Sequential Uniform Price Auctions

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Abstract

When selling divisible goods such as energy contracts or emission allowances, should the entire supply be auctioned all at once, or should it be spread over a sequence of auctions? How does the expected revenue in a sequence of uniform price auctions compare to the expected revenue in a single uniform price auction? These questions come up when designing high-stake auctions, and this paper answers them and other relevant questions regarding divisible good auctions. In uniform price auctions, large bidders have an incentive to reduce demand in order to pay less for their winnings. In a sequence of uniform price auctions, bidders also internalize the effect of their bidding in early auctions on the overall demand reduction in later auctions and discount their bids by the option value of increasing their winnings in later auctions. This paper shows that a sequence of two uniform price auctions yields lower expected revenue than a single uniform price auction particularly when competition is not very strong.

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1 Introduction

When designing high-stake auctions, such as auctions for energy contracts or emission allowances, one of the first questions that come up is whether to have a single auction or to spread the supply (or demand in a procurement case) over a sequence of auctions. More often than not the decision has been to have a sequence of auctions. The Regional Greenhouse Gas Initiative (RGGI) which comprises the 10 northeastern states in the U.S. allocates CO_2 emission allowances among electricity generators within the region by means of a sequence of uniform price auctions. The supply of a given vintage of CO_2 emission allowances is spread over four annual auctions and four quarterly auctions.¹ Electricity supply contracts are sold quarterly by Electricitè de France, Endesa and Iberdrola (Spain) and were sold by Electrabel (Belgium) through the so called virtual power plant auctions.² Gas release programme auctions is the name used for the annual auctions of natural gas contracts used by Ruhrgas, Gas de France (GDF) and Total among others.³ The New York Independent System Operator allocates installed capacity payments through a sequence of monthly uniform price auctions;⁴ and the Colombian system operator will procure forward electricity supply contracts to match the annual forecast electricity demand by means of a sequence of four quarterly auctions.⁵

The seller looks for the auction format that is best suited for achieving her main goals of revenue maximization and efficiency. Sometimes, the seller is also interested in the market that results after the auction, like in spectrum auctions, and prefers an auction that yields a diverse pool of winners even at the expense of revenue maximization and efficiency. There are several features of the market that should be considered when deciding between a single auction and a sequence of auctions such as transaction costs, budget or borrowing constraints, private information and bidders's risk aversion.

When the transaction costs of bidding in an auction are high relative to the profits bidders can expect to make in that auction, participation in the auction can be expected to be low, which tends to have a negative effect on expected revenues. For this reason, the seller might prefer a single auction over a sequence of auctions to keep transaction costs low. In

¹See Holt et al. (2007) for more details on the auction design for CO_2 selling emission allowances under the RGGI.

²See www.powerauction.com and Milgrom (2004) for more details on virtual power plant auctions.

³See www.powerauction.com for more details on gas release programme auctions.

⁴See Installed Capacity Manual (2008), NYISO for more details on installed capacity auctions.

⁵See Cramton (2007) and www.creg.gov.co for more details on the Colombian electricity market.

the event that bidders face budget or borrowing constraints a single auction might limit the quantity they can buy, while in a sequence of auctions bidders have the chance to raise more capital if needed. A sequence of sealed-bid auctions is somewhere between a single sealed-bid auction and an ascending auction in terms of the private information revealed through the auctions. Hence, when there is private information about the value of the good being auctioned, a sequence of sealed-bid auctions improves the discovery of the collective wisdom of the market relative to a single sealed-bid auction, possibly increasing expected revenues. Since the price in an auction might be too high or too low due to some unexpected events, risk averse infra-marginal bidders (i.e. bid-takers) prefer a sequence of auctions over a single auction. If there is a single auction, infra-marginal bidders might end up paying too high or too low a price for all their purchases. But, in a sequence of auctions this risk is reduced since the prices bidders pay for their purchases are determined at several points in time. In the presence of risk averse bidders the seller might also prefer a sequence of sealed-bid auctions, since such auction format might increase the seller's expected revenues not only by increasing participation of risk averse bidders, particularly bid-takers, but also by encouraging marginal bidders to bid more aggressively due to a weaker winner's curse in a case with affiliated information.⁶

In addition, the effect of strategic bidding on revenue generation and efficiency should be considered when deciding between a single auction and a sequence of auctions. There is an extensive literature that studies equilibrium bidding, revenue generation and efficiency in sequences of single object auctions, such as sequences of first price, second price or even English auctions.⁷ However, there is no theoretical nor empirical research that studies sequences of divisible good auctions. But, in several real-world cases where sequences of auctions are used, such as those mentioned before, the auctioneer sells a divisible good. Moreover, we know from the case of a single auction, that divisible good auctions are not a trivial extension of single object auctions; hence one should not expect the results from sequential single object auctions to extend over to the case of sequential divisible good auctions. Therefore, studying strategic bidding in a sequence of divisible good auctions as well as the efficiency and revenue generation properties of this type of auctions is not only relevant from an academic perspective, but also from a practical standpoint.

⁶In the case of common-values with affiliated signals, the extra information that is revealed through the sequence of auctions reduce the winner's curse and the real risk imposed by aggressive bidding.

⁷See Weber (1983), Milgrom and Weber (1999), Ashenfelter (1989), McAfee and Vincent (1993), Bernhardt and Scoones (1994), Jeitschko (1999), Katzman (1999).

This paper studies a sequence of two uniform price auctions for a divisible good in a pure common value model with symmetric information and aggregate uncertainty. The unique profile of equilibrium bid functions in the second auction is fully characterized, as well as the entire set of equilibrium bid functions in the first auction. Using the characterization of equilibrium bidding, the revenue generation properties of the sequence of two uniform price auctions are compared with those of a single uniform price auction. A sequence of uniform price auctions was chosen over a sequence of pay-as-bid auctions because uniform price auctions are more widely used in energy and emission allowance markets, and there is a growing trend toward the use of this type of auctions in other markets.

Ausubel and Cramton (2002) show bidders in a uniform price auction have an incentive to shade their bids (i.e. reduce demand) in order to lower the price they pay for their purchases. This incentive grows with the quantity demanded and is inversely related to the size of bidders, measured by the maximum quantity they want to buy. In each auction of a sequence of two uniform price auctions bidders have the same incentive to shade their bids, since spreading the supply over two auctions does not change the fact that a bidder behaves like a residual monopsonist. At the first auction of the sequence, bidders know that if they do not buy all the quantity they want in that auction, they still have another opportunity to do so in the second auction. Therefore, bidders discount their first auction bids by the option value of increasing their purchases in the second auction. This is similar to the case of a sequence of single object auctions, where bidders discount their bids in an auction by the option value of participating in later auctions (Milgrom and Weber (1999), Weber (1983), Bernhardt and Scoones (1994) and Jeitschko (1999)).

In a single uniform price auction or in the first auction of the sequence, the maximum quantity each bidder wants to buy (i.e. his demand) is exogenous. However, in the second auction of the sequence bidders' demands are endogenous, because they depend on the quantities bought in the first auction. Since the bid shading in the second auction depends on bidders' demands in that auction, bidders have an incentive to shape the bid shading in the second auction through their bidding in the first auction. In equilibrium, one bidder holds back in the first auction, by bidding lower prices than his competitors. In that way, this bidder reduces his competition in the second auction by letting the other bidders buy larger quantities in the first auction than otherwise. This feature of equilibrium will be called *dynamic* bid shading to differentiate it from the *static* bid shading described by Ausubel and Cramton (2002). The bidder who benefits the most from this strategic behavior is the largest bidder, because by having a larger demand he can profit the most from the more intense bid

shading in the second auction.

The *static* and *dynamic* bid shading together with the discounting of the option value of increasing the quantity purchased in the second auction reduce the seller's expected revenue when using a sequence of two uniform price auctions. The *dynamic* bid shading and the option value discounting, which are not present in single uniform price auction, are particularly strong when there are few bidders and at least one of them demands a small share of the supply. These features of equilibrium bidding are even stronger when the supply is split evenly between the two auctions of the sequence. Hence, in those cases it is certainly more profitable for the seller to use a single uniform price auction than a sequence of two uniform price auctions. These results are in line with the finding that it is better for the seller to use a sealed-bid auction than a dynamic auction when competition is not very strong (Cramton (1998) and Klemperer (2004)).

This is the first paper that studies a sequence of divisible good auctions. The benefit of modeling sequential divisible good auctions is that it allows for the study of strategic forward looking bidding, which could have not been done by modeling a sequence of single object auctions with either unit or multi-unit demands, or even a sequence of multi-unit auctions with unit demands. Bidders bid in the first auction not only to buy some quantity at that stage, but also to improve their strategic position in the second auction. The improvement in a bidder's strategic position is not a consequence of the bidder strategically revealing information to manipulate his opponents' beliefs, but a consequence of the bidding and the quantity bought in the first auction.

When bidders have private information and multi-unit demands or non trivial demands in the case of divisible goods, bidders' beliefs might become asymmetric in any auction after the first one. This asymmetry might be problematic when analyzing sequential auctions. Most of the literature on sequential auctions, which studies sequence of single object auctions, avoids this problem by assuming unit demands, since the winner of an auction does not bid in subsequent auctions.⁸ Exceptions to this are Katzman (1999) and Donald, Paarsch and Robert (2006). Katzman (1999) assumes two bidders with demand for two units, and deals with asymmetric bidders' beliefs by studying a sequence of two second price auctions, where the beliefs are irrelevant after the first auction, since the second price auction has a dominant strategy. Donald, Paarsch and Robert (2006) study a sequence of single-unit

⁸See for example Milgrom and Weber (1999), Weber (1983), Bernhardt and Scoones (1994) and Jeitschko (1999).

English auctions with multi-unit demands. They assume that the distribution of valuations is symmetric and remains identical across players, regardless of the number of units they have purchased in previous auctions. Another way of avoiding the problem of asymmetric beliefs is assuming pure common values with symmetric information. This assumption includes two different cases. In one case the value of the good on sale is known by every bidder. In the other case, the value is unknown but every bidder receives the same signal about it.

This paper relates to a broad literature on how to create and enhance market power.⁹ In any market, there are different ways of creating or enhancing market power. For example, firms can create barriers to entry, or create sub-markets either by independently differentiating their products from their competitors' products, or by explicitly coordinating on some type of market segmentation. The underlying idea on the different strategies to create or enhance market power is to profitably differentiate yourself from your potential or actual competitors. This is exactly what happens in a sequence of two uniform price auctions. *Dynamic bid shading* is a strategy that allows non-cooperative bidders to optimally differentiate themselves by splitting up the market into two less competitive markets.

The literature on auctions for split-award contracts studies the case in which a buyer divides the purchases of its input requirements into several (usually two) contracts that are awarded to different suppliers in separate auctions (Anton and Yao (1989, 1992), Perry and Sákovics (2003)). In a sequence of two uniform price auctions, the split or market segmentation, which is endogenous, is not complete (i.e. all bidders buy in both auctions) because of the uncertainty about the residual supply in the second auction. However, as Herrera Dappe (2012) shows for the case of forward trading ahead of a procurement uniform price auction, if bidders' expected profits from the first auction or market are zero, then one bidder, usually the largest one, will wait for the second auction or market even with uncertain residual supply.

This paper also relates to a branch of the auction literature that studies auctions with aggregate uncertainty. On one side, Klemperer and Meyer (1989), Holmberg (2004, 2005) and Aromí (2006) study procurement uniform price auctions where firms sell a divisible good and demand is uncertain. This framework is known as the supply function framework since firms compete by submitting supply functions. On the other side, Wang and Zender (2002) study standard divisible goods auctions in a common values model with random noncompetitive demand. The model in this paper is closer to Wang and Zender's (2002)

 $^{^{9}}$ See Tirole (1988) for a survey on creation or enhancement of market power.

model than to the supply function models, not only because it studies a standard auction where the seller is the auctioneer, but also because it assumes a common values model with random noncompetitive demand.

The structure of the paper is as follows. Section 2 describes the model of two sequential uniform price auctions. Section 3 characterizes equilibrium bidding in the second auction, while section 4 does the same for the first auction of the sequence. Then section 5 compares the expected revenue in a sequence of two uniform price auctions with the expected revenue in a single uniform price auction. Finally, section 6 concludes.

2 Model

The seller has a quantity, normalized to one, of a perfectly divisible good for sale. She uses a sequence of two uniform price auctions, selling a quantity S_1 in the first auction and a quantity S_2 in the second auction, with $S_1 + S_2 = 1$. The price paid and the quantities bought by each bidder in the first auction are revealed before the second auction takes place. Resale between auctions is not allowed and it is also assumed the discount factor between both auctions is one.

Each bidder has a constant marginal value for the good, up to the maximum quantity he wants to consume.¹⁰ Moreover, this marginal value, v, is the same for all bidders and no bidder has private information. This last assumption includes two different cases. In one case, every bidder knows the true value of the good. In the other case, the value is unknown, but every bidder receives the same signal about the value of the good, and winning any quantity in the first auction does not provide any extra information. In this last case, v can be reinterpreted as the expected value conditional on the signal. For simplicity, it is assumed the seller derives no value for this good.¹¹

There are $N = \{2, 3\}$ strategic bidders, each acting to maximize his expected profit. Strategic bidder l wants to consume any quantity, q_l , up to λ_l , where $\lambda_l > 0$. Define $\tilde{\lambda}$ as the second highest λ , and assume that $\tilde{\lambda} \leq \frac{S_2}{N}$. In Wilson's (1979) seminal study of divisible good auctions, he demonstrated that uniform price auctions have a continuum of equilibria. As it will become clear later, the last assumption is key in reducing the set of possible equilibria up

¹⁰The constant marginal value assumption is made just for tractability. As it will become clear along the paper, the results would hold even if the marginal values were decreasing.

¹¹The results will not change as long as the seller has a lower value for the good than the bidders.

to the point of having a unique profile of equilibrium bid functions on the second auction. A strategy for strategic bidder l is a pair of piece-wise twice continuously differentiable demand functions, one for each auction, $(d_{l1}(p), d_{l2}(p))$, with $d_{lt} : [0, \infty) \to [0, \lambda_l]$.

There is also a continuum of measure 1 of non-strategic bidders, who can consume any quantity up to one. The bid of a non-strategic bidder is just a quantity.¹² Each one of these bidders has probability S_t of being assigned to auction t, with t = 1, 2. Once a non-strategic bidder is assigned to an auction he can only bid in that particular auction. All non-strategic bidders in auction t receive the same demand shock X_t , with $x_t \in [0, 1]$. Therefore, a non-strategic bidder in auction t bids for a quantity X_t . The demand shocks X_1 and X_2 are *i.i.d.* with G(x) representing the cumulative distribution function. The aggregate demand from non-strategic bidders in auction t is given by S_tX_t with $S_tx_t \in [0, S_t]$. Hence, strategic bidders' residual supply at auction t, $Y_t = S_t(1 - X_t)$, is uncertain with $F(y_t)$ representing its cumulative distribution function over the interval $[0, S_t]$. As a consequence of this uncertainty, most, if not all, of the points on the strategic bidders' equilibrium demand functions will be characterized by equilibrium conditions.¹³

Since the auctions used by the seller are uniform price auctions, the price paid by bidders at an auction is the clearing price, which is defined as the highest losing bid. This price depends on strategic bidders' residual supply, y_t , and the demand functions submitted by all strategic bidders, $p_t = \inf \{p \mid \sum_l d_{lt}(p) \leq y_t\}$. If $\sum_l d_{lt}(p_t) = y_t$, then each strategic bidder lis assigned a quantity $q_{lt}(y_1) = d_{lt}(p_t)$. If $\sum_l d_{lt}(p_t) > y_t$, then the demand curves of some bidders are discontinuous at p_t and they will be proportionally rationed at such price.

Before each auction, both types of bidders submit their demand functions for that auction to the auctioneer, who aggregates them and find the clearing price for that auction. The main difference between both auctions is that bidders know the outcome of the first auction before they bid in the second auction; allowing them to condition their bidding in the second auction on the outcome of the first auction.

Given the information structure and the timing of the game, an equilibrium of this model is a profile of strategies, one for each strategic bidder, that defines a subgame perfect equilibrium

¹²This can be interpreted as a non-strategic bidder submitting a flat bid at a price of v, or just submitting a quantity and telling the auctioneer he will buy that quantity at whichever is the clearing-price.

¹³All the results would hold if instead of assuming the presence of non-strategic bidders it were assumed the supply is uncertain. However, in that case it would be hard to conceptualize the idea that the seller can spread the supply over a sequence of auctions.

(SPE) of the entire game. Consequently, the analysis first focuses on the second auction and once equilibrium bidding in that auction is fully characterized, the focus shifts to the first auction. As the reader probably has already realized the most interesting findings of this paper are regarding equilibrium in the first auction and their effects on expected revenue. This is so because the incentives bidders face in the last auction of the sequence are indistinguishable from the incentives they would face in an otherwise identical single uniform price auction. From now on, the word bidders by itself will be used when referring to strategic bidders, while the expression non-strategic bidders will still be used when referring to this other type of bidders.

3 Second Auction

Once the auctioneer has announced the outcome of the first auction, but before the residual supply in the second auction, y_2 , is known, bidders simultaneously choose their demand functions for the second auction. When doing this, bidder l maximizes his expected profit from the second auction conditional on the quantities purchased by each bidder in the first auction. Define $q_{l1}(y_1)$ as the quantity bought by bidder l in the first auction when the residual supply was y_1 . Bidder l's optimization problem becomes:

$$\max_{d_{l2}(p_2)} E_2\left[\left(v - p_2 \right) d_{l2}\left(p_2 \right) \right] \tag{1}$$

s.t.
$$d_{l2}(p_2) \le \lambda_l - q_{l1}(y_1)$$
 (2)

The most important source of uncertainty in equation (1) is non-strategic bidders' demand in the second auction, which translates into uncertainty about the clearing price, p_2 .

As mentioned above, a demand function for bidder l can be any piece-wise twice continuously differentiable, decreasing function mapping from \Re_+ to $[0, \lambda_l]$. However, as the following lemmas show, equilibrium demand functions in the second auction are smooth functions in the interval (0, v), strictly monotonic in the same interval for all bidders when N = 2, and for at least two bidders when N = 3.¹⁴

Lemma 1 Equilibrium demand functions in the second auction are continuous for every price $p \in (0, v)$.

 $^{^{14}}$ The ideas for the proofs of the first three lemmas, or part of them, follows Aromí (2006).

Proof. First, clearly no bidder will bid more than v, and all bidders will bid v for their first unit. Now, define $\underline{d}_{l2}(p^*) = \lim_{p \to p^{*+}} d_{l2}(p)$, $\overline{d}_{l2}(p^*) = \lim_{p \to p^{*-}} d_{l2}(p)$, and similarly for the aggregate demand, $D_2(p)$. Without loss of generality assume bidder 1's demand is discontinuous at $p^* \in (0, \overline{p}_2)$. Then $(\overline{d}_{12}(p^*) - \underline{d}_{12}(p^*)) > 0$. For any interval $[p^* - \epsilon, p^*]$ at least bidder 2 or 3 must demand additional quantity, otherwise bidder 1 can profitably deviate by withholding demand at p^* . Assume bidder 2 demands additional quantity at those prices and define $p^{\epsilon}(p^*) = \sup\{p \mid d_{22}(p) \ge \underline{d}_{22}(p^*) + \epsilon\}$.

Bidder 2 can increase his expected profit by deviating and submitting the following demand function:

$$\widetilde{d}_{22}^{\epsilon}(p) = \begin{cases} \underline{d}_{22}(p^*) + \epsilon & if \quad p \in (p^{\epsilon}(p^*), p^* + \epsilon) \\ d_{22}(p) & otherwise \end{cases}$$
(3)

The effect of this deviation on bidder 2's expected profits can be split in two parts, an expected loss from higher prices and an expected gain from larger purchases. The expected loss is bounded above by $(p^* + \epsilon - p^{\epsilon}(p^*))(\underline{d}_{22}(p^*) + \epsilon)Pr^{\epsilon}(\Delta p)$. Where $Pr^{\epsilon}(\Delta p)$ is the probability that the price changes due to the deviation by bidder 2; which clearly converges to zero as ϵ does so. Moreover, the derivative of the upper bound is zero at $\epsilon = 0$.

Now, the expected gain is bounded below by $(v - p^* - \epsilon)\Delta E^{\epsilon}(q_{22})$, where $\Delta E^{\epsilon}(q_{22})$ is the expected change in quantity bought by bidder 2 in the second auction. Clearly, the lower bound of the expected gain is zero at $\epsilon = 0$, and its derivative is positive.

Bidder l is a residual monopsonist whose residual supply is given by the residual supply strategic bidders face and the demand from bidders other than l: $rs_{l2}(p_2) = y_2 - \sum_{h \neq l} d_{h2}(p_2)$. Even knowing the demand from all other bidders, bidder l's residual supply is uncertain due to the uncertainty about y_2 . The goal of bidder l is to find the demand function that maximizes his expected profit conditional on bidder l's demand function. If bidder l could find the price-quantity points, $(p_2, rs_{l2}(p_2))$,¹⁵ that maximize his ex-post profit for every possible realization of y_2 , and that set of points could be characterized by a weakly decreasing demand function, then clearly that demand function would maximize his expected profit. Since the uncertainty only affects the location of bidder l's residual supply and not its slope, there is always a weakly decreasing demand function that characterizes the set of ex-post optimal price-quantity points.

When deciding how much to buy, a monopsonist looks for the quantity such that the marginal

¹⁵Bidder l selects a price-quantity point on his residual supply curve for each realization of y_2 . Hence, the price bidder l selects is the clearing price.

addition to his costs equals the marginal addition to his revenue. Since the price he pays is determined by the residual supply he faces, which is his average cost, a monopsonist pays a price lower than his marginal revenue. Now, a standard result in auctions with uniform pricing rules is that bidders reduce their demands or shade their bids. The reason for this behavior is found on the incentives faced by a monopsonist. The marginal revenue for a bidder is the marginal value he has for the good, and the marginal cost of his purchases is higher than his average cost (i.e. his residual supply). Equation (4), which is the first order condition for bidder l, shows that the more inelastic is bidder l's residual supply, the more he shades his bids.

$$v - p_2 = \frac{d_{l2}(p_2)}{-d'_{-l2}(p_2)} \tag{4}$$

In equilibrium, no bidder demands a strictly positive quantity at prices above v, or bid more than v for any quantity. The first order conditions for all bidders define a system of differential equations, which defines interior equilibrium bidding in the second auction.¹⁶ However, since the only difference between bidders is the maximum quantity each bidder wants to buy, represented by the λ s, the system of first order conditions for an interior solution is symmetric, and defines the following differential equation:

$$(n-1)d'_{2}(p_{2}) = -\frac{d_{2}(p_{2})}{v-p_{2}}$$
(5)

where *n* represents the number of bidders whose demand constraint is not binding at p_2 . Define $\mu_l = \lambda_l - q_{l1}(y_1)$ as bidder *l*'s demand in the second auction. The subscripts *i*, *j* and *k* will be used to label bidders according to their demands in the second auction: $\mu_i \ge \mu_j \ge \mu_k$.

Lemma 2 Equilibrium demand functions in the second auction are strictly decreasing at every price in (0, v) for bidders i and j, and at every price in $(d_k^{-1}(\mu_k), v)$ for bidder k.

Proof. Because interior equilibrium bidding is symmetric, if bidder l demands the same positive quantity at every $p \in [p', p'']$, then no bidder will demand additional quantity at that range of prices. In that case, bidder l can increase his expected profit by withholding demand at every price in $(p', p_{\epsilon}(p''))$, where $p_{\epsilon}(p'') = \inf\{p \mid d_{l2}(p) \leq d_{l2}(p'') - \epsilon\}$.

¹⁶Interior bidding means $d_{l2}(p_2) \in (0, \mu_l)$.

For example, bidder l can deviate by submitting:

$$\hat{d}_{l2}^{\epsilon}(p) = \begin{cases} d_{l2}(p_{\epsilon}(p'')) & if \quad p \in (p', p_{\epsilon}(p'')) \\ [d_{l2}(p_{\epsilon}(p'')), d_{l2}(p'')] & if \quad p = p' \\ d_{l2}(p) & otherwise \end{cases}$$
(6)

The effect of this deviation on bidder *l*'s expected profit can be split in two parts, an expected loss from lower purchases and an expected gain from lower prices. The expected loss is bounded above by $(v - p')\epsilon [F(D_2(p'')) - F(D_2(p_{\epsilon}(p'')))]$, which converges to zero as ϵ converges to zero, and its derivative is also zero at $\epsilon = 0$. Now, the expected gain is bounded below by $(p'' - p') d_{l_2}(p_{\epsilon}(p')) [F(D_2(p'')) - F(d_{l_2}(p_{\epsilon}(p'')) + d_{-l_2}(p''))]$, which also converges to zero as ϵ converges to zero, but is strictly increasing in ϵ at $\epsilon = 0$. Hence, equilibrium demand functions are strictly decreasing at any price in $(d_l^{-1}(\mu_l), v)$. Moreover, because of symmetric interior equilibrium bidding, $d_l^{-1}(\mu_l) = 0$ at least for l = i, j.

Lemma 3 In the second auction, only the equilibrium demand function of the bidder with the largest demand in that auction could be discontinuous at p = 0.

Proof. Clearly, at a price of zero, every bidder demands the largest quantity he wants to consume. Because of symmetric interior equilibrium bidding, the strict monotonicity of bidders i and j's equilibrium demand functions at any price in (0, v), and the strict monotonicity of bidder k's equilibrium demand function at any price in $(d_k^{-1}(\mu_k), v)$, only the equilibrium demand function of the bidder with the largest demand in the second auction (i.e. bidder i) could be discontinuous at p = 0. If the demand functions of more than one bidder were discontinuous at p = 0, then any of them could use a deviation like the one proposed in the proof of lemma 1 and increase his expected profit.

The intuition behind lemma 3 can be explained as follows. If $\mu_i > \mu_j$, in equilibrium, bidder i will not demand more than the second largest demand in that auction, μ_j , at any positive price, or bid more than zero for any quantity above μ_j . If the residual supply in the second auction happens to be larger than $\mu_k + 2\mu_j$, then bidder i, who has a strictly positive value for a quantity larger than μ_j , becomes the marginal bidder, the one setting the price. Hence, his optimal strategy is to bid a price of zero for any quantity above μ_j . If $\mu_i = \mu_j$, then bidders i and j are the marginal bidders when the residual supply in the second auction falls in $(3\mu_k, \mu_k + 2\mu_j)$, and no equilibrium demand function is discontinuous at p = 0.

The differential equation in (5) has multiple solutions, one for each possible initial conditions. However, as corollary of lemmas 1-3, the initial conditions are unique. Therefore, there exists only one set of demand functions in the second auction that can be part of an equilibrium. These equilibrium demand functions can be inverted to obtain the following equilibrium bid function:

$$b_{l2}(q_{l2}; \mathbf{q_1}) = \begin{cases} v \left[1 - \frac{q_{l2}^2}{\mu_j \mu_k} \right] & if \quad q_{l2} \in [0, \mu_k) \\ v \left[1 - \frac{q_{l2}}{\mu_j} \right] (1 - I(k)) & if \quad q_{l2} \in [\mu_k, \mu_j) \\ 0 & \text{otherwise} \end{cases}$$
(7)

where $\mathbf{q_1} = (q_{i1}(y_1), q_{j1}(y_1), q_{k1}(y_1))$ and I(k) is an indicator function that equals one if l = k, and zero otherwise. When there are only two bidders, $\mu_k = 0$ and the first line of (7) disappears.

As discussed before, all active bidders (i.e. $d_{l2}(p) < \mu_l$) bid symmetrically. The demand reduction or bid shading in the second auction increases with the quantity demanded. But most importantly, it depends on the number of bidders and the size of either the two smallest bidders when there are three bidders, or the smallest bidder when there are only two of them. When there are two bidders, a decrease in the smallest demand in the second auction makes competition in this auction less intense, the smallest bidder becomes smaller. Similarly, in the three bidder case, a decrease in either the smallest or the second smallest demand in the second auction turns competition in that auction less intense, at least for some realizations of the residual supply. Hence, the residual supply that each bidder faces becomes more inelastic, which increases bid shading. This last feature of equilibrium bidding in the second auction is particularly interesting. In a single auction, the maximum quantity bidders want to buy is exogenous; however, such quantity becomes endogenous through out a sequence of auctions. Therefore, bidders can, and will, influence bid shading in the second auction through their bidding in the first auction.

4 First Auction

Bidders simultaneously and independently choose the demand functions they will submit for the first auction. As in the case of the second auction analyzed before, bidders make their choices without knowing the demand from non-strategic bidders in the first auction, which means bidders do not know the supply left for them in that auction, y_1 .

For a given realization of bidders residual supply, as long as $y_1 < \sum_l \lambda_l$, an increase in bidder *l*'s purchases in the first auction implies a decrease in purchases from at least one of the other bidders in that same auction. Moreover, since equilibrium bidding in the second

auction depends on the smallest demand (N = 2) or the two smallest demands (N = 3), bidder *l*'s profit from the second auction depends on the demand functions submitted in the first auction. For that reason, when selecting the demand function for the first auction, bidder *l* does not look for the demand that maximizes his expected profit from that auction, but looks for the one that maximizes the expected value of his entire stream of profit. Hence, bidder *l*'s optimization problem becomes:

$$\max_{d_{l1}(p_1)} E_1\left[\left(v - p_1 \right) d_{l1}\left(p_1 \right) + E_2\left[\pi_{l2}\left(\mathbf{q_1} \right) \right] \right]$$
(8)

s.t.
$$d_{l1}(p_1) \le \lambda_l$$
 (9)

In order to start characterizing the first auction equilibrium demand functions, the marginal change in bidder l's expected profit from the second auction due to a marginal change in his own purchases in the first auction needs to be defined. Because $d_{l1}(p_1) = y_1 - \sum_{-l} d_{-l1}(p_1)$ in equilibrium,¹⁷ this change can be expressed in terms of either q_{l1} or Q_{-l1} , with $Q_{-l1} = \sum_{h \neq l} q_{h1}$.¹⁸ But, as it will become clear later, it is more convenient to express the change in terms of q_{-l1} . Evidently, the effect of a change in demand reduction depends on the relative size of bidder l's demand in the second auction.

Define
$$a_{lh} = E_2 \left[\frac{\partial \pi_{l2}}{\partial q_{h1}} \right]$$
 for $l \neq h$ as the elements of the vector A_l for $l = i, j, k$.

$$\begin{pmatrix} a_{ij} \\ a_{ik} \end{pmatrix} = \begin{pmatrix} \int_0^{3\mu_k} \frac{y_2^3 v}{27\mu_j^2 \mu_k} \, \mathrm{d}F(y_2) + \int_{3\mu_k}^{\mu_k + 2\mu_j} \frac{v(y_2 - \mu_k)^2}{4\mu_j^2} \, \mathrm{d}F(y_2) + \int_{\mu_k + 2\mu_j}^{S_2} v \, \mathrm{d}F(y_2) \\ \int_0^{3\mu_k} \frac{y_2^3 v}{27\mu_j \mu_k^2} \, \mathrm{d}F(y_2) + \int_{3\mu_k}^{\mu_k + 2\mu_j} \frac{v(y_2 - \mu_k)}{2\mu_j} \, \mathrm{d}F(y_2) + \int_{\mu_k + 2\mu_j}^{S_2} v \, \mathrm{d}F(y_2) \\ \begin{pmatrix} a_{ji} \\ a_{jk} \end{pmatrix} = \begin{pmatrix} -a_{ij} + 2\int_{\mu_k + 2\mu_j}^{S_2} v \, \mathrm{d}F(y_2) \\ a_{ji} + \int_0^{3\mu_k} \frac{y_2^3 v}{27\mu_j \mu_k^2} \, \mathrm{d}F(y_2) + \int_{3\mu_k}^{\mu_k + 2\mu_j} \frac{v(y_2 - \mu_k)}{2\mu_j} \, \mathrm{d}F(y_2) \end{pmatrix} \\ \begin{pmatrix} a_{ki} \\ a_{kj} \end{pmatrix} = \begin{pmatrix} -\int_0^{3\mu_k} \frac{y_2^3 v}{27\mu_j \mu_k^2} \, \mathrm{d}F(y_2) + \int_{3\mu_k}^{\mu_k + 2\mu_j} \frac{v(y_2 - 2\mu_k)}{4\mu_j} \, \mathrm{d}F(y_2) + \int_{\mu_k + 2\mu_j}^{S_2} v \, \mathrm{d}F(y_2) \\ a_{ki} + \int_0^{3\mu_k} \frac{y_2^3 v}{27\mu_j \mu_k^2} \, \mathrm{d}F(y_2) + \int_{3\mu_k}^{\mu_k + 2\mu_j} \frac{v(y_2 - \mu_k)}{4\mu_j} \, \mathrm{d}F(y_2) + \int_{\mu_k + 2\mu_j}^{S_2} v \, \mathrm{d}F(y_2) \end{pmatrix}$$

When N = 2, only a_{ij} and a_{ji} are relevant and $\mu_k = 0$. In this case, the intuition behind the change in bidder l's profit from the second auction due to a change in his purchases in the first auction is the following. If the quantity purchased by bidder l in the first auction decreases $(q_{-l1} \text{ increases})$, there are two effects on bidder l's expected profit from the second auction.

¹⁷This is true as long as $y_1 \leq \sum_l \lambda_l$. In the case that y_1 is greater than $\sum_l \lambda_l$, the equilibrium price in the first auction will be zero and each bidder will buy the maximum quantity he wants to consume.

¹⁸Consequently, $\frac{\partial \pi_{l2}}{\partial q_{l1}} = -\frac{\partial \pi_{l2}}{\partial Q_{-l1}}$.

On one side, bidder *l*'s expected profit from the second auction increases, as the last term of a_{ji} and a_{ij} indicate. By decreasing the quantity he purchases in the first auction, bidder *l* increases the maximum quantity he wants to buy in the second auction, μ_l . Moreover, if y_2 is greater than $2\mu_j$, bidder *l* gets that extra quantity for free in the second auction, since the clearing price is zero. On the other side, bidder *l*'s expected profit from the second auction increases or decreases depending on whether bidder *l* has the largest demand or not after the first auction. In the case that $\mu_j = \mu_{-l}$, the clearing price in the second auction decreases when y_2 is smaller than $2\mu_j$, increasing bidder *l*'s expected profit from the second auction, as the second term of a_{ij} indicates (the first term is zero). However, when $\mu_j = \mu_l$ and y_2 is smaller than $2\mu_j$, the effect on bidder *l*'s expected profit is the opposite since the clearing price increases.

When N = 3, the change in a bidder's purchase also implies a balancing change in the purchases of other bidders. However, how that change is allocated among the other bidders depends on the elasticity of their demand functions. In this case, the expected change in bidder l's profit from the second auction due to a change in the quantities purchased by the other bidders in the first auction can be written as follows:

$$E_2\left[\frac{\partial \pi_{l2}}{\partial Q_{-l1}}\right] = \frac{\sum_{h \neq l} d'_{h1}(p_1)a_{lh}}{\sum_{h \neq l} d'_{h1}(p_1)} \tag{10}$$

For example, if bidder *i* decreases the quantity he buys in the first auction, the aggregate quantity bought by bidders *j* and *k* increases. Moreover, bidder *i's* expected profit increases because a_{ij} and a_{ik} are positive.

The following lemmas start characterizing the equilibrium demand functions in the first auction, by stating the conditions for them to be smooth and strictly monotonic. Define $\bar{p}_1 = p_1(0)$.

Lemma 4 Equilibrium demand functions in the first auction are continuous at any price $p \in (0, \overline{p}_1)$, as long as $D_1(p) < S_1$.

Proof. The proof of this lemma is an extension of the proof of lemma 1. Therefore, only the differences between both cases will be developed. Assume, without loss of generality, bidder 1's demand function is discontinuous at $p^* \in (0, \overline{p}_1)$. Then, as in the proof of lemma 1, assume bidder 2 can deviates by submitting a demand function with the same structure as that in equation (3).

The upper bound for the expected loss from the first auction due to higher prices and the

lower bound for the expected gain, also from the first auction, due to larger purchases are the same as those on the proof of lemma 1 with the subscripts referring to the auction changed to 1. However, since the deviation now takes place in the first auction, it also triggers a change in expected profit from the second auction. The change in bidder 2's expected profit caused by the impact this deviation has in equilibrium bidding in the second auction can be written as:

$$\Delta E_1\left[\pi_{22}\right] = \int_{D_1(p^*+\epsilon)}^{\overline{D}_1(p^\epsilon(p^*))} A_2 \cdot \mathbf{\Delta q_{-21}}(y_1) \,\mathrm{d}F(y_1) \tag{11}$$

Where $\Delta \mathbf{q}_{-21}(y_1)$ is a vector. The derivatives of bidder 2's expected profit from the second auction with respect to q_{-21} can take any sign. Hence, bidder 2 can suffer an expected loss or an expected gain from the second auction due to his deviation. The expected gain is bounded below by zero, by definition, and it is weakly increasing in ϵ . Bidder 2's expected loss is bounded above by: $\overline{M}_2(\underline{d}_{21}(p^*) + \epsilon - d_{21}(p^* + \epsilon)) \left[F(\overline{D}_1(p^\epsilon(p^*))) - F(D_1(p^* + \epsilon))\right]$. When N = 2, \overline{M}_2 is the max_{y1} a_{ji} when $y_1 \in [D_1(p^* + \epsilon), \overline{D}_1(p^\epsilon(p^*))]$. When N = 3, because $a_{ik} > a_{ij}, a_{jk} > a_{ji}$ and $a_{kj} > a_{ki}$, then $\overline{M}_2 = \max_{y_1} a_{ik}$, if i = 2; $\overline{M}_2 = \max_{y_1} a_{jk}$, if j = 2; and $\overline{M}_2 = \max_{y_1} a_{kj}$, if k = 2, when $y_1 \in [D_1(p^* + \epsilon), \overline{D}_1(p^\epsilon(p^*))]$. Hence, the upper bound and its derivative with respect to ϵ converge to zero as ϵ does so, even if $\overline{D}_1(p^\epsilon(p^*)) > S_1 > \underline{D}_1(p^*)$. Consequently, the deviation by bidder l is profitable.

Lemma 5 When N = 2, equilibrium demand functions in the first auction are strictly decreasing at every price $p \in (0, \overline{p}_1)$, as long as $D_1(p) \leq S_1$.

Proof. If bidder l demands the same positive quantity at every $p \in [p', p'']$, there are two possible cases. First, if bidder -l demands additional quantity for that range of prices, then he can increase his expected profit by withholding demand at prices in [p', p'']. Second, if no bidder demands additional quantity at that range of prices, bidder l can increase his expected profit by deviating and submitting a demand function with the same structure as that in equation (6)

The lower bound for the expected loss due to smaller purchases and the upper bound for the expected gain due to lower prices are the same as those on the proof of lemma 2 with the subscripts referring to the auction changed to 1. However, since the deviation now takes place in the first auction, it also triggers a change in expected profits from the second auction. The change in bidder l's expected profits caused by the impact this deviation has in equilibrium bidding in the second auction can be written as:

$$\Delta E_1 [\pi_{l2}] = \int_{D_1(p')-\epsilon}^{D_1(p')} A_l \cdot \Delta q_{-l1}(y_1) \,\mathrm{d}F(y_1) \tag{12}$$

In this case $\Delta q_{-l1}(y_1)$ is positive. As mentioned before, A_l can take any sign. Hence, bidder l can suffer an expected loss or an expected gain from the second auction due to his deviation. In this case, the expected gain is also bounded below by zero, by definition, and it is weakly increasing in ϵ . Bidder l's expected loss is bounded above by: $-\underline{M}\epsilon \left[F(D_1(p')) - F(D_1(p') - \epsilon)\right]$, where \underline{M} is the $\min_{y_1} a_{ji}$ when $y_1 \in [D_1(p') - \epsilon, D_1(p')]$. This upper bound and its derivative with respect to ϵ converge to zero as ϵ does so. Hence, when N = 2, equilibrium demand functions are strictly decreasing at any price in $(0, \overline{p}_1)$ as long as $D_1(p) \leq S_1$.

When three bidders participate in the sequence of two uniform price auctions, it is possible that first auction equilibrium demand functions are constant at some price range. However, at most a single demand function can be constant for a given price range. Otherwise, at least one bidder would have the incentive to deviate in the same way as it was explained in the proof of lemma 5.¹⁹

Bidder l is not only a residual monopsonist in the second auction, but also in the first auction. As a consequence, bidder l can construct the demand function for the first auction that maximizes the expected value of his stream of profits by finding all the price-quantity points $(p_1, rs_{l_1}(p_1))$ that maximize his ex-post stream of profits for each possible realization of the residual supply in the first auction, y_1 .²⁰ Another implication of bidder l being a residual monopsonist is that bidder l has the incentive to shade his bids in the first auction for the same reason as he does in the second auction of the sequence. Since that behavior also comes up in single uniform price auction, from now on it will be referred to as *static* bid shading or *static* demand reduction.

Bidder *l*'s optimal interior bidding in the first auction, conditional on other bidders' demand functions, is characterized by the following equations for all $y_1 \leq \min\{S_1, \sum_l \lambda_l\}$. Define $\underline{p}_1 = p_1(S_1)$.

(i) If $d_{l1}(p)$ is strictly decreasing (N = 2, 3):

$$-\sum_{h\neq l} d'_{h1}(p_1)v - (d_{l1}(p_1) - p_1\sum_{h\neq l} d'_{h1}(p_1)) = -\sum_{h\neq l} d'_{h1}(p_1)a_{lh}$$
(13)

¹⁹In the second auction, interior inelastic segments on the equilibrium demand functions were ruled out because of the symmetric bidding for interior quantities. However, that is not necessarily the case in the first auction.

 $^{^{20}}$ Ex-post in the first auction means after the realization of the residual supply in the first auction, but before the realization of the residual supply in the second auction.

(ii) If $d_{l1}(p)$ is constant for $p \in (p', p'') \subset (\underline{p}_1, \overline{p}_1)$ (N = 3):

$$\int_{D_1(p')}^{D_1(p)} \left[-\sum_{h \neq l} d'_{h1}(p_1)(v - p_1 - a_{lh}) - d_{l1}(p_1)\right] \mathrm{d}F(y_1) \le 0 \quad \forall p \in (p', p'') \tag{14}$$

$$\int_{D_1(\bar{p}_1)}^{D_1(\underline{p}_1)} \left[-\sum_{h\neq l} d'_{h1}(p_1)(v-p_1-a_{lh}) - d_{l1}(p_1)\right] \mathrm{d}F(y_1) = 0 \tag{15}$$

The intuition behind equation (13) is better understood in terms of the ex-post maximization where bidder l selects the first auction clearing price, p_1 , that maximizes his stream of ex-post profits conditional on y_1 and other bidders' demand functions.²¹ For a given y_1 , if bidder lmarginally increases p_1 , the quantity he buys in the first auction increases by $-\sum_{h\neq l} d'_{h1}(p_1)$. Hence, the left hand side of equation (13) represents the marginal change in profits from the first auction due to a marginal increase in p_1 . The first term represents the marginal increase in value, while the terms inside the brackets represent the marginal increase in cost. When the clearing price in the first auction increases, bidders other than l buy a smaller aggregate quantity in that auction, which affects the demand reduction in the second auction. The right hand side of equation (13) represents the expected marginal change in profit from the second auction due to the marginal change in the first auction clearing price. If there were a single auction, or this were the last auction of the sequence, then the last term on the right-hand side would be zero. Hence, when selecting his bid for the first auction, bidder l balances the marginal change in profit from the first auction with the expected marginal change in profit from the second auction.

In a single auction or in the last auction of a sequence, a bidder knows that it is his last chance to buy the quantity he wants. In the first auction of the sequence, a bidder knows that if he does not buy at that time all the quantity he wants, he still have another opportunity to buy some quantity, the second auction. In other words, in the first auction bidders have the option of buying later. Hence, bidders discount their bids in the first auction by the option value of increasing their purchases in the second auction. The option value for bidder l is given by the expected marginal change of his profit from the second auction due to a change in the quantity he buys in the first auction, which is given by equation (10). In the two-bidder case, the option value of increasing purchases in the second auction is larger for the bidder reaching the second auction with the largest demand than for the other bidder, due to the asymmetric effect on bid shading $(a_{ij} > a_{ji})$.

²¹Since in the ex-post maximization bidder l selects a price-quantity point on his residual supply curve, $rs_{l1}(p_1)$, it is equivalent to thinking bidder l selects a clearing price or a quantity.

The *F.O.C.s* for all bidders define a system of differential equations, which characterizes interior equilibrium bidding. This system of differential equations does not have explicit solutions. Hence, the next step will be to characterize equilibrium bidding in as much detail as possible. The following proposition states that, as long as the residual supply in the first auction is non greater than $\min\{S_1, \sum_l \lambda_l\}$, at the beginning of the second auction the demand of one of the bidders is always smaller than the demand from his competitors. As the proof of the proposition shows, the cause of the asymmetry can be found on bidders incentive to optimally intensify the demand reduction in the second auction.

Proposition 1 In a sequence of two uniform price auctions, at least one of the bidders always reach the second auction with a different demand than the others. Moreover, $\mu_i > \mu_j \ge \mu_k$, $\forall y_1 < \min\{S_1, \sum_l \lambda_l\}$, with $i, j, k \in \{1, 2, 3\}$ and $i \ne j \ne k$.

Proof. Assume without loss of generality that bidders 2 and 3 submit the functions $d_{21}(p)$ and $d_{31}(p)$, and also $\lambda_2 - q_{21}(y_1) \ge \lambda_3 - q_{31}(y_1)$. In equilibrium, $d_{11}(p_1) = y_1 - d_{21}(p_1) - d_{31}(p_1)$. Then, bidder 1's demand after the first auction can be written as $\lambda_1 - d_{11}(p_1) = \lambda_1 - y_1 + d_{21}(p_1) + d_{31}(p_1)$.

(i) If $\lambda_2 - q_{21}(y_1) > \lambda_3 - q_{31}(y_1)$, then define \tilde{p}_1 as the clearing price in the first auction such that $\mu_1 = \mu_2$. Then, $\lim_{p_1 \to \tilde{p}_1^-} a_{12} > \lim_{p_1 \to \tilde{p}_1^+} a_{12}$ and $\lim_{p_1 \to \tilde{p}_1^-} a_{13} > \lim_{p_1 \to \tilde{p}_1^+} a_{13}$, which implies $\lim_{p_1 \to \tilde{p}_1^-} \frac{\partial \Pi_{11}}{\partial p_1} < \lim_{p_1 \to \tilde{p}_1^+} \frac{\partial \Pi_{11}}{\partial p_1}$. Hence, it is never optimal for bidder 1 to select \tilde{p}_1 when $d'_{11}(\tilde{p}_1) < 0$. Moreover, since at most one equilibrium demand function can be constant at a given price range, bidders 1 and 2 reach the second auction with asymmetric demands.

(*ii*) If $\lambda_2 - q_{21}(y_1) > \lambda_3 - q_{31}(y_1)$, then define \hat{p}_1 as the clearing price in the first auction such that $\mu_1 = \mu_3$. Then, $\lim_{p_1 \to \hat{p}_1^-} a_{12} = \lim_{p_1 \to \hat{p}_1^+} a_{12}$ and $\lim_{p_1 \to \hat{p}_1^-} a_{13} = \lim_{p_1 \to \hat{p}_1^+} a_{13}$, which implies $\lim_{p_1 \to \hat{p}_1^-} \frac{\partial \Pi_{11}}{\partial p_1} = \lim_{p_1 \to \hat{p}_1^+} \frac{\partial \Pi_{11}}{\partial p_1}$. Hence, in equilibrium, bidders 1 and 3 could reach the second auction with symmetric demands if $\lambda_2 - q_{21}(y_1) > \lambda_3 - q_{31}(y_1)$. Actually, if $\mu_1 = \mu_3$, then $a_{12} = a_{32}$ and $a_{13} = a_{31}$, which implies $d_{11}(p) = d_{31}(p)$. Hence, $\mu_1 = \mu_3$ only happens if $\lambda_1 = \lambda_3$.

(*iii*) If $\lambda_2 - q_{21}(y_1) = \lambda_3 - q_{31}(y_1)$, then for the same reason as in (*i*) bidders 1, 2 and 3 can not reach the second auction with symmetric demands.

When N = 2, $\mu_3 = 0$ and symmetry is ruled out by the argument in (i). Finally, continuity of equilibrium demand functions in the first auction (for N = 2 or 3) ensures the ranking of bidders according to their demands after the first auction is the same for all $y_1 < \min\{S_1, \sum_l \lambda_l\}. \quad \blacksquare$

A corollary of proposition 1 is that there is no symmetric equilibrium in the first auction when bidders are symmetric. When $\lambda_i = \lambda_j = \lambda_k$, bidder *i* demands the smallest quantity at every price, followed by bidder *j* and then bidder *k*, $d_{i1}(p) < d_{j1}(p) \leq d_{k1}(p)$ for all $p \in (\underline{p}_1, \overline{p}_1)$, where $\overline{p}_1 = p_1(0)$ and $\underline{p}_1 = p_1(S_1)$. Bidders use bidding in the first auction to optimally shape bid shading in the second auction. The next proposition shows asymmetric bidding also arises under more general conditions.

Proposition 2 When N = 2 and in partially symmetric equilibria when N = 3 (i.e., $\mu_i > \mu_j = \mu_k$), bidder *i* bids less aggressively than bidder *j* in the first auction: $d_{i1}(p) < d_{j1}(p) \forall p \in (\underline{p}_1, \overline{p}_1)$. Moreover, in partially symmetric equilibria bidders *j* and *k* bid symmetrically: $d_{j1}(p) = d_{k1}(p) \forall p \in (\underline{p}_1, \overline{p}_1)$

Proof. First, assume N = 3 and $\mu_j = \mu_k$. Then $a_{ji} = a_{ki}$ and $a_{jk} = a_{kj}$, which implies $d_{j1}(p) = d_{k1}(p)$. Hence, $\mu_j = \mu_k$ only happens if $\lambda_j = \lambda_k$. When $\lambda_k = \lambda_j = \lambda_i$, the result comes trivially from proposition 1. When $\lambda_k = \lambda_j > \lambda_i$, there is no partially symmetric equilibria because a contradiction would arise for low realizations of y_1 . So, the case that needs to be proved is when $\lambda_k = \lambda_j < \lambda_i$.

Remember $\overline{p}_1 = \inf\{p \mid d_{i1}(p) = 0 \text{ and } d_{j1}(p) = 0\}$. All demand functions can not be discontinuous at \overline{p}_1 otherwise any bidder can deviate by using a deviation like the one on lemma 4. Now, assume $d_{j1}(p)$ and $d_{k1}(p)$ are continuous at \overline{p}_1 . If $d_{i1}(p)$ is either continuous or discontinuous at \overline{p}_1 , then $\lim_{p\to\overline{p}_1^-} \frac{\partial \Pi_{i1}}{\partial p_1} = -2(v-\overline{p}_1-a_{ij})\lim_{p\to\overline{p}_1^-} d'_{j1}(p) - \lim_{p\to\overline{p}_1^-} d_{i1}(p) = 0$. Now, if $d_{i1}(p)$ is continuous (discontinuous) at \overline{p}_1 , then $(v-\overline{p}_1-a_{ij})$ is zero (positive). Hence, $(v-\overline{p}_1-a_{ji})$ and $(v-\overline{p}_1-a_{jk})$ are strictly positive since $a_{ij} > a_{jk} > a_{ji}$, which implies $\lim_{p\to\overline{p}_1^-} \frac{\partial \Pi_{j1}}{\partial p_1} > 0$. Therefore, $d_{j1}(p)$ and $d_{k1}(p)$ cannot be continuous at \overline{p}_1 , but $d_{i1}(p)$ is continuous at \overline{p}_1 .

Now, assume all bidders' demands are identical (i.e., quantity and slope) at \tilde{p} . Then,

$$\frac{\partial \Pi_{j1}}{\partial p_1}\Big|_{\widetilde{p}} = -d_{j1}\left(\widetilde{p}\right) - d'_{i1}\left(\widetilde{p}\right)\left(v - \widetilde{p} - a_{ji}\right) - d'_{k1}\left(\widetilde{p}\right)\left(v - \widetilde{p} - a_{jk}\right) \\
> -d_{i1}\left(\widetilde{p}\right) - 2d'_{j1}\left(\widetilde{p}\right)\left(v - \widetilde{p} - a_{ij}\right) \\
> \frac{\partial \Pi_{i1}}{\partial p_1}\Big|_{\widetilde{p}}$$

The inequality arises because $a_{ij} > a_{ji}$ and $a_{ik} > a_{jk}$. Hence, equilibrium demand functions cannot be symmetric at any price in $(\underline{p}_1, \overline{p}_1)$. Moreover, bidder j's (and k's) equilibrium demand function in auction one cannot cross that of bidder *i* from above (i.e., $d_{k1}(\tilde{p}) = d_{j1}(\tilde{p}) = d_{i1}(\tilde{p})$ and $d'_{k1}(\tilde{p}) = d'_{j1}(\tilde{p}) > d'_{i1}(\tilde{p})$). Hence, $d_{i1}(p) < d_{j1}(p) = d_{k1}(p)$ for all $p \in (\underline{p}_1, \overline{p}_1)$. The same reasoning applies in the two-bidder case.

Propositions 1 and 2 state an interesting feature of equilibrium bidding in a sequence of uniform price auctions which is not found in a sequence of single object auctions. In the first auction of a sequence of two uniform price auctions bidders not only internalize they have another option for buying their desired quantity, but also internalize they can affect the intensity of bid shading in the second auction through their bidding in the first auction. Hence, in the first auction of the sequence bidder i bids lower prices than the other bidders, allowing them to buy a larger quantity in that auction than otherwise. This strategy is profitable for bidder i because even though he buys a lower quantity in the first auction, he then benefits from weaker competition in the second auction, which translates into larger bid shading in the last auction of the sequence.²² This characteristic of equilibrium bidding will be called *dynamic bid shading* since it is a consequence of the dynamic feature of a sequence of auctions, and also, to differentiate it from the *static bid shading* that comes up even in a single uniform price auction.

The idea behind dynamic bid shading relates to a broad literature on how to create or enhance market power. In any market, there are different ways of creating or enhancing market power. For example, firms can create barriers to entry, or create sub-markets either by independently differentiating their products from their competitors' products, or by explicitly coordinating on some kind of market segmentation. The underlying idea on the different strategies to create or enhance market power is to profitably differentiate yourself from your potential or actual competitors. This is exactly what happens in a sequence of two uniform price auctions. *Dynamic bid shading* is a strategy that allows bidders to optimally differentiate themselves by splitting up the market into two less competitive markets. There is no full market segmentation, where bidder j (and also bidder k in the three-bidder case) buys only in the first auction and bidder i waits for the second auction, because of the uncertainty about the residual supply in the second auction. However, as Herrera Dappe (2012) shows for the case of forward trading ahead of a uniform price auction, if bidders make no profits from the first auction or market, then bidder i will wait for the second auction or market

 $^{^{22}}$ Bidder *i* not only buys a lower quantity in the first auction, but also pays a lower price in that auction. However, what makes this strategy profitable is the higher expected profit bidder *i* can reap from the second auction. Otherwise, there would be asymmetric bid shading in the last auction of the sequence and even in single uniform price auction.

even with uncertain residual supply.

If the highest possible residual supply in the first auction, S_1 , is smaller than the aggregate quantity bidders want to buy, $\sum_l \lambda_l$, then the system of equations defined by the *F.O.C.s* in (13)–(15) only characterizes the equilibrium demand functions for prices in the interval $[\underline{p}_1, \overline{p}_1)$.²³ Then all demand functions have to be extended over $(0, \underline{p}_1)$ in a way that none of these prices become clearing prices. This can be achieved by using any decreasing twice continuously differentiable functions $(\tilde{d}_{k1}(p), \tilde{d}_{j1}(p), \tilde{d}_{i1}(p))$ defined over the interval $(0, \underline{p}_1]$, that satisfy $\tilde{d}_{l1}(\underline{p}_1) = d_{l1}(\underline{p}_1)$ as well as the following inequality for all $p \in (0, \underline{p}_1)$ and for l = i, j, k:

$$-(S_1 - \sum_{h \neq l} \tilde{d}_{h1}(p)) - \sum_{h \neq l} \tilde{d}'_{h1}(p)(v - p - a_{lh}) > 0$$
(16)

The left-hand side is the derivative of bidder *l*'s ex-post stream of profits with respect to the price in the first price auction,²⁴ evaluated using the market-clearing condition and $y_1 = S_1$.

According to proposition 2 bidder j bids more aggressively than bidder i in the first auction. However, nothing has been said about the identity of these bidders when N = 2. If bidders are symmetric when there are only two bidders, clearly for every pair of equilibrium bid functions there will be two almost identical equilibria, where the only difference between them will be bidders' identity. However, as the maximum quantities bidders want to buy become more asymmetric, bidding lower prices in the first auction becomes less profitable for the smaller bidder. For example, when $\lambda_i < \lambda_j$ and bidder i's demand decreases while bidder j's remains constant, bidder i's second auction demand becomes smaller, leaving him with less quantity to profit from the more intense bid shading in the second auction. On the other side, when $\lambda_i < \lambda_j$ but bidder j's demand increases while bidder i's remains constant, even though bidder j buys larger quantities in the first auction, his second auction demand increases, weakening the bid shading in the second auction.

Define $\underline{\lambda}_i$ as the lowest demand of bidder *i* for an equilibrium to exist. Clearly, $\underline{\lambda}_i$ depends on the demand of bidder *j*, the marginal value of the good, the split of the supply and the distributions of the second auction residual supply. If bidders demands are such that $\lambda_1 \in [\underline{\lambda}_1, \underline{\lambda}_2^{-1}(\lambda_2)]$, then there are two equilibria, one with j = 1 and another with j = 2.

²³When \underline{p}_1 equals zero, the interval is open at \underline{p}_1 ; because the equilibrium demand functions are not necessarily continuous at zero and $d_{l1}(0) = \lambda_l$ for all l.

²⁴Remember ex-post in this case means after the realization of the residual supply in the first auction, but before the realization of the residual supply in the second auction.

But, when bidders are so asymmetric that λ_1 lies outside of that interval, then there is only one equilibrium, and the bidder holding back in the first auction (i.e. bidder *i*) is the larger bidder.²⁵ As Table 1 shows, bidders do not have to be too different for only one equilibrium to exist.

S_1	v	$\underline{\lambda_i}^a$	λ_j	\overline{p}_1	$d_{j1}(\overline{p}_1)$	$d_{i1}(0)$	$\overline{\Delta}^{b}$	$\underline{\Delta}^{c}$
0.5	10	0.165	0.18	3.83	0.036	0.111	0.069	0.036
0.5	10	0.092	0.10	2.13	0.020	0.061	0.039	0.020
0.5	10	0.230	0.25	5.32	0.053	0.154	0.096	0.060
0.5	20	0.165	0.18	7.67	0.036	0.111	0.069	0.036
0.5	20	0.092	0.10	4.26	0.020	0.061	0.039	0.020
0.5	20	0.230	0.25	10.65	0.053	0.154	0.096	0.060
0.36	10	0.165	0.18	2.99	0.036	0.111	0.069	0.036
0.36	10	0.092	0.10	1.66	0.020	0.061	0.039	0.020
0.36	10	0.229	0.25	4.16	0.053	0.154	0.096	0.060
0.64	10	0.165	0.18	5.32	0.036	0.111	0.069	0.036
0.64	10	0.092	0.10	2.96	0.020	0.061	0.039	0.020

Table 1: Equilibrium First Auction Bidding when N=2 and $\mathbf{Y_2}\sim \mathbf{U}[0,1-\mathbf{S_1}]$

^{*a*}Lowest λ_i for the equilibrium to exist. ^{*b*} $\overline{\Delta} = \max[d_{j1}(p) - d_{i1}(p)].$

 $^{c}\underline{\Delta} = \min[d_{j1}(p) - d_{i1}(p)].$

Table 2: First Auction Bidding in Partially Symmetric Equilibria ${\rm Y_2}\sim {\rm U}[0,1-{\rm S_1}]$

S_1	v	$\underline{\lambda_i}^a$	λ_j	λ_k	\overline{p}_1	$\overline{p}_1^{i^b}$	$d_{i1}(0)$	$\overline{\Delta}^c$
0.5	10	0.10	0.10	0.10	5.99	2.94	0.067	0.035
0.5	10	0.15	0.15	0.15	8.99	4.42	0.099	0.052
0.5	20	0.10	0.10	0.10	11.98	5.89	0.067	0.035
0.5	20	0.15	0.15	0.15	17.98	8.83	0.099	0.052
0.3	10	0.10	0.10	0.10	4.28	2.10	0.067	0.035
0.3	10	0.15	0.15	0.15	6.42	3.16	0.099	0.052
0.7	10	0.10	0.10	0.10	9.99	4.90	0.067	0.035

^{*a*}Lowest λ_i for the partially symmetric equilibrium to exist.

 ${}^{b}\overline{p}_{1}^{i}$ is the $\inf\{p \mid d_{i1}(p) = 0\}, \ {}^{c}\overline{\Delta} = \max[d_{j1}(p) - d_{i1}(p)].$

²⁵If $\lambda_1 < \underline{\lambda_1}$, then j = 1 and i = 2. But, if $\lambda_1 > \underline{\lambda_2}^{-1}(\lambda_2)$, then j = 2 and i = 1.

When there are three bidders, partially symmetric equilibria exist when bidder i' demand is not smaller than bidder j and k's demands. Table 2 presents some partially symmetric equilibria when bidders' residual supply is uniformly distributed.

When there are only two bidders, the equation $v - p - a_{ji} = 0$ defines a locus of price-quantity points, $(p, q_1(p))$, where equilibrium bidding has a particular feature. If the equilibrium demand function of bidder *i* is perfectly elastic at price p, then bidder *j* will demand $q_1(p)$ at p. Following Klemperer and Meyer (1989), this locus will be called bidder *i*'s Bertrand locus. Similarly, the equation $v - p - a_{ij} = 0$ defines bidder *j*'s Bertrand locus of pricequantity points, $(p, q_1(p))$. If in equilibrium bidder *j* demands $q_1(p)$ at p and $d_{i1}(p) > 0$, then the equilibrium demand function of bidder *j* will be perfectly elastic at price p.

Bidder j equilibrium demand function in the first auction cannot go through any point above or to the right of bidder j's Bertrand locus; otherwise, bidder i would be demanding negative quantities or bidder j's demand function would be increasing. Because $a_{ij} > a_{ji}$, bidder j's Bertrand locus is lower than bidder i's Bertrand locus. Hence, bidder j's Bertrand locus defines an upper bound of bidder j's equilibrium bids in the first auction.²⁶ Defining λ as the smallest λ , the upper bound of bidder j's first auction equilibrium bids, when N = 2, becomes:²⁷

$$\hat{b}_{1}^{j}(q_{1}) = \begin{cases} v\left(F\left(2\underline{\lambda}\right) - \int_{0}^{2\underline{\lambda}} \frac{y_{2}^{2}}{4\underline{\lambda}^{2}} \,\mathrm{d}F\left(y_{2}\right)\right) & \text{if } q_{1} \in [0, \lambda_{j} - \underline{\lambda}] \\ v\left(F\left(2\left(\lambda_{j} - q_{1}\right)\right) - \int_{0}^{2\left(\lambda_{j} - q_{1}\right)} \frac{y_{2}^{2}}{4\left(\lambda_{j} - q_{1}\right)^{2}} \,\mathrm{d}F\left(y_{2}\right)\right) & \text{if } q_{1} \in (\lambda_{j} - \underline{\lambda}, \lambda_{j}) \\ 0 & \text{otherwise} \end{cases}$$
(17)

In addition, since bidder j buys a larger quantity than bidder i in the first auction, and the difference is at least $\lambda_j - \underline{\lambda}$, an upper bound of bidder i's first auction equilibrium bids, when N = 2, can be defined as $\hat{b}_1^i(q_1) = \hat{b}_1^j(q_1 + \lambda_j - \underline{\lambda})$.

²⁶The way demand functions were extended over the whole domain of prices ensures bids are also bounded above by bidder j's Bertrand locus for all $q_1 \in (d_{j1}(\underline{p}_1), \lambda_j)$. However, this is actually irrelevant since those bids are never going to be realized.

²⁷Since bidder j is the bidder with the smallest second auction demand, if $\lambda_j > \underline{\lambda}$, then there is no equilibrium with $q_{j1} \in [0, \lambda_j - \underline{\lambda}]$. Hence, $\hat{b}_1^j(q_1) = \hat{b}_1^j(\lambda_j - \underline{\lambda})$ for all $q_1 \in [0, \lambda_j - \underline{\lambda}]$.



Figure 1: Bidder *j*'s Bertrand locus when $N = 2^{28}$

When there are three bidders and bidders j and k reach the second auction with symmetric demands, it is also possible to define an upper bound of the the first auction equilibrium bids similar to the one defined in the two-bidder case. When each one of the three bidders bid for positive quantities, the equation $v - p - a_{ij} = 0$ defines bidders j and k's Bertrand locus. Since $d_{i1}(p) < d_{j1}(p) = d_{k1}(p)$, there is a range of prices at which only j and k bid for positive quantities. In that case, the equation $v - p - a_{jk} = 0$ defines bidder j and k's Bertrand locus. Finally, the equation $v - p - a_{ji} = 0$ defines bidder i's Bertrand locus. Since $a_{ij} > a_{jk} > a_{ji}$, when all bidders bid for positive quantities, $p = v - a_{ij}$ defines an upper bound of each bidder's equilibrium bid. When bidder i does not bid for positive quantities, $p = v - a_{jk}$ defines an upper bound of bidders j and k's equilibrium bids. However, without an analytic solution for the equilibrium, the latter defines the lowest upper bound of bidders' j and k bids for all q_1 , while the former defines the upper bound of bidder i's bids in the first auction.

$$\hat{b}_{1}^{jk}(q_{1}) = \begin{cases} vF(3(\underline{\lambda} - q_{1})) & if \quad q_{1} \in [0, \underline{\lambda}) \\ 0 & otherwise \end{cases}$$
(18)

$$\hat{b}_{1}^{i}(q_{1}) = \begin{cases} v \left(F(3(\underline{\lambda} - q_{1})) - \int_{0}^{3(\underline{\lambda} - q_{1})} \frac{y_{2}^{3}}{27(\underline{\lambda} - q_{1})^{3}} \, \mathrm{d}F(y_{2}) \right) & if \quad q_{1} \in [0, \underline{\lambda}) \\ 0 & otherwise \end{cases}$$
(19)

When there are only two bidders, the upper bounds of first auction bids take into account the discounting of bidder i's option value of increasing his purchases in the second auction,

 $^{^{28}}$ This is just a representation of the Bertrand locus. However, when the demand shock in the second auction is uniformly distributed, the locus is linear like in figure 1.

 a_{ij} . However, they do not fully take into account the *dynamic* bid shading that takes place in the first auction, and they completely ignore the *static* bid shading in that auction. The same applies for the upper bound of bidder *i*'s bids when there are three bidders.²⁹ In the case of bidders *j* and *k*, the upper bound of their first auction bids only takes into account their option value of increasing purchases in the second auction when bidder *i* does not buy any quantity in the first auction. This option value is smaller than bidder *i*'s own option value; hence the higher upper bounds for *j* and *k*'s bids. Furthermore, bidders *i* and *j*'s upper bounds are higher in the three-bidder case than in the two-bidder case, because of the higher competition.

Since the option value of buying a higher quantity in the second auction is positive, the upper bounds of equilibrium bids in (17)–(19) are smaller than v. Moreover, when bidder j buys his maximum demand, λ_j , in the first auction, bidder i's option value equals the value of the good; because if he were to buy some quantity in the second auction, he would pay zero for it, since he would be the only bidder in that auction. In addition, the highest equilibrium bid in the first auction is not higher than $v\left(F\left(2\underline{\lambda}\right) - \int_0^{2\underline{\lambda}} \frac{y_2^2}{4\underline{\lambda}^2} dF\left(y_2\right)\right)$, when N = 2, and not higher than $vF(3\underline{\lambda})$, when N = 3.

As mentioned before, the *F.O.C.s* define a system of differential equations, which characterizes interior equilibrium bidding. There is not one but multiple pairs of demand functions that solve that system of differential equations. The same problem came up in the second auction with equation (5). In that case, the existence of a unique equilibrium was ensured by assuming the smaller bidder will be able to buy, with strictly positive probability, as much as he wanted in the second auction ($\underline{\lambda} \leq \frac{S_2}{N}$). The following proposition states the conditions for a similar result in the first auction when N = 2.

Proposition 3 When N = 2, for any given pair of demands (λ_i, λ_j) , there exists a unique profile of equilibrium demand functions in the first auction $(d_{i1}(p), d_{j1}(p))$, if $S_1 \ge \lambda_j + \underline{\lambda}$ and $f(u) \ge \int_0^u \frac{y_2^2}{u^3} dF(y_2)$.

Proof. Assume $S_1 \ge \lambda_j + \underline{\lambda}$. Clearly, at a price of zero every bidder demands the largest quantity he wants to consume. Since both demand functions can not be discontinuous at a price of zero and $\mu_i > \mu_j$, then $\underline{d}_{i1}(0) \equiv \lim_{p\to 0^+} d_{i1}(p) < \lambda_i$ and $\underline{d}_{j1}(0) \equiv \lim_{p\to 0} d_{j1}(p) = \lambda_j$. Also, $d_{j1}(p) < \lambda_j$ for all strictly positive prices. Otherwise, bidder j's demand function would cross the upper bound. Hence, the price-quantity points $(0, \underline{d}_{i1}(0))$ and $(0, \lambda_j)$ are

²⁹When N = 2 and $\lambda_j = \underline{\lambda}$ bidder *i*'s upper bound fully overlooks *dynamic* bid shading, while in the other cases considered the upper bound takes into account a fraction of it.

the bottom conditions for the equilibrium demand functions of bidders *i* and *j* respectively. Each profile of equilibrium demand functions also has a pair of top conditions $(\overline{p}, 0)$ and $(\overline{p}, d_{j1}(\overline{p}))$ defined by the equation $v - \overline{p} - a_{ij} = 0$, where $\overline{p} = \inf\{p \mid d_{i1}(p) = 0\}$.

Assume $(d_{i1}(p), d_{j1}(p))$ is a pair of equilibrium demand functions with top conditions $(\overline{p}^a, d_{j1}(\overline{p}^a))$. Also, assume $(\tilde{d}_{i1}(p), \tilde{d}_{j1}(p))$ is another pair of equilibrium demand functions, but with top conditions $(\overline{p}^b, \tilde{d}_{j1}(\overline{p}^b))$, where $\overline{p}^a > \overline{p}^b$ and $d_{j1}(\overline{p}^a) < \tilde{d}_{j1}(\overline{p}^b)$. Because $\tilde{d}_{j1}(\overline{p}^b) > d_{j1}(\overline{p}^b)$ and $a_{ji}(\tilde{d}_{j1}(\overline{p}^b)) > a_{ji}(d_{j1}(\overline{p}^b))$, with the last inequality coming from the assumption that $f(u) \ge \int_0^u \frac{y_2^2}{u^3} dF(y_2)$, then $\tilde{d}'_{i1}(\overline{p}^b) < d'_{i1}(\overline{p}^b)$.

Therefore, there exists a price $p^* \in (0, \overline{p}^b)$ such that $\tilde{d}_{i1}(p^*) = d_{i1}(p^*)$. Moreover, at that price $\tilde{d}'_{i1}(p^*) < d'_{i1}(p^*)$, which implies $\tilde{d}_{j1}(p^*) > d_{j1}(p^*)$ because the elasticity of bidder *i*'s demand function at a given price increases with the quantity demanded by bidder *j*. Finally, since a_{ij} is increasing in the quantity purchased by bidder *j*, then $\tilde{d}'_{j1}(p^*) < d'_{j1}(p^*)$. Hence, since the slopes of the demand functions are monotonic to the top conditions, there is a unique set of bottom conditions for each set of top conditions. Therefore, there is a unique pair of equilibrium demand functions in the first auction.

5 Revenue Comparison

When choosing among several auction formats, the seller looks for the auction format best suited for achieving the main objectives of revenue maximization and efficiency. Sometimes, the seller is also interested in the market that results after the auction, like in spectrum auctions, and prefers an auction that yields a diverse pool of winners even at the expense of revenue maximization and efficiency. In this paper the seller is assumed to be unconcerned about the after auction market. Also, efficiency is not an issue for this seller because all the bidders are assumed to have the same value for the good.

When the transaction costs of bidding in an auction are high relative to the profits bidders can expect to make in the auction, low participation in the auction can be expected, which tends to have a negative effect on expected revenues. For this reason, the seller might prefer a single auction over a sequence of auctions to keep transaction costs low. In the event that bidders face budget or borrowing constraints a single auction might limit the quantity they can buy, while in a sequence of auctions bidders have the chance to raise more capital if needed. A sequence of sealed-bid auctions is somewhere between a single sealed-bid auction and an ascending auction, in terms of the private information revealed through the auctions. Hence, when there is private information about the value of the good being auctioned, a sequence of sealed-bid auctions improves the discovery of the collective wisdom of the market relative to a single sealed-bid auction, possibly increasing expected revenues. Since the price in an auction might be too high or too low due to some unexpected events, risk averse infra-marginal bidders (i.e. bid-takers) prefer a sequence of auctions over a single auction. If there is a single auction, infra-marginal bidders might end up paying too high or too low a price for all their purchases. But, in a sequence of auctions this risk is reduced since the prices bidders pay for their purchases are determined at several points in time. In the presence of risk averse bidders the seller might also prefer a sequence of sealed-bid auctions, since such auction format might increase the seller's expected revenues not only by increasing participation of risk averse bidders, particularly bid-takers, but also by encouraging marginal bidders to bid more aggressively due to a weaker winner's curse in a case with affiliated information.³⁰

The characterization of equilibrium bidding in the sequence of two uniform price auctions showed that even in an environment without transaction costs, budget or borrowing constraints, where bidders are risk neutral and the revelation of information is not an issue, a single uniform price auction and a sequence of two uniform price auctions most likely differ in terms of the expected revenues they yield.

The following propositions use the upper bounds of equilibrium bids defined above and state sufficient conditions for the expected revenue in a sequence of two uniform price auctions to be smaller than that from a single uniform price auction when the demand shocks are uniformly distributed.³¹

Proposition 4 When N = 2 and the demand shocks are uniformly distributed, the sequence of two uniform price auctions yields lower expected revenues than a single uniform price auction if:

$$\frac{\frac{S_1+3}{6(1-S_1)\underline{\lambda}}\left(\lambda_j+\underline{\lambda}-\frac{S_1}{2}-\frac{(\lambda_j-\underline{\lambda})^2}{2S_1}\right)<1\quad for\quad \lambda_j-\underline{\lambda}< S_1<\lambda_j+\underline{\lambda}}{\frac{S_1+3}{3(1-S_1)S_1}\lambda_j<1\quad for\quad S_1\geq\lambda_j+\underline{\lambda}$$

Proof. In the first auction, bidder j buys more than $(y_1 + \lambda_j - \underline{\lambda})/2$ as long as the residual

 $^{^{30}}$ In the case of common-values with affiliated signals, the extra information that is revealed through the sequence of auctions reduce the winner's curse and the real risk imposed by aggressive bidding.

³¹There is not much that can be said regarding the comparison of expected revenues without assuming a distribution for the demand shocks received by non-strategic bidders, G(x).

supply in that auction is smaller than $\lambda_j + \underline{\lambda}$. Furthermore, the expected price in the second auction, for a given y_1 , is decreasing in q_{j1} . Therefore, the upper bound of the expected revenue in the second auction of the sequence comes from assuming bidder j buys $(y_1 + \lambda_j - \underline{\lambda})/2$ in the first auction. The upper bound of the expected revenue in the first auction of the sequence can be constructed using the upper bounds of individual bids in that auction (17).

The left-hand side on both inequalities on proposition 4 are the upper bound of the expected revenue in the sequence of two uniform price auctions as a proportion of the expected revenue in a single uniform price auction, which is $v\underline{\lambda}$.

The conditions on proposition 4 translate into the following statements. (i) When the smallest bidder is the one who bids higher prices in the first auction (i.e. $\lambda_j = \underline{\lambda}$), and he demands less than 18.75% of the aggregate supply, a single uniform price auction yields higher expected revenue than any equilibrium of a sequence of two uniform price auctions. (ii) However, in any other case (i.e. $\lambda_j = \underline{\lambda} > 0.1875$ or $\lambda_j > \underline{\lambda}$) the upper bound of the expected revenue in a sequence of two uniform price auctions is higher than the expected revenue in a single uniform price auction for at least some values of S_1 . Furthermore, if $\lambda_j \geq 0.215$, the upper bound of the expected revenue in a single than the expected revenue in a single uniform price auction for any split between both auctions.

When bidder j has the smallest demand of both bidders and his demand increases, bidder i's option value of increasing his purchases in the second auction decreases. The reason is bid shading in the second auction will be smaller and its response to changes on the quantities purchased in the first auction will also be weaker. As a consequence, the upper bound of the expected revenue in the sequence of uniform price auctions increases more than the expected revenue in a single uniform price auction. The main difference between the cases where $\lambda_j \leq \lambda_i$ and $\lambda_j > \lambda_i$ is that in the latter case the expected revenue in a single auction is smaller than the upper bound of the expected revenue in a sequence of auction decreases, because the smallest bidder becomes smaller, and so does the expected revenue. The upper bounds of the expected equilibrium prices in the first and second auctions of a sequence also decrease as bid shading in the second auction is small the ratio of the upper bound of the expected revenue in a single the ratio of the upper bound of the expected revenue in a single uniform price auction decreases as bid shading in the second auction increases and so does bidder i's option value. However, when the first auction is small the ratio of the upper bound of the expected revenue in a single the ratio of the upper bound of the expected revenue in a single the ratio of the upper bound of the expected revenue in a single the ratio of the upper bound of the expected revenue in a single the ratio of the upper bound of the expected revenue in a single the ratio of the upper bound of the expected revenue in a single the ratio of the upper bound of the expected revenue in a single the ratio of the upper bound of the expected revenue in a single the ratio of the upper bound of the expected revenue in a single the ratio of the upper bound of the expected revenue in a single the ratio of the upper bound of the expected revenue in a single the ratio of the upper bound of the expected revenue in a single

auction increases as the largest decreases in price are not captured.³²

Remember the upper bound of the expected revenue in a sequence of two uniform price auctions ignores the *static* bid shading that takes place in the first auction, and it does not fully take into account the *dynamic* bid shading in that auction. Consequently, it does not fully take into account the *static* bid shading in second auction either. Hence, the upper bound of expected revenue in a sequence of two uniform price auctions does not convey the full picture, which is particularly relevant in case (*ii*). As Figure 3 shows, even when the upper bound of expected revenue in a sequence is higher than the expected revenue in a single auction, there are some equilibria of the sequence of auctions that yield lower expected revenue than a single uniform price auction. Moreover, since the uniform distribution satisfies the condition in proposition 3, the equilibria in Figure 3 are not just random equilibria, but the unique equilibria for $S_1 \ge 0.44$. The same is true about the equilibria in Figure 2 for values of S_1 greater than or equal to 0.2.



Figure 2: Ratio of $E[Rev_{Seq}]$ to $E[Rev_{Single}]$ when $\lambda_i \ge \lambda_j = 0.1$

³²Remember that bid shading and the option value increases with y_1 .



Figure 3: Ratio of $E[Rev_{Seq}]$ to $E[Rev_{Single}]$ when $\lambda_i \geq \lambda_j = 0.22$

Proposition 5 When N = 3 and the demand shocks are uniformly distributed, partially symmetric equilibria of a sequence of two uniform price auctions yield lower expected revenue than a single uniform price auction, if:

$$\frac{\frac{6\lambda_j - S_1}{6\lambda_j} + \frac{3(4\lambda_j - S_1)S_1}{8(1 - S_1)\lambda_j} < 1 \qquad for \quad S_1 \in [0, \frac{1}{2}\lambda_j)}{\frac{(20 + 7S_1)(6\lambda_j - S_1)}{120(1 - S_1)\lambda_j} + \frac{3\lambda_j}{80(1 - S_1)} < 1 \qquad for \quad S_1 \in [\frac{1}{2}\lambda_j, 3\lambda_j)}{\frac{3\lambda_j}{2S_1} + \frac{33\lambda_j}{16(1 - S_1)} < 1 \qquad for \quad S_1 \in [3\lambda_j, \infty)$$

Proof. In a partially symmetric equilibrium bidders j and k buy the same quantity in the first auction, with each of them buying more than $\frac{y_1}{3}$, as long as the residual supply is smaller than $3\lambda_j$. Therefore, like in the proof of proposition 4, the upper bound of the expected revenue in the second auction of the sequence is defined by assuming bidders j and k buy $\frac{y_1}{3}$ in the first auction. In this case, the upper bound of the expected revenue in the first auction of the sequence defined by using (18) and (19).

The left-hand side on the three inequalities in proposition 5 are also the upper bound of expected revenue in a sequence of two uniform price auctions as a proportion of the expected revenue in a single uniform price auction, which in this case is $2v\lambda_j$.

The conditions on proposition 5 translate into the following statements. (i) When there are two symmetric bidders, each one demanding less than 1/9 of the aggregate supply and the third bidder is not smaller than them, a single uniform price auction yields higher expected revenue than any partially symmetric equilibrium of a sequence of two uniform price auctions. (ii) However, if the two symmetric bidders are the smallest, but they demand more than 1/9 of the aggregate supply, then the upper bound of the expected revenue of partially symmetric equilibria in a sequence of two uniform price auctions is higher than the expected revenue in a single uniform price auction at least for some values of S_1 .

Attracting one more bidder increases the expected revenue in the sequence of auctions, and the increase is larger than in a single auction. When a third bidder participates in the sequence of auctions, both *static* and *dynamic* bid shading decrease as well as the option value in the first auction. As the conditions in propositions 4 and 5 and the equilibria in Figures 2 and 4 show, the share of expected revenue from a single uniform price auction which is lost by spreading the supply over a sequence of two uniform price auctions is smaller than when there are only two bidders.



Figure 4: Ratio of $E[Rev_{Seq}]$ to $E[Rev_{Single}]$ when $\lambda_i \ge \lambda_j = \lambda_k = 0.1$

Lastly, the conditions in propositions 4 and 5 and the equilibria in Figures 2-4 tell us that the worst for the seller is to spread the supply fairly evenly over the two auctions in the sequence. As the supply in the first auction increases, the expected price in the second auction conditional on the residual supply in the first auction increases. Consequently, the option value of increasing the quantity purchased in the second auction decreases, which increases the price in the first auction for a given y_1 . At the same time, a larger first auction supply increases the probability of low prices in both auctions at the expense of a reduction in the probability of high prices, also in both auctions. Hence, since Y_1 and Y_2 are identically distributed, and the uniform distribution is symmetric, these effects offset each other when the supply is evenly split between both auctions.

6 Conclusion

When choosing among several auction formats, the seller looks for the auction format that is best suited for achieving her main objectives of revenue maximization and efficiency. Sometimes, the seller is also interested in the market that results after the auction, like in spectrum auctions, and prefers an auction that yields a diverse pool of winners even at the expense of revenue maximization and efficiency. One decision that needs to be made by the seller when she has a divisible good for sale is whether to sell the entire supply in one auction or to spread it over several auctions. There are several features of the market that should be considered when deciding between a single auction and a sequence of auctions such as transaction costs, budget or borrowing constraints, private information and bidders's risk aversion.

The seller might prefer a single auction over a sequence of auctions when the transaction costs of bidding in an auction are high relative to the profits bidders can expect to make in that auction. In the event that bidders face budget or borrowing constraints a single auction might limit the quantity they can buy, while in a sequence of auctions bidders have the chance to raise more capital if needed. When there is private information about the value of the good being auctioned, a sequence of sealed-bid auctions improves the discovery of the collective wisdom of the market relative to a single sealed-bid auction, possibly increasing expected revenue. If some bidders are risk averse, the seller might also prefer a sequence of sealed-bid auctions, since that auction format reduces bidders' risk which might increase the seller's expected revenue by increasing participation and encouraging bidders to bid more aggressively.

In addition, the effect of strategic bidding on revenue generation and efficiency should be considered when deciding between a single auction and a sequence of auctions. There is an extensive literature that studies equilibrium bidding, revenue generation and efficiency in sequences of single object auctions, such as sequences of first price, second price or even English auctions. However, there is no theoretical nor empirical research that studies sequences of divisible good auctions. This paper filled that gap in the literature for the case of divisible good auctions with a uniform pricing rule by studying a sequence of two uniform price auctions and comparing its revenue generation properties with those of a single uniform price auction.

In auctions where bidders pay the clearing price for the quantity won, bidders have an

incentive to reduce demand (i.e. shade their bids) to pay less for their winnings. This incentive grows with the quantity demanded and is inversely related to bidders' demands. In a sequence of two uniform price auctions, bidders internalize that their bidding in the first auction has an effect on the demand reduction in the later auction. Bidders reduce their demands even more in the first auction with one bidder, usually the largest one, reducing it more than the others and thus strengthening the bid shading or demand reduction in the second auction. Hence, in a sequence of uniform price auctions there is not only *static* demand reduction but also *dynamic* demand reduction.

In any auction within a sequence of single object auctions with the exception of the last, bids are discounted by the option value of participating in later auctions. In the case of a sequence of two uniform price auctions, bids in the first auction are also discounted respect to what they would be in a single uniform price auction. The discount this time represents the option value of increasing the quantity purchased in the later auction.

In a sequence of two uniform price auctions with non-strategic bidders who bid randomly and strategic bidders with, equilibrium bidding in the second auction was shown to be unique and symmetric for any supply split with $S_2 \geq N\tilde{\lambda}$. However, this was not the case in the first auction. Nevertheless, first auction equilibrium bids are bounded above by the value of the good discounted by the option value of increasing the quantity purchased in the second auction.³³ Using this upper bound of equilibrium bids, an upper bound of the expected revenue in a sequence of two uniform price auctions was defined.

The *static* and *dynamic* bid shading together with the discounting of the option value of increasing the quantity purchased in the second auction reduce the seller's expected revenue when using a sequence of two uniform price auctions. The *dynamic* bid shading and the option value discounting, which are not present in single uniform price auction, are particularly strong when there are few bidders and at least one of them demands a small share of the supply. These features of equilibrium bidding are even stronger when the supply is split evenly between the two auctions of the sequence. Hence, in those cases it is certainly more profitable for the seller to use a single uniform price auction than a sequence of two uniform price auctions is not very strong.

³³If bidders do not know the actual value of the good and they all receive the same signal about it, then the upper bound is given by the expected value of the good discounted by the option value of increasing the quantity purchased in the second auction.

Appendix

Second Order Conditions

The F.O.C. for interior bidding are:

$$-(y_t - d_{-lt}(p_t)) - \left(v - p_t - I(t)E_{t+1}\left[\frac{\partial \pi_{lt+1}\left(\mathbf{q}_t\right)}{\partial d_{-lt}}\right]\right) d'_{-lt}(p) = 0$$
(20)

$$\frac{\partial^2 \Pi_{l1}}{\partial p_1^2} = \sum_{-l} d'_{-l1}(p_1) + \sum_{-l} (1 + I(t)E_2 \left[\frac{\partial^2 \pi_{l2}}{\partial d_{-l1}\partial p_1}\right]) d'_{-l1}(p_1) \\ - \sum_{-l} (v - p_1 - I(t)E_2 \left[\frac{\partial \pi_{l2}}{\partial d_{-l1}}\right]) d''_{-l1}(p_1)$$
(21)

Evaluating the F.O.C. in (20) at the equilibrium, and then totally differentiating it with respect to p_1 to obtain an expression for $\sum_{-l} (v - p_1 - I(t)E_2\left[\frac{\partial \pi_{l2}}{\partial d_{-l1}}\right])d''_{-l1}(p_1)$ and using it in (21) gives:

$$\frac{\partial^2 \Pi_{l1}}{\partial p_1^2} = d'_{l1}(p_1) + \sum_{-l} d'_{-l1}(p_1) < 0$$

Therefore, any solution to equation (20) would define a global maximum if bidders had unlimited demands.

Now, when t = 2 and bidder l wants to consume any quantity up to λ_l , only one demand function that solves (20) is a global maximum, and that is the one that also satisfy the end conditions described in lemmas 2 and 3. For every $y_2 \in [0, N\mu_j]$, with N = 2, 3, the demand functions are characterized by (20), where the global *S.O.C.s* are satisfied. For $y_2 \in (N\mu_j, S_2]$ all but one bidder buy all they want to consume, therefore, the best the other bidder can do is to choose a price of zero. Hence, the profiles of second auction bid functions given by equation (7) are Nash equilibrium of the second auction.

The sum of ex-post profit from the first auction and expected profit from the second auction, Π_{l1} , is twice continuously differentiable with respect to the first auction clearing price at every price besides \tilde{p}_{1} .³⁴ Hence, when t = 1, any solution to (20) locally maximizes Π_{l1} either on

³⁴Remember \tilde{p}_1 is defined by $\lambda_i - d_{i1}(\tilde{p}_1) = \lambda_j - d_{j1}(\tilde{p}_1)$.

 $(\underline{p}_1, \tilde{p}_1)$ or $(\tilde{p}_1, \overline{p}_1)$. Obviously, if the left-hand side and right-hand side derivatives of Π_{l1} with respect to p_1 , evaluated at \tilde{p}_1 , have the same sign, then any solution to (20) when t = 1will globally maximize Π_{l1} . If $S_1 < \sum_l \lambda_l$, the way demand functions were extended over the interval $[0, \underline{p}_1)$ guarantees none of these prices will maximize Π_{l1} . If $S_1 \ge \sum_l \lambda_l$, then for some realizations of y_1 all but one bidder buy all the quantity they want to consume, and the best the other bidder can do is to choose a price of zero for those realizations of y_1 . Hence, at least the local S.O.C.s are satisfied.

Since the system of differential equations defined by the set of F.O.C.s does not have analytical solutions when t = 1, the only way to find profiles of demand functions which are solutions to that system is through numerical methods. In that case it can be easily checked whether each bidder's demand function is a global maximum conditional on the other bidders' demand functions (i.e. if the profiles are Nash equilibria of the first auction). Tables 1 and 2 present some equilibria and show the set of equilibria is not the empty set.

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