\frac{1}{\gamma}} e^{-\frac{\delta_h t}{\gamma}}}.$$

When γ tends to zero, this function of t can be approximated by a downward step function with step levels at δ_h and δ_l.[4] In Figure 1, we draw this function for $\delta_h = 20\%$, $\delta_l = 5\%$ and $\gamma = 0.1$.

These two examples provide an additional intuition for why the social rate of impatience should be decreasing. Consider in particular example 2 which is illustrated by Figure 1. Consider a marginal investment by the cohort that would move some cohort's income from time t to $t - \Delta t$. If t is small, this change in the structure of the cash-flows will mostly benefit those who consume the largest share of the cohort's cake early in life. These are the more impatient agents. It is then intuitive that the social planner uses the (high) rate of impatience of these agents when performing the cost-benefit analysis of this investment project. On the contrary, for an investment

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[4]The step occurs at time horizon $t = (\ln \delta_h - \ln \delta_l)/(\delta_h - \delta_l)$.

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project moving some of the cohort's income from a larger time t to $t - \Delta t$, it will be the more patient members who will mostly benefit from this change, because they consume the larger share of the cake at those time horizons. The social planner will thus use their smaller rate of impatience to perform the cost-benefit analysis of this alternative investment project. In short, the social planner will use a rate of impatience which is decreasing with the time horizon from δ_h to δ_l.

The next example is interesting because it generates a fuctional form for the social discount rate that fits some of those that are already existing in the literature.

Example 3 *Suppose that $\nu(c, \theta) = c^{1-\gamma}/(1 - \gamma)$ and that $\delta(\theta) = \theta$. Suppose moreover that discount rates θ are distributed following a negative exponential law $\widetilde{\theta} \sim f(\theta) = e^{-\theta/\mu}/\mu$ on $\Theta \equiv [0, +\infty[$, with a mean $E\widetilde{\theta} = \mu$. We consider the Pareto-efficient allocation that corresponds to the weighting function λ such that $\lambda(\theta) = \theta^\eta$ for some scalar η.[5] In this illustration, it can be verified that*

$$\delta_v(z, t) = \frac{\eta + \gamma}{t + \frac{\gamma}{\mu}}, \tag{28}$$

which is independent of z. When relative risk aversion γ tends to infinity, δ_v tends to μ uniformly for all t. When γ tends to zero, $\overline{\delta}(z, t)$ tends uniformly to η/t. In Figure 2, we draw the maximum discount rate δ_v as a function of time when $\gamma = 2$ (relative risk aversion) and $\mu = 5\%$ (mean discount rate). As seen in (28), the socially efficient discount rate δ_v declines with time t as $1/(t + \frac{\gamma}{\mu})$. The discount factor $\beta(t)$ can be written as

$$\beta(t) = \exp\left[-\int_0^t \delta_v(z, \tau)d\tau\right] = \left[1 + \frac{\mu t}{\gamma}\right]^{-(\eta + \gamma)}. \tag{29}$$

This is the functional form suggested by Loewenstein and Prelec (1992), who generalized earlier proposals made by Herrnstein (1981) and Mazur (1987).

It is useful to examine how consumption is allocated in this economy. The set of first-order conditions (7) combined with the feasibility constraints can

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[5] One conception of fairness when all agents have the same utility function would set $\eta = 1$. This implies that the mean weight of individuals' felicity over their (infinite) lifetime is the same for everyone: $\int_0^\infty \lambda(\theta)e^{-\theta t}dt = 1$ for all θ.

be solved analytically to yield

$$c(z,t,\delta) = \frac{\mu z}{\Gamma\left(\frac{\gamma+\eta}{\gamma}\right)} \left(\frac{t}{\gamma} + \frac{1}{\mu}\right)^{\frac{\gamma+\eta}{\gamma}} \delta^{\eta/\gamma} e^{-\frac{\delta t}{\gamma}}, \qquad (30)$$

where $\Gamma(x) = \int_0^\infty \delta^{x-1} e^{-\delta} d\delta$ is the Gamma function. In Figure 3, we draw the efficient consumption plan for a few types when the average earnings in the population remain constant over time and are normalized to unity. We see again what drives the declining term structure of δ_v: At $t = 0$, individual consumption levels and individual degrees of tolerance are positively related to the individual rates of impatience. This weighting leads to a social rate of impatience that is greater than μ. As time goes forward, most resources go to those with low discount rates, and the social rate of impatience falls below μ. Notice that, following condition (18) with $r \equiv \delta_v$, the consumption profile of agent θ is locally increasing as long as $\delta(\theta)$ is less than $\delta_v(z,t)$. Because δ_v is decreasing in t, consumption profiles of all agents with a rate of impatience $\delta(\theta)$ less than $\delta_v(z,0) \simeq 7.5\%$ are hump-shaped, whereas those agents with a rate of impatience greater than 7.5% have decreasing consumption throughout. In general, efficient consumption profiles are either decreasing or hump-shaped under the assumptions of Proposition 6.[6]

It follows immediately from equation (23) that one can relax the assumption that members of the cohort use exponential discounting. If $\partial\delta/\partial t$ is nonpositive for all $\theta \in \Theta$, that will just reinforce the negativity of the right-hand side of this equation.

Corollary 2 *Suppose that agents have multiplicatively separable utility functions with hyperbolic discounting: $u(c,t,\theta) = k \exp(-\delta(t,\theta)t) h(c,\theta)$ and $\partial\delta/\partial t \leq 0$. The term structure of the social rate of impatience δ_v is decreasing if all utility functions $h(.,\theta)$ exhibit decreasing absolute risk aversion.*

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[6]The figure does not extend far enough to show the consumption for the individual with $\delta = 1$ to begin to fall. The limit case is the agent with $\delta = 0$; she is the only one to have a steadily increasing consumption plan.
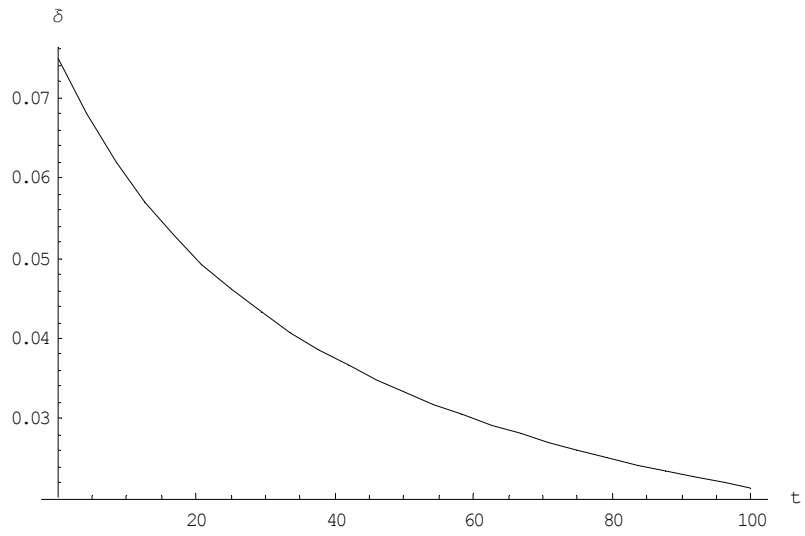
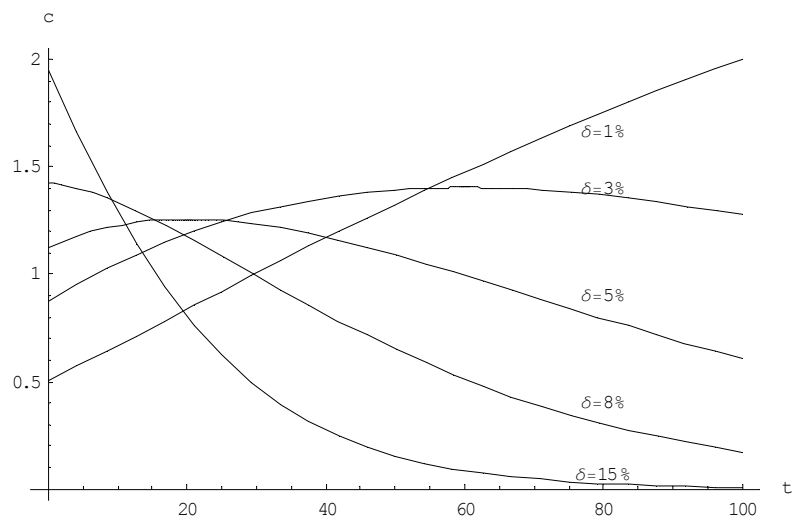Figure 2: The social rate of impatience δ_v as a function of time.

Figure 3: Efficient consumption path for agents with different discount rates δ.

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7   A wealth effect on the group's impatience

In the standard model of consumption, saving and growth, rates of impatience are assumed to be independent of consumption levels: $\partial\delta/\partial c \equiv 0$. However, it is often observed that wealthier economies are more patient. In our notation, this would mean that δ_v is decreasing in z. In this section, we examine whether these two assumptions can be compatible.

Observe that we found that δ_v is independent of z in our three examples. These examples also illustrate the following Proposition.

Proposition 7 *Suppose that u is multiplicatively separable in such a way that $u(c,t,\theta) = k\beta(t,\theta)h(c,\theta)$. The following conditions are equivalent:*

1. *For any distribution of individual discount factors and of Pareto weights, the social rate of impatience is independent of the consumption per capita z.*

2. *The members of the cohort have ISHARA preferences (24), i.e., $h(c,\theta) = \left(\frac{c-a(\theta)}{\gamma}\right)^{1-\gamma}$.*

Proof: Fully differentiating equation (21) with respect to z and using property (10) yields

$$(ET)\frac{\partial\delta_v}{\partial z}(z,t) = \widehat{E}\delta T' - (\widehat{E}\delta)(\widehat{E}T'), \qquad (31)$$

where T, δ and their derivative are evaluated at $(c(z,t,\widetilde{\theta}),t,\widetilde{\theta})$, and where $\widehat{E}$ is the "risk-neutral" expectation operator defined by (22). For ISHARA preferences, T' is a constant. This implies that the right-hand side of the above equality is zero. To prove the necessity of ISHARA, suppose that there exist (c_a,θ_a) and (c_b,θ_b) such that $T'(c_a,\theta_a) > T'(c_b,\theta_b)$. Consider individual discount functions β such that $\delta(t,\theta_a) > \delta(t,\theta_b)$. Let z denote the consumption per capita in the cohort when all agents who have type θ_a consume c_a and all agents who have type θ_b consume c_b. Then, select $\lambda(.)$ such that $\lambda(\theta)$ tends to zero for all $\theta \notin \{\theta_a,\theta_b\}$. Then, select $\lambda(\theta_a)$ and $\lambda(\theta_b)$ such that

$$\lambda(\theta_a)\beta(t,\theta_a)h'(c_a,\theta_a) = \lambda(\theta_b)\beta(t,\theta_b)h'(c_b,\theta_b).$$

This means that at time t with wealth per capita z, it is socially efficient for agents θ_i to consume c_i, $i = a, b$. Because of the positive correlation between

δ and T', we get that the right-hand side of equation (31) is positive. This implies that δ_v is increasing in wealth locally at z, a contradiction.∎

Except in the ISHARA case, the representative agent need not have a multiplicatively separable utility function even when each member in the cohort has a multiplicatively separable utility function. Rubinstein (1974) obtained the same wealth irrelevancy property in the special case of exponential and logarithmic utility functions. In the following Proposition, by contrast, we assume that more patient people have a tolerance to fluctuations that is more sensitive to changes in consumption. In such a situation, the social rate of impatience will be decreasing with the consumption per capita in the economy, despite the fact that consumers' impatience does not depend on consumption.

Proposition 8 *Suppose that the members of the cohort have a rate of impatience that is independent of their consumption: $\partial\delta/\partial c \equiv 0$. Suppose also that their tolerance to fluctuations is linear with respect to their consumption: $\partial T'/\partial c \equiv 0$. The social rate of impatience at t will be decreasing with the consumption level per capita if δ and T' evaluated at t are anti-comonotone: $\forall(\theta,\theta') \in \Theta^2 : [\delta(c,t,\theta) - \delta(c,t,\theta')]\,[T'(c,t,\theta) - T'(c,t,\theta')] \leq 0$.*

Proof: Under these assumptions, we have that

$$\widehat{E}\frac{\partial\delta}{\partial c}T = 0$$

and

$$\widehat{E}\delta T' \leq (\widehat{E}\delta)(\widehat{E}T').$$

From (31), this implies that δ_v is decreasing in z at t.∎

A simple intuition supports this result. It comes from the fact that the social rate of impatience is a weighted mean of individual rates of impatience. When δ and T' are anti-comonotone, an increase in wealth differentially increases the weights associated with the lower rates of impatience. An increase in z then pushes δ_v downwards.

In the special case of utility functions exhibiting constant relative risk aversion (CRRA), viz. $u'(c,t,\theta) = \beta(t,\theta)c^{-\gamma(\theta)}$, there is a negative relationship between relative risk aversion $\gamma(\theta)$ and $T'(c,t,\theta) = 1/\gamma(\theta)$. Thus, the above Proposition applied to the case of CRRA utility function implies that the rate of impatience is a decreasing function of societal wealth if more patient people are also less risk-averse. The following example illustrates this point.
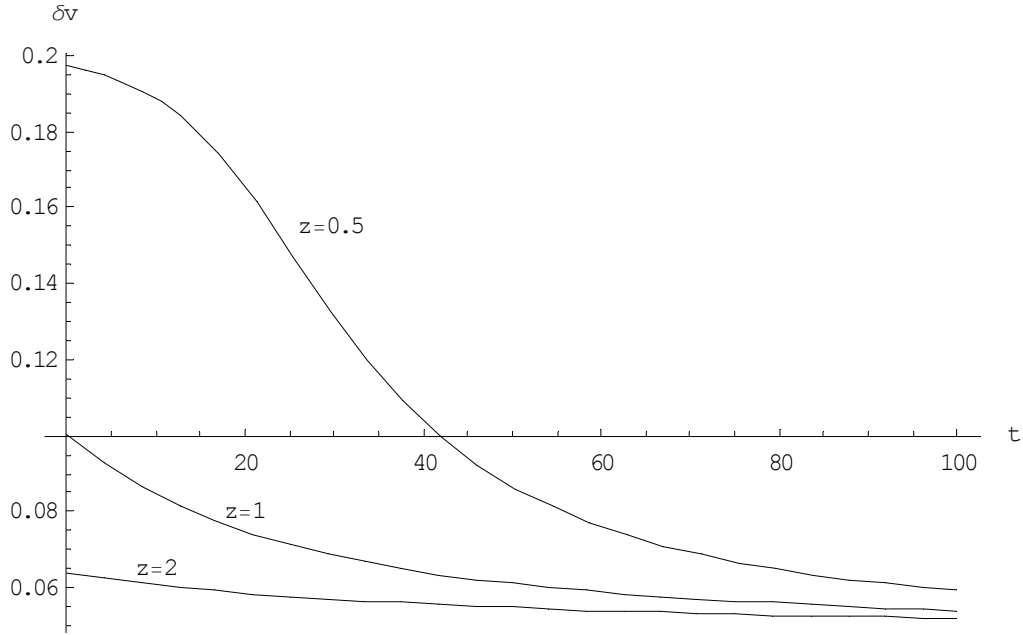
Figure 4: The term struture of the social rate of impatience when $u'(c, t, \theta_l) = e^{-0.05t}c^{-1}$ and $u'(c, t, \theta_h) = e^{-0.2t}c^{-10}$.

Example 4 *Consider an economy with two groups of equal size. Rates of impatience are constant, hence independent of time and consumption levels. The first group has low impatience rate δ_l and a logarithmic felicity function ($\gamma_l = 1$). The second group has a larger rate of impatience $\delta_h > \delta_l$ and a larger constant relative risk aversion $\gamma_h > 1$. We derived numerically the Pareto-efficient allocation correponding to equal Pareto weights $\lambda_l = \lambda_h$, in the case of $\gamma_h = 10$, $\delta_l = 0.05$ and $\delta_h = 0.20$. Figure 4 shows the term structure of the social rate of impatience when the consumption per capita z equals $0.5, 1$ and 2. We see that a larger per capita consumption yields a smaller rate of impatience for all time horizons, as proved by Proposition 8.*

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8 The interest rate and its term structure with a single generation

Up to now, we characterized the intertemporal preferences of a cohort. In the remaining of the paper, we use these characteristics to determine the optimal saving and investment choices and equilibrium interest rates. We will consider two different economic settings. In this section, we assume that there is a single generation of infinitely lived agents, whereas we consider the case of overlapping two-period generations in the next section.

We consider here an economy with a single cohort of infinitely lived agents. The collective choice problem is described by the opportunity to invest in various projects indexed by $j \in J$. Project j yields a per-capita flow of incomes $\zeta_j(.) : R^+ \to R$. The collective decision problem is to select the project that maximizes the lifetime utility of the cohort:

$$\max_{j \in J} \quad \int_0^\infty v(\omega(t) + \zeta_j(t), t)dt, \tag{32}$$

where $\omega(t)$ is the endowment per capita of the single consumption good at date t. Suppose that $z(.)$ is the optimal flow of consumption per capita in this economy. Consider in particular a marginal investment project that costs dx at time t and that yields a benefit $e^{rdt}dx$ at time $t + dt$.[7] Parameter r is the return of this investment project. Assuming that the consumption path is continuously differentiable, z is indeed optimal if the lifetime social utility is unaffected by this marginal investment, that is if

$$-v'(z(t), t) + e^{rdt}v'(z(t + dt), t + dt) = v'(z(t), t)\left[-1 + e^{rdt}e^{-\delta_v dt}e^{-A_v z'(t)dt}\right]$$

is zero, where $A_v = 1/T_v = -v''/v'$ is the collective degree of aversion to consumption fluctuations. This is equivalent to r being equal to

$$r(z, t) = \delta_v(z, t) + A_v(z, t)\frac{\partial z}{\partial t}. \tag{33}$$

This equation characterizes the equilibrium forward interest rate corresponding to date t, given the consumption per capita z and its growth z'. Condition

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[7]Obviously, these flows occur over a period of time of measure 0, with no effect on $\mathcal{U}$. In reality, costs and benefits are incurred during respectively period $[t, t + \varepsilon]$ and period $[t + dt, t + dt + \varepsilon]$.

(33) usefully extends the well-known property of optimal consumption when preferences exhibit exponential discounting. It shows the relationship between the psychological discount factor δ and the financial discount rate r appropriately employed to discount monetary cash flows. The shadow price of time, r, is the sum of two terms. The first is δ_v, the collective rate of impatience. The second term comes from the collective preference for consumption smoothing. When large consumption growth is expected, a large interest rate is required to induce agents to save. Otherwise, they would want to borrow money today to smooth the expected increase in their future incomes.[8]

The yield curve in this economy is characterized by $\rho(t) = r(z(t), t)$. When there are no aggregate fluctuations, socially efficient interest rates just equal the social discount rates whose term structure was examined in the previous two sections. Observe in particular that in the absence of growth, the yield curve will be decreasing when consumers have heterogeneous rates of impatience and decreasing absolute aversion to fluctuations. When the economic growth rate z'/z is not zero, the second term in the right-hand side of (33) comes into play. In this section, we examine the properties of this consumption smoothing effect. We first consider the direct effect of time on the consumption-smoothing term $A_v z'$. Because by definition we have that

$$\frac{\partial A_v}{\partial t}(z, t) = \frac{\partial \delta_v}{\partial c}(z, t) \tag{35}$$

for all (z, t), the previous section gives us some information on this problem. Consider for example the case where agents have ISHARA preferences. Proposition 7 combined with property (35) tells us that T_v is independent of t in that case. It implies that the interest rate formula (33) is time invariant in this case. In the next Proposition, we consider a special case of this where all agents have the same constant relative aversion γ, but they differ about their constant rate of impatient $\delta(\theta)$.

Proposition 9 *Suppose that $u(c, t, \theta) = ke^{-\delta(\theta)t}c^{1-\gamma}/(1-\gamma)$ for all (c, t, θ).*

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[8]Reversing the argument, if the equilibrium interest rate r is given by a production technology with constant returns to scale, the equilibrium growth of the economy is written as

$$\frac{\partial z}{\partial t} = T_v(z, t) \left[r - \delta_v(z, t) \right]. \tag{34}$$

This is the Keynes-Ramsey rule extended to heterogeneous time preferences in the economy.

Suppose also that the growth rate $z'(t)/z(t)$ of the economy is independent of time. If there exists at least one pair (θ, θ') such that $\delta(\theta) \neq \delta(\theta')$, the yield curve is decreasing.

Proof: Because this is a special case of ISHARA preferences, we know from Proposition 7 that δ_v is independent of z. Using equation (14), we have that

$$T_v(z,t) = ET(\widetilde{c}, t, \widetilde{\theta}) = E\widetilde{c}/\gamma = z/\gamma.$$

This implies that

$$\rho(t) = r(z(t), t) = \delta_v(t) + \gamma \frac{z'(t)}{z(t)}.$$

Because z'/z is independent of t, and because δ_v is decreasing in t because of Proposition 6, we obtain that ρ is decreasing in t. $\blacksquare$

In the next Proposition, we consider the alternative case where agents have heterogenous CRRA preferences. In a static context with uncertainty, Hara and Kuzmics (2002) show that this implies that the representative agent has an R_v which is decreasing with respect to z.[9] We now provide a shorter proof of this result. As shown by Gollier (2002a), it implies that the yield curve is decreasing under the standard assumptions.

Proposition 10 *Suppose that agents have constant, identical rates of impatience, and that consumption per capita is growing at a constant rate. Then if agents have heterogenous but constant rates of relative risk aversion, the yield curve is decreasing.*

Proof: Because we assume that agents have homogenous rates of impatience, the first term in the right-hand side of (33) is independent of t. Moreover, we know that it also implies that $\partial A_v/\partial t$ vanishes. We are done if $\partial z A_v/\partial z$ is nonpositive. Differentiating $zA_v(z,t) = z/T_v(z,t)$ with respect to z implies that this is the case if

$$ET(c(\widetilde{\theta}), t, \widetilde{\theta}) \leq z \frac{ET'(c(\widetilde{\theta}), t, \widetilde{\theta}) T(c(\widetilde{\theta}), t, \widetilde{\theta})}{ET(c(\widetilde{\theta}), t, \widetilde{\theta})},$$

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[9]Calvet, Grandmont and Lemaire (2001) obtain a similar result.

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or equivalently, since $T(c, t, \theta) = c/\gamma(\theta)$, if

$$\left[E \frac{c(\widetilde{\theta})}{\gamma(\widetilde{\theta})} \right]^2 \le Ec(\widetilde{\theta}) E \frac{c(\widetilde{\theta})}{(\gamma(\widetilde{\theta}))^2}.$$

The Cauchy-Schwarz inequality implies that this is always true.∎

9 A simple overlapping generation model

In this section, we consider a more realistic economy with overlapping generations of savers/consumers. In any stationary equilibrium of credit markets, the short-term interest rate must be constant through time, and the yield curve must be flat. Our analysis of an OLG model will address how the heterogeneity of time preferences affects the growth of the economy and the level of interest rates at equilibrium. We will consider the classical discrete-time OLG model in which each cohort lives only two periods. This implies that credit markets only allow for resource exchanges within cohorts. It implies in turn that the heterogeneity of time preferences affects the dynamics of the economy only through its effect on the time preferences of the representative agents of the successive generations. We assume that the distribution of preferences within each cohort remains stable through time.

In the discrete version of the model analyzed earlier in this paper, an agent of type θ has a lifetime utility described by

$$\mathcal{U}(\theta) = u(c_1, \theta) + \beta(\theta)u(c_2, \theta),$$

where c_t is her consumption at age $t = 1$ or 2. Observe that we consider here the simplifying assumption of a multiplicatively separable discount factor β. The representative agent of a cohort is given by

$$v(z, 1) = \max_{c_1(z,.)} E \left[\lambda(\widetilde{\theta}) u(c_1(z, \widetilde{\theta}), \widetilde{\theta}) \right] \text{ s.t. } Ec_1(z, \widetilde{\theta}) = z, \qquad (36)$$

for the cake sharing problem at young age, and by

$$v(z, 2) = \max_{c_2(z,.)} E \left[\lambda(\widetilde{\theta}) \beta(\widetilde{\theta}) u(c_2(z, \widetilde{\theta}), \widetilde{\theta}) \right] \text{ s.t. } Ec_2(z, \widetilde{\theta}) = z, \qquad (37)$$

for the old age. We assume that the ISHARA condition holds with $T'_u(c, \theta) = \gamma^{-1}$ for all (c, θ). This implies that there exists a scalar β_v such that $v(z, 2) =$

$\beta_v v(z,1)$. Adapting Appendix A in Gollier (2003) to this intertemporal context, the discount factor β_v of the representative agent equals

$$\beta_v = \left[\frac{E\left[\lambda(\widetilde{\theta})^{1/\gamma}\beta(\widetilde{\theta})^{1/\gamma}\right]}{E\left[\lambda(\widetilde{\theta})^{1/\gamma}\right]} \right]^{\gamma}. \tag{38}$$

To sum up, in the ISHARA case, the heterogeneity of time preferences has no further effect on the classical OLG growth model than modifying the discount rate of each cohort's representative agent. The simplest case is in the logarithmic case $u(c,\theta) = \ln(c)$ where $\beta_v = E\beta(\widetilde{\theta})$. When consumers are logarithmic, the heterogeneity of impatience has no effect at all on the equilibrium growth, and the economy evolves over time as in an homogeneous economy in which all agents would have the same discount factor $E\beta(\widetilde{\theta})$.

A more realistic case has $u(c,\theta) = c^{1-\gamma}/(1-\gamma)$ with an index of relative risk aversion γ larger than unity. Applying Jensen's inequality to equation (38) implies that

$$\beta_v \leq \frac{E\left[\lambda(\widetilde{\theta})^{1/\gamma}\beta(\widetilde{\theta})\right]}{E\left[\lambda(\widetilde{\theta})^{1/\gamma}\right]}.$$

In this alternative case, the heterogeneity of time preferences raises the collective degree of impatience of each cohort. This implies that the interest rate will be larger, that the equilibrium growth rate of the economy will be reduced, and that the steady state will entail a smaller consumption per capita. Notice that for non-logarithmic utility functions, each cohort's rate of impatience will depend upon the Pareto weights λ. If the allocation of consumption within each cohort is decentralized through competitive credit markets, this allocation will depend upon the interest rate.

10 Conclusion

Groups do not generally behave towards time as do individual consumers. For example, the group's rate of impatience is not independent of the group's wealth level. However, the basic property of additivity of individual preferences is transmitted to the preferences of the representative agent. This implies that the representative agent of the group has no consumption habits

and no anticipatory feelings if its members don't also have such psychological traits.

The main objective of the paper was to aggregate heterogeneous time preferences. It shows that the local collective rate of impatience is a weighted mean of the members' local rates of impatience. Each member's weight is proportional to her degree of absolute tolerance to consumption fluctuations. This aggregation rule implies that the collective rate of impatience is decreasing with respect to the time horizon when wealthier consumers are less averse to consumption fluctuations, a common assumption. For long horizons, any transfer of the group's wealth across time will mostly affect the most patient agents because they are those who have the largest stake on aggregate wealth. Thus, when considering investments affecting cash flows corresponding to these long horizons, the group should use the lower rate of impatience in the group for cost-benefit analysis. On the contrary, for short-time horizons, transferring wealth across time affects mainly the consumption flow of the most impatient agents. In the collective cost-benefit analysis for such investments, the larger rate of impatience of these agents should be employed. This reasoning presupposes, of course, that the group is able to redistribute consumption within the group in response to each agent's degree of impatience.

References

Barsky, R.B., F.T. Juster, M.S. Kimball and M. Shapiro, (1997), Preference parameters and behavioral heterogeneity: An experimental approach in the health and retirement study, *Quarterly Journal of Economics*, 537-79.

Benabou, R., and J. Tirole, (2000), Self-confidence: Intertemporal strategies, Princeton University discussion paper 209.

Calvet, L., J.-M. Grandmont and I. Lemaire, (2001), Aggregation of heterogenous beliefs and asset pricing in complete financial markets, mimeo, Harvard University and CREST (Paris).

Carrillo, J.D., and T. Mariotti, (2000), Strategic ignorance as a self-disciplining device, *Review of Economic Studies*, 67:3, 529-544.

Caroll, C.D. and M. Kimball, (1996), On the concavity of the consumption function, *Econometrica*, 64, 981-92.

Cass, D. and J. Stiglitz, (1970), The structure of investor preferences and asset returns, and separability in portfolio allocation, *Journal of Economic Theory*, 2, 122-60.

Cochrane, J., (2001), *Asset Pricing*, Princeton University Press.

Constantinides, G.M., (1982), Intertemporal asset pricing with heterogeneous consumers and without demand aggregation, *Journal of Business*, 55, 253-67.

Cox, J., J. Ingersoll and S. Ross, (1985a), A theory of the term structure of interest rates, *Econometrica*, 53, 385-403.

Cox, J., J. Ingersoll and S. Ross, (1985b), An intertemporal general equilibrium model of asset prices, *Econometrica*, 53, 363-384.

Frederick, S., G. Loewenstein and T. O'Donoghue, (2002), Time discounting and time preference: A critical review, *Journal of Economic Literature*, 40, 351-401.

Gollier, C., (2002a), Discounting an uncertain future, *Journal of Public Economics,* 85, 149-166.

Gollier, C., (2002b), Time horizon and the discount rate, *Jouranl of Economic Theory*, 107, 463-473.

Gollier, C., (2003), Collective risk-taking decisions with heterogeneous beliefs, CESifo Discussion Paper.

Hansen, L. and K. Singleton, (1983), Stochastic consumption, risk aversion and the temporal behavior of assets returns, *Journal of Political Economy*, 91, 249-268.

Hara, C., and C. Kuzmics, (2001), Representative consumer's risk aversion and efficient risk-sharing rules, mimeo, Cambridge University.

Herrnstein, R., (1981), Self-control as response strength, in *Quantification of steady-state operant behavior*, C.M. Bradshaw, E. Szabadi and C.F. Lowe Eds, Elsevier/North Holland.

Koopmans, T.C., (1960), Stationary ordinal utility and impatience, *Econometrica* 28, 287-309.

Laibson, D.I., (1997), Golden eggs and hyperbolic discounting, *Quarterly Journal of Economics*, 62, 443-479.

Leland, H.E., (1980), Who Should Buy Portfolio Insurance?, *Journal of Finance*, 35, 581-596.

Loewenstein, G., and D. Prelec, (1991), Negative time preference, *American Economic Review*, 81, 347-352.

Mazur, J.E., (1987), An adjustment procedure for studying delayed reinforcement, in *The effect of delay and intervening events on reinforcement value*, M.L. Commons, J.E. Mazur, J.A. Nevin and H. Rachlin Eds, Hillsdale, NJ: Erlbaum.

O'Donoghue, T., and M. Rabin, (2001), Choice and procrastination, *Quarterly Journal of Economics*, 116:1, 121-160.

Phelps, E.S., and R. Pollack, (1968), On second-best national saving and game equilibrium growth, *Review of Economic Studies*, 35, 185-199.

Rubinstein, M., (1974), An aggregation theorem for securities markets, *Journal of Financial Economics*, 1, 225-244.

Samuelson, P.A., (1937), A note on the measurement of utility, *Review of Economic Studies*, 4, 155-161.

Strotz, R.H., (1956), Myopia and inconsistency in dynamic utility maximization, *Review of Economic Studies*, 23:3, 165-180.

Warner, J.T., and S. Pleeter, (2001), The personal discount rate: Evidence from military downsizing programs, *American Economic Review*, 95:4, 547-580.

Weitzman, M.L., (2001), Gamma discounting, *American Economic Review*, 91, 260-271.

Wilson, R. (1968). "The theory of syndicates", *Econometrica* 36, 113-132.