Habit Persistence and Beliefs:

Lessons for the Effects of Monetary Policy

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First draft: November 1999
This draft: Janauary 2001

Abstract

The paper introduces habit persistence in consumption decisions in an infinitely–lived agents monetary model with a cash–in–advance constraint. We first show that strong enough habit persistence yields indeterminate equilibria. We then check the robustness of this result against alternative specifications, among which the introduction of other assets to escape the inflation tax. We however show that real indeterminacy is not *per se* sufficient to obtain an empirically relevant representation of the effects of monetary policy. The form of the beliefs matters. Money may be neutral when agents do not trust in money, whereas the monetary transmission mechanism is retrieved when beliefs are positively correlated with money supply shock. In the latter case, the model is also capable of generating a liquidity effect.

Keywords: Habit persistence, cash–in–advance, monetary transmission mechanism, beliefs, real indeterminacy, price stickiness, liquidity effect.

JEL Class.: E21, E32, E4

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†We would like to thank D. Andolfato, P. Beaudry, J.P. Benassy, S. Bosi, F. de Fiore, X. Fairise, R. Farmer, M. Guillard, P.Y. Hénin, J. J. Laffont, F. Langot, D. Laskar and G. Saint-Paul for helpful comments and discussions. This paper also benefited from discussions during presentations at Cepremap seminar, T2M conference (Nanterre, 2000), French–British Workshop on Macroeconomics (Paris, 2000) and Toulouse seminar (2001). The traditional disclaimer applies.

"All money is a matter of belief." Adam Smith

Introduction

The empirical literature that attempts to uncover the effects of monetary policy over the business cycle usually find a non–neutrality property of money in the short–run. More specifically, three main stylized facts seem to emerge from empirical studies, in particular from the VAR literature : following a contractionary monetary policy, *(i)* there is a persistent decline in real GDP; *(ii)* prices are almost non responsive in the very short–run but decrease and *(iii)* the nominal interest rate rises. *(i)* together with *(ii)* constitute the so–called monetary transmission and *(i)* together with *(iii)* define the liquidity effect (see e.g. Christiano [1991]). These results seem to be robust to different identification schemes (see e.g. Sims [1992], Christiano, Eichenbaum and Evans [1996] and [1999] , Sims and Zha [1995] and Leeper, Sims and Zha [1996]). Consequently, any structural model that could plausibly be used for monetary policy analysis should be able to account for these two mechanisms.

Flexible price monetary models, using either a cash–in–advance constraint (see e.g. Lucas and Stokey [1983] or Cooley and Hansen [1989] (for an application)) or the money in the utility function, imply that following a positive money injection output drops and the nominal interest rate rises. Indeed, in these models, the individuals attempt to escape the inflation tax the money injection creates by decreasing their consumption and increasing their leisure. Therefore the output drops. Further, since households postpone consumption and save more, the nominal interest rate rises. This contrasts with the aforementioned stylized facts. However, for a large part, this deceiving result may be explained by the existence of the inflation tax. Hence, most of the models that have attempted to provide with a better representation of effects of monetary on aggregate dynamics have tried to weaken the inflation tax.

A first approach has been to impose price stickiness, which prevents firms from instantaneously adjusting prices in response to monetary policy shocks (see e.g. Hairault and Portier [1993b], Rotemberg and Woodford [1992], Chari, Kehoe and McGrattan [1996], Christiano, Eichenbaum and Evans [1997] among others). This assumption breaks the inflation tax so that following a positive money injection the output rises and prices do not — obviously — fully respond on impact. But price stickiness remains to be explained. Another route that has been pursued is to assume limited participation in the model (see e.g. Lucas [1990], Christiano [1991], Fuerst [1992]), implying that households cannot adjust their behavior to any changes in financial market circumstances. Then any money injection disproportionately translates in higher supply of loanable funds, which puts downward pressure on the nominal interest rate. Access to credit being cheaper, firms can increase their scale of operations and the output rises. However, as noticed by Christiano [1991], the liquidity effect is not robust to as persistent money injections as the one found in the data. Common to these two classes of models is that they both assume the existence of frictions that either affect the price–setting behavior or the revelation of information to obtain a satisfying monetary transmission mechanism. More recently a third route has been pursued by Matheny [1998] or Benhabib and Farmer [2000] among others. They basically propose to go back to the initial monetary models approach and keep the *ex–ante* prices flexibility and complete information assumptions, and introduce other phenomena capable of generating the monetary transmission mechanism. Matheny [1998] examines the potential of Pareto substitutability for explaining the monetary transmission mechanism, and Benhabib and Farmer [2000] allows for important externalities in transaction. In this paper, we propose to investigate the role of intertemporal substitution in monetary transmission. Indeed, as aforementioned, part of the negative effects of the inflation tax are related to intertemporal substitution motives in consumption decisions, since households postpone consumption to escape the inflation tax. After all, "*it is nonsense that successive consumptions are independent; the normal condition is that there is a strong complementarity between them.*" (Hicks [1965]) We introduce intertemporal complementarities in consumption decisions in a monetary model and study its local dynamics properties. Money is held in the economy because households face a cash–in–advance constraint. More important is the fact that households' preferences are characterized by habit persistence, introducing time non–separability in the model. Habit persistence has proven to be a relevant assumption for representing preferences, and helpful for understanding puzzles related to the permanent income model (see e.g. Deaton [1992]), solving the equity premium puzzle (see e.g. Constantidines [1990], Campbell and Cochrane [1995]), and improving the ability of business cycle models to account for aggregate fluctuations (see Beaudry and Guay [1996], Lettau and Uhlig [1995] or Boldrin, Christiano and Fisher [1999]). An important feature of our modeling of habit persistence, is that it is internalized by individuals whenever they decide upon their consumption plans. Hence, the model exhibits only one externality: the inflation tax.

We first show that high enough habit persistence generates real indeterminacy in our monetary economy. It results from the interplay between habit persistence and the cash–in–advance constraint, given a specific environment on the labor and asset markets. Indeed, when individuals face the same positive belief on future inflation, higher expected inflation leads to them to substitute current for future consumption, thus increasing their habits. This translates into higher money demand for tomorrow when habit persistence is strong enough, putting upward pressure on prices. Then, inflation expectations become self–fulfilling. We then show that real indeterminacy is robust against the introduction of additional goods and/or asset that can be used to avoid the inflation tax, and against more general specification of the technology than the one considered in our benchmark model. Nevertheless, we show that real indeterminacy is not sufficient *per se* to generate the monetary transmission mechanism we are mainly interested in, the form of the beliefs matters a lot. Indeed, when beliefs are not correlated with money injection — when individuals do not trust in money — the model generates perfect price flexibility and money is neutral. Conversely, when beliefs are perfectly correlated with money injections — when individuals trust in money — the model displays endogenous price stickiness associated with a positive and persistent response of output. Hence the model can account for points *(i)* and *(ii)*, and therefore is capable of generating the monetary transmission mechanism provided agents have confidence in money. We also consider the ability of the model to account for points *(i)* and *(iii)*, that is to generate a liquidity effect. Like for the monetary transmission mechanism, we find that real indeterminacy is not sufficient to generate the liquidity effect, individuals must trust in money.

The paper is organized as follows. A first section presents our benchmark model economy, insisting on the individuals behavior. Section 2 characterizes the local dynamic properties of the model and discusses the conditions under which real indeterminacy occurs. It then check the robustness of our results against alternative specifications. Section 3 illustrates the theoretical results by exhibiting a showcase and discuss the role of beliefs in generating the monetary transmission mechanism. Section 4 then investigates the nominal interest rate behavior and evaluate the ability of our mechanisms to generate a liquidity effect. A last section offers some concluding remarks.

1 The model economy

This section describes the main ingredients that characterize our model economy, putting particular emphasis on the underlying mechanisms that drive the individuals' behavior.

1.1 The household behavior

The economy is comprised of a unit mass continuum of identical infinitely lived agents, so that we will assume that there exists a representative household in the economy. This household enters period t with real balances m_t/P_t brings into period t from the previous period, as a mean to transfer wealth from one period to another. The household supplies her hours on the labor market at the real wage W_t. During the period, she also receives a lump–sum transfer from the monetary authorities in the form of cash equal to N_t/P_t. All these revenues are then used to purchase a consumption bundle c_t and money balances for the next period. Therefore, the budget constraint simply writes as

$$\frac{m_{t+1}}{P_t} + c_t \leqslant W_t h_t + \frac{m_t}{P_t} + \frac{N_t}{P_t} \tag{1}$$

At a first glance, it may seem restrictive to assume that money is the only asset that appears explicitly in the budget constraint. Nevertheless, this formulation exploits the assumption that all households are identical, therefore although agents may freely trade a complete set of contingent claims, equilibrium asset prices will adjust such that agents will voluntarily choose a zero net position in these claims. We will relax this assumption in section 2.2 by introducing capital and in section 4 where we also consider the case of a non–zero net supply of public bonds.

Money is held because the household must carry cash — money acquired in the previous period and the money lump sum transfer — in order to purchase goods. She therefore faces a standard cash–in–advance constraint of the form:

$$c_t \leqslant \frac{m_t + N_t}{P_t} \tag{2}$$

Note that below, we will focus on equilibria in which the constraint binds. Implicit in this is that we will consider equilibria where the gross nominal interest rate exceeds unity or equivalently where the inflation rate is strictly positive, which will turn out to be the case for the specification of the shock we will consider.

Each household has preferences over consumption and leisure represented by the following intertemporal utility function:

$$E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \left[u\left(s_\tau\right) - v(h_\tau) \right] \tag{3}$$

where $\beta \in (0,1)$ is the discount factor, h_t denotes the number of hours supplied by the household. s_t is a consumption index on which we will come back and from which the household derives utility. E_t denotes the expectation operator conditional on the information set available in period t. We further impose the following assumption on $u(.)$ and $v(.)$.

Assumption 1 *$u : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a $\mathcal{C}^2$ function strictly increasing ($u'(.) > 0$) and concave ($u''(.) < 0$) in its argument, and satisfying $\lim_{x\to 0} u'(x) = +\infty$ and $\lim_{x\to\infty} u'(x) = 0$.*

Assumption 2 *$v : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a $\mathcal{C}^2$ function strictly increasing ($v'(.) > 0$) and convex ($v''(.) > 0$) in its argument, and satisfying $\lim_{x\to 0} v'(x) < \infty$ and $\lim_{x\to\infty} v'(x) = +\infty$ that represents the disutility of labor.*

This specification of the utility function assumes that consumption and leisure are separable. Although restrictive, this assumption eliminates one potential source of real indeterminacy in the cash–in–advance model stemming from the Pareto substitutability between consumption and leisure. Indeed, as already shown by Matheny [1998], when consumption and leisure are Pareto substitutes, real equilibrium in a cash–in–advance economy may be indeterminate. Indeed, when consumption and leisure are Pareto substitutes, any additional individual consumption increases the labor supply at a given real wage — because marginal utility of labor is a decreasing function of consumption — so that any prophecy that leads the household to consume more may be self–fulfilled. By forbidding this type of effect, our specification of the utility function implies that any potential instantaneous positive effect of consumption on the labor supply cannot be attributed to this mechanism.

We only depart from the standard cash–in–advance model, in that we allow for habit persistence in the consumption behavior, and therefore introduce time non–separability in the utility function. Habit persistence raises three main modeling issues: *(i)* the speed with which habit reacts to consumption, *(ii)* internal vs external habit and *(iii)* the functional form of habit formation (ratio vs. difference). In this paper, we follow Constantidines and Ferson [1991], Braun, Constantidines and Ferson [1993] and consider internal habit persistence specified in difference with one lag in consumption. More specifically, we assume that s_t takes the form

$$s_t = c_t - \theta c_{t-1} \text{ with } \theta \in (0, 1) \tag{4}$$

A more general specification (see Sundaresan [1989], Constantidines [1990]) would assume that habit reacts only gradually to changes in past consumption, such that $s_t = c_t - \theta \sum_{i=0}^{\infty}(1-\delta_s)^i c_{t-1-i}$, where $\delta_s \in [0, 1]$ represents the depreciation rate of habit stock. In order to found our analysis on one single parameter, we impose full depreciation $\delta_s = 1$ and retain the specification (4).[1] Further, since θ is restricted to

[1]It would be more satisfactory to consider incomplete habit stock depreciation. However this would lead to a third–order dynamic system for which we were not able to obtain analytical results. Nevertheless, we checked the robustness of our results against such a specification relying on numerical experiments. We considered several values for δ_s ranging from 0 to 1 (1 being our case), and found identical qualitative results. These results are available from the authors upon request.

lie between 0 and 1, we do not investigate the possibility of a durability effect.[2] It is also noteworthy that setting θ to zero allows to retrieve the standard cash–in–advance model. The consequences of such a specification for habit persistence on the dynamic properties of a monetary economy have not been, to our knowledge, investigated, although it may be of importance as it creates some intertemporal complementarity in the money demand behavior. It is also worth noting that we are ruling out external habit, as we want to keep externalities as small as possible, and do not want to introduce any other form of complementarity.[3] Finally, note that the choice of the functional form for habit formation does not really matter as far as our qualitative results are concerned.[4]

The household then determines its optimal consumption/money holdings and labor supply plans maximizing (3) subject to the budget (1) and cash–in–advance (2) constraints. Consumption behavior together with labor supply yield

$$\frac{v'(h_t)}{W_t} = \beta E_t \frac{P_t}{P_{t+1}} \left[u'\left(c_{t+1} - \theta c_t\right) - \beta\theta u'\left(c_{t+2} - \theta c_{t+1}\right) \right] \tag{5}$$

In order to provide with some intuition about the main specificity of our model economy, let us first set θ to zero to retrieve the standard cash–in–advance model. In this case, (5) reduces to

$$\frac{v'(h_t)}{W_t} = \beta E_t \frac{P_t}{P_{t+1}} u'\left(c_{t+1}\right) \tag{6}$$

It follows that any change in current consumption behavior cannot directly affect the labor supply behavior. Consider a situation where for any reason the household is willing to consume more today because she expects higher inflation tomorrow. Since the labor supply does not respond, her revenues remain constant for a given real wage. Therefore, either the household decides not to consume the extra amount she was willing to and the situation remains unchanged, or she decides to consume it and has to reduce the amount for money she will carry toward tomorrow and future consumption will decrease hence putting downward pressure on future inflation. It follows that higher future inflation expectations cannot be fulfilled. If we now consider our specification ($\theta > 0$, eq. (5)), today's consumption affects the level of future marginal utility

[2]This assumption is not as restrictive as it might seem at a first glance. See footnote 6 for a discussion.

[3]In a companion paper, we look at the case of external habit and find that our results are strengthened. However, this is largely due to the additional externality created by the external habit. Since we want to keep externalities to a minimum (only consider the externality created by money), we do not consider this case and focus on the situation where households internalize habit.

[4]We checked the robustness of our results against a ratio specification, and found very similar results. However, persistence only occurs for much larger values of the habit parameter.

of consumption. Since $u(.)$ is concave and given that c_t is negatively related to s_{t+1}, any increase in the level of consumption in the current period translates into an increase in future marginal utility, just reflecting the effects of habit persistence. Inada conditions impose that future consumption has to be at least greater than habit (θc_t), and thus creates some irreversibility in intertemporal consumption behavior. This makes higher money holdings compulsory for the household, by the cash–in–advance constraint. Consequently, the household is faced with a double constraint: *(i)* rise current consumption and *(ii)* increase money holdings. This can only be achieved if and only if her revenues increase, which triggers an increase in her labor supply, such that the willingness to consume more can be supported, therefore putting upward pressure on the inflation rate. In others, inflation expectations can be fulfilled provided that the habit persistence effect is high enough. It is noteworthy that we exhibit a similar pattern in the comovement of consumption and labor supply to the one found in Matheny [1998]. However, the underlying mechanism does not stem from a high intratemporal substitutability between consumption and leisure but from a weak intertemporal substitutability between current and future consumption.

1.2 The Equilibrium

The technology is described by the constant return to scale production function $Y_t = h_t$, such that in equilibrium the real wage is $W_t = 1$. Money is exogenously supplied by the central bank according to the following money growth rule:

$$M_{t+1} = g_t M_t \tag{7}$$

where $g_t \geqslant 1$ is the exogenous gross rate of growth of money, such that $N_t = M_{t+1} - M_t = (g_t - 1)M_t$. Finally, goods and labor market clearing condition implies:

$$y_t = c_t = h_t \tag{8}$$

and the money market clearing requires

$$m_{t+1} = M_{t+1} = M_t + N_t \tag{9}$$

Therefore, plugging (8), (9) and the definition of inflation in (5), the equilibrium is defined by the following second order finite difference equation written in terms of output.

$$v'(y_t) = E_t \frac{\beta}{g_{t+1}} \frac{y_{t+1}}{y_t} \left[u'\left(y_{t+1} - \theta y_t\right) - \beta\theta u'\left(y_{t+2} - \theta y_{t+1}\right)\right] \tag{10}$$

This last equation determines the dynamic properties of output, and therefore constitutes the key relation that we will be dealing with in the sequel. Also note that, as can be seen from equation (10), the model does not involve any externalities nor distorsions other than the inflation tax, that standard cash–in–advance models possess.

2 Dynamic properties of the economy

This section establishes the dynamic properties of our model economy, and more precisely characterizes conditions on the level of habit persistence for real indeterminacy, and discuss these results.

2.1 Habit persistence and indeterminacy

The dynamic properties of output are strongly related to the perfect foresight version of (10). First of all, note that the steady state value of output solves:

$$v'(y^\star) = \frac{\beta}{g}(1-\beta\theta)u'((1-\theta)y^\star)$$

The following proposition establishes the existence and uniqueness of the deterministic steady state.

Proposition 1 *Under assumptions 1 and 2, the deterministic steady state exists and is unique.*

Note that in this economy, as in other cash–in–advance economies, g, the rate of growth of money, exerts a negative effect on long–run output. Money is therefore non super–neutral in this economy.

Given existence and uniqueness of the steady state, we now turn to the analysis of the local dynamic properties of our model economy taking a first order log-linear approximation about its steady state. Holding the rate of growth of money supply constant, the log–linear approximation of the deterministic version of (10) about $y^\star$ yields the following linear second order finite difference equation:

$$\widehat{y}_{t+2} + \frac{(1-\theta)(1-\beta\theta)-\sigma_s(1+\beta\theta^2)}{\beta\theta\sigma_s}\widehat{y}_{t+1} + \left[\frac{1}{\beta} - \frac{(1-\theta)(1-\theta\beta)(1+\sigma_h)}{\beta\theta\sigma_s}\right]\widehat{y}_t \quad = (11)$$

where $\widehat{y}_t = (y_t - y^\star)/y^\star$. The parameter $\sigma_h = v''(h)h/v'(h) \geqslant 0$ is the inverse of the elasticity of Frishian labor supply. The parameter $\sigma_s = -u''(s)s/u'(s) \geqslant 0$ denotes the curvature parameter of the utility function. Note that, because of habit persistence, σ_s does not anymore correspond to the coefficient of relative risk aversion, nor does it correspond to the inverse of the intertemporal elasticity of substitution in consumption behavior, which, in our case, is given by (see Constantidines [1990] and Constantidines and Ferson [1991]):

$$\sigma_c = -\frac{cu''}{u'} \simeq \frac{\sigma_s c_t}{c_t - \theta c_{t-1}}$$

In the deterministic steady state, σ_c reduces to $\sigma_s/(1-\theta)$. It is then obvious that any increase in habit persistence decreases intertemporal substitutability motives.

Note that g — the steady state money growth — exerts no effect on the local dynamic properties of our model economy, as it does not appear in (11). Equation (11) can then be expressed in the more compact form

$$(1-\lambda L)(1-\mu L)\widehat{y}_{t+2} = 0$$

where L denotes the lag operator. The position of λ and μ around the unit circle characterizes the local dynamic properties of the log–linear economy. The model satisfies a saddle path property iff both λ and μ are of modulus greater than one. Conversely, if at least one of the eigenvalues lies inside the unit circle the equilibrium is locally indeterminate, *i.e.* there exists a continuum of equilibria in the neighborhood of the steady state. First of all, we can establish the following property.

Proposition 2 *The eigenvalues characterizing the dynamics of the economy are real.*

Proposition 2 rules out the possibility of persistent oscillations when the economy is converging back to its steady state, as the eigenvalues are real. Therefore, if the convergence path — in the neighborhood of the steady state — displays persistence, then it has to be the case that it is also monotone. This property does not rule out oscillatory equilibria, since it leaves room for negative eigenvalues. Nevertheless, the process is then negatively serially correlated. The following property now raises the possibility for real indeterminacy.

Proposition 3 *Provided $\sigma_s < 2+\sigma_h$, there exists $\theta^\star \in (0,1)$ such that for all $\theta \geqslant \theta^\star$ one and only one eigenvalue lies inside the unit circle.*

Proposition 3 establishes that — although money growth rule is exogenous[5] — for a given value of the discount factor (β) and the labor supply elasticity (σ_h) — there exists a value of the weight of habit persistence above which prophecies become self–fulfilling provided that σ_s is not too large,[6] that is when intertemporal substitution in excess consumption over habit is not too large. This restriction has appealing empirical counterpart as if we impose $\sigma_h = 0$, which corresponds to Hansen's [1985] assumption of labor indivisibility, σ_s has to be lower than 2, which lies within the range of point estimates for this parameter in a habit persistence model (see e.g. Constantidines and

[5] [?] establish that in a plausibly calibrated monetary model with explicit production but without habit persistence, exogenous money growth rules ensure real determinacy of equilibrium.

[6] This restriction stems from the fact that we are, by assumption, ruling out the possibility of durability. Nevertheless, if we allow for durability ($\theta < 0$), this restriction can be relaxed and it can be shown that indeterminacy occurs for low enough durability effect.

Ferson [1991] or Braun et al. [1993]). Note however, that contrary to most existing infinitely–lived agents general equilibrium models that display real indeterminacy (see e.g. Benhabib and Farmer [1999] for a survey), this model always keeps one forward looking dimension, as one of the two eigenvalue is always greater than 1. One way to think of this result is to restate the optimality conditions in terms of surplus consumption (s_t). The first order condition on current consumption is:

$$z_t - \beta\theta z_{t+1} = \chi_t$$

where $z_t = u'(s_t)$ and χ_t is the sum of the Lagrange multipliers associated with the budget and the cash–in–advance constraint. Clearly, since $\beta \in (0, 1)$ and $\theta \in (0, 1)$, the former difference equation is always forward looking, such that internal habit[7] preserves one of the permanent income properties of the model.

Figure 1: Roots of the characteristic polynomial

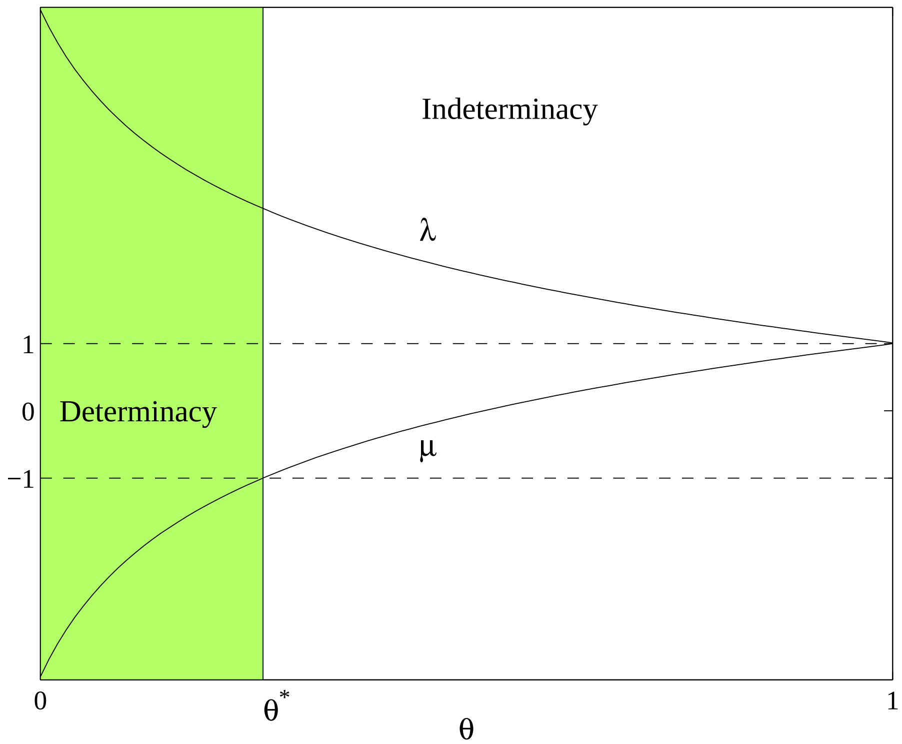


Figure 1 illustrates proposition 3 and the above discussion. The two curves represents the evolution of the two roots of the characteristic polynomial and shows that

[7] It is worth noting that an external habit version of this model would imply no discounting of future marginal utility, such that we would get

$$u'(s_t) = \chi_t$$

Therefore the external habit model only keeps track of the backward looking component under real indeterminacy.

one of the two roots always remain greater than unity. The shaded area corresponds to values of θ for which the equilibrium is totally determinate. Above $\theta^\star$ the equilibrium becomes indeterminate. It is worth noting that as θ tends to 1, the stable root tends to one. This can be easily seen from the characteristic polynomial associated with equation (11) and setting $\theta = 1$. The two roots are then $1/\beta$ (the unstable root) and 1 (the formerly stable root). Further, as can be seen from figure 1, the stable root is positive for high level of habit persistence. This leads to the following proposition.

Proposition 4 *There exists a threshold* $\widetilde{\theta} \in (\theta^\star, 1)$ *such that the stable root is strictly positive.*

An implication of this last result is that there exists values for the habit persistence parameter such that output is positively serially correlated, which is consistent with the observed persistence in aggregate data. This contrasts with the standard cash–in–advance model that generates no persistence, when the money growth process — if assumed to be exogenous — is serially uncorrelated. The latter proposition therefore establishes that the cash–in–advance model, when coupled with habit persistence, possesses internal propagation mechanisms strong enough to generate persistence. Beyond, this proposition states that, in this model, persistence goes together with real indeterminacy, as $\widetilde{\theta} > \theta^\star$. Another interesting result is that oscillatory sunspot equilibria may also occur as for values of $\theta \in (\theta^\star, \widetilde{\theta})$, the stable eigenvalue is strictly negative. It should however be noted, that, in this case, output is negatively serially correlated.

The two last properties have established existence of threshold values for θ, to show the possibility of indeterminacy and persistence. We now investigate some properties of these threshold values. More precisely, the next proposition characterizes the sensitivity of $\theta^\star$ and $\widetilde{\theta}$ to the two other preferences parameters σ_s and σ_h.

Proposition 5 *The thresholds* $\theta^\star$ *and* $\widetilde{\theta}$ *satisfy*

(i) $d\theta^\star/d\sigma_s < 0$ *and* $d\widetilde{\theta}/d\sigma_s < 0$

(ii) $d\theta^\star/d\sigma_h > 0$ *and* $d\widetilde{\theta}/d\sigma_h > 0$

(iii) $\lim\limits_{\sigma_h \longrightarrow \infty} \theta^\star = 1$ *and* $\lim\limits_{\sigma_h \longrightarrow \infty} \widetilde{\theta} = 1$

Result *(i)* states that low intertemporal substitutability in the consumption surplus (a large σ_s) implies that real indeterminacy occurs for lower values of the habit parameter. This should not be surprising as a high value for σ_s leads to a larger intertemporal complementarity in preferences, that plays in the same direction as habit persistence. In face of such results, one may question the relevance of habit persistence, since real indeterminacy can occur with a very low σ_s. But, in this case, the local dynamic properties are of lesser interest, since persistence does not occur. For instance, let us consider the extreme case where $\theta = 0$ and $\sigma_s = \sigma_c$ (in this case σ_c is the inverse of intertemporal elasticity of substitution in consumption), and $\sigma_h = 0$ (which, following proposition 5, actually favors indeterminacy). Then the equivalent to equation (11) writes as

$$\widehat{y}_{t+1} = \frac{1}{1-\sigma_c}\widehat{y}_t$$

As now well–known (see e.g. Woodford [1994], Carlstrom and Fuerst [1998], Farmer [1999], Bloise, Bosi and Magris [1999]), real indeterminacy occurs for values of σ_c greater than 2. Then, the root is always negative and tends toward 0 from below as σ_c tends to ∞. This type of dynamics is therefore of relatively low interest, as whatever value we take for σ_c the model will never generates persistence.

Result *(ii)* establishes that a low elasticity of Frishian labor supply makes real indeterminacy less likely, in the sense it imposes larger values for habit persistence. This is largely related to our last discussion in section 1.1 (last paragraph), and will be analyzed in the next section.

2.2 Discussion

This section attempts to shed light on the underlying forces that are at work in generating indeterminacy.

First note that whatever happens in this economy, labor demand takes the simple form $W_t = 1$. Therefore, the only way for an individual to increase her income is to supply more labor. In an equilibrium, the dynamics of the economy may be rewritten as

$$v'(c_t) = \beta E_t \tau(c_{t+1}, c_t, g_{t+1}) \left[u'\left(c_{t+1} - \theta c_t\right) - \beta\theta u'\left(c_{t+2} - \theta c_{t+1}\right)\right] \tag{12}$$

where $\tau(c_{t+1}, c_t, g_{t+1})$ denotes the inflation tax in the economy — taken as given by the individuals when determining their optimal plans.[8] The inflation tax can therefore be viewed as an increasing marginal tax function of future aggregate consumption — external to individuals. Let us assume that individuals behavior is characterized by a high intertemporal elasticity of substitution ($\theta \simeq 0$ and/or $\sigma_s \simeq 0$) and that they all have the same positive belief on future inflation. This leads every individual to increase current consumption. But, as intertemporal substitution is high, individual consumption drops in the next period. Since all individuals are identical and face the same belief, aggregate consumption drops in the next period. Therefore, the inflation

[8] Note that in equilibrium, $\tau(c_{t+1}, c_t, g_{t+1}) = c_{t+1}/(g_{t+1}c_t)$

tax shall decrease, which cannot support inflation beliefs. Any changes in beliefs can only be due to monetary policy, and is therefore related to fundamental shocks.

Let us now consider the case where intertemporal substitution is low ($\theta \gg 0$ and/or $\sigma_s \gg 0$) and that all individuals again face the same positive belief on future inflation. Like in the previous case, individuals consume more today. ut, contrary to the preceding case, the irreversibility in consumption decisions associated with habit persistence leads the agents to increase their future individual consumption too. Since, they are all identical and face the same belief, aggregate future consumption eventually increases. It follows that the aggregate inflation tax increases, therefore supporting the initial individual beliefs. Note that these beliefs may now depart from fundamentals — even though they can be arbitrarily correlated to fundamentals. Note that if the labor supply does not respond sufficiently — a large value for σ_h — the willingness to consume more cannot be supported and this may weaken the mechanism. Conversely, a more elastic labor supply (low σ_h) will enable the household to consume what she demands. Inflation beliefs will be easier to fulfill. This actually illustrates our previous results established in proposition 5.

The above discussion has shown how the interplay between habit persistence and cash–in–advance, given a specific environment on the labor and asset markets, can give rise to real indeterminacy and persistence. One may then question the robustness of our results to modifications in the labor and asset markets arrangements. We now investigate this issue.

As aforementioned, the response of the labor income is crucial in generating real indeterminacy. One of the implications of our technology on the labor market is that the real wage is constant in equilibrium. Therefore, labor income can shift upward following an increase in the labor supply. One may question the robustness of our previous results to a non–constant endogenous real wage. In order to address this issue, we now investigate the case of a more general production function given by $y_t = f(h_t)$, where $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a $\mathcal{C}^2$ function strictly increasing ($f'(.) > 0$) and concave ($f''(.) < 0$) in its argument. It further satisfies the Inada conditions $\lim_{x \to 0} f'(x) = +\infty$ and $\lim_{x \to \infty} f'(x) = 0$. We further introduce the notations $\alpha = f'(h)h/f(h) \in (0,1)$ and $\zeta = -f''(h)h/f'(h) > 0$ which denote the elasticities of, respectively, output and marginal product to hours. Note that, since returns to scale are decreasing, pure profits are non zero, and are redistributed to the households in a lump–sum fashion. Therefore, this leaves unaffected the first order conditions and the market clearing conditions. In such a case, it is straightforward to show that equation (11) rewrites as

$$\widehat{y}_{t+2} + \frac{(1-\theta)(1-\beta\theta) - \sigma_s(1+\beta\theta^2)}{\beta\theta\sigma_s}\widehat{y}_{t+1} + \left[\frac{1}{\beta} - \frac{(1-\theta)(1-\theta\beta)(1+\widetilde{\sigma}_h)}{\beta\theta\sigma_s}\right]\widehat{y}_t = 0 \tag{13}$$

where $\widetilde{\sigma}_h = (\sigma_h + \zeta)/\alpha$. In other words, the real indeterminacy results are left qualitatively unaffected by the non–constancy of the endogenous real wage, unless $\alpha \longrightarrow 0$. In order to provide with some intuition about the result, let us specify a Cobb–Douglas production function of the form: $y_t = h_t^{\alpha}$ (then $\zeta = 1 - \alpha$). $\widetilde{\sigma}_h$ is then given by $\widetilde{\sigma}_h = (\sigma_h + 1 - \alpha)/\alpha$. when α is close to 1, the real wage is not very responsive to increases in hours worked, and we retrieve the preceding results. Let us now consider a rather extreme experiment, where α is set close to 0. In this case, the real wage drops by a huge amount in face of an increase in hours worked, such that the labor income does not respond. This implies that θ has to be large to support inflation beliefs. But real indeterminacy will continue to occur, as shown in item *(iii)* of proposition 5. Hence, as long as α is strictly greater than 0, there exists a threshold value $\theta^{\star}$ such that real indeterminacy occurs. Also note that even if σ_h is set to zero — a infinite elasticity of labor supply corresponding to the labor indivisibility assumption — the effects of $\widetilde{\sigma}_h$ does not vanish as the real wage keeps on varying in face of fluctuations in hours worked: $\widetilde{\sigma}_h = (1 - \alpha)/\alpha$.

The previous discussion illustrated the robustness of our results when the assumption of a constant wage was relaxed. We now check the robustness of our results against the introduction of a new good (asset), as a mean to escape the inflation tax. Indeed, in our simple framework, the household can only use leisure to avoid paying the tax. We now consider an economy where the household can use another asset to avoid it: physical capital accumulation. We use a monetary optimal growth model *à la* Cooley and Hansen [1989] augmented with habit persistence. Each household has preferences over consumption and leisure represented by the following intertemporal utility function:

$$E_t \sum_{\tau=0}^{\infty} \beta^{\tau-t} \log(s_t) - \gamma h_t \tag{14}$$

Note that following Cooley and Hansen [1989], we imposed, for simplicity and tractability sakes, σ_s=1 and σ_h=0. We now allow for capital accumulation and assume a constant depreciation rate ($\delta \in (0,1)$), so that the intertemporal budget constraint of the household rewrites as

$$\frac{m_{t+1}}{P_t} + c_t + k_{t+1} \leq (Q_t + 1 - \delta)k_t + W_t h_t + \frac{m_t + N_t}{P_t} \tag{15}$$

where Q_t is the real rental rate of capital. As in the previous model, money is held because the household faces a cash–in–advance constraint (equation (2)). The problem of the representative household is then to choose her consumption–savings, labor and real balances plans to maximize (14) subject to (2) and (15). Monetary arrangements are assumed to be the same as in our benchmark framework.

The representative firm produces an homogeneous good that can be either invested or consumed using the constant returns to scale technology, represented by the Cobb–Douglas production function:

$$y_t \leq A k_t^{\alpha} h_t^{1-\alpha}$$

$A > 0$ is a constant scale parameter. The firm determines its production plans maximizing its profit. Note that we keep the preceding assumption concerning the determination of the labor demand, implying that the real wage does not remain constant when hours vary. Finally market clearing imposes $y_t = c_t + i_t$.

First of all, note that the labor supply writes

$$\gamma h_t = (1-\alpha)\lambda_t y_t$$

where λ_t is the lagrange multiplier associated with the budget constraint. Together with the production function, it implies that the output/capital ratio is a function of λ_t only:

$$\frac{y_t}{k_t} = \kappa_0 \lambda_t^{\frac{1-\alpha}{\alpha}} \text{ where } \kappa_0 = \left(\frac{1-\alpha}{\gamma}\right)^{\frac{1-\alpha}{\alpha}} A^{1/\alpha}$$

The Euler equation associated with capital decisions ($Q_t = \alpha y_t/k_t$) writes

$$\lambda_t = \beta E_t \lambda_{t+1}\left(\alpha \frac{y_{t+1}}{k_{t+1}} + 1 - \delta\right)$$

Plugging the labor market clearing condition in the Euler equation, we obtain

$$\lambda_t = \beta E_t \lambda_{t+1}\left(\alpha \kappa_0 \lambda_{t+1}^{\frac{1-\alpha}{\alpha}} + 1 - \delta\right)$$

which can be solved for λ_t independently from the rest of the dynamic system. Then, this model with labor indivisibility implies the same conditions for real indeterminacy with $\sigma_s = 1$ and $\sigma_h = 0$. Hence all our previous results still apply. In other words, letting the agent escape the inflation tax using another asset[9] does not eliminate the possibility of real indeterminacy of the equilibrium.

[9]We also obtain the same conclusion when we allow the agent to escape the inflation tax by arbitrating between cash (submitted to habit persistence) and credit goods. It should however not be surprising since, as pointed out by Lucas and Stokey [1987], leisure can also be viewed as a credit good, since the household can use it to escape the inflation tax. Expressing the model in terms of leisure, the utility rewrites $u(c_t) - v(1-\ell_t)$ and the budget constraint $\frac{m_{t+1}}{P_t} + c_t + w_t \ell_t \leqslant W_t + \frac{m_t}{P_t} + \frac{N_t}{P_t}$. ℓ_t can then be reinterpreted in terms of credit good. Further note that, although we do not restrict our attention to this case, in the case of labor indivisibility ($\sigma_h = 0$) the model including cash goods and credit goods yields exactly the same dynamics as the simple model, because credit goods are constant in equilibrium (when $\sigma_h > 0$ this result still holds but higher habit persistence parameters are needed). Results are not reported but are available from the authors upon requests.

3 A showcase

As shown in the previous section, the qualitative features of the model seem to be robust to the overall economic environment the agents face. In this section, we present some quantitative implications of our simple model, which illustrates its ability to account for persistent real positive effects of monetary policy and potential beliefs based price stickiness.

Since we are interested in the quantitative properties of the model, we have to specify an explicit functional form for the utility function. For tractability purposes and to keep the exposition simple,[10] we use

$$u(s_t) - v(h_t) = \log(s_t) - h_t$$

which corresponds to $\sigma_s = 1$ and $\sigma_h = 0$. Further, it should be noticed that previous empirical study by Braun et al. [1993] report point estimates for σ_s around 1, in a model with habit persistence of the kind we are studying. Given this specification, the output dynamics is described by the second–order finite difference equation

$$E_t\widehat{y}_{t+2} - \frac{1+\beta}{\beta}E_t\widehat{y}_{t+1} + \left[\frac{1}{\beta} - \frac{(1-\theta)(1-\theta\beta)}{\beta\theta}\right]\widehat{y}_t = \frac{(1-\theta)(1-\theta\beta)}{\beta\theta}E_t\widehat{g}_{t+1}$$

where $\widehat{g}_t$ denotes the percent change in the money supply. In this case, the threshold values[11] for θ, $\theta^\star$ and $\widetilde{\theta}$, take the simple forms

$$\begin{aligned} \theta^\star &= \frac{3(1+\beta) - \sqrt{9(1+\beta)^2 - 4\beta}}{2\beta} \\ \tilde{\theta} &= \frac{2+\beta-\sqrt{\beta^2+4}}{2\beta} \end{aligned}$$

Reasonable values for β (β close to unity) imply that $\theta^\star \approx 0.17$ and $\widetilde{\theta} \approx 0.38$. Therefore, real indeterminacy occurs rather easily in this economy, but more remarkable is that the model generates positive serial correlation in output dynamics with a value of θ which is not too high with respect to existing point estimates. Indeed, empirical studies suggest parameter estimates for θ that exceed significantly the minimal value that yields indeterminacy. For instance, Constantidines and Ferson [1991] and Braun

[10]It should be clear to the reader that our specification for the utility function yields an infinite labor supply elasticity, while microeconomic evidence on variations of hours worked suggest an elasticity lower than unity. This would amount to use a value for σ_h greater than 1. Nevertheless, as shown in the previous section, we would be able to get real indeterminacy even if we were to use $\sigma_h > 1$ (for $\sigma_h = 2$, we obtain $\theta^\star$=0.335 and $\widetilde{\theta}$=0.567), but this would lead to further, and unnecessary, complications in the exposition.

[11]Note that the threshold values are exactly the same in a model with capital accumulation and labor indivisibility (see section 2.2).

et al. [1993] obtain an estimated value of θ that lies within $[0.5; 0.9]$ on macro data. Habit persistence appears to be lower but still significant on micro data. Naik and Moore [1996] report estimates for habit persistence on food consumption data of 0.486 which is far above the threshold value of θ yielding indeterminacy and exceeds that needed for positive persistence. An important issue related to the previous discussion is the level of σ_c it implies. Neither $\theta^\star$ nor $\widetilde{\theta}$ leads to unreasonable values for σ_c, as $\sigma_c(\theta^\star)$=1.20 and $\sigma_c(\widetilde{\theta})$=1.62. Note however, that our specification implies that σ_c has to be necessarily greater than 1 to generate real indeterminacy. High (greater than 5) — and maybe unreasonable — values for σ_c only appear for $\theta > 0.8$. Also note that the model generates real indeterminacy for $\sigma_c < 2$, the required value in a model without habit persistence.

Finally, a monetary policy has to be specified. To keep things simple we use Cooley and Hansen's [1989] simple specification and assume that g_t evolves as an exogenous AR(1) process

$$\widehat{g}_{t+1} = \rho_g \widehat{g}_t + \varepsilon^g_{t+1}$$

where $|\rho_g| < 1$ and ε^g is a centered gaussian white noise with variance σ^2_ε.

We now focus on solutions associated with indeterminate equilibrium. In such cases, we have

$$\widehat{y}_t = \mu \widehat{y}_{t-1} - \frac{\rho_g}{\lambda - \rho_g} \frac{(1-\theta)(1-\theta\beta)}{\beta\theta} \widehat{g}_{t-1} + \varepsilon^y_t$$

where ε^y_t denotes a martingale difference sequence that can be related to fundamental shocks (money shocks), depending on individuals beliefs about monetary policy. In others, the random variable ε^y_t writes

$$\begin{aligned} \varepsilon^y_t &= b(g_t - E_{t-1} g_t) + \nu_t \\ &= b\varepsilon^g_t + \nu_t \qquad (16) \end{aligned}$$

with $E_{t-1}\nu_t = 0$ and $|b| < \infty$. ν_t denotes pure extrinsic beliefs that are unrelated to fundamentals. The parameter b rules the dependency of agents beliefs to fundamentals. It is worth noting that this parameter is an extrinsic characteristic of agents beliefs. Like in Benhabib and Farmer [2000] and Matheny [1998], the value of b is critical for the properties of the equilibrium. A glance at output and inflation dynamics illustrates this point. Given (16), inflation and output dynamics are given by

$$\begin{aligned} \widehat{y}_t &= \mu \widehat{y}_{t-1} - \frac{\rho_g}{\lambda - \rho_g} \frac{(1-\theta)(1-\theta\beta)}{\beta\theta} \widehat{g}_{t-1} + b\varepsilon^g_t + \nu_t \qquad (17) \\ \widehat{\pi}_t &= (1-\mu)\widehat{y}_{t-1} + \rho_g \left[1 + \frac{1}{\lambda - \rho_g} \frac{(1-\theta)(1-\theta\beta)}{\beta\theta}\right] \widehat{g}_{t-1} + (1-b)\varepsilon^g_t - \nu_t \qquad (18) \end{aligned}$$

To keep the exposition simple, let us consider the case where $\rho_g = 0$ and $\nu_t = 0$, $\forall t$. Then, (17) and (18) reduce to

$$\begin{aligned}
\widehat{y}_t &= \mu\widehat{y}_{t-1} + b\varepsilon^g_t \\
\widehat{\pi}_t &= (1-\mu)\widehat{y}_{t-1} + (1-b)\varepsilon^g_t = \mu\widehat{\pi}_{t-1} + (1-b)\varepsilon^g_t + (b-\mu)\varepsilon^g_{t-1}
\end{aligned}$$

Note that the two last equations just show that the model can generate persistence, provided $\mu > 0$. But they also show that real indeterminacy is not *per se* sufficient to generate the monetary transmission mechanism (output increases in face of a positive money injection), additional assumption are to be placed on individuals' beliefs — in particular how they comove with money supply shocks— as we now illustrate. Let us first consider the case where b=0, such that the above system reduces to

$$\begin{aligned}
\widehat{y}_t &= \mu\widehat{y}_{t-1} \\
\widehat{\pi}_t &= \varepsilon^g_t
\end{aligned}$$

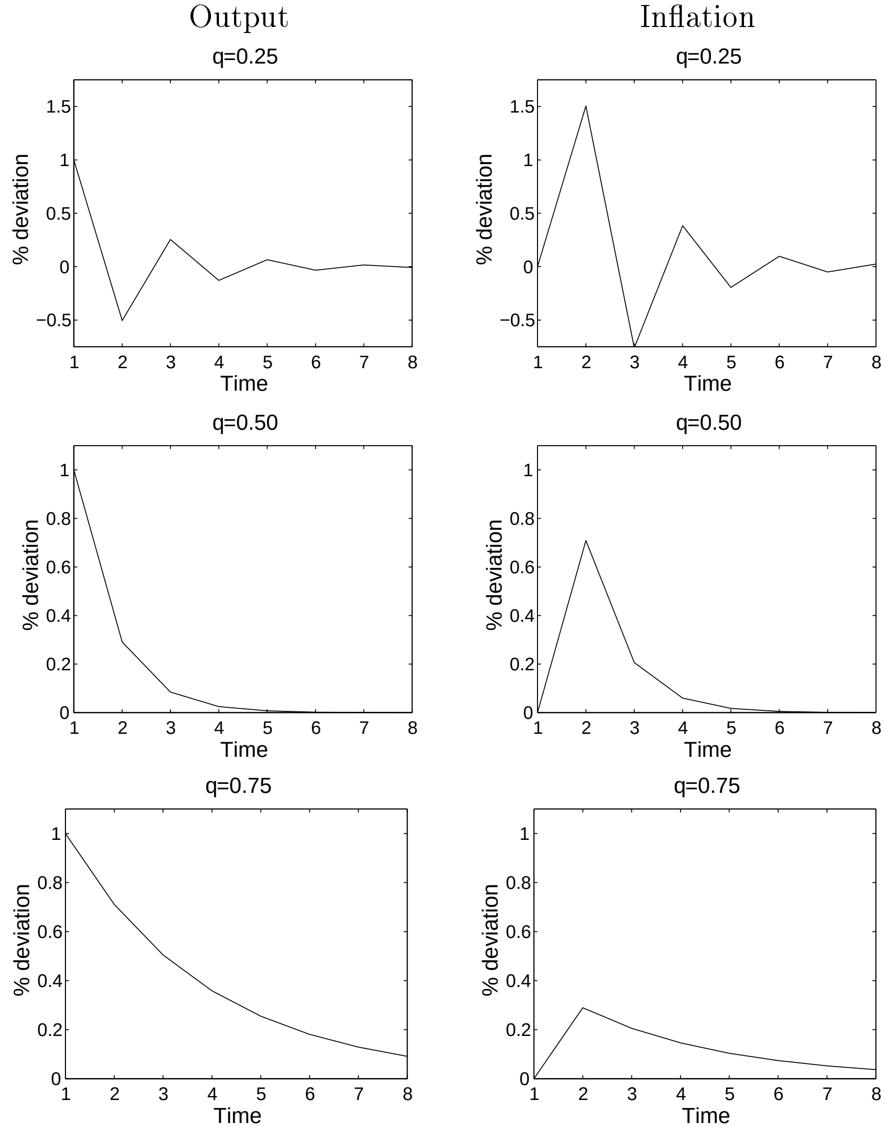
which implies that money is neutral in this model. Indeed, following a money injection, output remains at its steady state level ($\widehat{y}_t = 0$ and $\widehat{y}_t$ is a degenerated stochastic variable). Conversely, the inflation rate responds one for one to a money injection, hence fully absorbs the shock. This case corresponds to a full price flexibility situation and we retrieve the quantitative theory of money, which can then be associated with a particular form of beliefs where agents do not trust in money. It is worth noting that this results is similar to the pattern displayed by the standard cash–in–advance model without habit persistence ($\widehat{y}_t = 0$ and $\widehat{\pi}_t = \varepsilon^g_t$).

We now investigate a situation where individuals' beliefs are perfectly positively correlated with the money supply shock — agents trust in money — such that b=1. The output/inflation dynamics then rewrites

$$\begin{aligned}
\widehat{y}_t &= \mu\widehat{y}_{t-1} + \varepsilon^g_t \\
\widehat{\pi}_t &= (1-\mu)\widehat{y}_{t-1} = \mu\widehat{\pi}_{t-1} + (1-\mu)\varepsilon^g_{t-1}
\end{aligned}$$

Then, output responds instantaneously one for one to a one percent change in money supply. But, even more interesting, is that the model generates price stickiness, as the inflation rate does not respond instantaneously to the current money supply shock, but rather takes one lag to adjust partially. Note that, in this case, price stickiness does not result from any assumptions on the good market (like predetermined price setting, or nominal price contracts), but rather derives from particular individual beliefs concerning the monetary policy. This can be seen from figure (2) which reports the impulse response functions of output and the inflation rate with respect to a one percent

Figure 2: Output and Inflation responses to 1% money growth shock



Note : These figures are drawn for $\sigma_s = 1$, $\sigma_h = 0$, $\beta = 0.99$, $\rho_g = 0$ and $b = 1$.

money supply shock, for several levels of habit persistence. Another noteworthy feature that emerges from figure 2 is that the model possesses internal propagation mechanisms provided that θ is high enough ($\theta > \widetilde{\theta} \approx 0.38$). Indeed, let us recall that although $\rho_g = 0$, such that monetary policy is not long–lived, output and inflation remain above their steady state value for more than one period. After, their initial shift, they go back steadily to their long–run value, this convergence being monotone. Further, the higher θ is, the longer it takes to output and inflation to go back to their steady state. Conversely, when θ lies in $[\theta^\star; \widetilde{\theta}]$, the model generates negative serial correlation both in output and the inflation rate.

Another important result is that the higher the degree of habit persistence is, the less the inflation rate rises following a positive money injection. The intuition for that result lies in the fact that higher θ is associated with lower intertemporal substitutability in consumption plans, which in turns yields a lower response of intertemporal prices and therefore inflation. This is illustrated in table 1 that reports the first order autocorrelation of output and the inflation rate for the two polar views of price flexibility in the economy and various levels of monetary persistence and habit persistence.

Table 1: First–order autocorrelation

	b=0				b=1			
	ρ_g=0		ρ_g=0.5		ρ_g=0		ρ_g=0.5	
θ	(y)	(π)	(y)	(π)	(y)	(π)	(y)	(π)
0.25	–	0.00	-0.01	0.26	-0.50	-0.50	-0.62	-0.45
0.50	–	0.00	0.69	0.56	0.29	0.29	0.08	0.29
0.75	–	0.00	0.89	0.53	0.71	0.71	0.66	0.56

Note: These numbers were obtained for $\sigma_s = 1$, $\sigma_h = 0$, $\beta = 0.99$ and $\sigma_g = 0.009$. Finally we assumed $\nu_t = 0\ \forall t$

A first implication of our model is that under the flexible price regime ($b = 0$) output does not respond to money injection and inflation does not display any persistence, when money growth is i.i.d. Conversely, under sticky price beliefs ($b = 1$), both output and the inflation rate display persistence. In particular, when θ=0.75, output persistence is of about 0.70 and the inflation rate displays the same high degree of persistence. If we now have a look at the implications of a persistent money growth, things changes significantly as the sticky price economy experiments less persistence than the flexible price economy. Indeed, in the flexible price economy, agents have no confidence in money, so that a persistent money injection leads them to cut down their consumption. Habit persistence then implies that this cut will persist, and therefore output persistence rises. In the sticky price economy, individuals respond positively to a money injection as they trust in money. This actually corresponds to an anti inflation tax effect. In the subsequent periods, this effect does not play anymore and individuals expect a persistent inflation tax effect, which leads them — as in the perfectly flexible price economy — to cut down their consumption, even though they have habit persistence. The persistent inflation tax then leads them to be a turncoat, and therefore to weaken the persistence mechanism.

4 Habit persistence and the liquidity effect

The preceding models have not investigated the properties of the nominal interest rate in this economy so far. However, the nominal interest rate, and more particularly its behavior, is at the core of the monetary transmission mechanism and should be accounted for by the model. We thus now address this issue. In order to make the nominal interest rate explicit, we consider an augmented version of our initial model and introduce a government that issues nominal bonds to finance open market operations. These nominal bonds could also be used to finance government consumption. Nevertheless, this issue is beyond the scope of this paper, such that we ignore it. Households enter period t with real balances m_t/P_t carried over the previous period and nominal bonds b_t. The household supplies her hours on the labor market at the real wage W_t. During the period, the household also receives a lump–sum transfer from the monetary authorities in the form of cash equal to N_t/P_t and interest rate payments from bond holdings ($(R_{t-1}-1)b_t/P_t$). All these revenues are then used to purchase a consumption bundle c_t, money balances and nominal bonds for the next period. Therefore, the budget constraint simply writes as

$$\frac{b_{t+1}}{P_t}+\frac{m_{t+1}}{P_t}+c_t=W_t h_t+R_{t-1}\frac{b_t}{P_t}+\frac{m_t}{P_t}+\frac{N_t}{P_t} \tag{19}$$

Money is held because the household must carry cash — money acquired in the previous period and the money lump sum transfer — in order to purchase goods. She therefore faces a cash–in–advance constraint of the form

$$c_t \leqslant \frac{m_t+N_t+R_{t-1}b_t-b_{t+1}}{P_t} \tag{20}$$

The government budget constraint is

$$M_{t+1}+B_{t+1}=M_t+R_{t-1}B_t+N_t \tag{21}$$

with M_0 and B_0 are given. The household now determines her optimal consumption/saving, labor supply and money and bond holdings plans maximizing (3) subject to (19)–(20). It turns out that labor and consumption decisions are left unaffected by this slight modification Hence, in equilibrium, making use of the government budget constraint (21) into the cash–in–advance constraint (20), we retrieve equation (10). Thus, propositions 1–5 still hold in this economy. Nominal return of bond holdings is then determined, in equilibrium, by

$$v'(y_t)R_t = u'\left(y_t - \theta y_{t-1}\right) - \beta\theta E_t u'\left(y_{t+1} - \theta y_t\right) \tag{22}$$

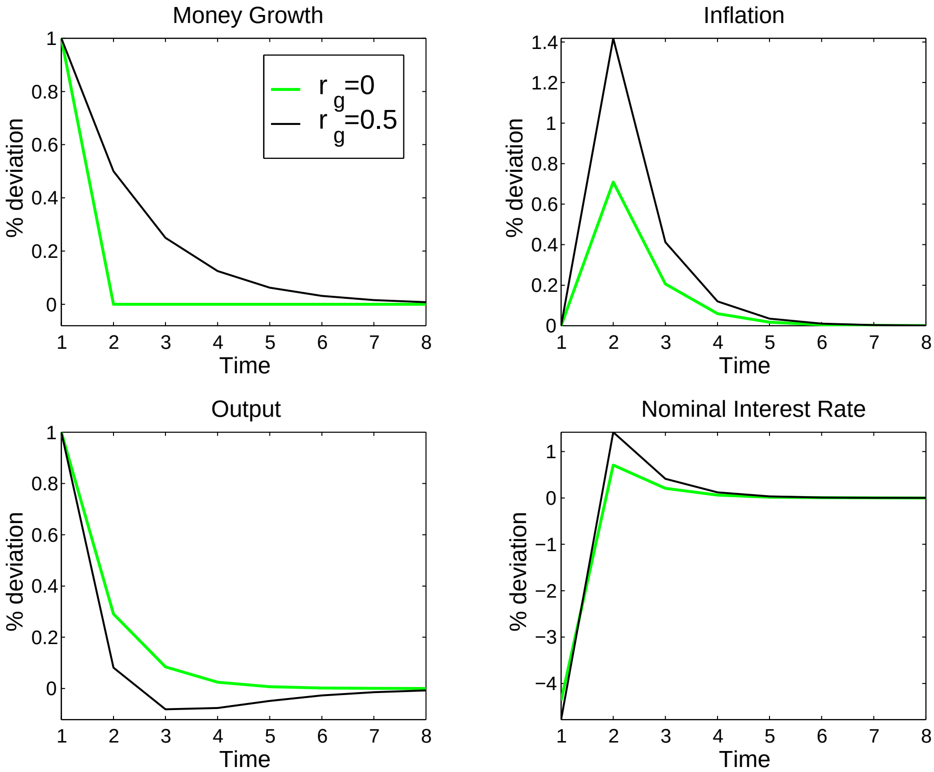
Using the solution for $\widehat{y}_t$, the local dynamics of the nominal interest rate in the neighborhood of the deterministic steady state can be expressed as a linear function of $\widehat{y}_{t-1}, \widehat{g}_t, \widehat{g}_{t-1}, \varepsilon_t^g, \nu_t$. More precisely, in the special case $\sigma_s = 1$, $\sigma_h = 0$, $\rho_g = 0$ and $\nu_t = 0\ \forall t$, the nominal interest rate behavior simply writes

$$\widehat{R}_t = \frac{\theta(1+\beta\mu^2) - \mu(1+\beta\theta^2)}{(1-\theta)(1-\beta\theta)}\widehat{y}_{t-1} + b\frac{\beta\theta\mu - 1 - \beta\theta^2}{(1-\theta)(1-\beta\theta)}\varepsilon_t^g$$

Since μ (the stable root) is strictly less than one in modulus, the quantity in front of ε_t^g is negative provided $b > 0$ — *i.e.* individuals trust in money. The model thus generates a liquidity effect: the nominal interest rate drops following a money injection, while output increases. This is illustrated by figure 3, which reports the impulse response functions of output, inflation and the nominal interest rate to money injection. As can be seen from the lower right panel of figure 3, the nominal interest rate drops following a positive i.i.d. ($\rho_g = 0$) money injection, and output responds one for one to the same injection. Interestingly, the liquidity effect generated by the model is robust to higher serial correlation ($\rho_g = 0.5$), contrary to standard limited participation models where the liquidity effect vanishes as ρ_g increases (see Christiano [1991]).

As already discussed in the previous section, when beliefs are positively related to money growth — *i.e.* when individuals trust in money — a positive money injection leads individuals to increase their current consumption. This triggers higher consumption in the next period, because of the intertemporal complementarity that habit persistence generates. The households is therefore willing to transfer wealth toward the future in order to support this future additional consumption. This can be achieved either by demanding more money for tomorrow and/or by purchasing bonds. But, as shown in equation (20), buying nominal bonds today, B_{t+1}, reduces the cash–in–hand available for current consumption. It is then optimal for the household to substitute money for bonds, which puts downward pressure on the nominal interest rate, therefore generating the observed liquidity effect. Once again, this is the interplay between the cash–in–advance constraint and the habit persistence hypothesis which is at the core of the phenomenon. This liquidity effect would not occur in an economy with no habit

Figure 3: The Liquidity effect



Note : These figures are drawn for $\sigma_s = 1$, $\sigma_h = 0$, $\beta = 0.99$, $\theta = 0.5$ and $b = 1$.

persistence, as money injection would then trigger a cut in consumption, such that the household would have less incentive to hold money. She would then essentially rely on bonds to transfer wealth intertemporally, putting upward pressure on the nominal interest rate.

Table 2: Correlation with output (money supply shocks)

	b=0				b=1			
	ρ_g=0		ρ_g=0.5		ρ_g=0		ρ_g=0.5	
θ	(g)	(R)	(g)	(R)	(g)	(R)	(g)	(R)
0.25	–	–	-0.45	-0.94	0.86	-0.99	0.28	-0.99
0.50	–	–	-0.48	-0.77	0.96	-0.89	0.86	-0.93
0.75	–	–	-0.38	-0.51	0.70	-0.69	0.97	-0.71

Note: These numbers were obtained for $\sigma_s = 1$, $\sigma_h = 0$, $\beta = 0.99$ and $\sigma_g = 0.009$. Finally we assumed $\nu_t = 0\ \forall t$

In order to illustrate the potential liquidity effect the model can generate, we report in table 2, the correlation of output with money growth and the nominal interest rate in an economy where only money shocks are considered.[12] As can be seen, the model generates a strong liquidity effect when beliefs are positively correlated with the money injection: output rises following a positive injection, while the nominal interest rate drops. As should be expected for low values of θ, i.i.d. shocks generates a higher correlation between output and money growth since being non persistent, the shocks do not generate expected inflation tax. On the contrary, when money growth shocks are persistent then expected inflation tax effects weakens the correlation since output may pass below its steady state for a while. Even more worth noting is the fact that the model always generates a negative correlation between output and the nominal interest rate in face of money supply shocks. This should however be interpreted with caution, as it may lead to spurious conclusions. Indeed, when $b = 0$, this negative correlation cannot be associated with a liquidity effect since output is negatively correlated with persistent money injection. When $b = 0$ and ρ_g=0, output and the nominal interest rate remain constant along the business cycle. Conversely, when $b = 1$, the correlation between the nominal interest rate and output is strongly negative and corresponds to a liquidity effect, as the output/money growth correlation is positive. Note that this effect remains robust to higher persistence in money supply injection, provided habit persistence is strong enough. For instance, when $\theta = 0.75$ and ρ_g=0.5, the output/money growth correlation is of about 0.97 and the output/nominal interest rate correlation is -0.71. It is even stronger than that obtain when shocks are i.i.d. since *(i)* shocks are more persistent and *(ii)* the habit persistence mechanism is strong enough to break the persistent inflation tax effect. Note that habit persistence plays an important role in generating this effect. Indeed, when habit persistence is low, the monetary transmission is far weaker as the inflation tax regains in strength. For instance, when $\theta = 0.25$, the output/money growth correlation is much lower ($\rho(y, g) = 0.28$).

5 Conclusions

The paper has shown that, introducing time non–separability in consumption decisions, an infinitely–lived agents monetary model with a cash–in–advance constraint may account for the so–called monetary transmission mechanism. We assume that one period lagged consumption produces service flows, that are perfectly internalized by the representative household. Hence, the model only exhibit one potential source of externality: money. We first show — in a simple model where fiat money is the only disposable

[12]This explains the high level of correlations (positive for output/money growth and negative for output/nominal interest rate when b=1) the model generates. The introduction of other sources of disturbances may weaken the correlations and allow to math the data. This is however beyond the scope of the paper.

asset to allocate wealth intertemporally — that high enough habit persistence yields self–fulfilling prophecies. Further, the dynamic equilibrium can then be supported by a sunspot. When they face a positive belief on future inflation, individuals are willing to achieve higher current consumption purchases. The irreversibility in consumption decisions associated with habit persistence implies that individuals increase their future individual consumption too. Since, all individuals are identical and face the same belief, aggregate future consumption eventually increases and so does the inflation rate. Prophecies are self–fulfilling. We then check the robustness of the indeterminacy result against alternative specifications, among which the introduction of a good that can be used to avoid the inflation tax. It appears that the results are quite robust. A simple showcase illustrates that real indeterminacy occurs for parameterization of the habit persistence in the range of existing estimation results (see e.g. Braun et al. [1993] or Constantidines and Ferson [1991]).

Besides our results show the potential of this approach. First, depending on the form of the beliefs, the model can generate either money neutrality — retrieving the quantitative theory of money — or price stickiness associated with a positive and persistent response of output when beliefs are positively related to the money supply shock. It therefore constitutes a framework which is endogenously consistent with the two main theories of prices. Further, it can also generate a liquidity effect, which is again dependent on the form of beliefs (when beliefs are positively related to the money supply shock). Implicit in this discussion is that real indeterminacy is not *per se* sufficient to generate the monetary transmission mechanism. Two conditions have to be full–filled: *(i)* beliefs should matter and *(ii)* beliefs should be positively related to money injection. *(i)* essentially states that real indeterminacy is a necessary condition to obtain the monetary transmission mechanism in our framework. *(ii)* states that there must be some confidence in money. In other words, our setting can account for the monetary transmission mechanism provided that individuals trust in money.

Several issues may be then worth considering. First of all, one may check the robustness of our results against other specifications for the money demand, in particular with money in the utility function. Our intuition is that when money and consumption are gross complement, our results should still hold as a cash–in–advance constraint reveals a strong complementarity between money and consumption. Another route that may be worth pursuing is to provide with a systematic quantitative evaluation of the time series implications of the mechanism we discussed. In particular, it may be interesting to assess the ability of the monetary economy we consider to account quantitatively for price stickiness and the liquidity effect in a more general model. Besides these robustness and quantitative evaluations, one may then wonder the kind of implications such a mechanism may have for the conduct of monetary policy. For instance, if the central bank were to pursue a countercyclical monetary policy, this would play against the intertemporal complementarity at work to generate multiplicity of equilibrium paths. This raises the question of the "merit of monetary policy rules" which may be used to rule out real indeterminacy and eliminate sunspots as a potential source of uncertainty.

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Appendix

Proof (Proposition 1): First recall that $v'(.)$ is strictly increasing, while $u'(.)$ is strictly decreasing from the convexity and concavity assumptions posed on both function. Hence, if they cross, they only cross once. Further, from the Inada conditions we have that $\lim_{x\to 0} u'(x) = +\infty$ and $\lim_{x\to 0} v'(x) < \infty$, such that $u'(.)$ and $v'(.)$ cross at least once, such that the unique steady state exists.

$\square$

Proof (Proposition 2): The characteristic polynomial associated with equation (11) is given by:

$$P(\lambda) = \lambda^2 + \frac{(1-\theta)(1-\beta\theta) - \sigma_s(1+\beta\theta^2)}{\beta\theta\sigma_s}\lambda + \left[\frac{1}{\beta} - \frac{(1-\theta)(1-\theta\beta)(1+\sigma_h)}{\beta\theta\sigma_s}\right]$$

The discriminant of $P(.)$ is given, after some algebra, by:

$$\begin{aligned}
\Delta &= \frac{\left((1-\theta)(1-\beta\theta) - \sigma_s(1-\beta\theta^2)\right)^2 + 4\beta\theta\sigma_s(1-\theta)(1-\beta\theta)(1-\theta+\sigma_h)}{(\beta\theta\sigma_s)^2} \\
&\equiv \frac{a^2+b}{c^2}
\end{aligned}$$

a^2 and c^2 are clearly positive. Now, as we restrict ourselves to situations where the economy is characterized by habit persistence ($\theta \in (0,1)$) and since $\beta \in (0,1)$ and $\sigma_h \geqslant 0$, b is positive as well. Therefore, the discriminant is strictly positive. Hence, $P(.)$ admits two real roots.

$\square$

Proof (proposition 3): Real indeterminacy occurs as at least one eigenvalue lies inside the unit circle. We therefore seek conditions for which $|\lambda| = 1$, where λ is an eigenvalue of the log–linearized dynamic equation, holds. The characteristic polynomial associated with the log–linearized version of the economy, $P(.)$ (see proof of proposition 2), satisfies

$$\begin{aligned}
P(1) &= \frac{1+\beta}{\beta} - \frac{\sigma_h(1-\theta)(1-\beta\theta) + \sigma_s(1+\beta\theta^2)}{\beta\theta\sigma_s} \\
P(-1) &= \frac{1+\beta}{\beta} + \frac{\sigma_s(1+\beta\theta^2) - (2+\sigma_h)(1-\theta)(1-\beta\theta)}{\beta\theta\sigma_s}
\end{aligned}$$

As can be easily checked, only two values of θ are compatible with $P(1) = 0$: $\theta = 1$ and $\theta = 1/\beta$. As we are interested in situations where $\theta \in (0,1)$ none of them is relevant for our purpose.

We now study the possibility for real indeterminacy to be associated with $P(-1) = 0$. Solving this condition for θ amounts to solve

$$Q(\theta) \equiv \theta^2 + \frac{1+\beta}{\beta}\left[\frac{\sigma_s+\sigma_h+2}{\sigma_s-\sigma_h-2}\right]\theta + \frac{1}{\beta} = 0$$

First note that:

$$Q'(\theta) < 0 \text{ for } \theta < \theta_0 \equiv \frac{1+\beta}{2\beta}\left[\frac{\sigma_s+\sigma_h+2}{\sigma_h+2-\sigma_s}\right]$$

Then, as $\beta \in (0,1)$, $\sigma_h > 0$, $\sigma_s > 0$ and $\sigma_s < \sigma_h + 2$, $\theta_0 > 1$. This implies that $Q(\theta)$ is strictly decreasing for $\theta \in (0,1)$. Further, $Q(0) = 1/\beta > 0$ and $Q(1) = 2\sigma_s(1+\beta)/\beta(\sigma_s - \sigma_h - 2) < 0$ as long as $\sigma_s < \sigma_h + 2$. In others, there exists a unique value $\theta^\star \in (0,1)$ such that $Q(\theta) = 0$. Finally, since $Q(\theta)$ is strictly decreasing, $Q(\theta) < 0$ (identically $P(-1) > 0$) for all $\theta > \theta^\star$.

□

Proof (proposition 4): Determining conditions for which the "stable" eigenvalue is positive essentially amounts to find conditions on θ for $P(0) = 0$ to hold. This is actually equivalent to study:

$$R(\theta) \equiv \theta^2 - \left[\frac{(1+\beta)(1+\sigma_h)+\sigma_s}{\beta(1+\sigma_h)}\right]\theta + \frac{1}{\beta} = 0$$

First note that:

$$R'(\theta) < 0 \text{ for } \theta < \theta_1 \equiv \frac{(1+\beta)(1+\sigma_h)+\sigma_s}{2\beta(1+\sigma_h)}$$

Then, as $\beta \in (0,1)$, $\sigma_h > 0$, $\sigma_s > 0$ and $\sigma_s < \sigma_h + 2$, $\theta_1 > 1$. This implies that $R(\theta)$ is strictly decreasing for $\theta \in (0,1)$. Further, $R(0) = 1/\beta > 0$ and $R(1) = -\sigma_s/\beta(1+\sigma_h) < 0$ such that there exists a unique value $\widetilde{\theta} \in (0,1)$ such that $R(\theta) = 0$. Finally, as $R(\theta)$ is strictly decreasing, $R(\theta) < 0$ (identically $P(0) > 0$) for all $\theta > \widetilde{\theta}$. This establishes the existence of $\widetilde{\theta}$.

In order to verify that $\widetilde{\theta} \geqslant \theta^\star$, we study the sign of $R(\theta) - Q(\theta)$. Recall that $Q(\theta)$ is strictly decreasing for all $\theta > \theta^\star$ (see proof of proposition 2.3) and that $R(\theta)$ is strictly decreasing for all $\theta > \widetilde{\theta}$. Then,

$$R(\theta) - Q(\theta) = -\left[\frac{(1+\beta)(1+\sigma_h)+\sigma_s}{\beta(1+\sigma_h)}\right]\theta - \frac{1+\beta}{\beta}\left[\frac{\sigma_s+\sigma_h+2}{\sigma_s-\sigma_h-2}\right]\theta$$

After some algebra, the latter expression reduces to

$$\frac{-\theta\sigma_s(\sigma_h(1+2\beta)+2\beta+\sigma_s)}{\beta(1+\sigma_h)(\sigma_s-\sigma_h-2)}$$

which, as $\beta \in (0,1)$, $\sigma_h > 0$, $\sigma_s > 0$, and $\sigma_s < \sigma_h + 2$ is strictly positive. Hence, $R(\theta) - Q(\theta) > 0$. Now since both $R(\theta)$ and $Q(\theta)$ are decreasing, we necessarily have $\widetilde{\theta} > \theta^\star$.

□

Proof (proposition 5): We first study the properties of $\theta^\star$. Let us first recall that it is defined by (see proof of proposition 3)

$$\theta^2 - \frac{1+\beta}{\beta}\left[\frac{\sigma_s+\sigma_h+2}{\sigma_s-\sigma_h-2}\right]\theta + \frac{1}{\beta} = 0$$

Total differentiation of the latter equation yields

$$\alpha_\theta \mathrm{d}\theta + \alpha_{\sigma_s}\mathrm{d}\sigma_s + \alpha\sigma_h \mathrm{d}\sigma_h = 0$$

where

$$\begin{aligned}
\alpha_\theta &= 2\theta + \frac{1+\beta}{\beta}\left[\frac{\sigma_s+\sigma_h+2}{\sigma_s-\sigma_h-2}\right] \\
\alpha_{\sigma_s} &= \frac{-2(1+\beta)}{\beta}\left[\frac{2+\sigma_h}{(\sigma_s-\sigma_h-2)^2}\right]\theta \\
\alpha_{\sigma_h} &= \frac{1+\beta}{\beta}\left[\frac{2\sigma_s}{(\sigma_s-\sigma_h-2)^2}\right]\theta
\end{aligned}$$

Since $\theta \in (0,1)$, $\sigma_s > 0$, $\sigma_h > 0$ and $\beta \in (0,1)$, it is clear that $\alpha_{\sigma_h} > 0$, $\alpha_{\sigma_s} < 0$. Studying the sign of σ_θ is a bit more demanding, as θ is itself a function of the σ_s and σ_h. Let's first rewrite α_θ as $\alpha_\theta = 2\theta + b$ where $b = \left(\frac{1+\beta}{\beta}\right)\left(\frac{\sigma_s+\sigma_h+2}{\sigma_s-\sigma_h-2}\right)$. As $\theta^\star = -\frac{1}{2}\left(b + \sqrt{b^2 - \frac{4}{\beta}}\right)$, (from $Q(\theta) = 0$), $\alpha_\theta = -\left(b + \sqrt{b^2 - \frac{4}{\beta}}\right) < 0$. Therefore, $\mathrm{d}\theta^\star/\mathrm{d}\sigma_h > 0$ and $\mathrm{d}\theta^\star/\mathrm{d}\sigma_s < 0$. This establishes points *(i)* and *(ii)* for $\theta^\star$. In order to establish *(iii)*, let's take the limit of $Q(\theta)$ as σ_s tends toward infinity

$$\lim_{\sigma_h \longrightarrow \infty} Q(\theta) = \theta^2 - \left(\frac{1+\beta}{\beta}\right)\theta + \frac{1}{\beta} = 0$$

which admits 1 and $1/\beta$ as a solution. Therefore, remembering that we are interested by the stable eigenvalue, we have $\lim\limits_{\sigma_h \longrightarrow \infty} \theta^\star = 1$.

We now investigate the properties of $\widetilde{\theta}$, which is defined by $R(\theta) = 0$ (see proof of proposition 4)

$$\theta^2 - \frac{(1+\beta)(1+\sigma_h)+\sigma_s}{\beta(1+\sigma_h}\theta + \frac{1}{\beta} = 0$$

Total differentiation of the latter equation yields

$$\gamma_\theta \mathrm{d}\theta + \gamma_{\sigma_s}\mathrm{d}\sigma_s + \gamma_{\sigma_h}\mathrm{d}\sigma_h = 0$$

where

$$\begin{aligned}
\gamma_\theta &= 2\theta - \frac{(1+\beta)(1+\sigma_h)+\sigma_s}{\beta(1+\sigma_h)} \\
\gamma_{\sigma_s} &= \left[\frac{-1}{\beta(1+\sigma_h)}\right]\theta \\
\gamma_{\sigma_h} &= \left[\frac{\beta\sigma_s}{\beta^2(1+\sigma_h)^2}\right]\theta
\end{aligned}$$

As $\theta \in (0,1)$, $\sigma_s > 0$, $\sigma_h > 0$ and $\beta \in (0,1)$, $\gamma_{\sigma_h} > 0$, $\gamma_{\sigma_s} < 0$. Like in the previous case, α_θ is a bit more demanding, and we rewrite γ_θ as $\gamma_\theta = 2\theta - b$ with $b =$

$\frac{(1+\beta)(1+\sigma_h)+\sigma_s}{\beta(1+\sigma_h}$. As $\widetilde{\theta} = \frac{1}{2}\left(b - \sqrt{b^2 - \frac{4}{\beta}}\right)$, we obtain $\gamma_\theta = -\left(b + \sqrt{b^2 - \frac{4}{\beta}}\right) < 0$. Therefore, $\mathrm{d}\widetilde{\theta}/\mathrm{d}\sigma_h > 0$ and $\mathrm{d}\widetilde{\theta}/\mathrm{d}\sigma_s < 0$, establishing *(i)* and *(ii)* for $\widetilde{\theta}$. To establish *(iii)*, let's take the limit of $R(\theta)$ as σ_s approaches infinity

$$\lim_{\sigma_h \longrightarrow \infty} R(\theta) \equiv \theta^2 - \left(\frac{1+\beta}{\beta}\right)\theta + \frac{1}{\beta} = 0$$

which is eventually the same limit as $Q(\theta)$. The same result therefore holds. $\square$