Attribute Choices and Structural Econometrics of Price Elasticity of Demand

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Abstract

The identification of demand parameters from individual data may be impossible due to the lack of price variation within the sample. Even if panel data are available, the slow modification of price within the menu and the usual small number of observed periods make very hazardous the estimation of prices effects. The aim of this paper is to deliver an empirical methodology for the treatment of this kind of data. The approach relies on a simple hedonic model of consumer behavior wherein aggregate demand and expenditure depend on an heterogeneity factor. Using the restrictions created by this structural model, we consider the identification of the price elasticity. We first show that the price elasticities as well as other parameters that summarize the consumers'preferences are identified in a two–period case. An empirical illustration with actual data thus illustrates the potential of the approach. The paper then propose several extensions of the model – multi–products, non–linear demand – and determine conditions for identification.

Keywords: hedonic model, structural econometrics, heterogeneity factor, identification, price elasticity

JEL Class.: C1, C2, D1

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Introduction

This paper considers the identification of demand parameters over different periods with heterogenous consumers and unobserved product characteristics. This paper thus follows the application of hedonic models (see Rosen (1974), Brown and Rosen (1982), Epple (1987), Bartik (1987) among others). These models allow to analyze the demand or the supply of quality associated for example to a good, a job or environment. As noted by Ekeland, Heckman and Nesheim (2002), the problem of identification in hedonic models constitutes a reference for the identification problem in a variety of economic models in which the individual characteristics are unobservable to the econometrician. Moreover, the recovery of the structural parameters that summarize the preferences allows researchers to analyze the quantitative implications and the welfare effects that follow a policy change.

However, many things differentiate this paper from the literature on hedonic models. First, the paper focuses on the demand side without any particular attention to the supply side of the market. Second, it is essentially concerned with the identification of a summary of preferences, *i.e.* the price elasticity of demand. Third, the identification issue is analyzed in a poor data set environment: individual characteristics as well as product characteristics are not observed by the econometrician (see Bajari and Benkard (2001) for a similar setup), individual total demand and total expenditure are the only observable variables, the time period can be short and price variation may be small.

The identification of price elasticity of demand from cross section individual data is impossible due to the lack of prices variation in the sample. Even if panel data are available, the slow modification of tariffs and the usual small number of observed periods make very hazardous the estimation of prices effects. In many cases however (see De Rycke, Florens and Marcy, 2001) artificial prices, different over consumers, can be observed. These prices are computed as the ratio between the observed consumers' expenditure and demand. However, these prices actually reflect some composition effects of demand and expenditure for a differentiated product. Let us review some empirical results on mail demand, as it is the focus of the empirical part of this paper. Cazals and Florens (2002) presented a brief summary of the results in the existing literature about the estimation of price elasticities of mail demand. They noted that the estimated price elasticities are lower in dynamic models – where data comes from aggregation over individuals – than those estimated in cross–section demand models – where data comes from aggregation over mail products –. They provided some simulation experiments from a fully disaggregated model with heterogeneity that support these results and they shew that this model mimics results from actual data – higher price elasticities with cross–section data–. About the cross–section analysis, they mainly concluded that the use of an artificial mail price for each consumer – defined as the ratio of expenditure over demand – is unsatisfactory for empirical application because there exists an aggregation bias that they are not able to control.

The aim of this paper is to present a simple methodology for the treatment of this kind of data and to show that the observation of choices between attributes of a product over periods may contain useful information about the effect of price variations. The methodology is based on a hedonic structural model of the consumer behavior. We choose to adopt a structural approach as the restrictions created by the model allows to identify – under some conditions about the preferences and the change in price – some structural parameters that summarize the consumers'behavior. Although any structural and parametric modeling can suffer from a well known lack of robustness, a structural approach – compared to descriptive one – allows to recover parameters of demand – or supply – at the realization of the market equilibrium (see Reiss and Wolak (2003) for a discussion). The general principle of our structural model is to consider several possible attributes of a product. A consumer selects a vector of quantity of this product for each possible attribute by maximizing utility that depends on an unobservable – for the econometrician – heterogeneity factor, which is randomly generated across consumers. The first order conditions of this program determine the demand functions from which the total demand and the revenue are generated as a function of the heterogeneity factor. Using the restrictions created by this structural model, we then consider the identification and the estimation of the structural parameters (more precisely, the price elasticity of demand). In the one period case, the structural parameters are not identified but parametric restrictions – although arbitrary – about the distribution of the heterogeneity factor are sufficient to recover the identification of the demand parameters. Another – less arbitrary and more robust – way to circumvent the identification problem is to take into account for different time periods (see Brown (1983), Brown and Rosen (1982), Kahn and Lang (1988), Berry, Levinsohn and Pakes (1995) and Tauchen and Witte (2001)). The motivation of such an approach is that if preferences – or technology – and the distribution of tastes – or productivities – remains constant over periods – or are the same across market –, exogenous prices changes across periods allows to identify the preferences – technology – parameters and thus recover the price elasticity. Although unexplained (see Ekeland, Heckman and Nesheim (2002) and (2003) for a critical survey), shifts across time in technology when preferences are stable over periods will offer an opportunity to identify the demand parameters.

The paper also considers other identification issues. They first concern the identification of the other demand parameters and the effect of aggregate random shocks. Moreover, we show that demand parameters can be identified even in the case of unstable preferences. The shift in demand includes now two components: a change in price and a change in preferences. If the shift in preferences affects the average demand behavior and the covariance structure of demand and expenditure, the price elasticity of demand can be identified. An empirical illustration with mail data in a two period case illustrates the potential of this approach. We then extend the original model to a multiple product case and we deliver new conditions for identification. This new conditions impose that the number of periods must strictly exceed the number of observed goods. This result is a direct consequence of asymmetry in the multiple goods demand function. Second, we investigate the case of non–linear demand and we determine under which conditions, previous results about identification in the linear case still apply.

The paper is organized as follows. In a first section, we introduce a simple structural hedonic model of demand and we discuss some modeling issues. In section two, we determine the conditions for identification of price elasticity of demand in a two period case. In section three, we present empirical results with mail data. Section four considers two extensions of the benchmark model. A last section offers some concluding remarks. Proofs are given in appendix.

1 The Structural Model

We use a simplified hedonic model and we determine the consumers' total demand and expenditure for a single and differentiated product. The model is standard and the functional form of the utility function is chosen in order to yield an analytic solution from which we can deliver some conditions for identification in the next section. We also discuss some empirical issues and various specifications of the heterogeneity factor.

1.1 The Hedonic Model

The hedonic model is devoted to the identification of structural parameters – more precisely a summary of them – associated to a simple representation of consumers'preferences. The supply side is exogenous and considered as given. Moreover, we assume that there is no price discrimination, so every consumer faces the same price menu. Consumers and firms match on the vector of $m < \infty$ possible values for an attribute $Q = (Q_1, ..., Q_m)'$. We denote $p = (p_1, ..., p_m)'$ the associated vector of prices. The preferences are described by the following linear–quadratic utility function

$$U(Q) = \frac{1}{2}Q'AQ + Q'\Theta + X_o$$

The matrix A is $m \times m$ symmetric and negative definite. It is the same for all consumers. The variable $\Theta = (\Theta_1, ..., \Theta_m)'$ represents shifts in preferences and enters linearly in the utility function. It is assumed to be specific to each consumer and randomly generated. Consumers thus differ in their preference vector Θ. Some elements of Θ can be observed by both the consumer and the econometrician. In what follows, we consider that the elements of Θ are observed only by the consumer. For the simplicity of the presentation, we omit for the moment individual i and time t indexes. The mean and the variance of the vector Θ are denoted Θ_o and V_o.

The budget constraint is given by

$$Q'p + X_o = R$$

X_o represents all the other goods and enters linearly into the utility function. Total expenditure over the m possible values for an attribute is given $Q'p$. R is the disposable income of consumers and the price of X_o is normalized to one. After substitution for X_o, the utility function rewrites – up to a constant term –

$$U(Q) = \frac{1}{2}Q'AQ + Q'\Theta - Q'p$$

The first order condition of the maximization of utility is[1]

$$AQ + \Theta - p = 0$$

The demand is thus deduced from the previous equation

$$Q = A^{-1}(p - \Theta)$$

[1]The second order conditions are equal to A and are also negative definite.

The demand is a linear function of the difference between the price p and the random variable Θ. Let us now define $B = A^{-1}$. The matrix B is thus negative definite and we deduce that the parameter $\beta = \sum_{j=1}^{m} \sum_{j'=1}^{m} b_{j,j'} \equiv \mathbf{1}'_m B \mathbf{1}_m$ is negative. $\mathbf{1}_m = (1, \ldots, 1)'$ is the aggregator over the m possible values for an attribute.

In order to illustrate the effect of price variations, suppose that all prices $\{p_1, ..., p_m\}$ of Q increase in the same amount, *i.e.* $dp_1 = ... = dp_m \equiv dp > 0$. It follows that the variation of total demand $\sum_{j=1}^{m} Q_j = \mathbf{1}'_m Q \equiv x$ verifies

$$\begin{aligned} dx &= (\mathbf{1}'_m B) \begin{pmatrix} dp_1 \\ \vdots \\ dp_m \end{pmatrix} \\ &= (\mathbf{1}'_m B \mathbf{1}_m)\, dp \\ &\equiv \beta dp \end{aligned}$$

Following the same increase in all prices, the total demand will decrease of $-\beta > 0$. This parameter allows to measure the sensitivity of total demand to an uniform price variation and thus to determine further the price elasticity. Note that the demand function introduces a non–additive error term Θ, making the distance between the model and the data as the result of a structural random term[2] – the heterogeneity parameter in the utility function – (see, Brown and Walker (1989) and Lewbel (2001) for models of consumer behavior and McElroy (1987) and Brown and Walker (1995) concerning the production side). The structural models implies

$$\left| \frac{\partial Q}{\partial p'} \right| = \left| \frac{\partial Q}{\partial \Theta'} \right|$$

The shift in the preference parameter Θ allows to measure the sensitivity of demand to price variation as their derivatives – in absolute value – are equal. In what follows, we will take advantage from this structural restriction.[3]

[2]Following Heckman (1974), Heckman and Willis (1977), Mc Fadden (1974) and Lancaster (1979), the introduction of an unobservable random term within a model has received an increasing attention.

[3]This will appear clearer from the covariance structure of demand and expenditure, provided some additional

1.2 Modeling Issues

Let us consider now the more general case of many consumers $i = 1, ..., N$ for different time periods $t = 1, ..., T$. The total demand – or the aggregate demand over the m possible values for an attribute – of each consumer at any period t is given by

$$x_{i,t} = \sum_{j=1}^{m} Q_j(i,t) \equiv \mathbf{1}'_m Q_{i,t}$$

whereas the total expenditure is

$$y_{i,t} = \sum_{j=1}^{m} p_j(t) Q_j(i,t) \equiv p'_t Q_{i,t}$$

In what follows, we assume that total demand $x_{i,t}$ and total expenditure $y_{i,t}$ are observable by the econometrician for each individual i at any period t. Conversely, the m demands $\{Q_j(i,t)\}_{j=1}^{m}$ and the m prices $\{p_j(i,t)\}_{j=1}^{m}$ are not observable. However, in some cases, the different prices that compose the menu can be partially observable. For example, the tariffs of many postal services are known (for example, the prices associated to weight or type of pre–sorting). The observation of all prices of a menu over periods is not essential as our identification of price elasticity relies on the observation of aggregate demand and expenditure. Nevertheless, the observation of prices allows to evaluate the empirical relevance of conditions for identification.[4] Finally, the structural model shows that different and *artificial* prices over consumers can be obtained dividing the total expenditure by the total demand (when $m > 2$). Nevertheless, these *artificial* prices only reflect some composition effects of demand and expenditure for a differentiated product.

The important modeling issue that we consider in this paper concerns the specification of the preference parameter Θ. In order to determine total demand and revenue, we have now to specify this random variable – or heterogeneity factor – for each consumer at any period. Let the heterogeneity

conditions on preferences or on the nature of changes in price.

[4] See the discussion about the conditions for identification in section 2.

factor $\Theta_j(i,t)$ represents the preference of consumer i at period t for the product j. For illustrative purpose, we use the following series of examples.

Example 1.1: Individual effect only

In this first example, we consider that the random factor is only composed of an unobserved – by the econometrician – individual effect:

$$\Theta_j(i,t) = \Theta_j(i)$$

The aggregation over m yields:

$$x_{i,t} = \mathbf{1}'_m B p_t - \mathbf{1}'_m B \Theta_i$$

where $\Theta_i = (\Theta_{1,i}, \ldots, \Theta_{m,i})'$. Note that this random variable is time invariant. The average demand is:

$$E(x_{i,t}) = \mathbf{1}'_m B p_t - \mathbf{1}'_m B \Theta_o$$

It follows that the average change in demand over consumers results only in change in price over periods.

$$E(\Delta x_{i,t}) = \mathbf{1}'_m B \Delta p_t$$

Figure 1 illustrates the effect of change in price on average demand. This figure represents the demand with respect to the heterogeneity factor. For simplicity, the figure assumes that all prices of the menu vary in the same direction and the same amount. It follows that the average change in demand rewrites:

$$E(\Delta x_{i,t}) = \mathbf{1}'_m B \mathbf{1}_m dp \equiv \beta dp$$

The decrease in price implies an upward shift in demand but change in average demand does not depend on the average value of Θ. The non–stationarity of demand over periods is the direct consequence of the non–stationarity of the price.

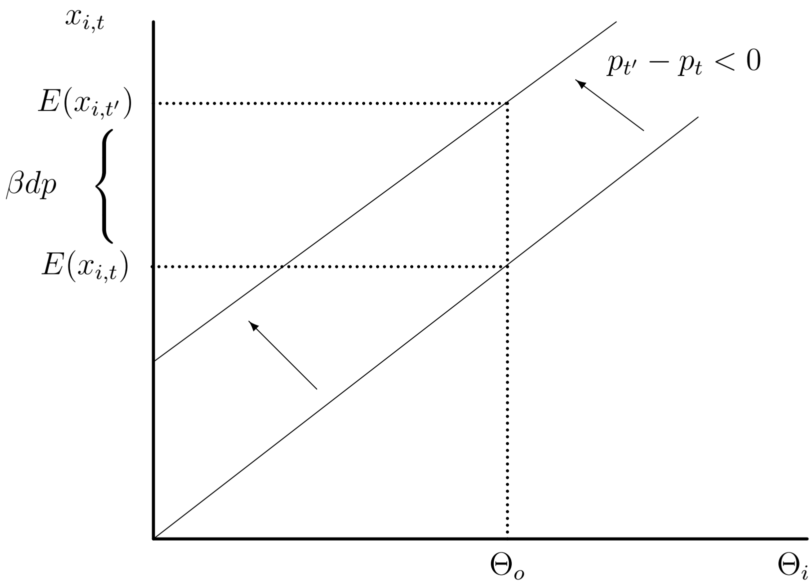

Figure 1: Change in demand and price

Example 1.2: Individual effect + Aggregate random shock

The random factor includes now an unobserved aggregate shock, denoted ε_t. This shock is common to all consumers:

$$\Theta_j(i,t) = \Theta_j(i) + \mathbf{1}_m \varepsilon_t$$

The aggregate shock is assumed to be *iid* with zero mean. The aggregation yields

$$x_{i,t} = \mathbf{1}_m' B p_t - \mathbf{1}_m' B \Theta_i - \mathbf{1}_m' B \mathbf{1}_m \varepsilon_t$$

The average demand is exactly the same as in example 1 as $E(\varepsilon_t) = 0 \ \ \forall t$

$$E(x_{i,t}) = \mathbf{1}_m' B p_t - \mathbf{1}_m' B \Theta_o$$

Again, the average change in demand over consumers results only in change in price over periods.

Example 1.3: (Individual effect + Aggregate random shock) $\times$ Trend

We introduce now a multiplicative trend in the specification of the heterogneity factor. This trend shifts upward – or downward – the preference shocks of all individuals:

$$\Theta_j(i,t) = \mu^{t-1}\left(\Theta_j(i) + \mathbf{1}_m \varepsilon_t\right)$$

The trend component is normalized to one in first period. When $\mu \neq 1$, this specification of the heterogeneity factor accounts for increase (decrease) in demand independently from price variations. This trend summarizes a set of omitted and unexplained – by the simple structural model – variables that shift demand over periods. Note that the specification of the trend component implies an exponential growth[5] ($\mu > 1$). Moreover, the growth factor μ^{t-1} can be replaced by $\mu(t) = \prod_{i=1}^{t-1} \mu_i$. In this case, the trend component is not imposed to be regular. The aggregate demand rewrites:

$$x_{i,t} = \mathbf{1}_m' B p_t - \mu^{t-1} \mathbf{1}_m' B \Theta_i - \mu^{t-1} \mathbf{1}_m' B \mathbf{1}_m \varepsilon_t$$

The average demand is

$$E(x_{i,t}) = \mathbf{1}_m' B p_t - \mu^{t-1} \mathbf{1}_m' B \Theta_o$$

and change in average demand is now

$$E(\Delta x_{i,t}) = \mathbf{1}_m' B \Delta p_t - \mu^{t-2}(\mu - 1) \mathbf{1}_m' B \Theta_o$$

Now, consider the normalization of the growth factor in first period and the change in demand between periods 1 and 2. As in example 1.1, assume for simplicity that all prices of the menu varies in the same direction and the same amount. The average demand rewrites:

$$E(\Delta x_{i,t}) = \beta dp - (\mu - 1) \mathbf{1}_m' B \Theta_o$$

As illustrated by figure 2, the change in demand has now two components: a change in price over periods βdp and a change in preferences over periods $(\mu - 1)\mathbf{1}_m' B \Theta_o$.

[5] In the case of a linear trend, we have $\Theta_{i,t} = \mu(t-1)\left(\Theta_i + \mathbf{1}_m \varepsilon_t\right)$. In a two period case, these two specifications provide the same result.

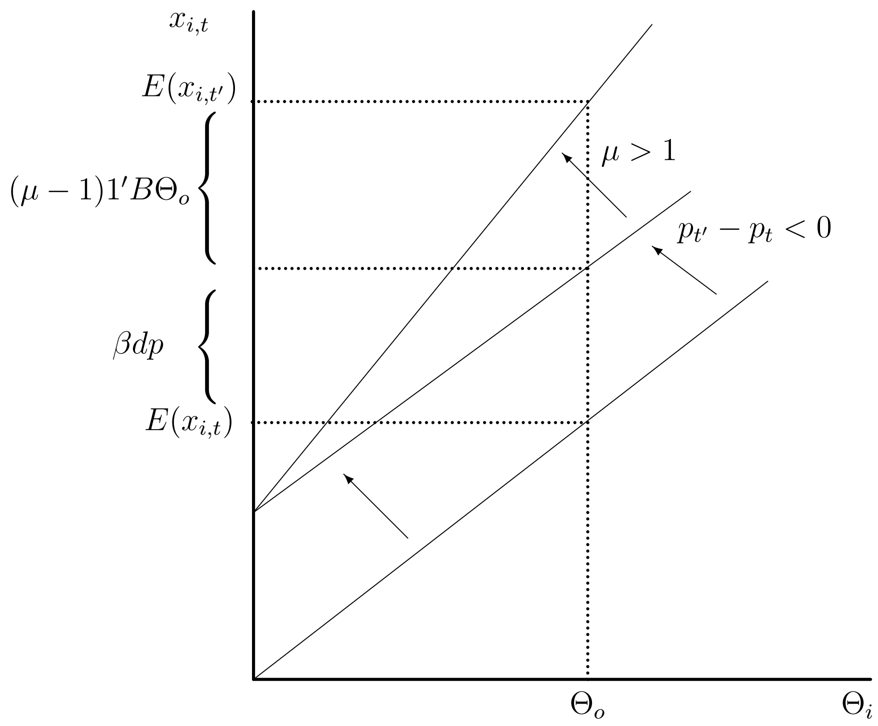

Figure 2: Change in demand and price with a trend

We do not pursue here our discussion about the specification of the heterogeneity factor as many other empirical issues may be worth considering. For example, as previously mentioned, the econometrician can observe some individual data (for example, individual characteristics) that allow to usefully represent the preferences. In the same way, the distribution of tastes can be specified as a function of aggregate observed variables (see Petrin (2002), Berry, Levinsohn, and Pakes (2003)). Moreover, various functional forms of the heterogeneity factor can also be considered. We do not consider these situations of a richer data environment, where some characteristics about the differentiated product and/or the consumer are available. In what follows, we use example 1.1 as a benchmark for the detailed study of the identification of demand parameters (see section 2.1), whereas example 1.3 is discussed in section 2.2 and is used in the empirical application (see section 3).

2 Identification

In this section, we determine conditions that allow to identify the demand parameters. Two types of conditions[6] are considered in a two period case ($t = 1, 2$). The first condition imposes some – rather weak – restrictions on preferences, without any restriction about the change in price, except that at least one price in the menu must vary. The second one imposes strong restriction on the change in price, *i.e* all the prices of the menu must vary in the same direction and in the same amount. Conversely, no restriction about preferences is introduced. However, these two conditions share the same aggregate implication, that is relative co–movement over periods between demand and expenditure can be represented by a single common factor.

2.1 Identification in a Basic Setup

A suggested way to identify the structural parameters in hedonic models is to take into account for different time periods. Brown (1983), Brown and Rosen (1982), Kahn and Lang (1988), Berry, Levinsohn and Pakes (1995) and Tauchen and Witte (2001) consider the hedonic price model either with mutlimarket data with a single period data or single market with multiperiod data. The motivation of such an approach is that if preferences – or technology – and the distribution of tastes – or productivities – remains constant over periods – or are the same across market –, exogenous prices changes across periods allows to identify the preferences parameters and thus recover the price elasticities. Although unexplained (see Ekeland, Heckman and Nesheim (2002) and (2003) for a critical survey), shifts across time in technology when preferences are stable across time will offer an opportunity to identify preference parameters. Note that figures 1 and 2 illustrate this issue. Following a change in price, the average demand shifts and thus allows to determine the sensitivity of average demand to price variation. Nevertheless, change in demand that follows a change in price

[6]In a the single period case, these conditions do not allow to identify the structural parameter (see the discussion at the end of section 2.2).

is not sufficient for identification of structural parameters. The relative co–movements over periods between demand and expenditure have to be considered. We now exploit this idea in the specific case of example 1.1, where only individual heterogeneity matters. The demand and expenditure for $t = 1, 2$ are given by

$$\begin{aligned} x_{i,t} &= \alpha_t - \mathbf{1}_m' B \Theta_i \\ y_{i,t} &= \gamma_t - p_t' B \Theta_i \end{aligned}$$

where $\alpha_t = \mathbf{1}_m' B p_t = p_t' B \mathbf{1}_m$ (by symmetry of B) and $\gamma_t = p_t' B p_t$. Let the moment conditions associated to demand and expenditures

$$\begin{aligned} E(x_{i,t}) &= \alpha_t - \mathbf{1}_m' B \Theta_o \\ E(y_{i,t}) &= \gamma_t - p_t' B \Theta_o \\ V(x_{i,t}) &= \mathbf{1}_m' B V_o B \mathbf{1}_m \\ V(y_{i,t}) &= p_t' B V_o B p_t \\ Cov(x_{i,t}, y_{i,t}) &= \mathbf{1}_m' B V_o B p_t \\ Cov(x_{i,t}, x_{i,t'}) &= \mathbf{1}_m' B V_o B \mathbf{1}_m \\ Cov(y_{i,t}, y_{i,t'}) &= p_t' B V_o B p_{t'} \\ Cov(x_{i,t}, y_{i,t'}) &= \mathbf{1}_m' B V_o B p_{t'} \end{aligned}$$

where $t \neq t'$ and $t = 1, 2$.

It is worth noting that we are not concerned with the identification and the estimation of all the parameters that describe the demand behavior. In other words, all the parameters that enter in the $m \times m$ matrices B and V_o and in the $m \times 1$ vectors Θ_o and p_t are not identified, except if we assume very particular forms of the demand function. This is a direct consequence of the lack of observation by the the econometrician of the vector of prices p_t. Only the aggregation over m of demand and expenditure are observed. As we are interested in the identification of the price elasticity, the empirical methodology that we propose is essentially concerned with the identification of the parameter α_t for each period. Indeed, this parameter represents a useful summary of the demand behavior, as previously shown in examples 1.1–1.3. From the identification of this parameter, we can identify and estimate the price elasticity for every period:

$$\begin{aligned}\mathcal{E}_t^p &= \frac{\partial E(x_{i,t})}{\partial p_t}\frac{p_t}{E(x_{i,t})} \\ &= \frac{\mathbf{1}_m' B p_t}{E(x_{i,t})} \\ &= \frac{\alpha_t}{E(x_{i,t})}\end{aligned}$$

Note that this elasticity is computed using the average demand over consumers. The average demand can be obtained either from the actual data or from the structural model.

We formulate now the following conditions for identification. These two conditions concern primary the parameter α_t, but the remaining parameters can be directly identified from α_t. Note that only one of these two conditions has to be satisfied.

Condition (a): $\mathbf{1}_m$ is eigenvector of $V_o B$ and $\mathbf{1}_m' B\,(p_2 - p_1) \neq 0$

Condition (b): $p_2 = (1+\delta)p_1$ and $\delta \mathbf{1}_m' B p_1 \neq 0$

These two conditions appeal several remarks. We first concentrate on the first part of each condition.

Condition (a) says that the aggregator $\mathbf{1}_m$ must be the eigenvector of the scaled sensitivity of demand to change in price $V_o B$, where the scaling factor V_o is the inverse of the precision matrix of the heterogeneity factor. The condition (a) rewrites

$$V_o B \mathbf{1}_m = k \mathbf{1}_m$$

where k is a scalar.

We may question whether this condition is restrictive about the specification of preferences. The matrices V_o and B include both $m(m+1)/2$ parameters and condition (a) implies that $m-1$ restrictions must be satisfied. As we are not interested in the identification and the estimation of all the parameters that enter in B and V_o, but rather in a useful summary α_t of preferences, this means that there exist many representations of preferences in the linear-quadratic setup and many configurations of the covariance matrix that satisfy this restriction. It follows that the restrictions about B and V_o are rather weak if we are essentially concerned with the identification of the price elasticity of demand. As illustrations, we present three examples that satisfy the first part of condition (a).

Example 2.1: Assume that V_o takes the form:

$$V_o = v_o \begin{pmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{pmatrix} \equiv v_o \mathbf{1}_m \mathbf{1}_m'$$

with $v_o > 0$. This form of the covariance matrix implies that there is no heterogeneity about the variance of the random factor over the m possible values for an attribute. Using this form, condition (a) rewrites

$$\begin{aligned} V_o B 1 &= v_o \mathbf{1}_m \mathbf{1}_m' B \mathbf{1}_m \\ &= \beta v_o \mathbf{1}_m \end{aligned}$$

where $\beta = \sum_{j=1}^{m} \sum_{j'=1}^{m} b_{j,j'} < 0$. The eigenvalue is given by

$$k = \beta v_o < 0$$

In this example, we introduce a particular form for the covariance matrix.[7] Conversely, we do not introduce additional restrictions on the other parameters that describe the preferences, except that B must be negative definite and symmetric.

[7]In this example, V_o is not inversible.

Example 2.2: Assume now that V_o takes the form:

$$V_o = v_o I_m$$

and denote

$$b(1) = \sum_{k=1}^{m} b_{1,k}, \ldots, b(m) = \sum_{k=1}^{m} b_{m,k}$$

the sum of the elements of the rows (or the lines by symmetry) of the matrix B. The condition (a) rewrites

$$b(1) = \ldots = b(m) = k/v_o$$

The condition (a) imposes stronger restrictions on the matrix B, compared to example 2.1. To see this, assume for simplicity that B is diagonal. Condition (a) imposes

$$b_{1,1} = \ldots = b_{m,m} \equiv b = k/v_o$$

This restriction implies that the sensitivity of each $\{Q_1, \ldots, Q_m\}$ to each price $\{p_1, \ldots, p_m\}$ must be the same.

Example 2.3: Assume that V_o takes the form:

$$V_o = \begin{pmatrix} \sigma_1^2 & 0 & \ldots & 0 \\ 0 & \sigma_2^2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & \sigma_m^2 \end{pmatrix}$$

In this example, the variances of the heterogeneity factor over the m values of a attribute differs. The condition (a) rewrites:

$$\sigma_1^2 b(1) = \ldots = \sigma_m^2 b(m) = k$$

where $b(j)$ $(j = 1, \ldots, m)$ is defined as in example 2.2. Note that this example differs from the previous one as it does not impose that the elements $b(j)$ have to be equal. For example, when B is diagonal, the restriction becomes:

$$\sigma_1^2 b_{1,1} = \ldots = \sigma_m^2 b_{m,m} = k$$

In these three examples, the parameters that compose the matrix B cannot be separately identified, except in very particular cases. Example 2.1 involves the sum of all the elements of B and example 2.3 the sum of the diagonal elements. Conversely, in example 2.2 where B has identical diagonal elements, the lack of heterogeneity over m allows to identify the single parameter b.

Let us now consider the first part of condition (b). This condition imposes that all prices $\{p_1, ..., p_m\}$ of the menu must change in the same direction and the same proportion

$$\frac{p_{2,1}}{p_{1,1}} = \ldots = \frac{p_{2,m}}{p_{1,m}} = 1 + \delta$$

This condition thus eliminates heterogeneity among the m values for an attribute, but it does not introduce any restriction about the specification of preferences.

The first part of the two conditions can be summarize as follows:

(i) Condition (a) imposes some restrictions about the preferences and the variance of the heterogeneity factor Θ without any conditions on uniform price changes. This means that preferences and the heterogeneity factor must be specified such that demand over $\{Q_1, Q_2, ..., Q_m\}'$ must change uniformly with respect to changes in prices $\{p_1, p_2, ..., p_m\}'$, *i.e.* the aggregator $\mathbf{1}_m$ must be the eigenvector of the scaled demand parameters.

(ii) Condition (b) imposes restrictions on price changes without any conditions on preferences. Conversely, prices $\{p_1, p_2, ..., p_m\}'$ must change uniformly over periods.

(iii) Conditions (a) and (b) are rather similar in the sense that they eliminate any composition effect in the covariance structure of demand and expenditure over periods.

Let us concentrate now on the second part of conditions (a) and (b). For condition (a), we have

$$\mathbf{1}_m' B \left(p_2 - p_1\right) \neq 0$$

Provided $b(k) \neq 0$ ($\forall k \in [1, m]$), this conditions means that if one price variation in the menu is non–zero, the condition for identification holds. In this case, this condition writes

$$b(k)\,(p_{2,k} - p_{1,k}) \neq 0$$

In condition (a), we do not need that all the prices of the menu must change in the same direction. We only need that only one price in the menu must vary. To be more precise, consider again the simple case where B is diagonal. As B is definite negative, $b_{j,j}$ is strictly negative for $j = 1, \ldots, m$. It follows that if one price $k \in [1, m]$ in the menu satisfies $(p_{2,k} - p_{1,k}) \neq 0$, the condition for identification is verified. Note that if there exists some extra information about the modification of the menu over periods, this condition can be easily checked.

For condition (b), we have

$$\delta \mathbf{1}_m' B p_1 \neq 0$$

It follows immediately that a sufficient condition for a lack of identification is $\delta = 0$, *i.e.* there is no uniform price variation in the menu. In other words, all the prices of the menu must uniformly vary. The observation of prices of the menu for successive periods allows again to evaluate the empirical relevance of this condition.

Given these two conditions and their related discussions, we turn now to the identification of the price elasticity of demand.

Theorem 1 *Under condition (a) or condition (b), the price elasticity of demand is identified for each period.*

A simple sketch of proof relies on the following linear and time invariant relationship:

$$Cov(x_{i,t}, y_{i,t}) = k\alpha_t$$

This relation is directly deduced from condition (a) or (b) and the moment conditions. The covariance structure over individual demands $x_{i,t}$ and expenditures $y_{i,t}$ is a linear function of the price effect on demand α_t for all periods $t = 1, 2$. If $Cov\,(x_{i,t}, y_{i,t})$ is a linear and time invariant function of α_t, the ratio of the covariance for two successive periods is equal to the ratio α_t/α_{t-1}. Moreover, as the change in average demand over periods 1 and 2 (see the example 1.1 for an illustration) is equal to $\alpha_t - \alpha_{t-1}$, the parameters α_1 and α_2 are identified. In the case of condition (a), the time invariant relationship results in restriction of matrix V_o and B, whereas condition (b) implies that the covariance ratio over the two periods is equal to the uniform price change.

Using the condition (a), the structural model delivers a simple expression of the price elasticity of demand:

$$\mathcal{E}_1^p = \left(\frac{\mathbf{1}'_m B p_1}{\mathbf{1}'_m B p_2 - \mathbf{1}'_m B p_1}\right) \frac{E(x_{i,2} - x_{i,1})}{E(x_{i,1})}$$

$$\begin{aligned} \mathcal{E}_2^p &= \left(\frac{\mathbf{1}'_m B p_2}{\mathbf{1}'_m B p_2 - \mathbf{1}'_m B p_1}\right) \frac{E(x_{i,2} - x_{i,1})}{E(x_{i,2})} \\ &= \varepsilon_1^p \frac{\mathbf{1}'_m B p_2}{\mathbf{1}'_m B p_1} \frac{E(x_{i,2})}{E(x_{i,1})} \end{aligned}$$

The price elasticity of demand in period 1 – structurally – corresponds to the relative change in demand following a relative change in price. The price elasticity in period 2 is equal to the one of period 1 after correcting for change in price and demand.[8]

[8]Under condition (b), the expression is simpler although it provides a similar interpretation:

$$\begin{aligned} \mathcal{E}_1^p &= \frac{1}{\delta} \frac{E(x_{i,2} - x_{i,1})}{E(x_{i,1})} \\ \mathcal{E}_2^p &= \varepsilon_1^p (1 + \delta) \frac{E(x_{i,2})}{E(x_{i,1})} \end{aligned}$$

2.2 Discussion

We now discuss various econometric issues. They concerns the identification of demand parameters, other specifications of the heterogeneity factor and the introduction of over–identifying restrictions.

Identification of the structural parameters

Theorem 1 states that the parameters α_t for $t = 1, 2$ are identified from the moment conditions if we use either condition (a) or condition (b). Note that the other parameters that summarize preferences are also identified. Under condition (a), the moment conditions on demand and expenditure rewrite

$$\begin{aligned}
E(x_{i,t}) &= \alpha_t - \eta \\
E(y_{i,t}) &= \lambda_t \\
V(x_{i,t}) &= k\beta \\
V(y_{i,t}) &= \omega_t^2 \\
Cov(x_{i,t}, y_{i,t}) &= k\alpha_t \\
Cov(x_{i,t}, x_{i,t'}) &= k\beta \\
Cov(y_{i,t}, y_{i,t'}) &= \rho_{t,t'} \\
Cov(x_{i,t}, y_{i,t'}) &= k\alpha_{t'}
\end{aligned}$$

The parameters $\{\lambda_t, \omega_t^2, \rho_{t,t'}\}$ for $t, t' = 1, 2$ and $t \neq t'$ in these moment conditions are of less interest for the representation of demand behavior. These parameters are directly identified from their associated moment conditions: λ_t is obtained from the average expenditure $E(y_{i,t})$, ω_t^2 is given by the variance of expenditure $V(y_{i,t})$ and $\rho_{t,t'}$ corresponds to the cross-covariance of expenditure over periods. The other parameters $\{\alpha_t, \eta, k, \beta, \}$ are more structural as they represent a summary of preferences. The parameter $\beta < 0$ is the sum of all the elements of the matrix B and it represents the sensitivity of aggregate demand to uniform price changes. As previously mentioned, α_t allows to quantify the price elasticity of demand each period. The parameter k is a direct consequence of condition (a) and it corresponds to the eigenvector of $V_o B$. Finally, the parameter η is proportional to the average value of the heterogeneity factor – or preference shock – over m. These parameters enter simultaneously in different moment conditions and this creates cross–equation restrictions. However, the identification of α_t allows to identify the others demand parameters. First, the parameter η is obtained from the average demand

$$\eta = \alpha_t - E(x_{i,t}) \equiv \frac{Cov(x_{i,t-1}, y_{i,t-1})E(x_{i,t}) - Cov(x_{i,t}, y_{i,t})E(x_{i,t-1})}{Cov(x_{i,t}, y_{i,t}) - Cov(x_{i,t-1}, y_{i,t-1})}$$

Second, the parameter k is deduced from the covariance of demand and expenditure

$$k = \frac{Cov(x_{i,t}, y_{i,t})}{\alpha_t} \equiv \frac{Cov(x_{i,t}, y_{i,t}) - Cov(x_{i,t-1}, y_{i,t-1})}{E(x_{i,t} - x_{i,t-1})}$$

Finally, the parameter β is identified from the variance of demand

$$\beta = \frac{V(x_{i,t})}{k} \equiv V(x_{i,t}) \frac{E(x_{i,t} - x_{i,t-1})}{Cov(x_{i,t}, y_{i,t}) - Cov(x_{i,t-1}, y_{i,t-1})}$$

Note that we can use other moment conditions in order to determine these parameters. This simply suggests to introduce over–identifying restrictions (see the discussion below).

Aggregate shocks

The original model with individual effect only can be easily extended to the case of aggregate random shocks in the line of example 1.2. This shock is assumed to be *iid* with zero mean and variance σ_ε^2. Taking into account for aggregate shocks, the moment conditions rewrite:

$$\begin{aligned} E(x_{i,t}) &= \alpha_t - \mathbf{1}_m' B\Theta_o \\ E(y_{i,t}) &= \gamma_t - p_t' B\Theta_o \end{aligned}$$

$$
\begin{aligned}
V(x_{i,t}) &= \mathbf{1}'_m B V_o B \mathbf{1}_m + \beta^2 \sigma_\varepsilon^2 \\
V(y_{i,t}) &= p'_t B V_o B p_t + \alpha_t^2 \sigma_\varepsilon^2 \\
Cov(x_{i,t}, y_{i,t}) &= \mathbf{1}'_m B V_o B p_t + \alpha_t \beta \sigma_\varepsilon^2 \\
Cov(x_{i,t}, x_{i,t'}) &= \mathbf{1}'_m B V_o B \mathbf{1}_m \\
Cov(y_{i,t}, y_{i,t'}) &= p'_t B V_o B p_{t'} \\
Cov(x_{i,t}, y_{i,t'}) &= \mathbf{1}'_m B V_o B p_{t'}
\end{aligned}
$$

where $t \neq t'$ and $t, t' = 1, 2$. As previously mentioned in example 2.1, the average demand is left unaffected by the introduction of the aggregate shock. Using condition (a), these moment conditions become[9]

$$
\begin{aligned}
E(x_{i,t}) &= \alpha_t - \eta \\
E(y_{i,t}) &= \lambda_t \\
V(x_{i,t}) &= k\beta + \beta^2 \sigma_\varepsilon^2 \\
V(y_{i,t}) &= \omega_t^2 + \alpha_t^2 \sigma_\varepsilon^2 \\
Cov(x_{i,t}, y_{i,t}) &= \alpha_t \left(k + \beta \sigma_\varepsilon^2 \right) \\
Cov(x_{i,t}, x_{i,t'}) &= k\beta \\
Cov(y_{i,t}, y_{i,t'}) &= \rho_{t,t'} \\
Cov(x_{i,t}, y_{i,t'}) &= k\alpha_{t'}
\end{aligned}
$$

As the average demand is left unaffected by the introduction of the aggregate random shock, the change in the average demand is always equal to $\alpha_t - \alpha_{t-1}$. Moreover, the covariance of demand and expenditure is always a proportional and time invariant function of α_t. The ratio of the covariance over the two periods is thus equal to α_t / α_{t-1}. The other parameters can thus be deduced from the

[9]We choose to present the results with condition (a). Results are the same with condition (b).

other moment conditions.

Heteroscedasticity

Comments about Theorem 1 state that demand parameters are identified if the covariance of demand and expenditure is a linear and time invariant relation of the price. However, the lack of time invariant relationship between $Cov(x_{i,t}, y_{i,t})$ and α_t is not really problematic if the covariance matrix of $x_{i,t}$ and $y_{i,t}$ is appropriately scaled. To see this, consider the case where the aggregate shock is heteroscedastic, *i.e.* $V(\varepsilon_t) = \sigma^2_{\varepsilon,t}$. The moment conditions are the same, except for $V(x_{i,t})$, $V(y_{i,t})$ and $Cov(x_{i,t}, y_{i,t})$. Using condition (a), these moment conditions rewrite

$$\begin{aligned} V(x_{i,t}) &= \beta\left(k + \beta\sigma^2_{t,\varepsilon}\right) \\ V(y_{i,t}) &= \omega^2_t + \alpha^2_t\sigma^2_{t,\varepsilon} \\ Cov(x_{i,t}, y_{i,t}) &= \alpha_t\left(k + \beta\sigma^2_{t,\varepsilon}\right) \end{aligned}$$

The covariance is now a time varying function of α_t because of heteroscedasticity of ε_t. However, the covariance in period t scaled by the variance of demand at the same period is time invariant

$$\frac{Cov(x_{i,t}, y_{i,t})}{V(x_{i,t})} = \frac{\alpha_t}{\beta}$$

It follows that the ratio of the scaled covariance over the two periods is equal to α_t/α_{t-1}.

Accounting for growth

In example 1.3, we introduce a growth component that accounts for change in demand independently from price variation. The growth component $\mu(t)$ captures a set of unobserved variables that affects the consumers'behavior. This model can be useful in order to identify demand parameters, even when preferences are not necessary stable across periods. Recall that demand and expenditure are given by

$$
\begin{aligned}
x_{i,t} &= \alpha_t - \mu(t)\mathbf{1}_m' B\Theta_o - \mu(t)\mathbf{1}_m' B\mathbf{1}_m\varepsilon_t \\
y_{i,t} &= \gamma_t - \mu(t)p_t' B\Theta_o - \mu(t)p_t' B\mathbf{1}_m\varepsilon_t
\end{aligned}
$$

The aggregate shock is again assumed to be *iid* with zero mean and constant variance σ_ε^2. The moment conditions are

$$
\begin{aligned}
E(x_{i,t}) &= \alpha_t - \mu(t)\mathbf{1}_m' B\Theta_o \\
E(y_{i,t}) &= \gamma_t - \mu(t)p_t' B\Theta_o \\
V(x_{i,t}) &= \mu(t)^2\left(\mathbf{1}_m' BV_oB\mathbf{1}_m + \beta^2\sigma_\varepsilon^2\right) \\
V(y_{i,t}) &= \mu(t)^2\left(p_t' BV_oBp_t + \alpha_t^2\sigma_\varepsilon^2\right) \\
Cov(x_{i,t}, y_{i,t}) &= \mu(t)^2\left(\mathbf{1}_m' BV_oBp_t + \beta\alpha_t\sigma_\varepsilon^2\right) \\
Cov(x_{i,t}, x_{i,t'}) &= \mu(t)\mu(t')\mathbf{1}_m' BV_oB\mathbf{1}_m \\
Cov(y_{i,t}, y_{i,t'}) &= \mu(t)\mu(t')p_t' BV_oBp_{t'} \\
Cov(x_{i,t}, y_{i,t'}) &= \mu(t)\mu(t')\mathbf{1}_m' BV_oBp_{t'}
\end{aligned}
$$

where $t \neq t'$ and $t, t' = 1, 2$. Note that the growth factor $\mu(t)$ enter both in the mean and the variance. The model implies heteroscedasticity as in the previous example, but the specification of the growth component takes advantage from the cross–equation restrictions because shifts in preferences over the two periods both affect the mean and the variance of demand and expenditure.

Proposition 1 *Under condition (a) or condition (b), the price elasticity of demand is identified for each period .*

The demonstration of Proposition 1 closely follows the one of Theorem 1, except that we need to identify the growth factor in order to correct for the change in demand between period 1 and 2. The growth factor μ can be easily identified from the variance of demand in period 1 and 2, given the normalization in first period.

The specification choice of the growth factor is of importance for identification purpose. In example 1.3, we take advantage from that $\mu(t)$ enters both in the mean and the variance od demand and expenditure, as the average value and the variance of Θ will change over periods. Now, assume that the growth factor is separable:

$$\Theta_j(i,t) = \Theta_j(i) + \mathbf{1}_m \mu(t) + \mathbf{1}_m \varepsilon_t$$

The average demand will again change independently from price variation, but the variance of demand will remain constant over periods. In this case, we cannot identify the parameter $\mu(t)$ and thus the other parameters of demand.

Identification in a single period case

We assume for identification purpose that the number of period strictly exceed one. We may assess if our conditions for identification are sufficient in a single period case. The moment conditions associated to demand and expenditures are $E(x_i) = \alpha - 1'B\Theta_o$, $E(y_i) = \gamma_t - p'B\Theta_o$, $V(x_i) = 1'BV_oB1$, $V(y_i) = p'BV_oBp$ and $Cov(x_i, y_i) = 1'BV_oBp$. Under condition (a), these moment conditions rewrite[10] $E(x_i) = \alpha - \eta$, $E(y_i) = \lambda$, $V(x_i) = k\beta$, $V(y_i) = \omega^2$ and $Cov(x_i, y_i) = k\alpha$. It appears that the number of parameters is six and exceeds the number of moment conditions. Condition (a) does not allow to identify the demand parameters. Consider now a more restrictive case where the parameter Θ is the same over $j = 1, \ldots, m$

$$\Theta_i = \theta_i \mathbf{1}_m$$

where $E(\theta_i) = \theta_o$ and $V(\theta_i) = v_o$. It is worth noting that the variance of Θ_i corresponds to the one

[10]Obviously, we do not use condition (b) as it assumes that the number of periods exceeds strictly one.

of example 2.1, *i.e.* $V(\Theta_i) = V(\theta_i)\mathbf{1}_m'\mathbf{1}_m \equiv v_o\mathbf{1}_m'\mathbf{1}_m$. This specification of Θ_i thus verifies condition (a). The moment conditions rewrites $E(x_i) = \alpha - \beta\theta_0$, $E(y_i) = \gamma - \alpha\theta_o$, $V(x_i) = \beta^2 v_o$, $V(y_i) = \alpha^2 v_o$ and $Cov(x_i, y_i) = \alpha\beta v_o$. We have now to identify the following five parameters

$$\psi = \{\alpha, \beta, \gamma, \theta_o, v_o\}$$

from five moment conditions. Condition for identification appears at a first glance verified. This is not the case. First note that the parameters θ_o and γ are only identified from equations about the mean of demand and expenditure. So we discard these two parameters from identification study. The identification of the other parameters $\widetilde{\psi} = \{\alpha, \beta, v_o\}$ is thus based on the remaining moment conditions. The condition for identification relies on the rank of $f(\widetilde{\psi})$ where

$$f(\widetilde{\psi}) = \begin{pmatrix} \beta^2 V_o - V(x_i) \\ \alpha^2 V_o - V(y_i) \\ \beta\alpha V_o - Cov(x_i, y_i) \end{pmatrix}$$

Let the first derivative of $f(\widetilde{\psi})$ with respect to $\widetilde{\psi}$

$$\frac{\partial f(\widetilde{\psi})}{\partial \widetilde{\psi}'} = \begin{pmatrix} 0 & 2\beta V_0 & \beta^2 \\ 2\alpha V_o & 0 & \alpha^2 \\ \beta V_o & \alpha V_o & \alpha\beta \end{pmatrix}$$

The determinant of $\partial f(\widetilde{\psi})/\partial \widetilde{\psi}'$ is zero and thus $f(\widetilde{\varphi})$ is not full rank. It follows that conditions for identification are not fulfilled.[11]

Over–identifying restrictions

Finally, in the series of examples, we introduce a set of moment conditions, but only some of them are essentials for identification. In each example, the number of moment conditions exceeds the number

[11]Note that the demand parameters can be identified when particular assumptions about the distribution of the heterogeneity factor are formulated Consider for example an exponential distribution. Let $\theta_i \sim f_{\theta_o}$ where f is an exponential distribution of parameter $\theta_o > 0$. Moreover, we assume here for simplicity that $m = 1$. The moments are $E(\theta_i) = \theta_o$ and $V(\theta_i) = \theta_o^2$. It follows that $E(x_i) = \alpha - \beta\theta_o$ and $V(x_i) = \beta^2\theta_o^2$. We obtain $\sigma_{x_i} = -\beta\theta_o$ and after substitution into the mean of demand, we obtain $\alpha = E(x_i) - \sigma_{x_i}$. The price elasticity is thus given by $\mathcal{E} = 1 - \sigma_{x_i}/E(x_i)$. This approach relies on arbitrary choice of the distribution and it is not robust (see Cazals, Feve, Feve and Florens (2003) for a discussion in a simple setting)

of parameters to be estimated. This suggests to implement a GMM (see Hansen (1982)) and thus to introduce over–identifying restrictions in order to estimate the demand parameters. Moreover if the model is extended to a larger number of period, the number of moment conditions increases rapidly compared to the number of moment to be estimated. To see this, consider only the identification of the parameter α_t in a three period case. For $t = 1, 2$, the parameters α_1 and α_2 are given by

$$\begin{aligned}
\alpha_1 &= \frac{Cov(x_{i,1}, y_{i,1})}{Cov(x_{i,2}, y_{i,2}) - Cov(x_{i,1}, y_{i,1})} E(x_{i,2} - x_{i,1}) \\
\alpha_2 &= \frac{Cov(x_{i,2}, y_{i,2})}{Cov(x_{i,2}, y_{i,2}) - Cov(x_{i,1}, y_{i,1})} E(x_{i,2} - x_{i,1})
\end{aligned}$$

and for $t = 2, 3$

$$\begin{aligned}
\alpha_2 &= \frac{Cov(x_{i,2}, y_{i,2})}{Cov(x_{i,3}, y_{i,3}) - Cov(x_{i,2}, y_{i,2})} E(x_{i,3} - x_{i,2}) \\
\alpha_3 &= \frac{Cov(x_{i,3}, y_{i,3})}{Cov(x_{i,3}, y_{i,3}) - Cov(x_{i,2}, y_{i,2})} E(x_{i,3} - x_{i,2})
\end{aligned}$$

There exists two ways to estimate α_2 and this imposes an additional moment condition that can be the basis of a global specification test of the structural model. Table 1 reports the number of moment conditions, the number of parameters and the associated number of over–identifying restrictions with respect to the number of periods T for various cases we discussed previously (see the examples 1.1,1.2 and 1.3, aggregate shock, growth). The fast increase with T of the number of moment conditions is essentially due to the covariance of demand and expenditure. Conversely, the condition (a) or (b) for identification of the price elasticity lower the number of structural parameters that represent the covariance structure of demand and expenditure.

3 Application with Mail Data

In order to illustrate the potential of the approach, we present empirical results about the estimation of price elasticity of mail demand in a very short – two years – panel. We first briefly describe the

Table 1: Over–identifying restrictions

		Number of Moment Conditions	Number of Parameters	Number of over–identifying restrictions
Example 1.1		$2T^2 + 3T$	$\frac{6+5T+T^2}{2}$	$\frac{3T^2+T-6}{2}$
Example 1.2	(a)	$2T^2 + 3T$	$\frac{8+5T+T^2}{2}$	$\frac{3T^2+T-8}{2}$
	(b)	$2T^2 + 3T$	$\frac{6+7T+T^2}{2}$	$\frac{3T^2-T-6}{2}$
Example 1.3	(a)	$2T^2 + 3T$	$\frac{10+5T+T^2}{2}$	$\frac{3T^2+T-10}{2}$
	(b)	$2T^2 + 3T$	$\frac{6+7T+T^2}{2}$	$\frac{3T^2-T-6}{2}$

Note : T denotes thne number of periods. Example 1.2 (a) corresponds to the case of homoscedastic aggregate random shock, Example 1.2 (b) corresponds to the case of heteroscedastic aggregate random shock, Example 1.3 (a) corresponds to the case of regular growth, Example 1.3 (a) corresponds to the case of irregular growth.

data and the econometric method. We then present the empirical results.

3.1 Data and Method

The original data set include 200 observations on mail data.[12] We discard some observations – associated to unexplained and/or unknown zero price or non available quantities – from the original data set, so the final data set includes 186 observations for two successive periods – end of year 1999/begin of year 2000 and end of year 2000/begin of year2001 –, denoted hereafter period 1 and period 2. The estimations are performed using moment conditions associated to means and the covariance matrix across quantities and expenditures and across periods. Tables 2 and 3 report these moments.

Table 2: Sample Mean

Period	$x_{i,t}$	$y_{i,t}$
1	0.87316	0.21784
2	0.92571	0.23212

[12]The strategic component of the results bans us to communicate the true origin of the data, although estimation of the demand parameters are obtained from real mail data. We choose to work with this data set as it displays the empirical features that we have discussed (observation of demand ans expenditure, only). It is worth noting that many other data sets may display similar empirical features. We apologize the reader for this lack of information.

Table 3: Covariance Matrix

	$x_{i,1}$	$x_{i,2}$	$y_{i,1}$	$y_{i,2}$
$x_{i,1}$	3.65737			
$x_{i,2}$	3.74976	3.92211		
$y_{i,1}$	0.89949	0.92157	0.22444	
$y_{i,2}$	0.92374	0.96670	0.23002	0.24156

Table 2 shows that the average demand and expenditure increase over the two periods. The problem is that the non–stationarity in the mean – if considered alone – is not sufficiently informative about the nature of change, *i.e* the change can result either in change in price or shift in preferences. However, the covariance matrix (see table 3) shows that the variances of demand and revenues increase over periods. This suggests to use the model with a trend that affects the random factor (see example 3.1). Given the chosen structural model, the set of parameters to be estimated are

$$\psi = \left\{\alpha_1, \alpha_2, \mu, \eta, \tilde{\lambda}_1, \tilde{\lambda}_2, k, \beta, \omega_1, \omega_2, \sigma_\varepsilon, \rho_{1,2}\right\}$$

so, the number of parameters is equal to 12. The moment conditions we use in a two period case are the theoretical counterparts of the empirical moments in tables 2 and 3. The moment conditions are the following:

$$f_N\left(x_{i,t}, y_{i,t}; \psi\right) = \begin{pmatrix} x_{i,1} - \alpha_1 + \eta \\ x_{i,2} - \alpha_2 + \mu\eta \\ y_{i,1} - \tilde{\lambda}_1 \\ y_{i,2} - \tilde{\lambda}_2 \\ x_{i,1}^2 - (k\beta + \beta^2\sigma_\varepsilon^2) - (\alpha_1 - \eta)^2 \\ x_{i,2}^2 - \mu^2\left(k\beta + \beta^2\sigma_\varepsilon^2\right) - (\alpha_2 - \eta)^2 \\ y_{i,1}^2 - (\omega_1^2 + \alpha_1^2\sigma_\varepsilon^2) - \tilde{\lambda}_1^2 \\ y_{i,2}^2 - \mu^2\left(\omega_2^2 + \alpha_2^2\sigma_\varepsilon^2\right)\tilde{\lambda}_2^2 \\ x_{i,1}y_{i,1} - (k\alpha_1 + \beta\alpha_1\sigma_\varepsilon^2) - (\alpha_1 - \eta)\tilde{\lambda}_1 \\ x_{i,2}y_{i,2} - \mu^2\left(k\alpha_1 + \beta\alpha_1\sigma_\varepsilon^2\right) - (\alpha_2 - \eta)\tilde{\lambda}_2 \\ x_{i,1}x_{i,2} - \mu k\beta - (\alpha_1 - \eta)(\alpha_2 - \eta) \\ y_{i,1}y_{i,2} - \mu\rho_{12} - \tilde{\lambda}_1\tilde{\lambda}_2 \\ x_{i,1}y_{i,2} - \mu k\alpha_2 - (\alpha_1 - \eta)\tilde{\lambda}_2 \\ x_{i,2}y_{i,1} - \mu k\alpha_1 - (\alpha_2 - \eta)\tilde{\lambda}_1 \end{pmatrix}$$

The number[13] of moment conditions is equal to 14. We define these moment conditions in a more

[13]Without aggregate shocks, the number of moment conditions is equal to 13.

compact form

$$g_N(\psi) = \frac{1}{N}\sum_{i=1}^{N} f_N(x_{i,t}, y_{i,t}; \psi)$$

The GMM estimate $\widehat{\psi}_N$ of ψ minimizes the following loss function

$$\widehat{\psi}_N = \arg\min_{\psi} g_N' W_N g_N$$

where W_N is a weighting matrix given by

$$W_N = V(f_N(x_{i,t}, x_{i,t}; \psi))^{-1}$$

Given an estimate of ψ, the price elasticity for each period is computed as follows:

$$\mathcal{E}_t^p = \frac{\alpha_t}{E(x_{i,t})} \qquad t = 1, 2$$

where the average demand can be either obtained from the data – $E(x_{i,t}) = (1/N)\sum_{i=1}^{N} x_{i,t}$ $(t = 1, 2)$ – or from the model – $E(x_{i,1}) = \alpha_1 - \eta$ and $E(x_{i,2}) = \alpha_2 - \mu\eta$–. For identification sake, we impose that the number of moments exceeds the number of parameters. This enables us to conduct a global specification test (Hansen (1982)), which is asymptotically distributed as a chi–square, with the number degrees of freedom equals to the number of over–identifying restrictions.

3.2 Empirical Results

Table 4 reports the parameters estimates of the structural model and the global specification test statistic (J–stat). First of all, the model is not globally rejected by the data, as the p–values associated to the J–stat are 96.18% for the model without aggregate shock and 86.95% for the model with aggregate shock. A second feature that emerges from the table is that the sign of each estimated parameter is in accordance with the theory. Third, the parameters are precisely estimated, notably for the model without aggregate shock. The parameter μ exceeds significantly unity, indicating the relevance of the growth factor in order to estimate consistently the preferences parameters. We can thus decompose the change in demand that is due to change in price and change in preferences. We present this decomposition when there is no aggregate shock:

$$\underbrace{E(x_{i,2} - x_{i,1})}_{\text{Change in demand}} = \underbrace{\alpha_2 - \alpha_1}_{\text{Change in price}} - \underbrace{\eta(\mu - 1)}_{\text{Change in preferences}}$$

$$0.055 = -0.002 + 0.057$$

It appears that the price change is relatively small compared to the shift in the preferences between period 2 and 1. Beyond the global specification test, we also look at each moment separately in order

Table 4: Parameters Estimates

	Without aggregate shock			With aggregate shock		
ψ	$\widehat{\psi}$	$\widehat{\sigma}(\psi)$	t-stat	$\widehat{\psi}$	$\widehat{\psi}(\psi)$	t-stat
α_1	-0.3570	0.045840	-7.7888	0.4114	0.5465	-0.7528
α_2	-0.3590	0.045948	-7.8122	-0.4138	0.5510	-0.7509
μ	1.0468	0.01401	74.7121	1.0453	0.0200	52.383
η	-1.2140	0.1881	-6.4548	-1.2666	0.5511	-2.2986
$\tilde{\lambda}_1$	0.2144	0.0395	5.4180	0.2140	0.0316	6.7816
$\tilde{\lambda}_2$	0.2297	0.0418	5.4946	0.2294	0.0333	6.8839
k	-2.4076	0.1777	-8.1787	-2.0816	2.8839	-0.7206
β	-1.4458	0.1767	-8.1787	-1.6652	2.2052	-0.7551
ω_1	0.4642	0.0907	5.1162	0.4635	0.0707	6.5589
ω_2	0.4664	0.0916	5.0996	0.4658	0.0712	6.5410
σ_ε	-	-	-	0.1055	0.1408	0.7489
ρ_{12}	0.2163	0.0848	2.5499	0.2156	0.0659	3.27323
$J-stat$		0.2907			0.2798	
P-value		96.18%			86.95%	

Note : N=186, standard–errors computed from heteroscedastic–consitent matrix.

to locate some potential failures of the structural model in reproducing empirical moments. The main idea is that each element of $g_N(\psi)$ measures the discrepancy between the moments computed from the data and those computed from the model. A small value for a given element in g_N indicates that the structural model is able to account for this specific feature of the data, while large values may reveal some failures. This simple diagnostic is constructed from the following vector of *t-statistics* $T_N = \left\{diag\left[\Omega_N\right\}^{-1/2}\right\} \sqrt{N} g_N\left(\widehat{\psi}_N\right)$, which is asymptotically distributed as a $\mathcal{N}(0,1)$. The test statistics is computed replacing $\Omega_N = W_N^{-1}$ by a consistent estimate. Table 5 reports observed and theoretical values of moments and the diagnostic test. First of all, all observed moments are significant, making this set of historical moments demanding for the model. Second, the theoretical moments match well their empirical counterparts. This is confirmed by the last column of the table 5 that reports the diagnostic test. It clearly indicates that, when taken one by one, the model generates moments that are significantly equal to those observed on actual data.

Table 5: Moments from both Observed Data and Model

		Without aggregate shock		Without aggregate shock	
	Actual data	Model	Test	Model	Test
$E(x_{i,1})$	0.8732 (0.1129)	0.8569	0.0085	0.8552	0.0093
$E(x_{i,1})$	0.9257 (0.1171)	0.9118	0.0070	0.9103	0.0078
$E(y_{i,1})$	0.2178 (0.0279)	0.2144	0.0073	0.2140	0.0081
$E(y_{i,2})$	0.2321 (0.0290)	0.2297	0.0050	0.2294	0.0056
$E(x_{i,1}^2)$	4.4001 (1.069)	4.2460	0.0082	4.2285	0.0091
$E(x_{i,2}^2)$	4.7579 (1.1645)	4.6794	0.0039	4.6496	0.0053
$E(y_{i,1}^2)$	4.5379 (1.1045)	4.4250	0.0058	4.4017	0.0071
$E(y_{i,2}^2)$	0.2707 (0.0676)	0.2634	0.0062	0.2625	0.0070
$E(x_{i,1}y_{i,1})$	0.2941 (0.0742)	0.2932	0.0008	0.2917	0.0019
$E(x_{i,2}y_{i,2})$	0.2793 (0.0699)	0.2756	0.0030	0.2745	0.0040
$E(x_{i,1}x_{i,2})$	1.0848 (0.2677)	1.0510	0.0072	1.0471	0.0081
$E(y_{i,1}y_{i,2})$	1.1764 (0.2921)	1.1648	0.0023	1.1582	0.0036
$E(x_{i,1}y_{i,2})$	1.1215 (0.2768)	1.1014	0.0041	1.0964	0.0052
$E(x_{i,2}y_{i,1})$	1.1183 (0.2762)	0.7043	0.0048	0.7003	0.0089

Note : N=186, standard–errors computed from heteroscedastic–consitent matrix, standard–erros in parentheses.

From the parameters estimates, we compute the price elasticity for the two periods. Table 6 reports these estimated elasticity. Note that the estimates of price elasticity are very similar when the average demand is computed from the model or the actual data. This is explained by the very good match by the model of the mean of demand for period 1 and 2. The price elasticity of demand is relatively low – in absolute values – especially if we compare them to previous estimations conducted on individual data set. For comparison purpose, we briefly present some of the previous results with mail data. Using US annual data for households from 1986 to 1994, Wolak (1997) obtains values of estimated price elasticity of demand for postal delivery services that vary between -0.758 (in 1986) and -1.27 (in 1994). Santos and Lagoa (2001) focus on direct mail demand from firms in Portugal. The value of the price elasticity is estimated as being -0.84. De Rycke, Florens and Marcy (2001) estimate a mail demand model using cross-section data for a sample of small French firms observed in 1998. They also consider the demand for two classes of mail (first and second class). The estimated price elasticity of demand for firms using only first class letters is -0.8. For firms using both classes of mail, the estimated own-price elasticities are -0.82 for first class letters and -0.17 for second-class mail.

Table 6: Estimates of Price Elasticities

	Without aggregate shock		With aggregate shock	
Average demand	Period 1	Period 2	Period 1	Period 2
Data	-0.4089	-0.3878	-0.4712	-0.4470
Model	-0.4167	-0.3937	-0.4811	-0.4545

4 Extensions

We use examples 1.1–1.3 as benchmarks for the study of identification. We first extend the original model to a multiple product case and we deliver new conditions for identification. Second, we investigate the case of non–linear demand and we determine under which conditions, previous results about identification in the linear case still hold.

4.1 Multiple Goods

We consider a multi–period case ($t = 1, ..., T$ with $T \geq 2$) and we introduce now n different types of goods, denoted $\{Q_1, \ldots, Q_n\}$ with $n \geq 2$. The demands and expenditures about these n goods are observable by the econometrician. This model with n goods is a natural extension of the structural model of section 1. Note, that this framework is the same as in section 1, if we partition Q in n subsets and we assume that each of them is observable. The demand function is now

$$\begin{pmatrix} Q_1 \\ \vdots \\ Q_n \end{pmatrix} = \begin{pmatrix} B_{11} & \ldots & B_{1n} \\ \vdots & & \vdots \\ B_{n1} & \ldots & B_{nn} \end{pmatrix} \begin{pmatrix} p_1 - \Theta_1 \\ \\ p_n - \Theta_n \end{pmatrix}$$

where each demand is given by $Q_1 = (Q_{1,1}, ..., Q_{1,m_1})'$, ... , $Q_n = (Q_{n,1}, ..., Q_{n,m_n})'$, where $m = \sum_{l=1}^{n} m_l$. m_l is not constrained to be equal to $m_{l'}$ for $l, l' = 1, .., n$ and $l \neq l'$. The associated vector of prices is given by $p_1 = (p_{1,1}, ..., p_{1,m_1})'$, ... , $p_n = (p_{n,1}, ..., p_{n,m_n})'$. As previously, we first omit for simplicity the individual i and time t index. We express demand and expenditure in the following matrix notation:

$$\begin{aligned} X &= \mathcal{A}1_n - MB\Theta \\ Y &= C - PB\Theta \end{aligned}$$

where X and Y are composed of the different demands and expenditures for the n goods,

$$X = \begin{pmatrix} Q_1 \\ \vdots \\ Q_n \end{pmatrix} \equiv \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad\qquad Y = \begin{pmatrix} p_1'Q_1 \\ \vdots \\ p_n'Q_n \end{pmatrix} \equiv \begin{pmatrix} p_1'x_1 \\ \vdots \\ p_n'x_n \end{pmatrix}$$

C is an unimportant – for the purpose of identification of the price elasticity – $n \times 1$ vector. The matrix A is given by:

$$\mathcal{A} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \ldots & \ldots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \ldots & \ldots & \alpha_{2n} \\ \vdots & & & \vdots & \\ \alpha_{n1} & \alpha_{n2} & \ldots & \ldots & \alpha_{nn} \end{pmatrix}$$

where

$$\alpha_{ll'} = \mathbf{1}'_{m_l} B_{ll'} P_{l'} \qquad l, l' = 1, ..., n$$

This matrix $\mathcal{A}$ is not symmetric as we do not impose that $m_l = m_{l'}$ for $l, l' = 1, ..., n$ and $l \neq l'$. Moreover, as $P_l \neq P_{l'}$, the matrix is not symmetric even if $m_l = m_{l'}$. We will discuss further this point as it represents the key identification issue of the model with multiple goods. The matrix M is composed of the different aggregators over the different values of an attribute for the different observed goods:

$$M = \begin{pmatrix} 1'_{m_1} & 0 & 0 & 0 \\ 0 & 1'_{m_2} & 0 & 0 \\ \vdots & & & \vdots \\ 0 & 0 & 0 & 1'_{m_n} \end{pmatrix}$$

The matrix B corresponds to the matrix of the demand functions for the n goods

$$B = \begin{pmatrix} B_{11} & \dots & B_{1n} \\ \vdots & & \vdots \\ B_{n1} & \dots & B_{nn} \end{pmatrix}$$

where B_{ll} is a $m_l \times m_{l'}$ matrix. The vector Θ is given by

$$\Theta' = (\underbrace{\Theta_{1,1}, ..., \Theta_{1,m_1}}_{\Theta_1}, \dots, \underbrace{\Theta_{n,1}, ..., \Theta_{n,m_n}}_{\Theta_n})$$

This vector is randomly generated. As previously, the mean and the variance of the vector Θ are denoted Θ_o and V_o. We consider here only an individual effect. Finally, the matrix P is composed of the different prices of goods

$$P = \begin{pmatrix} p'_1 & 0 & \dots & 0 \\ 0 & p'_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & p'_n \end{pmatrix}$$

Let us consider again many consumers $i = 1, ..., N$ for different time periods $t = 1, \dots, T$. Demand and expenditure rewrite:

$$X_{i,t} = \mathcal{A}_t 1_n - MB\Theta_i$$

$$Y_{i,t} = C_t - PB_t\Theta_i$$

Consider the linear regression of $Y_{i,t}$ on $X_{i,t}$

$$\Psi_t = V(X_{i,t})^{-1} Cov(X_{i,t}, Y_{i,t})$$

$$= (MBV_oBM)^{-1} MBV_oBP_t$$

where Ψ_t is an $n \times n$ matrix. Now, express this matrix for two successive periods as follows:

$$\begin{aligned}\Psi_t^{-1}\Psi_{t-1} &= (MBV_oBP_t)^{-1} MBV_oBM (MBV_oBM)^{-1} MBV_oBP_t \\ &= (MBV_oBP_t)^{-1} MBV_oBP_{t-1}\end{aligned}$$

This equation shows immediately that if price variation is zero (*i.e.* if $P_t = P_{t-1}$), then $\Psi_t = \Psi_{t-1}$ and the covariance ratio remains identical over periods. In a such a case, the change in the covariance ratio over periods does not contain any information about the demand behavior. Conversely, when price variation is not zero, the model introduces a structural link between the actual and model moment conditions, provided some additional restrictions on structural parameters or on the prices change. We use the following useful property.

Theorem 2 *(a) If M is the eigenvectors matrix of MBV_o and the associated eigenvalues matrix is inversible or (b) if $P_t = (1+\delta_t)P_{t-1}$, then*

$$\Psi_t^{-1}\Psi_{t-1} = \mathcal{A}_t^{-1}\mathcal{A}_{t-1}$$

Theorem 2 states that if we impose some restrictions – again rather weak – on preferences or on the change in prices (as in Theorem 1), we can express the moments in terms of the structural parameters that compose the matrix $\mathcal{A}$. Theorem 2 is just an extension to the multiple good case of Theorem 1. Given this property, we can deduce the following moments for the n demands and n expenditures for different time period $t = 1, \ldots, T$. The change in mean over two periods is given by

$$\begin{aligned}(\mathcal{A}_T - \mathcal{A}_{T-1})\, 1_n &= D_{T,T-1} \\ (\mathcal{A}_T - \mathcal{A}_{T-2})\, 1_n &= D_{T,T-2} \\ \vdots \quad & \quad \vdots \\ (\mathcal{A}_T - \mathcal{A}_1)\, 1_n &= D_{T,1}\end{aligned}$$

where $D_{T,T-t} = E(X_{i,T}) - E(X_{i,T-t})$ for $t = 1, ..., T-1$. The ratio of scaled covariances over periods is given by $C_{T,T-t} = \Psi_T^{-1}\Psi_{T-t}$ for $t = 1, \ldots, T-1$ and from Theorem 2 we deduce

$$\begin{aligned} C_{T,T-1} &= \mathcal{A}_T^{-1}\mathcal{A}_{T-1} \\ C_{T,T-2} &= \mathcal{A}_T^{-1}\mathcal{A}_{T-2} \\ \vdots \quad & \quad \vdots \\ C_{T,1} &= \mathcal{A}_T^{-1}\mathcal{A}_1 \end{aligned}$$

From these moment conditions, we can thus determine the necessary conditions for identification of $\mathcal{A}_1, \ldots, \mathcal{A}_T$. Because $\mathcal{A}$ is in general not symmetric, each matrix $\mathcal{A}_t$ $(t = 1, ..., T)$ involves $n \times n$ parameters. The number of parameters to be identified is thus equal to $T \times n^2$. The number of moments that we can use is equal to the sum of $(T-1) \times n$ for the change in mean and $(T-1) \times n^2$ for the covariance ratio. This implies that a necessary condition for identification of the sequence $\{\mathcal{A}_1, \ldots, \mathcal{A}_T\}$ is

$$(T-1) \times n + (T-1) \times n^2 \geq T \times n^2$$

or equivalently

$$T \geq n + 1$$

This means that the number of periods must strictly exceed the number of goods. In order to illustrate this new condition for identification, consider for example a two good model. This implies that the two period case of the previous section does not allow to identify the demand parameter. In this case, a necessary condition for identification is that the number of period must be equal to three. This result comes from the asymmetry of the matrices $\mathcal{A}_1$ and $\mathcal{A}_2$. If these matrices are symmetric, the number of parameters that composes each matrix $\mathcal{A}_t$ is equal to $n(n+1)/2$. It follows that in this – very restrictive – case, a necessary condition for identification is satisfied as the number of periods verifies $T \geq 2$, *i.e.* the number of periods must be at least equal to the number of goods.

However, a two period case does not allow in general to identify the demand parameters in a two good model, because of the asymmetry of the matrix $\mathcal{A}$.

The identification of $\{\mathcal{A}_1, \ldots, \mathcal{A}_T\}$ imposes additional restriction on the covariance structure over goods and periods. First note that we have only to determine under which condition the matrix $\mathcal{A}_T$ is identified, as the sequence of matrices $\{\mathcal{A}_1, \ldots, \mathcal{A}_{T-1}\}$ can be directly deduced from

$$\mathcal{A}_{T-1} = \mathcal{A}_T C_{T,T-1}$$

Using the previous equation, we thus deduce

$$\begin{aligned} \mathcal{A}_T(I_n - C_{T,T-1})1_n &= D_{T,T-1} \\ \vdots \quad & \quad \vdots \\ \mathcal{A}_T(I_n - C_{T,1})1_n &= D_{T,1} \end{aligned}$$

These later equations form a system of $T-1 \times n$ equations with $n \times n$ unknown parameters that enter in $\mathcal{A}_T$. Denote $\mathcal{IC}_{T,T-1} = I_n - C_{T,T-1}$ and

$$\mathcal{IC}_{T-1,n} = \begin{pmatrix} \mathbf{1}_n' \mathcal{IC}_{T,T-1} \\ \mathbf{1}_n' \mathcal{IC}_{T,T-2} \\ \vdots \\ \mathbf{1}_n' \mathcal{IC}_{T,1} \end{pmatrix}$$

The system of equations rewrites

$$\mathcal{C}_{T-1,n} \begin{pmatrix} \alpha_{\bullet 1,T} \\ \alpha_{\bullet 2,T} \\ \vdots \\ \alpha_{\bullet n,T} \end{pmatrix} = \begin{pmatrix} D_T(1) \\ D_T(2) \\ \vdots \\ D_T(n) \end{pmatrix}$$

where the matrix $\mathcal{C}_{T-1,n}$ has the form

$$\mathcal{C}_{T-1,n} = \begin{pmatrix} \mathcal{IC}_{T-1,n} & 0 & \ldots & 0 \\ 0 & \mathcal{IC}_{T-1,n} & \ldots & 0 \\ 0 & 0 & \ddots & \\ 0 & 0 & 0 & \mathcal{IC}_{T-1,n} \end{pmatrix}$$

whereas $D_T(l)$ for $l = 1, ..., n$ are given by

$$D_T(l) = (D_{T,T-1}(l), D_{T,T-2}(l), \ldots, D_{T,1}(l))' \qquad l = 1, \ldots, n$$

and finally the vectors $\alpha_{\bullet l,T}$ for $l = 1, ..., n$ are

$$\alpha_{\bullet l,T} = \{\alpha_{1,l}, \alpha_{2,l}, \ldots, \alpha_{n,l}\} \qquad l = 1, \ldots, n$$

As $\mathcal{C}_{T-1,n}$ is block–diagonal, its determinant reduces to the determinant of $\mathcal{IC}_{T-1,n}$

$$|\mathcal{C}_{T-1,n}| = |\mathcal{IC}_{T-1,n}|^n$$

Provided $A_t \neq A_t' \quad t \neq t' \in [1, T]$ – the prices must vary –, a necessary and sufficient condition for identification is that the rank of $\mathcal{IC}_{T-1,n}$ must be greater than the number of goods:

$$rank\,(\mathcal{IC}_{T-1,n}) \geq n$$

When $T = n+1$, the condition for identification imposes $|\mathcal{IC}_{n,n}| \neq 0$. This means that there does not exist colinearity over periods in the covariance structure of the n goods, $i.e$ the changes over periods in the covariance structure of the all demands and expenditures must be sufficiently informative.

Finally, given the identification of the sequence of $\{\mathcal{A}_1, \ldots, \mathcal{A}_{T-1}\}$, we can determine the own price and the cross-price elasticity of demand for each period $t = 1, ..., T$. The own price elasticity is given by

$$\begin{aligned} \mathcal{E}^p_{ll,t} &= \frac{\partial E(x_{i,l,t})}{\partial p_{l,t}} \frac{p_{l,t}}{E(x_{i,l,t})} \qquad l = 1, ..., n \\ &= \frac{\alpha_{ll,t}}{E(x_{i,l,t})} \end{aligned}$$

whereas the cross price elasticity for $l, l' = 1, ..., n$, $l \neq l'$ is

$$\begin{aligned} \mathcal{E}^p_{ll',t} &= \frac{\partial E(x_{i,l,t})}{\partial p_{l',t}} \frac{p_{l',t}}{E(x_{i,l,t})} \\ &= \frac{\alpha_{ll',t}}{E(x_{i,l,t})} \end{aligned}$$

where $\alpha_{ll',t}$ are obtained from the matrix

$$\mathcal{A}_t = \begin{pmatrix} \alpha_{11,t} & \alpha_{12,t} & \dots & \dots & \alpha_{1n,t} \\ \alpha_{21,t} & \alpha_{22,t} & \dots & \dots & \alpha_{2n,t} \\ \vdots & & & \vdots & \\ \alpha_{n1,t} & \alpha_{n2,t} & \dots & \dots & \alpha_{nn,t} \end{pmatrix}$$

4.2 Non–Linear Model

The structural model assumes for simplicity that preferences are describe by a linear–quadratic utility function. This specification choice has the attractive feature of simplicity. However, this utility function has also unattractive properties. It does not exclude that negative quantities are demanded for many individual when the heterogeneity factor Θ is close to the price p and the variance of Θ is sufficiently large. Moreover, the linear–quadratic utility and the linear demand can be viewed has a local approximation of a more general model with non–linear demand. The question we investigate now, is to verify if our previous results about identification still holds if the demand function is non–linear.

Consider now that the demand takes the form

$$Q = \varphi\,(p - \Theta)$$

This type of non–linear demand function with non–additive error term imposes some restrictions about the representation of preferences. The specification of the utility function must be of the form

$$U(Q) = \mathcal{F}(Q) + Q'\Theta - Q'p$$

In this case of separability of the herteogeneity factor, the non–linear demand function is given by $\varphi = \mathcal{F}_Q^{-1}$, provided some inversibility conditions on $\mathcal{F}(Q)$.

We now turn to the identification of demand parameters in the non–linear case. The following theorem states new conditions for identification.

Theorem 3 *If the price variation is small and if* $\mathbf{1}_m$ *is eigenvector of*

$$V\left(\varphi\left(p_t-\Theta\right)\right)^{-1}E\left(\frac{\partial\varphi\left(p_t-\Theta\right)}{\partial p_t'}\right)$$

then the previous results with linear demand still apply.

Beyond the eigenvector condition that is similar to condition (a), the most important restriction is that this condition must hold every period. In the linear case, because of the linearity of the demand function, condition (a) imposes restriction on preferences parameters independently from price variation (except that price variation should not be zero). Here, the average derivative and the variance of demand are both indexed by time, because they depend on prices. This essentially means that the price variation must be sufficiently small in order to extend our previous to the non–linear case. We illustrate now these new conditions for identification.

Example 4.1 Let us consider the following non–linear model with $m = 2$ in a two period case ($t = 1, 2$). The utility function is given by:

$$U(Q) = \sum_{j=1}^{2}\left(a_i \log Q_j + \theta_j Q_j - p_j Q_j\right)$$

where $a_j > 0, \forall j = 1, 2$. The utility is assumed for simplicity to be separable across Q_1 and Q_2. The first order conditions of the maximization of the utility is

$$\frac{a_j}{Q_j} + \theta_j - p_j = 0 \qquad j = 1, 2$$

whereas the second order conditions verify

$$-\frac{a_j}{Q_j^2} < 0 \qquad j = 1, 2$$

We assume for tractability that each random factor θ_j is uniformly and independently distributed

$$\theta_j \sim U_{[0,\bar{\theta}_j]} \qquad j = 1, 2$$

where $\bar{\theta}_j < p_{j,t}$, $\forall t$ and $\forall j$. The average derivative over individuals for $j = 1, 2$ and $t = 1, 2$ is given by

$$\begin{aligned} E\left(\frac{\partial Q_{i,j,t}}{\partial p_{j,t}}\right) &= \frac{1}{\bar{\theta}_j}\int_0^{\bar{\theta}_j} -\frac{a_j}{\left(p_{j,t}-\theta_{i,j}\right)^2}1_{[0,\bar{\theta}_j]}(\theta_{i,j}) \\ &= \frac{1}{\bar{\theta}_j}\left(-\frac{a_j}{p_{j,t}-\bar{\theta}_j}+\frac{a_j}{p_{j,t}}\right) \\ &= -\frac{a_j}{p_{j,t}\left(p_{j,t}-\bar{\theta}_j\right)} \end{aligned}$$

We deduce the (2×2) matrix of average derivative:

$$E\left(\frac{\partial\varphi\left(p_t-\Theta\right)}{\partial p_t'}\right) = \begin{pmatrix} -\frac{a_1}{p_{1,t}\left(p_{1,t}-\bar{\theta}_1\right)} & 0 \\ 0 & -\frac{a_2}{p_{2,t}\left(p_{2,t}-\bar{\theta}_2\right)} \end{pmatrix}$$

The average demand over individuals for $j = 1, 2$ and $t = 1, 2$ is given by

$$\begin{aligned} E\left(Q_{i,j,t}\right) &= E\left(\frac{a_j}{p_{j,t}-\theta_{i,j}}\right) \\ &= \frac{1}{\bar{\theta}_j}\int_0^{\bar{\theta}_j} \frac{a_j}{p_{j,t}-\theta_{i,j}}1_{[0,\bar{\theta}_j]}(\theta_{i,j}) \\ &= -\frac{a_j}{\bar{\theta}_j}\left(\log\left(p_{j,t}-\bar{\theta}_j\right)-\log p_{j,t}\right) \\ &= \frac{a_j}{\bar{\theta}_j}\log\frac{p_{j,t}}{p_{j,t}-\bar{\theta}_j} \end{aligned}$$

Using the average demand, the average variance of individual demands for $j = 1, 2$ and $t = 1, 2$ is

$$\begin{aligned} V\left(Q_{i,j,t}\right) &= E\left(Q_{i,j,t}^2\right)-E\left(Q_{i,j,t}\right)^2 \\ &= \frac{a_j^2}{\bar{\theta}_j}\int_0^{\bar{\theta}_j} \frac{1}{\left(p_{j,t}-\theta_{i,j}\right)^2}1_{[0,\bar{\theta}_j]}(\theta_{i,j})-E\left(Q_{i,j,t}\right)^2 \\ &= \frac{a_j^2}{\bar{\theta}_j}\left(\frac{1}{p_{j,t}-\bar{\theta}_j}-\frac{1}{p_{j,t}}\right)-E\left(Q_{i,j,t}\right)^2 \\ &= \frac{a_j^2}{p_{j,t}\left(p_{j,t}-\bar{\theta}_j\right)}-E\left(Q_{i,j,t}\right)^2 \end{aligned}$$

From the expressions of the average derivatives, the variance of demand rewrites

$$V\left(Q_{i,j,t}\right) = -\left(a_j E\left(\frac{\partial Q_{i,j,t}}{\partial p_{j,t}}\right)+E\left(Q_{i,j,t}\right)^2\right)$$

The variance of total demand at periods 1 and 2 is given by the following diagonal matrix:

$$V\left(Q_{i,t}\right)=\left(\begin{array}{cc} -\left(a_1 E\left(\frac{\partial Q_{i,1,t}}{\partial p_{1,t}}\right)+E\left(Q_{i,1,t}\right)^2\right) & 0 \\ 0 & -\left(a_2 E\left(\frac{\partial Q_{i,2,t}}{\partial p_{2,t}}\right)+E\left(Q_{i,2,t}\right)^2\right) \end{array}\right)$$

Let denote $d_{j,t}=E\left(\frac{\partial Q_{i,j,t}}{\partial p_{j,t}}\right)$. Using the eigenvector condition of Theorem 2, we have to verify

$$\left(\begin{array}{cc} -\left(a_1 d_{1,t}+E\left(Q_{i,1,t}\right)^2\right) & 0 \\ 0 & -\left(a_2 d_{2,t}+E\left(Q_{i,2,t}\right)^2\right) \end{array}\right)^{-1}\left(\begin{array}{cc} d_{1,t} & 0 \\ 0 & d_{2,t} \end{array}\right)=\left(\begin{array}{c} k \\ k \end{array}\right)$$

for each period $t=1,2$. This implies that the following six restrictions for $t=1,2$ must hold:

$$\begin{aligned}
d_{1,t}\left(a_2 d_{2,t}+E\left(Q_{i,2,t}\right)^2\right) &= d_{2,t}\left(a_1 d_{1,t}+E\left(Q_{i,1,t}\right)^2\right) \\
d_{1,t-1}\left(a_2 d_{2,t-1}+E\left(Q_{i,2,t-1}\right)^2\right) &= d_{2,t-1}\left(a_1 d_{1,t-1}+E\left(Q_{i,1,t-1}\right)^2\right) \\
\frac{1}{\mathcal{D}_t} d_{1,t}\left(a_2 d_{2,t}+E\left(Q_{i,2,t}\right)^2\right) &= \frac{1}{\mathcal{D}_{t-1}} d_{1,t-1}\left(a_2 d_{2,t-1}+E\left(Q_{i,2,t-1}\right)^2\right) \\
\frac{1}{\mathcal{D}_t} d_{2,t}\left(a_1 d_{1,t}+E\left(Q_{i,1,t}\right)^2\right) &= \frac{1}{\mathcal{D}_{t-1}} d_{2,t-1}\left(a_1 d_{1,t-1}+E\left(Q_{i,1,t-1}\right)^2\right) \\
\frac{1}{\mathcal{D}_t} d_{1,t}\left(a_2 d_{2,t}+E\left(Q_{i,2,t}\right)^2\right) &= \frac{1}{\mathcal{D}_{t-1}} d_{2,t-1}\left(a_1 d_{1,t-1}+E\left(Q_{i,1,t-1}\right)^2\right) \\
\frac{1}{\mathcal{D}_t} d_{2,t}\left(a_1 d_{1,t}+E\left(Q_{i,1,t}\right)^2\right) &= \frac{1}{\mathcal{D}_{t-1}} d_{1,t-1}\left(a_2 d_{2,t-1}+E\left(Q_{i,2,t-1}\right)^2\right)
\end{aligned}$$

where $\mathcal{D}_t=\left(a_1 d_{1,t}+E\left(Q_{i,1,t}\right)^2\right)\left(a_2 d_{2,t}+E\left(Q_{i,2,t}\right)^2\right)$. These 6 restrictions are verified as the model implies 8 free parameters: $a_1, a_2, \bar{\theta}_1, \bar{\theta}_2, p_{1,t}, p_{1,t-1}, p_{2,t}, p_{2,t-1}$. Note that additional free parameters can be included, as the utility is separable over the quantities, the covariance matrix of the heterogeneity factor is assumed to be diagonal and the lower bound of the uniform distribution is set to zero. This simple example illustrates that the eigenvector condition for every period can be verified rather easily as there exist many free structural parameters. As we are not concerned with the identification of all structural parameters but rather with a summary of them, conditions for identification and restriction on preferences can be easily satisfied.

5 Concluding Remarks

In this paper, we propose a simple structural model that allows to identify and estimate the price elasticity of demand. The paper delivers conditions for identification and discusses various modeling issues. These conditions impose either some restrictions – although not restrictive – on preferences without any conditions on price change or on uniform price change without any conditions on preferences. The structural approach is especially useful when available data set concerns a very short panel. An empirical application illustrates the potential of this approach. Moreover, various extensions – multiple goods and non–linear demand – are introduced and new conditions for identification are determined. However, several issues may be then worth considering. First, we can mix some models that we study separately – shift in preferences, multiple goods, non-linear demand – and thus deliver their associated conditions for identification. Second, the models we consider do not take advantage either from individual or aggregate information in the specification of the heterogeneity factor. Finally, the model only considers an exogenous supply and obviously we must deliver new conditions for identification when supply is endogenous and random.

Appendix

A Proof of Theorem 1

Proof:

Under condition (a):

Using $V_oB1 = k1$, the moment conditions on demand and expenditure rewrite $E(x_{i,t}) = \alpha_t - \eta$, $E(y_{i,t}) = \lambda_t$, $V(x_{i,t}) = k\beta$, $V(y_{i,t}) = \omega_t^2$, $Cov(x_{i,t}, y_{i,t}) = k\alpha_t$, $Cov(x_{i,t}, x_{i,t'}) = k\beta$, $Cov(y_{i,t}, y_{i,t'}) = \rho_{t,t'}$ and $Cov(x_{i,t}, y_{i,t'}) = k\alpha_{t'}$. First subtract the demand between two successive periods t and $t-1$

$$E(x_{i,t}) - E(x_{i,t-1}) \equiv E(x_{i,t} - x_{i,t-1}) = \alpha_t - \alpha_{t-1}$$

This equation determines the change in demand due to change in price. Second, divide the covariance between demand and expenditure between two successive periods t and $t-1$

$$\frac{Cov(x_{i,t}, y_{i,t})}{Cov(x_{i,t-1}, y_{i,t-1})} = \frac{\alpha_t}{\alpha_{t-1}}$$

This equation represents the change in demand and expenditure co–movement due to change in price. These two equations allow to identify the two parameters α_t and α_{t-1}:

$$\begin{aligned} \alpha_{t-1} &= \frac{Cov(x_{i,t-1}, y_{i,t-1})}{Cov(x_{i,t}, y_{i,t}) - Cov(x_{i,t-1}, y_{i,t-1})} E(x_{i,t} - x_{i,t-1}) \\ \alpha_t &= \frac{Cov(x_{i,t}, y_{i,t})}{Cov(x_{i,t}, y_{i,t}) - Cov(x_{i,t-1}, y_{i,t-1})} E(x_{i,t} - x_{i,t-1}) \end{aligned}$$

The other parameters can be deduced directly from the other moment conditions.

Under condition (b):

Using $p_t = (1+\delta)p_{t-1}$ $(t = 1, 2)$, the moment conditions on demand and revenues are

$$\begin{aligned} E(x_{i,t}) &= (1+\delta)\alpha_{t-1} - 1B\Theta_o \\ Cov(x_{i,t}, y_{i,t}) &= (1+\delta)Cov(x_{i,t-1}, y_{i,t-1}) \end{aligned}$$

It follows that

$$E(x_{i,t} - x_{i,t-1}) = \delta\alpha_{t-1}$$

and

$$\frac{Cov(x_{i,t}, y_{i,t})}{Cov(x_{i,t-1}, y_{i,t-1})} = 1 + \delta$$

These two equations allow to identify the parameter α_{t-1}. The parameter α_t is obtained from $\alpha_t = (1+\delta)\alpha_{t-1}$.

□

B Proof of Proposition 1

Proof:

Under condition (a):
Using $V_oB1 = k1$, the moment conditions on demand and expenditure rewrite $E(x_{i,t}) = \alpha_t - \mu(t)\eta$, $E(y_{i,t}) = \tilde{\lambda}_t$, $V(x_{i,t}) = \mu(t)^2 k\beta + \beta^2\sigma_\varepsilon^2$, $V(y_{i,t}) = \mu(t)^2(\omega_t^2 + \alpha_t^2\sigma_\varepsilon^2)$, $Cov(x_{i,t}, y_{i,t}) = \mu(t)^2(k\alpha_t + \beta\alpha_t\sigma_\varepsilon^2)$, $Cov(x_{i,t}, x_{i,t'}) = \mu(t)\mu(t')k\beta$, $Cov(y_{i,t}, y_{i,t'}) = \mu(t)\mu(t')\rho_{t,t'}$ and $Cov(x_{i,t}, y_{i,t'}) = \mu(t)\mu(t')k\alpha_{t'}$. The growth factor is normalized to one in first period, *i.e.* $\mu(1) = 1$. First, divide the variance of demand the two successive periods:

$$\frac{V(x_{i,2})}{V(x_{i,1})} = \frac{\mu(2)^2}{\mu(1)^2}$$

From the normalization to one of the growth factor in first period, we deduce

$$\mu(2)^2 = \frac{V(x_{i,2})}{V(x_{i,1})}$$

Second, subtract the demand between period 1 and 2

$$E(x_{i,2}) - E(x_{i,1}) = \alpha_2 - \alpha_1 - \eta\left(\mu(2) - 1\right)$$

This equation determines the change in demand between period 1 and 2 due to change in price and the change in preferences. From the average demand in period 1, we deduce:

$$-\eta = E(x_{i,1}) - \alpha_1$$

After replacement of η into the change in average demand, we obtain

$$E(x_{i,2}) - \mu(2)E(x_{i,1}) = \alpha_2 - \mu(2)\alpha_1$$

where $\mu(2) = \sqrt{V(x_{i,2})/V(x_{i,1})}$. Third, divide the covariance between demand and expenditure by the variance of demand

$$\frac{Cov(x_{i,t}, y_{i,t})}{V(x_{i,t})} = \frac{\alpha_t}{\beta}$$

and form the ratio for two periods 1 and 2

$$\frac{Cov(x_{i,2}, y_{i,2})/V(x_{i,2})}{Cov(x_{i,1}, y_{i,1})/V(x_{i,1})} = \frac{\alpha_2}{\alpha_1}$$

This later equation together with the change in average demand allow to identify the two parameters α_1 and α_2:

$$\begin{aligned}
\alpha_1 &= \frac{E(x_{i,2}) - (\sigma_{x_{i,2}}/\sigma_{x_{i,1}})E(x_{i,1})}{Cov(x_{i,2}, y_{i,2})/V(x_{i,2}) - (\sigma_{x_{i,2}}/\sigma_{x_{i,1}})Cov(x_{i,1}, y_{i,1})/V(x_{i,1})}\frac{Cov(x_{i,1}, y_{i,1})}{V(x_{i,1})} \\
\alpha_2 &= \frac{E(x_{i,2}) - (\sigma_{x_{i,2}}/\sigma_{x_{i,1}})E(x_{i,1})}{Cov(x_{i,2}, y_{i,2})/V(x_{i,2}) - (\sigma_{x_{i,2}}/\sigma_{x_{i,1}})Cov(x_{i,1}, y_{i,1})/V(x_{i,1})}\frac{Cov(x_{i,2}, y_{i,2})}{V(x_{i,2})}
\end{aligned}$$

The other parameters can be deduced directly from the other moment conditions.
Under condition (b):
Using $p_t = (1 + \delta)p_{t-1}$, the change in average demand between period 1 and 2 is

$$\begin{aligned}
E(x_{i,2}) - E(x_{i,1}) &= \delta\alpha_1 - \eta(\mu(2) - 1) \\
&= \delta\alpha_1 - (E(x_{i,1}) - \alpha_1)(\mu(2) - 1)
\end{aligned}$$

and the covariance

$$\frac{Cov(x_{i,2}, y_{i,2})/V(x_{i,2})}{Cov(x_{i,1}, y_{i,1})/V(x_{i,1})} = 1 + \delta$$

These two equations allows to identify α_1, provided $\mu(2) = \sigma_{x_{i,2}}/\sigma_{x_{i,1}}$. Given α_1, the parameter α_2 is deduced using $\alpha_2 = (1 + \delta)\alpha_1$.

□

C Proof of Theorem 2

Proof:

Under condition (a):
M is eigenvector of BV_o rewrites as $MBV_o = KM$ where K is the matrix of eigenvalues. We assume that each eigenvalue is different from zero. Now using this condition, we deduce:

$$\begin{aligned}
\Psi_t^{-1}\Psi_{t-1} &= (MBV_oBP_t)^{-1} MBV_oBP_{t-1} \\
&= (KMBP_t)^{-1} KMBP_{t-1} \\
&= (K\mathcal{A}_t)^{-1} K\mathcal{A}_{t-1} \\
&= \mathcal{A}_t^{-1}\mathcal{A}_{t-1}
\end{aligned}$$

Under condition (b):
Condition (b) imposes $P_t = (1 + \delta_t)P_{t-1}$ and we deduce

$$\begin{aligned}
\Psi_t^{-1}\Psi_{t-1} &= (MBV_oBP_t)^{-1} MBV_oBP_{t-1} \\
&= \frac{1}{1+\delta_t} (KMBP_{t-1})^{-1} KMBP_{t-1} \\
&= \frac{I_n}{1+\delta_t} \\
&= \mathcal{A}_t^{-1}\mathcal{A}_{t-1}
\end{aligned}$$

□

D Proof of Theorem 3

Proof:
Let first introduce the average sensitivity to change in price of the non–linear demand function:

$$\begin{aligned}
\alpha_t &= \mathbf{1}_m' E\left(\frac{\partial\varphi\left(p_t - \Theta\right)}{\partial p_t'}\right) p_t \\
&= -\mathbf{1}_m' E\left(\frac{\partial\varphi\left(p_t - \Theta'\right)}{\partial\Theta'}\right) p_t \\
&= -\mathbf{1}_m' \left(\int_\Omega \frac{\partial\varphi\left(p_t - \Theta\right)}{\partial\Theta'} f\left(\Theta\right) d\Theta\right) p_t
\end{aligned}$$

Note also that if

$$\lim_{||\Theta||\to\infty} \varphi\left(p_t - \Theta\right) f\left(\Theta\right) = 0$$

the average sensitivity rewrites:

$$\begin{aligned} \alpha_t &= \mathbf{1}_m' \left(\int_\Omega \varphi(p_t - \Theta) \frac{\partial f(\Theta)}{\partial \Theta'} d\Theta \right) p_t \\ &= \mathbf{1}_m' \left(\int_\Omega \varphi(p_t - \Theta) \left(\frac{\partial f(\Theta)}{\partial \Theta'} / f(\Theta) \right) f(\Theta) d\Theta \right) p_t \\ &\equiv \mathbf{1}_m' E\left(\varphi(p_t - \Theta) g(\Theta)'\right) p_t \end{aligned}$$

Second, we determine the average change in demand

$$\begin{aligned} E(x_{i,t}) - E(x_{i,t-1}) &= \mathbf{1}_m' \int_\Omega \varphi(p_t - \Theta) f(\Theta) d\Theta - \mathbf{1}_m' \int_\Omega \varphi(p_{t-1} - \Theta) f(\Theta) d\Theta \\ &= \mathbf{1}_m' \int_\Omega \left(\varphi(p_t - \Theta) - \varphi(p_{t-1} - \Theta)\right) f(\Theta) d\Theta \\ &\simeq \mathbf{1}_m' \int_\Omega \frac{\partial \varphi(p_{t-1} - \Theta)}{\partial p_{t-1}'} (p_t - p_{t-1}) f(\Theta) d\Theta \quad \text{if } \Delta p_t \text{ is small} \\ &\simeq \mathbf{1}_m' \left(\int_\Omega \frac{\partial \varphi(p_{t-1} - \Theta)}{\partial p_t'} f(\Theta) d\Theta \right) p_t - \mathbf{1}_m' \left(\int_\Omega \frac{\partial \varphi(p_{t-1} - \Theta)}{\partial p_{t-1}'} f(\Theta) d\Theta \right) p_{t-1} \\ &\equiv \alpha_t - \alpha_{t-1} \end{aligned}$$

For small price variations, this simply means that the linear approximation of the demand function is accurate.

Now, recall that condition (a) for identification requires that the – standardized – covariance between demand and expenditure must be proportional to the average derivative α_t for each and every periods

$$Cov(x_{i,t}, y_{i,t}) = \kappa \alpha_t$$

The covariance matrix is given by

$$Cov(x_{i,t}, y_{i,t}) = \mathbf{1}_m' V(\varphi(p_t - \Theta)) p_t$$

We deduce from the above expression

$$\mathbf{1}_m' V(\varphi(p_t - \Theta)) p_t = \kappa \mathbf{1}_m' E\left(\frac{\partial \varphi(p_t - \Theta)}{\partial p_t'} \right) p_t$$

A sufficient condition for identification is

$$\mathbf{1}_m' V(\varphi(p_t - \Theta)) = \kappa \mathbf{1}_m' E\left(\frac{\partial \varphi(p_t - \Theta)}{\partial p_t'} \right)$$

that is equivalent to

$$V(\varphi(p_t - \Theta))^{-1} E\left(\frac{\partial \varphi(p_t - \Theta)}{\partial p_t'} \right) \mathbf{1}_m = \kappa \mathbf{1}_m$$

i.e $\mathbf{1}_m$ must be an eigenvector of

$$V(\varphi(p_t - \Theta))^{-1} E\left(\frac{\partial \varphi(p_t - \Theta)}{\partial p_t'} \right)$$

Now, using the change in average demand

$$E(x_{i,t}) - E(x_{i,t-1}) = \alpha_t - \alpha_{t-1}$$

and the ratio of the covariances between demand and expenditure over two periods

$$\frac{Cov(x_{i,t}, y_{i,t})}{Cov(x_{i,t-1}, y_{i,t-1})} = \frac{\alpha_t}{\alpha_{t-1}}$$

from which we can identify the parameters α_t and α_{t-1} and thus the price elasticity of demand every period.
□

References

Bajari, P. and Benkard, L. (2001) "Demand Estimation with Heterogenous Consumers and Unobserved Product Characteristics: A Hedonic Approach", NBER, *Technical Working Paper* 272.

Bartik, T. (1987) "The Estimation of Demand Parameters in Hedonic Price Models", *Journal of Political Economy*, 95 (1), pp. 81–88.

Berry, S., J. Levinsohn and and Pakes, A. (1995) "Automobile Prices in Market Equilibrium", *Econometrica*, 60 (4), pp. 889-917.

Berry, S., J. Levinsohn, and Pakes, A. (2003) "Differentiated Products Demand Systems from a Combination of Micro and Macro Data: The New Car Market', *working paper*, Yale University.

Brown, J. and Rosen, H. (1982) "On the Estimation of Structural Hedonic Price Models", *Econometrica*, 50, pp 765–769.

Brown, J. (1983) "Structural Estimation in Implicit Markets", in *The Measurement of Labor Cost*, edited by J. Triplett. Chicago: University Press for NBER.

Brown, B. and Walker, M. (1989) "The Random Utility Hypothesis and Inference in Demand Systems", *Econometrica*, 57(4), pp. 818–829.

Brown, B. and Walker, M. (1995) "Stochastic Specification in Random Production Models of Cost–Minimizing Firms", *Journal of Econometrics*, 66, pp. 175–205.

Cazals, C., Fève, F., Fève, P. and and Florens, J.P. (2003) "Simple Structural Econometrics of Price Elasticity", *mimeo* Gremaq–Idei.

Cazals C. and Florens J–P. (2002) "Econometrics of mail demand : a comparison between cross-section and dynamic data", in *Postal and Delivery Services: Delivering on Competition*, Crew M.A. and Kleindorfer P.R. eds., Kluwer Academic Publisher.

De Rycke M., Florens J–P. and Marcy S. (2001) "Mail use by firms", in *Future Directions in Postal Reforms*, Crew M.A. and Kleindorfer P.R. eds., Kluwer Academic Publisher.

Ekeland I., Heckman J. and Nesheim L. (2002) "Identifying Hedonic Models", *American Economic Review*, 92(2), 304-309.

Ekeland I., Heckman J. and Nesheim L. (2003) "Identification and Estimation of Hedonic Models", forthcoming in *Journal of Political Economy.*

Epple, D. (1987) "Hedonic Prices and Implicit Markets: Estimating Demand and Supply Functions for Differentiated Products", *Journal of Political Economy*, 95 (1), pp. 59–80.

Hansen, L. (1982) "Large Sample Properties of Generalized Method of Moment Estimators", *Econometrica*, 50, pp. 1029–1054.

Heckman, J. (1974) "Effects of Child Care Programs on Women's Work Effort", *Journal of Political Economy*, 82(2), Part II, pp. S 136–163.

Heckman, J. and Willis, R. (1977) "A Beta–Logistic Model for the Analysis of Sequential Labor Force Participation by Married Women", *Journal of Political Economy*, 85(1), pp. 27–58

Kahn, S. and Lang K. (1988) "Efficient Estimation of Structural Hedonic Systems", *International Economic Review*, 29, pp. 157–166.

Lancaster, T. (1979) "Econometric Methods for the Analysis of Unemployment", *Econometrica*, 47, pp. 939–956.

Lewbel, A. (2001) "Demand Systems with and without Errors", *American Economic Review*, 91(3), pp. 611–618.

McElroy, M. (1987) "Additive General Error Models for Production, Cost and Derived Demand or Share Systems", *Journal of Political Economy*, 95(4), pp. 737–757.

McFadden, D. (1974) "Conditional Logit Analysis of Qualitative Choice Behavior", in P. Zarembka, ed., *Frontiers in Econometrics*, ,pp. 105–142, Academic Press.

Petrin, A. (2002) "Quantifying the Benefits of New Products: the Case of Minivan", *Journal of Political Economy*, 110(4), pp. 705–729.

Reiss, P. and F. Wollak (2003) "Structural Econometric Modeling: Rationales and Examples from Industrial Organization", forthcoming in *Handbook of Econometrics*, J.Heckman and E. Leamer eds.

Rosen, S. (1974) “Hedonic Prices and Implicit Markets: Product differentiation in Pure Competition”, *Journal of Political Economy*, 82, PP 34–55.

Santos, R. and Lagoa, S. (2001) “The Demand for Direct Mail in Portugal”, in *Future Directions in Postal Reforms*, Crew M.A. and Kleindorfer P.R. eds., Kluwer Academic Publisher.

Tauchen, H. and Witte, A. (2001) “Estimating Hedonic Models: Implications of the Theory”, NBER, *Technical Working Paper* 271.

Wolak, F. (1997) “Changes in the Household Level Demand for Postal Delivery Services from 1986 to 1994”, in *Managing Change in the Postal and Delivery Industries*, Crew M.A. and Kleindorfer P.R. eds., Kluwer Academic Publisher.