Competing Matchmaking

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Abstract: We study how matchmakers use prices to create search markets and to compete with each other in two-sided matching environment, and how equilibrium outcomes compare with monopoly in terms of prices, market structure and sorting efficiency. The role of prices to facilitate sorting is comprised by the need to survive price competition. We show that the competitive outcome can be less efficient in sorting than the monopoly outcome. In particular, the need to survive price competition results in a smaller and less efficient quality difference in equilibrium search markets.

Keywords: Search market, complementarity, overtaking, market coverage, market differentiation

JEL codes: C7, D4

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1. Introduction

This paper studies how matchmakers use prices to create search markets and to compete with each other in two-sided matching environment, and how equilibrium outcomes compare with monopoly in terms of prices, market structure and sorting efficiency. The monopoly outcome is analyzed in Damiano and Li (2003). In a matching environment with agents that have private information about their one-dimensional type characteristics, a monopoly matchmaker uses a schedule of entrance fees to sort different types of agents on the two sides of a matching market into different search markets, where agents randomly form pairwise matches. The first-best matching outcome is positive assortative under the standard assumption that the match value function exhibits complementarities. The same outcome maximizes the monopolist's revenue if the complementarities are sufficiently strong relative to the rent lost due to elicitation of private type information.

Price competition in a matching environment differs from the standard Bertrand models because prices also play the role of sorting heterogeneous agent types into different search markets. Aside from the usual strategy of lowering price to steal rivals' market share, we identify a pricing strategy called "overtaking" that is unique to the sorting role of prices. Overtaking a rival is achieved by charging a price just higher than the rival does, and thus providing a market with a higher quality (average agent type). When the price difference is small enough, the rival's search market loses all its customers because quality difference dominates. The overtaking strategy is crucial for our result that the role of prices to facilitate sorting is comprised by the need to survive price competition. We show that the competitive outcome can be less efficient in sorting than the monopoly outcome.[1]

Our paper is related to a growing literature on competing marketplaces (Katz and Shapiro, 1985; Fujita, 1988; Gehrig, 1998; Caillaud and Jullien, 2000, 2001; Ellison, Fudenberg and Mobius (2002); Ellison and Fudenberg, 2002). For a comprehensive review of

[1] Sorting efficiency is the subject of a recent paper by McAfee (2002). He shows how most of the efficiency gains in sorting can be made with just two search markets. He does not consider the incentives of market participants.

this literature, see Armstrong (2002). These papers emphasize increasing returns, or thick market effect, which favor the dominance of a single marketplace, and ignore sorting by assuming that agents are homogeneous when they choose which market to participate in. A central question in the literature is whether and when multiple marketplaces can coexist in equilibrium. In contrast, in our model thick market effect is absent, and complementarity between heterogeneous agent types is the driving force behind sorting by prices. We do not claim that increasing returns are unimportant for competing marketplaces, but we abstract from these considerations to focus on the impact of price competition on sorting.

In section 2 we lay out the framework of duopoly price competition in a matching environment. We introduce the concept of market structure, and borrow from a refinement concept in sequential games to select a unique market structure for any price profile. Price competition in our model takes the form of overtaking to provide higher sorting quality with a higher price, as well as undercutting the rival's price as in standard Bertrand competition. The benchmark cases of efficient market structure for a social planner and optimal market structure for a monopolist, each with two search markets, are presented in section 3. We show that the incentives of the monopolist to differentiate the two search markets are aligned with those of the social planner, but total search market coverage by the monopolist may be smaller.[2] In section 4, we provide the main results of the paper about price competition and sorting. We show that no pure-strategy equilibrium exists in the simultaneous-move pricing game, because by overtaking each matchmaker can drive the rival out of the market and increase the revenue. We provide a sufficient condition for the two matchmakers to coexist in the equilibrium of the sequential-move version of the pricing game. The first-mover has to create a niche market for the low types in order to survive the overtaking strategy of the second-mover, which in equilibrium serves the higher types. The equilibrium outcome of the duopoly competition involves inefficient sorting, because the search markets created by the two matchmakers are insufficiently differentiated. We conclude the paper in section 5 with some discussions about robustness of our main results when more than two search markets are created.

[2] The monopolistic sorting problem in the present paper is not covered by the characterization in Damiano and Li (2003), because here we assume the monopolist is constrained in the number of search markets that can be used.

2. A Duopoly Model of Competing Matchmakers

Consider a two-sided matching environment. Agents of the two sides have heterogeneous one-dimensional characteristics, called "types." For simplicity, we assume that the type distribution function is F for both sides, with a support $[a, b] \subseteq \mathbb{R}_+$, and a continuous density function f. For notational convenience, we assume that b is finite, but our analysis applies to the case of infinite support with appropriate modifications. The two sides are assumed to have the same size.

A match between a type x agent and a type y agent from the other side produces a value of xy to both of them. This match value function satisfies the standard complementarity condition (positive cross partial derivatives). Agents are risk neutral and care only about the difference between the expected match value and the entrance fee they pay. Unmatched agents get a payoff of 0, regardless of type.

Two matchmakers, unable to observe types of agents, use entrance fees to create two search markets. For each $i = 1, 2$, let p_i be the entrance fee charged by matchmaker i. Each agent participates in only one search market, where they form pairwise matches randomly. In each search market, the probability that a type x agent meets a type y agent from the other side is given by the density of type y in that search market. For simplicity, we assume that search markets are costless to organize. The objective function of each matchmaker is to maximize the sum of entrance fees collected from agents.

2.1. Monopoly and duopoly market structures

For any pair of entrance fees p_1 and p_2, agents have three participation strategies: participate in matchmaker 1's search market, participate in 2's market, and not participate. We examine the Nash equilibria of the simultaneous move game played by the agents, and for concreteness refer to each equilibrium as a "market structure." Since our model is symmetric with respect to the two sides, we can restrict our attention to symmetric Nash equilibria, with the same size of participants in each search market.

For any two types $c, c' \in [a, b]$, with $c < c'$, let $\mu(c, c')$ be the mean type on the interval $[c, c']$. Without loss of generality, suppose that $p_1 < p_2 \leq b^2$. One "monopoly market structure," denoted as M_1, is no agents participating in search market 2. The participation threshold c_1 for search market 1 is determined by

$$\begin{cases} c_1\mu(c_1,b) = p_1 & \text{if} \quad p_1 \in [a\mu(a,b), b^2]; \\ c_1 = a & \text{if} \quad p_1 \in [0, a\mu(a,b)). \end{cases} \tag{2.1}$$

The above condition implies that the threshold participation type is either a type c_1 that is indifferent between participating in search market 1 and not participating, or the lowest type a, which strictly prefers participation. The other monopoly market structure, denoted as M_2, is no agents participating in search market 1; the participation threshold c_2 for search market 2 is similarly determined.

The prices p_1 and p_2 may also support a "duopoly market structure," denoted as D_{12}, where types between c_1 and c_2 participate in search market 1 and types above c_2 participate in search market 2.[3] This occurs if either there exist participation thresholds c_1 and c_2, with $a \leq c_1 < c_2$, such that

$$\begin{aligned} &c_1\mu(c_1,c_2) = p_1; \\ &c_2(\mu(c_2,b) - \mu(c_1,c_2)) = p_2 - p_1, \end{aligned} \tag{2.2}$$

or there is c_2 such that

$$\begin{aligned} &a\mu(a,c_2) > p_1; \\ &c_2(\mu(c_2,b) - \mu(a,c_2)) = p_2 - p_1. \end{aligned} \tag{2.3}$$

In both cases above, the threshold type c_2 is indifferent between participating in search market 2 and in search market 1. In the first case, the threshold type c_1 is indifferent between participating in search market 1 and not participating at all, while in the second case, type c_1 is the lowest type a, which strictly prefers participating in search market 1. Whether a pair of prices p_1, p_2 with $p_2 < p_1$ support a duopoly market structure, denoted D_{21}, is determined similarly.

The assumption of complementarity in the match value function implies that participation decisions are made according to threshold rules, such that the three choices, namely participating in search market 2, participating in search market 1 and not participating,

[3] When $p_1 = p_2$ we assume that the two matchmakers evenly split the types above the participation threshold; the analysis is unaffected by this assumption.

are ordered from high to low, with the preference of any type between a higher choice over a lower one implying the same preference for any higher type. As a result, the two monopoly market structures and the duopoly market structure, together with the "null market structure" where agents participate in neither search market, cover all possible equilibrium market structures.

We now make an assumption that rules out multiple duopoly market structures for given p_1 and p_2. This is necessary for characterizing the equilibrium market structure. A sufficient condition for there to be at most one duopoly market structure is that $\mu(t,x') - \mu(x,t)$ is a non-decreasing function in t for any $t \in (x,x') \subset [a,b]$: in equations (2.2), this condition guarantees that c_2 is positively related to c_1 from the second equation while they are negatively related from the first equation, so that there can be at most one solution to the two equations; in equations (2.3) this condition guarantees that there is a single c_2 that satisfies the second equation. The derivative of $\mu(t,x')$ with respect to t is given by

$$\frac{\partial \mu(t,x')}{\partial t} = \frac{f(t)(\mu(t,x') - t))}{F(x') - F(t)}. \tag{2.4}$$

This derivative converges to $\frac{1}{2}$ as x' approaches t.[4] Further, the derivative of $\partial\mu(t,x')/\partial t$ with respect to x' has the same sign as $\frac{1}{2}(t+x') - \mu(t,x')$, which is non-negative if $f'(\cdot) \leq 0$. Thus, $\partial\mu(t,x')/\partial t$ is non-decreasing in x' if $f'(\cdot) \leq 0$. Similarly, the derivative of $\mu(x,t)$ with respect to t, given by

$$\frac{\partial \mu(x,t)}{\partial t} = \frac{f(t)(t - \mu(x,t))}{F(t) - F(x)},$$

converges to $\frac{1}{2}$ as x approaches t, and is non-decreasing in x if $f'(\cdot) \leq 0$. Non-increasing density is sufficient to imply that $\mu(t,x') - \mu(x,t)$ is non-decreasing in t as $\partial\mu(t,x')/\partial t \geq \frac{1}{2} \geq \partial\mu(x,t)/\partial t$. We make the following assumption.

ASSUMPTION 1. *The density function f is non-increasing.*

For the analysis that we will carry out, we also need the standard assumption of monotone hazard rate. Let $\rho(\cdot)$ be the inverse hazard rate function. We assume that

[4] The derivative $\partial\mu(t,x')/\partial t$ at $x' = t$ can be calculated using L'Hospital rule and solving for it from the resulting equation. It is equal to $\frac{1}{2}$ because a continuous density is locally uniform.

$\rho'(\cdot) \leq 0$. This is equivalent to the assumption that the right tail distribution function $1 - F(\cdot)$ is log-concave, which implies that the conditional mean function $\mu(t, b)$ satisfies $\mathrm{d}\mu(t, b)/\mathrm{d}t \leq 1$ (An, 1998).

Assumption 2. *The hazard rate function of F is non-decreasing.*

For any $(x, x') \subset [a, b]$, let μ_l be the partial derivative of $\mu(x, x')$ with respect to x, and μ_r be the derivative with respect to x'. Then, the above two assumptions imply that $\frac{1}{2} \leq \mu_l \leq 1$, and $0 \leq \mu_r \leq \frac{1}{2}$.

Note that the uniform distribution and the exponential distributions are the two opposite extremes of distributions that satisfy Assumption 1 and Assumption 2. The uniform distribution on $[a, b]$ has a constant density function, while the hazard rate function is strictly increasing. The exponential distribution on $[a, \infty)$ has a strictly decreasing density function, while the hazard rate function is constant.

2.2. Selection of market structures

Unlike in standard Bertrand price competition, in a matching environment participation decisions of agents are not completely determined by prices. What an entrance fee buys for agents on one side of the search market depends on participation decisions by the agents on the other side of the market. Nash equilibrium alone does not pin down the market structure. It is possible to have multiple market structures for a given pair of prices. Indeed, from equations (2.1), for any $p_1, p_2 \in [0, b^2]$, either of the two monopoly market structures M_1 and M_2 can be supported as equilibrium. Loosely speaking, M_1 is an equilibrium because the "belief" that no agents participate in matchmaker 2's market is self-fulfilling. In contrast, the duopoly market structures cannot be supported by all price pairs. Before introducing a market structure selection criterion, we first determine the range of prices that allow the duopoly market structures to operate.

Fix any $p_1 \in [0, b^2]$. Using equations (2.2) and (2.3), first we determine the range of prices p_2 above p_1 that allow D_{12} to operate; a similar procedure determines the range of price pairs for D_{21}. Define $s(p_1)$ as the smallest $p_2 \geq p_1$ such that D_{12} obtains, given by

$$s(p_1) = \begin{cases} \sqrt{p_1}\mu(\sqrt{p_1}, b) & \text{if} \quad p_1 \geq a^2; \\ p_1 + a(\mu(a, b) - a) & \text{if} \quad p_1 < a^2. \end{cases} \tag{2.5}$$

At $p_2 = s(p_1)$, equation (2.2) is satisfied by $c_1 = c_2 = \sqrt{p_1}$ if $p_1 \geq a^2$, and equation (2.3) by $c_1 = c_2 = a$ if $p_1 < a^2$. The interpretation is that at $p_2 = s(p_1)$, if $p_1 \geq a^2$ then the duopoly market structure is such that only type $\sqrt{p_1}$ agents participate in search market 1 and all higher types participate in search market 2; if $p_1 < a^2$ then D_{12} is such that only type a agents participate in search market 1 and all other agents participate in search market 2. Under Assumption 1, there is no solution in c_1 and c_2 with $c_1 < c_2$ to equations (2.2) or (2.3) if $p_2 < s(p_1)$: duopoly market structure D_{12} cannot be sustained if p_2 is smaller than $s(p_1)$, because the price difference is too small for search market 1 to attract even a single low type when all higher types join search market 2. Intuitively, since the quality difference $(\mu(c_2, b) - \mu(c_1, c_2))$ between the two markets is strictly positive, a price difference is needed for the low quality market to have a positive market share.

Next, define $g(p_1)$ as the greatest $p_2 \geq p_1$ such that D_{12} obtains, according to:

$$g(p_1) = \begin{cases} p_1 + b(b - \mu(c_1, b)) & \text{if} \quad p_1 \geq a\mu(a, b); \\ p_1 + b(b - \mu(a, b)) & \text{if} \quad p_1 < a\mu(a, b), \end{cases} \tag{2.6}$$

where c_1 is uniquely determined by $c_1\mu(c_1, b) = p_1$ in the first case, and $c_1 = a$ in the second case. The interpretation is as follows. At $p_2 = g(p_1)$, equation (2.2) is satisfied by some $c_1 \geq a$ and $c_2 = b$ if $p_1 \geq a\mu(a, b)$, and equation (2.3) by $c_1 = a$ and $c_2 = b$ if $p_1 < a\mu(a, b)$. In either case, at $p_2 = g(p_1)$, only the highest type b agents participate in search market 2. Under Assumption 1, there is no solution in c_1 and c_2 to equations (2.2) or (2.3) if $p_2 > g(p_1)$: duopoly market structure D_{12} cannot be sustained if p_2 is greater than $g(p_1)$, because the price difference is too great for search market 2 to attract even the highest type. Intuitively, in any duopoly market structure, the quality difference between the two markets is bounded from above, so the high quality market cannot operate if the price difference is too large.

Since the agents in our matching model play a coordination game given the prices, a crucial ingredient in a selection criterion for monopoly market structures is the specification of "out-of-equilibrium beliefs." Consider the monopoly market structure M_1. Given the prices p_1, p_2 and the expected type μ_1 in search market 1, it would be profitable for a type x agent to deviate to search market 2 if search market 2 is believed to have an (out-of-equilibrium) expected type μ_2 such that $x\mu_1 - p_1 < x\mu_2 - p_2$. Borrowing from Banks and

Sobel's (1987) theory of refinement in sequential games ("universal divinity"), we say that a type x agent is "most likely to deviate" to search market 2 if the set of expected types μ_2 that induce the deviation is the largest among all types.[5] We introduce the following definition.

DEFINITION 1. *A monopoly market structure is unstable if the type that is the most likely to deviate is strictly better off with the deviation provided only agents of the same type on the other side of the market deviate.*

Fix any $p_1 < p_2$. Consider M_1 again. The expected type μ_1 and the threshold type c_1 in search market 1 are determined according to (2.1). We claim that type b is the most likely to deviate. Intuitively, since the price p_2 of the alternative search market is higher than the price p_1 of the existing search market, it will most likely attract the type that cares most about match quality, which is type b. More precisely, first note that among all types in $[a, c_1]$, type c_1 has the largest set of expected types μ_2 that induce the deviation, because $x\mu_2 - p_2 > 0$ implies that $c_1\mu_2 - p_2 > 0$ for any $x \in [a, c_1]$. Further, among all types in $[c_1, b]$, type b has the largest set of expected types μ_2 that induce the deviation because $x\mu_2 - p_2 > x\mu_1 - p_1$ implies that $b\mu_2 - p_2 > b\mu_1 - p_1$ for any $x \in [c_1, b]$. By Definition 1, M_1 is stable if and only if

$$b^2 - p_2 \leq b\mu(c_1, b) - p_1.$$

From the definition of $g(p_1)$ (equation 2.6), stability is equivalent to $p_2 \geq g(p_1)$.

Now consider M_2. The expected type μ_2 and the threshold type c_2 in search market 2 are determined according to (2.1). We claim that in this case the threshold type c_2 is most likely to deviate. The intuition is that the alternative search market now has a lower price p_1 than the currently operating search market does, so it will most likely attract the type that cares the least about match quality among all participating types, which is type c_2.

5 The concept of universal divinity of Banks and Sobel does not directly apply to our model, because we deal with a coordination game among different types of agents on the two sides of the market, as opposed to the problem of how to update belief after an out-of-equilibrium move. Nonetheless, the common feature in these two problems is that in equilibrium selection one has to impose restrictions on beliefs that are not restricted by rationality.

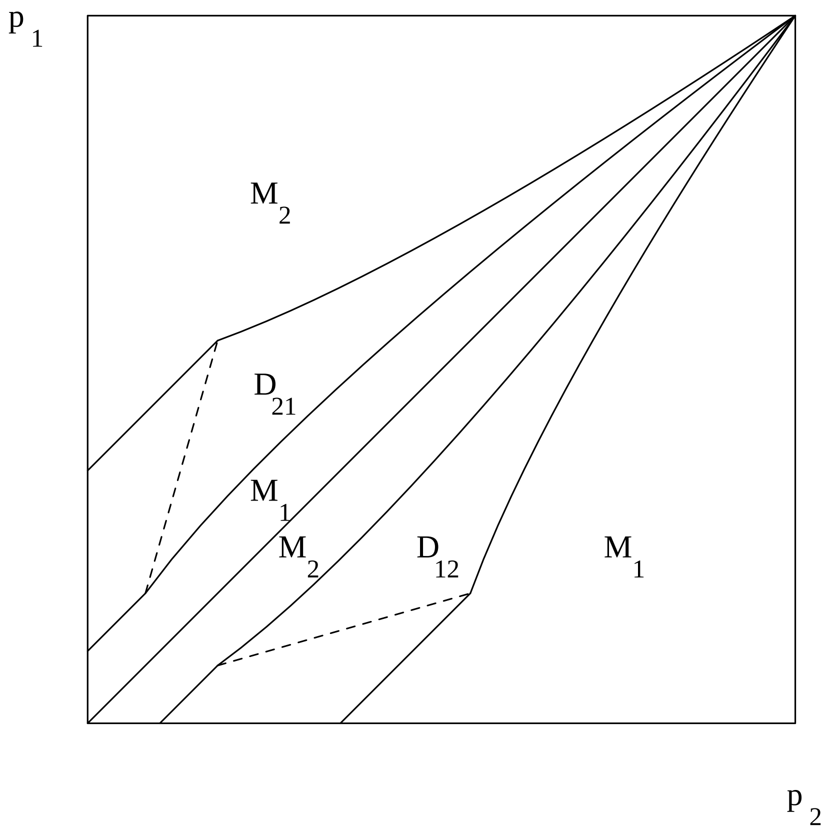

Figure 1

More precisely, note that among all types in $[a, c_2]$, type c_2 has the largest set of expected types μ_1 that induce the deviation. Among all types in $[c_2, b]$, type c_2 again has the largest set of expected types μ_1 that induce the deviation, because $x\mu_1 - p_1 > x\mu_2 - p_2$ implies that $c_2\mu_1 - p_1 > c_2\mu_2 - p_2$ for any $x \in [c_2, b]$. By Definition 1, for the case where $c_2 > a$, M_2 is stable if and only if

$$c_2^2 - p_1 \leq c_2\mu(c_2, b) - p_2 = 0,$$

and for the case where $c_2 = a$, M_1 is stable if and only if

$$a^2 - p_1 \leq a\mu(a, b) - p_2.$$

From the definition of $s(p_1)$ (equation 2.5), stability is equivalent to $p_2 \leq s(p_1)$.

Combining the results above, we conclude that (i) neither of the two monopoly market structures is stable when $p_2 \in [s(p_1), g(p_1)]$, (ii) M_1 is the only stable structure when $p_2 > g(p_1)$, and (iii) M_2 is the only stable structure when $p_2 \in (p_1, s(p_1))$. Note that when $p_2 \in [s(p_1), g(p_1)]$, our selection criterion picks out the duopoly structure D_{12} by excluding both M_1 and M_2. By symmetry a unique market structure is selected in the region where $p_1 > p_2$. We refer to case (ii) above as matchmaker 1 "undercutting" matchmaker 2, and case (iii) as matchmaker 2 "overtaking" matchmaker 1.

Figure 1 depicts the selected market structure for the case in which types are uniformly distributed on $[a, b]$. The dashed line in the duopoly region D_{12} represents the border between the section where $c_1 = a$ and the section where $c_1 > a$. It is implicitly defined by:

$$\begin{aligned} &a\mu(a, c_2) = p_1; \\ &c_2(\mu(c_2, b) - \mu(a, c_2)) = p_2 - p_1, \end{aligned} \tag{2.7}$$

where c_2 varies from a (at $p_1 = a^2$ and $p_2 = a\mu(a, b)$ on the s function) to b (at $p_1 = a\mu(a, b)$ and $p_2 = a\mu(a, b) + b(b - \mu(a, b))$ on the g function). The participation constraint of the lower threshold type c_1 is binding in the $c_1 > a$ section of D_{12}, and slack in the $c_1 = a$ section. The dashed line in the duopoly region D_{21} is symmetrically defined as in (2.7).

The strategy of overtaking is unique to the sorting role of prices. Overtaking a rival is achieved by charging a price slightly higher than the rival does to provide a search market with a higher quality. This induces deviation from the rival's search market by the highest type agents, which triggers further deviations by lower type agents. Thus, the overtaking strategy plays on the differences in willingness to pay for quality (average match type) between the highest and the lowest type agents participating in a market. When matchmakers are allowed to use more prices and create more search markets, the overtaking strategy becomes less effective because, as markets become shorter, the differences in willingness to pay between the highest and lowest participant in each market are reduced. A more detailed discussion on the robustness of overtaking in a sorting environment with many prices is provided in section 5.

Our selection criterion can be thought of as a strengthening of trembling hand perfection in strategic-form games (Selten, 1975). We will illustrate this point with the monopoly market structure M_1 (the lower-priced search market). A Nash equilibrium is trembling hand perfect if it is a limit of totally mixed "ϵ-equilibria" where players are constrained to choose non-optimal strategies (tremble) with increasingly small probabilities. Applied to our model, the convergence of non-optimal participation decisions of agents would generate an expected match quality μ_2 of matchmaker 2's market, which is what we have referred to as the out-of-equilibrium belief. Although the concept of trembling hand perfection itself does not impose any restrictions on how different types of agents might tremble, it is natural to require that a type x make the non-optimal decision of participating in market 2 more often than another type x', if x is more likely to deviate to market 2 than x' (in the sense of having a larger set of expected type μ_2 that would make deviation optimal for x). Any tremble that respects this monotonicity requirement will generate a μ_2 that lies between the unconditional mean $\mu(a, b)$, when all types tremble with the same probability, and b, when the type most likely to deviate (type b) trembles infinitely more often than other types, as in our selection criterion (Definition 1).[6] A stronger monotonicity requirement in terms of a greater rate of increase in trembling probabilities for types that are more likely to deviate, makes it more difficult to sustain M_1 as a stable equilibrium for small price differences, and brings the version of trembling hand perfection closer to our selection criterion. Further, our criterion is robust in the sense that independent of how strong the monotonicity requirement is, M_1 is stable if the price difference is large enough.

3. Monopolistic Sorting

In this section we examine the sorting efficiency achieved by a monopolist who can create at most two search markets. A benchmark is the second-best market structure, the choice of a planner who faces the same information constraints as the monopolist and who can

[6] Myerson's (1978) proper equilibrium is also a refinement of trembling hand perfection, and is similar in spirit to our selection criterion. In a proper equilibrium each player trembles on his second best strategy infinitely more often than the third best and so on. The general idea is that more costly mistakes are made less often. Properness imposes restrictions on how a given player trembles among his non-optimal strategies, while our selection criterion imposes restrictions on how trembling on a given non-optimal action varies across different types of agents. Because trembling probabilities are specified according to how costly mistakes are to individual agents, proper trembling generally does not satisfy the monotonicity requirement. Detailed characterization of proper equilibria in our model is available upon request.

use two prices to create two search markets to maximize the sum of expected match values. We consider the differences in the solutions for the planner and for the monopolist both in terms of total coverage of the two search markets and in terms of differentiation between the two markets.

The planner is assumed to maximize the total match value achieved by two search markets. The planner's maximization problem reflects the same two constraints faced by the matchmaker/s: the information constraint that agents' type is privately known and the technology constraint that at most two search markets can be created. As the monopoly matchmaker, the planner has to use prices to give proper incentives for agents to sort into two search markets. The objective function of the planner reflects the assumption that the planner gives equal weight to each agent's utility and to the revenue from matchmaking. This is a reasonable assumption since the planner could always redistribute the revenue through lump sum transfers.

In the analysis of this section, it is more convenient to use threshold types instead of prices as choice variables for the planner and for the monopolist. This change of variables is valid. For any prices p_1 and p_2, a unique market structure obtains with their threshold types c_1 and c_2, according to our selection criterion (Definition 1). Conversely, for any threshold types c_1 and c_2, say $c_1 \leq c_2$, the price p_1 is given by the indifference of type c_1 between the lower quality market and non-participation, and the price p_2 of the higher quality search market is given by the indifference of type c_2 between the two markets.[7]

3.1. Market coverage

To gather some intuition before plunging into the full problem of comparing the monopolist's optimal market structure with the efficient market structure, we consider how the two compare when only a single price is allowed.

The planner's single-price sorting problem is to choose a participation threshold c to maximize the total match value

$$(1 - F(c))\mu^2(c, b).$$

[7] When $c_1 = a$, there are multiple prices p_1 that implement the same market structure. However, the monopolist will always choose to bind the participation constraint of type c_1 in order to maximize the revenue, while any such price p_1 gives the planner the same total match value.

The first order condition is

$$f(c)(\mu^2(c,b) - 2c\mu(c,b)) = 0.$$

By Assumption 2, $\mu(c,b) - 2c$ is a decreasing function of c, implying that the derivative of the total match value function crosses 0 at most once and from above. Thus, the efficient threshold c^* is either a corner solution at a (if $\mu(a,b) \leq 2a$), or uniquely determined by the first order condition

$$\mu(c^*, b) = 2c^*. \tag{3.1}$$

In contrast, the monopolist chooses a threshold c to maximize the revenue

$$(1 - F(c))c\mu(c,b).$$

We will show that the above revenue function is quasi-concave, i.e. the derivative crosses zero at most once and from above. First, if the solution $\hat{c}$ is interior then it satisfies the first order condition

$$\rho(\hat{c})\mu(\hat{c}, b) = \hat{c}^2. \tag{3.2}$$

Since $\mu_l(c,b) \leq 1$ by Assumption 2, from equation (2.4) we have

$$\rho(c) \geq \mu(c,b) - c \tag{3.3}$$

for any c. Together with $\mu(c,b) > c$, the first order condition (3.2) implies

$$\mu(\hat{c}, b) - \hat{c} \leq \rho(\hat{c}) < \hat{c}, \tag{3.4}$$

and therefore $\mu(\hat{c}, b) < 2\hat{c}$. It follows that $\hat{c} > c^*$ if $\hat{c} > a$. Further, because $\mu_l(c,b) \leq 1$ and $\rho'(c) \leq 0$, the derivative of $\rho(c)\mu(c,b) - c^2$ is less than $\rho(c) - 2c$, which by (3.4) is less than 0 at any $\hat{c}$ that satisfies the first order condition (3.2). Thus, the above revenue function is quasi-concave. It follows that the second order condition of the monopolist's problem is satisfied, and equation (3.2) uniquely determines the solution $\hat{c}$ if it is interior. Finally, since the monopolist's revenue is 0 at $c = b$, if $\rho(a)\mu(a,b) \leq a^2$, the solution to the monopolist's problem is given by $\hat{c} = a$. But in this case the planner's solution c^* is also

a, because $\rho(a)\mu(a,b) \leq a^2$ implies that the inequalities (3.4) can be derived in the same way with $\hat{c} = a$, which in turn implies $\mu(a,b) < 2a$.

The monopolist's search market is smaller and more selective than the planner's because raising the participating threshold has different effects for the monopolist and for the planner. For both of them, raising the threshold has the same negative effect of reducing the size of the search market (decreasing $1 - F(c)$), but the positive effect of making the search market more selective is different. The monopolist is concerned with the change in the marginal type's willingness to pay the participation fee, whereas the planner cares about the change in the average expected type. In particular, under Assumption 2, increasing the threshold has a proportionally larger impact on the revenue $c\mu(c,b)$ than on the total expected match values $\mu^2(c,b)$. More precisely, the derivative of $\ln(c\mu(c,b))$ is greater than the derivative of $\ln(\mu^2(c,b))$.

The result that the monopolist's optimal market coverage $\hat{c}$ is smaller than the planner's coverage c^* is anticipated by Damiano and Li (2003). There we have shown that the monopolist maximizes the expected sum of virtual match values, defined as the product of virtual match type $x - \rho(x)$ for each type x and the match quality that type x gets.[8] It follows that the monopolist will never serve agents of negative virtual types. This does not change even though the monopolist is restricted in the number of prices and search markets that can be offered in the present model, as opposed to unrestricted schedules in Damiano and Li (2003). As shown in equation (3.4), at the monopolist's optimal threshold $\hat{c}$, the virtual type $\hat{c} - \rho(\hat{c})$ is strictly positive.[9] In contrast, there is no reduction from match type to virtual match type for the planner because the information rent needed to elicit private type information is internalized. Combining equation (3.1) with the inequality (3.3), we have that at the planner's efficient choice c^*, the virtual type is negative (with possible exception when $c^* = a$). Note that under Assumption 2, the virtual type function

[8] Damiano and Li (2003) study monopoly price discrimination in a general asymmetric matching environment, where the virtual match type functions are different for the two sides. Rayo (2002) studies how a monopolist can use price discrimination to sell status goods. His problem can be interpreted as a special case of the matching model of Damiano and Li (2003), where the two sides of the matching environment are identical.

[9] The result that the virtual type is non-negative for the monopolist does not depend on Assumption 2. It follows immediately from the first order condition (3.2) as $\mu(\hat{c},b) \geq \hat{c}$.

$c - \rho(c)$ is increasing, and so positive virtual type for the monopolist and negative virtual type for the planner confirm our result that the market coverage is generally smaller for the monopolist.

3.2. Market differentiation

Now we examine the original two-price problems of the planner and the monopolist. The planner chooses c_1 and c_2, with $c_1 < c_2$, to maximize the total match values:

$$(F(c_2) - F(c_1))\mu^2(c_1, c_2) + (1 - F(c_2))\mu^2(c_2, b). \tag{3.5}$$

The monopolist chooses c_1 and c_2 to maximize the revenue from the two search markets:

$$(1 - F(c_1))c_1\mu(c_1, c_2) + (1 - F(c_2))c_2(\mu(c_2, b) - \mu(c_1, c_2)).$$

Fix c_1 and consider how the planner and the monopolist choose c_2.

For the planner, the first order condition with respect to c_2 is

$$f(c_2)(\mu(c_2, b) - \mu(c_1, c_2))(\mu(c_1, c_2) + \mu(c_2, b) - 2c_2) = 0. \tag{3.6}$$

For any c_1, there exists at least one c_2^* that satisfies the above first order condition, as $\mu(c_1, c_1) + \mu(c_1, b) \geq 2c_1$ and $\mu(c_1, b) + \mu(b, b) \leq 2b$. Such c_2^* is unique too: under Assumption 2, $\mu_r(c_1, c_2) + \mu_l(c_2, b) \leq \frac{1}{2} + 1 < 2$. Moreover, an increase in c_1 leads to a greater c_2^*, and we can write:

$$\frac{\mathrm{d}c_2^*}{\mathrm{d}c_1} = \frac{\mu_l(c_1, c_2^*)}{2 - \mu_r(c_1, c_2^*) - \mu_l(c_2^*, b)}. \tag{3.7}$$

For the monopolist, the first order condition with respect to c_2 is

$$\frac{\mu(c_2, b) - \mu(c_1, c_2)}{c_2 - \mu(c_1, c_2)} = \frac{1 - F(c_1)}{\rho(c_2)} \frac{c_2 - c_1}{F(c_2) - F(c_1)}. \tag{3.8}$$

The right-hand-side of equation (3.8) approaches 1 while the left-hand-side becomes arbitrarily large when c_2 takes on the value of c_1, and the opposite happens when c_2 approaches b. As a result, for any c_1, there exists at least one $\hat{c}_2$ that satisfies the above first order condition. Further, the right-hand-side of equation (3.8) is increasing c_2, because Assumption 1 implies that the ratio $(c_2 - c_1)/(F(c_2) - F(c_1))$ increases with c_2 while $\rho(c_2)$ decreases with c_2 by Assumption 2. The left-hand-side is decreasing in c_2, because $\mu_l(c_2, b) \leq 1$ by Assumption 1. Thus, the $\hat{c}_2$ that satisfies (3.8) is unique. Finally, the right-hand-side of equation (3.8) is decreasing in c_1 because $\rho'(\cdot) \leq 0$ and $f'(\cdot) \leq 0$ imply that $(1 - F(c_1))(c_2 - c_1)/(F(c_2) - F(c_1))$ decreases with c_1, while the left-hand-side increases with c_1 because $\mu(c_2, b) \geq c_2$. Therefore, an increase in c_1 leads to a greater $\hat{c}_2$.

Comparison between c_2^* and $\hat{c}_2$ for fixed c_1 reflects different incentives for the planner and the monopolist to differentiate search market 2 from search market 1. For both the planner and the monopolist, increasing c_2 raises the qualities (expected match types) in both search markets at the expense of reducing the relative size of the higher quality market (search market 2). The size effect is the same for the planner and for the monopolist, but the effects on the qualities are different because the monopolist is concerned with the change in the marginal type's willingness to pay, whereas the planner cares about the change in the average expected type. Unlike the comparison of market coverages in the one-price problem, the comparison between c_2^* and $\hat{c}_2$ is sensitive to the type distribution. To see this, note that using the identity

$$(F(c_2) - F(c_1))\mu(c_1, c_2) + (1 - F(c_2))\mu(c_2, b) = (1 - F(c_1))\mu(c_1, b), \tag{3.9}$$

we can rewrite the objective function of the planner as

$$(1 - F(c_1))(\mu^2(c_1, b) + (\mu(c_1, b) - \mu(c_1, c_2))(\mu(c_2, b) - \mu(c_1, b))),$$

and the objective function of the monopolist as[10]

$$(1 - F(c_1))(c_1\mu(c_1, b) + (\mu(c_1, b) - \mu(c_1, c_2))(c_2 - c_1)).$$

The two expressions above conveniently isolate the effect of market coverage from the effect of market differentiation. In the objective function of the planner $(1 - F(c_1))\mu^2(c_1, b)$ is the total match value generated by just one search market with threshold c_1, representing the market coverage effect, while $(1 - F(c_1))(\mu(c_1, b) - \mu(c_1, c_2))(\mu(c_2, b) - \mu(c_1, b))$ is the

[10] The following restatement of the monopoly revenue shows that for any c_1 adding a second search market with any $c_2 \in (c_1, b)$ increases the revenue. That is, the monopolist will always create two search markets instead of a single one.

additional gain in total match value from dividing the covered types into two markets, which is the differentiation effect. Similarly, $(1 - F(c_1))(\mu(c_1, b) - \mu(c_1, c_2))(c_2 - c_1)$ is the additional revenue to the monopolist from dividing the total market (c_1, b) into two markets through the threshold c_2. Since c_1 and hence $\mu(c_1, b)$ are fixed, the difference between the problems of the planner and of the monopolist comes down to that between $\mu(c_2, b) - \mu(c_1, b)$ and $c_2 - c_1$. It follows that the planner's solution c_2^* is smaller than the monopolist's solution $\hat{c}_2$ if the proportional rate of increase of $\mu(c_2, b) - \mu(c_1, b)$ is smaller than the proportional rate of increase of $c_2 - c_1$ for all c_2. This is equivalent to

$$\mu_l(c_2, b) \leq \frac{\mu(c_2, b) - \mu(c_1, b)}{c_2 - c_1}. \tag{3.10}$$

For a host of special distributions which have a linear conditional mean function $\mu(\cdot, b)$, the condition (3.10) holds with equality for any c_1 and c_2, so that $c_2^* = \hat{c}_2$. For example, under the uniform distribution with $f(x) = 1/(b - a)$, we have $\mu(c, b) = \frac{1}{2}(c + b)$, and so both sides of (3.10) are equal to $\frac{1}{2}$ for any c_1 and c_2. Under the exponential distribution with $f(x) = \gamma \exp(-\gamma x)$, we have $\mu(c, b) = x + \frac{1}{\gamma}$, and so both sides of (3.10) are equal to 1 for any c_1 and c_2. For distributions with a decreasing linear density function $f(x) = 2(1 - (x - a)/(b - a))/(b - a)$, we have $\mu(c, b) = \frac{1}{3}(2c + b)$, and so both sides of (3.10) are equal to $\frac{2}{3}$ for any c_1 and c_2.

More generally, note that the left-hand-side of (3.10) is the slope of the conditional mean function $\mu(c, b)$ at c_2, while the right-hand-side is the slope of the line segment connecting the two points corresponding to c_1 and c_2 on the conditional mean function. Therefore, for distributions that have a strictly concave condition mean function $\mu(\cdot, b)$, condition (3.10) holds with strict inequality for any $c_2 > c_1$. Conversely, the inequality (3.10) is reversed in any region of the support $[a, b]$ for a distribution that has a strictly convex conditional mean function $\mu(\cdot, b)$ in the region.[11]

We regard the difference between c_2^* and $\hat{c}_2$ as secondary to the difference in terms of market coverage. We have already seen from the previous analysis of one-price planner and monopolist problems that market coverage is greater for the planner. Now we show

[11] Numerical examples of trapezoidal distributions (i.e., decreasing linear density with $f(b) > 0$) with either $c_2^* < \hat{c}_2$ or $c^* > \hat{c}_2$ for given c_1 are available upon request.

that the same is true for the two-price problems. The first order condition with respect to c_1 for the planner is:

$$f(c_1)(\mu(c_1, c_2) - 2c_1) = 0. \tag{3.11}$$

Thus, there is a unique solution c_1^* if

$$\mu_l(c_1, c_2^*) + \mu_r(c_1, c_2^*)\frac{\mathrm{d}c_2^*}{\mathrm{d}c_1} < 2.$$

Using the relation (3.7) between c_2^* and c_1, we can show that the above holds because $\mu_l(c_2, b) < 1$ and $\mu_r(c_1, c_2) \leq \frac{1}{2}$ by Assumption 1 and Assumption 2. Further, we can show that $c_1^* \leq c^*$ (with equality only if $c^* = a$), so that a planner with two prices available will expand the market coverage relative to a planner with a single price. This can be seen by comparing (3.11) to (3.1) and using the fact that $\mu(c_1, c_2^*) < \mu(c_1, b)$.[12] For the monopolist, the first order condition with respect to c_1 is:

$$\frac{1 - F(c_2)}{F(c_2) - F(c_1)}(c_2 - c_1) = \frac{\rho(c_1)\mu(c_1, c_2) - c_1^2}{\mu(c_1, c_2) - c_1}. \tag{3.12}$$

As in the case of one-price monopolist, the two-price monopolist will never serve any agent with a negative virtual type, i.e. at the optimal $\hat{c}_1$ we have $\hat{c}_1 \geq \rho(\hat{c}_1)$. To see this, note that if $\hat{c}_1 < \rho(\hat{c}_1)$, then the right-hand-side of equation (3.12) is greater than $\hat{c}_1$, implying

$$(1 - F(\hat{c}_2))\hat{c}_2 > (1 - F(\hat{c}_1))\hat{c}_1. \tag{3.13}$$

Since the revenue of the monopolist can be alternatively written as:

$$((1 - F(c_1))c_1 - (1 - F(c_2))c_2)\mu(c_1, c_2) + (1 - F(c_2))c_2\mu(c_2, b),$$

and since the monopolist can always choose c_1 just below c_2 to make the first term arbitrarily small, the inequality (3.13) implies that the monopolist is not choosing $\hat{c}_1$ optimally, a contradiction. Therefore, as in the case of one-price sorting, the market coverage remains smaller for the monopolist than for the planner. We summarize the findings so far as the following proposition:

[12] It follows from equation (3.6) that $c_2^* < b$ for any c_1.

Proposition 1. *(i) Total market coverage is at least as large for the planner as for the monopolist; (ii) monopolist market differentiation is efficient given the total market coverage if the type distribution is uniform.*

The restriction to two-price schedules in the present model exaggerates the extent to which market coverage differs for the planner and for the monopolist. At least for the uniform type distribution case, the difference in market coverage becomes smaller eventually when more search markets are feasible. Moreover, as long as $a/b > \frac{1}{2}$, i.e. the virtual match type of a is positive, all types will be optimally served for both the planner and the monopolist if the number of search markets allowed is sufficiently large.

Let the number of search markets be N. The planner chooses a sequence of thresholds $c_1, \ldots, c_N$, in increasing order, to maximize total match value

$$\sum_{i=1}^{N}(F(c_{i+1}) - F(c_i))\mu^2(c_i, c_{i+1}),$$

where $c_{N+1} = b$. The first order condition with respect to c_i, $i = 2, \ldots, N$, is

$$\mu(c_{i-1}, c_i) + \mu(c_i, c_{i+1}) = 2c_i, \tag{3.14}$$

and is given by equation (3.11) for c_1. As in the two-price case, the above first order conditions can be characterized recursively, starting from $i = N$. For the uniform type distribution case, these relations are given by:

$$c_i^* = c_{i-1}^* + \frac{1}{N}(b - c_1) \tag{3.15}$$

for each $i = 2, \ldots, N$. In other words, the thresholds are evenly spaced given any market coverage $[c_1, b]$. It then follows from the first order condition (3.11) with respect to c_1 that

$$c_1^* = \max\left\{\frac{1}{2N+1}b, a\right\}, \tag{3.16}$$

implying full market coverage eventually as N increases. For the monopolist, the objective is to maximize the revenue

$$\sum_{i=1}^{N}(1 - F(c_i))c_i\mu(c_i, c_{i+1}).$$

Under the uniform type distribution, market differentiation takes the same form as for the planner, so equation (3.15) holds for the optimal thresholds $\hat{c}_i$. The first order condition with respect to c_1 takes the same form as in the two-price case, given by (3.12), and for the uniform case reduces to

$$c_1^2 - c_1(b - c_1) = \frac{1}{2N^2}(b - c_1)^2. \tag{3.17}$$

As N increases, the optimal interior solution $\hat{c}_1$ decreases and becomes arbitrarily close to $\frac{1}{2}b$. Thus, if $a/b > \frac{1}{2}$, both the planner and the monopolist provide full market coverage when N is sufficiently large.

4. Competitive Sorting

In this section we analyze the equilibrium outcome under duopolistic competition. We assume that the two matchmakers each choose a single price to maximize revenue. It turns out that no pure-strategy equilibrium exists in the simultaneous-move pricing game, because by overtaking each matchmaker can drive the rival out of the market and increase the revenue. Characterization of the equilibrium in the sequential-move pricing game is complicated by the need to consider both the market structure where the first-mover serves the lower quality market and the symmetric structure. An additional complication arises, because unlike in the case of monopoly, we need to consider the sections in the duopoly regions where the participation constraint of the threshold type for the lower quality search market is not binding. Indeed, we will show that in some cases the only way for the first-mover to survive the overtaking strategy of the second-mover is to lower its price sufficiently to serve all lower types and leave some rents to the lowest type (type a).

The natural comparison is between the equilibrium outcome with two competing single-price matchmakers and the outcome with a two-price monopoly matchmaker. We will make comparisons in terms of both market coverage and market differentiation, as these are the two factors that determine sorting efficiency measured by the total match value. From Proposition 1, we know that the total market coverage is smaller for the monopolist than for the planner, while at least in the case of uniform type distribution, the market differentiation is the same. We will show that under competition the coverage is expanded but the quality difference is reduced relative to monopoly. The comparison in terms of sorting efficiency is thus ambiguous. We will give examples in which competition yields a smaller total match value than monopoly.

4.1. Simultaneous competition

Consider a static game where the two matchmakers choose prices simultaneously. In this subsection we show that feasibility of the overtaking strategy to each matchmaker implies that there is no pure-strategy Nash equilibrium in prices.

First, note that only a duopoly market structure is a candidate for equilibrium outcome. This is due to the fact that given the competitor's price, say p_1, the other matchmaker can always earn a positive profit by overtaking (charging a p_2 just above p_1). This strategy drives the competitor out of the market and guarantees a positive revenue.

Second, in any duopoly market structure, by using the overtaking strategy each matchmaker can earn a revenue strictly greater than the competitor, which is impossible. To see this point, without loss of generality suppose that $p_1 \leq p_2$ and the duopoly market structure D_{12} obtains. The participation threshold $\tilde{c}_1$ and $\tilde{c}_2$ are determined by equation (2.2), or in the case of $\tilde{c}_1 = a$, by equation (2.3). If matchmaker 2 charges a price just above p_1, say $p_1 + \epsilon'$, in the case of $\tilde{c}_1 > a$, matchmaker 2 becomes a monopoly and the participation threshold c' satisfies (2.1). Comparing the two equations $c'\mu(c', b) = p_1 + \epsilon'$ and $\tilde{c}_1\mu(\tilde{c}_1, \tilde{c}_2) = p_1$, we conclude that $c' < \tilde{c}_1$ for some ϵ' slightly greater than zero. In the case of $\tilde{c}_1 = a$, matchmaker 2 becomes a monopoly by charging a price just above p_1, and the participation threshold $c' = a$. In either case, matchmaker 2 earns a strictly greater revenue in deviation than matchmaker 1 does in the duopoly market structure through a higher price and a larger search market. Similarly, given p_2, if matchmaker 1 overtakes matchmaker 2 with a price just above p_2, say $p_2 + \epsilon''$, then matchmaker 1 becomes a monopolist and the participation threshold c'' satisfies (2.1). Comparing the two equations $c''\mu(c'', b) = p_2 + \epsilon''$ and $\tilde{c}_2\mu(\tilde{c}_2, b) = p_2 + \tilde{c}_2\mu(\tilde{c}_1, \tilde{c}_2) - p_1 > p_2$ (because $\tilde{c}_2\mu(\tilde{c}_1, \tilde{c}_2) - p_1 > \tilde{c}_1\mu(\tilde{c}_1, \tilde{c}_2) - p_1 = 0$), we conclude that $c'' < \tilde{c}_2$ for some ϵ'' slightly greater than zero. This implies that matchmaker 1 earns a strictly greater revenue in deviation than matchmaker 2 in the duopoly market structure. Thus, feasibility of the overtaking strategy implies the impossible scenario that, in equilibrium, each matchmaker should earn a revenue strictly greater than the competitor. We summarize the analysis in the following proposition:

Proposition 2. *There is no pure-strategy equilibrium in a simultaneous-move game.*

The non-existence of pure-strategy equilibria in the simultaneous-move game points to a difference between competing matchmaking and the standard Bertrand price competition. Payoff discontinuities exist in both competing matchmaking and in Bertrand competition, and tend to homogenize prices in the absence of any asymmetry between the competitors. While in Bertrand competition this leads to marginal cost pricing, in competing matchmaking prices cannot be determined in a similar fashion because they also play the role of sorting.

Existence of a mixed-strategy equilibrium can be established using the concept of payoff-security of Reny (1999). Our simultaneous-move game is discontinuous, because market structure switches from one monopoly structure to the other one when prices move from below the diagonal to above (Figure 1). However, by charging a slightly higher price each matchmaker can secure a payoff at worst only marginally lower against small perturbations of its rival's price. It follows that the mixed-extension of our simultaneous-move game is payoff-secure, and therefore a mixed strategy equilibrium in prices exists (Corollary 5.2 in Reny, 1999).[13]

4.2. Surviving overtaking

Rather than studying mixed-strategy equilibria in a simultaneous-move game, we look at pure-strategy (subgame perfect) equilibria in a sequential-move game. For the remainder of this section, we consider a game where matchmaker 1 first picks a price p_1, and matchmaker 2 then chooses p_2 after observing p_1. From our analysis of the simultaneous-move game we know that the sequential-move game gives an advantage to matchmaker 2 because it can

[13] An additional condition needed for Reny's result to apply, reciprocal upper semicontinuity, is satisfied in our model, because the sum of the payoffs to the matchmakers is continuous.

secure a revenue strictly higher than matchmaker 1 through overtaking. We are interested in finding out whether this advantage is so overwhelming that matchmaker 1 cannot survive as a first-mover.[14] One hope for matchmaker 1 to survive overtaking is to choose a price so low that matchmaker 2 finds more profitable creating another differentiated search market rather than overtaking matchmaker 1 and drive it out of the competition. We distinguish two cases: $a > 0$ and $a = 0$.

Assume $a > 0$. Fix any $p_1 < a^2$. First, note that by equation (2.6), $g(p_1)$ converges to $b(b - \mu(a, b))$ as p_1 approaches 0. Under Assumption 1, $\mu(a, b) \leq \frac{1}{2}(a + b)$, so that $b(b - \mu(a, b)) \geq a^2$. Thus, for $p_1 < a^2$, it is not possible for matchmaker 2 to drive matchmaker 1 out of the market by undercutting. Refer to Figure 1. It remains to show that for p_1 sufficiently small, it is not optimal for matchmaker 2 to drive matchmaker 1 out of the market by overtaking. By equation (2.1), the monopoly structure M_2 obtains for any $p_2 \in (p_1, s(p_1))$. The threshold type of participation is $c_2 = a$ because by assumption $p_1 < a^2 < a\mu(a, b)$. Matchmaker 2's revenue from overtaking is simply p_2 for any $p_2 \in (p_1, s(p_1))$, so any such p_2 is not optimal. For any $p_2 \in [s(p_1), g(p_1)]$, the duopoly market structure D_{12} obtains. By equation (2.3), $c_1 = a$, and c_2 satisfies

$$c_2(\mu(c_2, b) - \mu(a, c_2)) = p_2 - p_1.$$

Consider how matchmaker 2's revenue in the duopoly structure D_{12}, given by $p_2(1-F(c_2))$, changes at $p_2 = s(p_1)$. Since $c_2 = a$ at $p_2 = s(p_1)$, the derivative of matchmaker 2's revenue with respect to p_2 at $s(p_1)$ is given by

$$1 - \frac{f(a)s(p_1)}{\mu(a, b) - a + a\left(\mu_l(a, b) - \frac{1}{2}\right)}.$$

So the derivative is positive if and only if

$$\mu(a, b) - a + a\left(f(a)(\mu(a, b) - a) - \frac{1}{2}\right) > f(a)s(p_1). \tag{4.1}$$

As p_1 approaches 0, $s(p_1)$ approaches $a(\mu(a, b) - a)$. Thus, the derivative of matchmaker 2's revenue is positive at $p_2 = s(p_1)$ for p_1 approaching 0, if and only if $\mu(a, b) > \frac{3}{2}a$.

[14] If existence of such equilibrium fails, then there is a continuum of equilibria indexed by the price charged by the first-mover, where the first-mover earns zero revenue.

The analysis above does not cover the case $a = 0$, since it assumes $p_1 < a^2$. When $a = 0$, we have $s(p_1) = \sqrt{p_1}\mu(\sqrt{p_1}, b)$ for all $p_1 \geq 0$ (equation 2.5). In Figure 1, this means that the linear segments of the g and s functions and their inverses are absent. Since $s(0) = 0$, matchmaker 2's revenue from overtaking is arbitrarily close to zero when p_1 is sufficiently small. At the same time matchmaker 2's maximum revenue in the D_{12} region is bounded away from zero. Therefore for p_1 small enough overtaking cannot be matchmaker 2's best response.

We say that the type distribution is sufficiently diffused if $\mu(a, b) > \frac{3}{2}a$. When the type distribution is sufficiently diffused, there is room for two matchmakers to coexist. Note that the condition $\mu(a, b) > \frac{3}{2}a$ is automatically satisfied if $a = 0$. The following proposition summarizes our findings in this subsection.

PROPOSITION 3. *If the type distribution is sufficiently diffused, there exists a pure-strategy equilibrium with a duopoly market structure in a sequential-move game.*

A sufficiently diffused distribution allows the first-mover to survive overtaking by the second-mover by focusing on a lower quality "niche" market. Note that when $a > 0$, the survival strategy of charging $p_1 < a^2$ for the first-mover implies that all low types are served and some rents are left to the lowest type (type a) with a relaxed participation constraint. Further, the sufficient condition in the proposition for the existence of a duopoly market structure depends on the type distribution only through the unconditional mean $\mu(a, b)$. This is because at the boundary between the market structures M_2 and D_{12}, where p_1 is small and $p_2 = s(p_1)$, the behavior of matchmaker 2's revenue is independent of the type distribution, and is locally identical to the behavior under the uniform type distribution. Later in this section we show that the condition is also necessary for the sequential-move game to have an equilibrium duopoly market structure under the uniform distribution.

4.3. Niche market

Proposition 3 provides a sufficient condition for the two matchmakers to coexist in an equilibrium, by considering the second-mover's incentives to overtake the first-mover while ignoring the possibility that the second-mover may allow the first-mover to serve a higher quality search market in a duopoly structure. In this subsection we show that this latter possibility does not arise if the type distribution is uniform. The market structure is D_{12} in any duopoly equilibrium, i.e. matchmaker 1's niche market is the lower quality search market while matchmaker 2 serves the higher one. To establish the claim, we will show that regardless of p_1, it is never optimal for matchmaker 2 to allow matchmaker 1 to serve a higher quality search market.

For the following analysis, it is convenient to introduce some new notation. Recall that the optimal threshold $\hat{c}$ for a one-price monopolist, is either a, or if it is interior, satisfies equation (3.2). Denote the corresponding optimal price as $\hat{p}$, given by

$$\hat{p} = \hat{c}\mu(\hat{c}, b).$$

By equation (3.2) there is a one-to-one relation between the threshold type c and the price p. Since the one-price monopolist's revenue is quasi-concave in c, it is also quasi-concave in p. Depending on the comparison between p_1 and $\hat{p}$, we distinguish four cases. It would be helpful to refer to Figure 1 throughout this subsection. The first three cases are dealt with for any type distribution; only case 4 requires the assumption of the uniform type distribution.

Case 1. If $p_1 \in (g(\hat{p}), b^2]$, by charging $p_2 = \hat{p}$ matchmaker 2 can undercut matchmaker 1 and earn the maximal revenue of a one-price monopolist. This price is the best response to such p_1 and leaves zero revenue for matchmaker 1.

Case 2. If $p_1 \in (s^{-1}(\hat{p}), \hat{p})$, matchmaker 2 can overtake matchmaker 1 by charging a price $p_2 = \hat{p}$ and again earn the maximal revenue of a one-price monopolist.[15] This leaves zero revenue for matchmaker 1.

Case 3. If $p_1 \in [0, s^{-1}(\hat{p})]$, then we can show that matchmaker 2's best response is $p_2 \in [s(p_1), g(p_1))$. That is, it is optimal for matchmaker 2 either to charge the highest price that supports the monopoly market structure M_2 (overtaking matchmaker 1 by charging $s(p_1)$), or to choose a price that allows matchmaker 1 to serve the lower search market and supports the duopoly market structure D_{12}. This is a candidate for equilibrium

[15] It follows from $s(0) = a(\mu(a, b) - a)$ and $\hat{p} \geq a\mu(a, b)$ that $s^{-1}(\hat{p})$ is well-defined.

outcome with a duopoly market structure, because matchmaker 1 can potentially earn a positive revenue (Proposition 3). To see this, first note that charging $p_2 \in [g(p_1), b^2]$, or $p_2 \in [s^{-1}(p_1), p_1)$, is not optimal because matchmaker 2 would have zero revenue. Charging any price $p_2 \in [0, g^{-1}(p_1)]$ (undercutting), if such prices are feasible, or $p_2 \in (p_1, s(p_1))$ (overtaking) is not optimal because these prices make matchmaker 2 a monopolist but they are dominated by a higher price $p_2 = s(p_1)$, which is closer to $\hat{p}$ and where matchmaker 2 is also a monopolist but earns a greater revenue due to the quasi-concavity of the one-price monopolist's revenue function. It remains to show that any $p_2 \in [g^{-1}(p_1), s^{-1}(p_1)]$, if it is feasible, is non-optimal. For any p_2 in this region, the duopoly market structure D_{21} obtains and matchmaker 2 serves the lower quality search market. Notice that matchmaker 2's revenue would be higher if he could be a one price monopolist at the same price. This is because at the same price, as a monopolist, matchmaker 2 would be serving a larger market. But then, due to the quasi-concavity of the one-price monopolist's revenue function, the monopolist revenue at $p_2 = s(p_1)$ is larger that the monopolist revenue at any $p_2 \in [g^{-1}(p_1), s^{-1}(p_1)]$, hence overtaking dominates any price in the D_{21} region.

Case 4. If $p_1 \in [\hat{p}, g(\hat{p})]$, we can show that matchmaker 2's best response is either to undercut or overtake matchmaker 1, implying that this scenario is not a candidate for an equilibrium outcome with coexistence of the two matchmakers. For any price $p_1 \in [\hat{p}, g(\hat{p})]$, matchmaker 2 has maximum four viable options. (i) Matchmaker 2 can overtake by charging $p_2 \in (p_1, s(p_1)]$. Since $p_1 \geq \hat{p}$, the matchmaker 2's revenue decreases in the overtaking region. By charging a price close to p_1, the revenue from overtaking can be made arbitrarily close to

$$(1 - F(c_2))p_1, \tag{4.2}$$

where by (2.1) c_2 satisfies $c_2\mu(c_2, b) = p_1$. (ii) Matchmaker 2 can undercut by charging $p_2 \leq g^{-1}(p_1)$ (when $g^{-1}(p_1)$ is defined). Since $p_1 \leq g(\hat{p})$, the best undercutting price is $p_2 = g^{-1}(p_1)$. The corresponding revenue is given by

$$(1 - F(c_2))g^{-1}(p_1), \tag{4.3}$$

where by (2.1) c_2 satisfies $c_2\mu(c_2, b) = g^{-1}(p_1)$. (iii) Matchmaker 2 can allow duopoly and serve the higher quality market by charging $p_2 \in [s(p_1), g(p_1)]$. However, this option is dominated by the option of overtaking: any $p_2 \in [s(p_1), g(p_1)]$ leads to a duopoly revenue lower than the one-price monopolist revenue at the same price p_2, but since $s(p_1) > p_1 \geq \hat{p}$, the maximum monopolist revenue for matchmaker 2 when $p_2 \in [s(p_1), g(p_1)]$ is reached at $p_2 = s(p_1)$, which is in turn smaller than the revenue from overtaking. (iv) Lastly, matchmaker 2 can allow duopoly and serve the lower quality market by charging $p_2 \in (g^{-1}(p_1), s^{-1}(p_1))$. We want to show that this is never optimal because either undercutting or overtaking would generate a greater revenue. This is the only place where the assumption of uniform type distribution is used. As the argument mostly involves straightforward calculations, we leave the details to the appendix. We summarize the analysis in this subsection in the following proposition.

PROPOSITION 4. *Under the uniform type distribution, in any equilibrium of the sequential-move game with a duopoly market structure, the first-mover serves the lower quality search market.*

While the above result depends on the assumption of the uniform type distribution, the intuition behind it is more general. By charging a very low price, or a very high price, the first-mover makes the overtaking strategy unappealing to the second-mover. Both low and high prices deter overtaking and one would expect that, to survive overtaking, the first-mover will target a niche market, serving either a small fraction of high types or a small market of low types. A low price is also very effective against undercutting by matchmaker 2, while choosing a high price invites undercutting and makes the first-mover vulnerable. What we have shown in this subsection is that under the uniform type distribution, when the first-mover charges a price high enough to deter overtaking, the second-mover will find it optimal to undercut. Thus, to deter undercutting as well as overtaking by the second-mover, matchmaker 1 has to find its niche market with low prices.

4.4. Price competition and sorting

The preceding analysis has prepared us to consider the effects of price competition on equilibrium sorting. Proposition 3 provides a sufficient condition for there to be an equilibrium with a duopoly market structure by showing that with a sufficiently low price matchmaker

1 can induce matchmaker 2 to serve the higher quality search market rather than overtake matchmaker 1. Proposition 4 shows that with the uniform type distribution, in looking for an equilibrium with a duopoly market structure, we need not consider the alternative market structure where matchmaker 1 serves the higher quality search market. In this subsection, we ask how the equilibrium outcome compares with the optimal market structure and pricing for a two-price monopolist.

First, we consider search market differentiation. In the duopoly market structure D_{12}, matchmaker 2 takes matchmaker 1's price $p_1 \in [0, s^{-1}(\hat{p})]$ as given and chooses $p_2 \in [s(p_1), g(p_1)]$ to maximize the revenue from the higher search market:

$$(1 - F(c_2))p_2,$$

where c_2 is determined according to equation (2.2) or equation (2.3). Alternatively, we can think of matchmaker 2 taking p_1 as given and choosing c_2 to maximize

$$(1 - F(c_2))(p_1 + c_2(\mu(c_2, b) - \mu(c_1, c_2))),$$

where c_1 is determined by

$$c_1\mu(c_1, c_2) = p_1, \tag{4.4}$$

or $c_1 = a$ if $a\mu(a, c_2) \geq p_1$. Compare the incentive of matchmaker 2 to differentiate the higher search market from the competitor's search market, to that of a two-price monopolist to differentiate the higher search market from the lower one. The latter problem can be thought of as taking p_1 as given and choosing c_2 to maximize

$$(F(c_2) - F(c_1))p_1 + (1 - F(c_2))(p_1 + c_2(\mu(c_2, b) - \mu(c_1, c_2))),$$

where c_1 is determined in the same way as (4.4) or $c_1 = a$ if $a\mu(a, c_2) \geq p_1$. Equation (4.4) implies that the size of search market 1 shrinks as c_2 decreases. While the monopolist internalizes the cannibalization of lower search market, matchmaker 2 in the duopoly market structure has no such consideration. As a result, at the optimal choices $\hat{c}_1$ and $\hat{c}_2$ of the monopolist, the derivative of matchmaker 2's revenue with respect to c_2 is negative, implying that for the same total market coverage c_1 the equilibrium threshold for the higher quality market is lower than the corresponding monopoly threshold. In other words, there is less market differentiation under competition than under monopoly. Note that this comparison between monopolistic and competitive market differentiation is independent of the type distribution. Thus, we have established the following:

PROPOSITION 5. *In any equilibrium of the sequential-move game with a duopoly market structure where the first-mover serves the lower quality market, the equilibrium outcome has less market differentiation than the optimal structure of a two-price monopolist.*

The above result can be strengthened when the type distribution is uniform. In this case, Proposition 4 implies that we can state Proposition 5 for any equilibrium in the sequential-move game with a duopoly market structure. Further, by Proposition 1 the monopolist and the planner have identical incentives in choosing c_2 for a given p_1. Thus, competition between the two matchmakers induces a smaller, and less efficient, degree of search market differentiation. This result will be used below to compare the sorting efficiency under competition and under monopoly.

Next, we consider the effect of competition on search market coverage and show that under competition the total coverage is at least as large as under the two-price monopoly. We only need to establish the claim for the case where under competition c_1 is greater than a and determined by equation (4.4). Taking derivatives, we obtain from (4.4)

$$\frac{\mathrm{d}c_1}{\mathrm{d}c_2} = -\frac{c_1\mu_r(c_1,c_2)}{\mu(c_1,c_2)+c_1\mu_l(c_1,c_2)}. \tag{4.5}$$

We can then use the above expression to compute the derivative of matchmaker 2's revenue with respect to c_2. At the boundary between M_2 and D_{12} where $c_2 = c_1$ (and $p_2 = s(p_1)$), equation (4.5) becomes $\mathrm{d}c_1/\mathrm{d}c_2 = -\frac{1}{3}$. For matchmaker 2's revenue to increase with c_2 at the boundary, we need

$$\rho(c_1)\left(\mu(c_1,b)-\frac{4}{3}c_1\right) > c_1^2. \tag{4.6}$$

By the inequality (3.3), condition (4.6) implies that $\rho(c_1) > c_1$, i.e. the virtual type of c_1 is negative. Since the virtual type of $\hat{c}_1$ for the two-price monopolist is positive, it follows immediately that competition between two matchmakers leads to a greater market coverage. However, condition (4.6) is in general not necessary for a duopoly equilibrium because matchmaker 2's duopoly revenue function need not be quasi-concave. We resolve this issue by assuming that the type distribution is uniform. Under this assumption, condition (4.6) reduces to

$$c_1 < (\sqrt{19} - 4)b. \tag{4.7}$$

We can use (4.5) to verify that matchmaker 2's revenue maximization problem is concave in the $c_1 > a$ section of the duopoly region D_{12}. It follows that condition (4.7) is a necessary condition for matchmaker 2's best response in the duopoly market structure to be interior, or equivalently, for there to be an equilibrium outcome in the sequential-move game with a duopoly market structure.

Before stating the above result on the comparison of market coverage, we use the uniform type assumption to strengthen our previous finding in Proposition 3. In particular, the condition stated in Proposition 3, which reduces to $a/b < \frac{1}{2}$, is necessary as well as sufficient for an equilibrium duopoly market structure to exist. To establish this claim, we assume $a/b \geq \frac{1}{2}$ and consider two cases: $p_1 \leq a^2$ and $p_1 > a^2$. First, for any $p_1 \leq a^2$, by equations (2.5) and (2.6), a price of $p_2 \in [s(p_1), g(p_1)]$ leads to $c_1 = a$ in the duopoly structure D_{12}. As a result, one can verify that matchmaker 2's revenue as a function of c_2 is concave. Since $a/b \geq \frac{1}{2}$, equation (4.1) implies that matchmaker 2's revenue is decreasing at $p_2 = s(p_1)$ for all $p_1 \leq a^2$. Thus, the best response of matchmaker 2 to $p_1 \leq a^2$ is to charge $p_2 = s(p_1)$ and serve all types as a monopolist. Matchmaker 1 cannot survive the overtaking strategy of matchmaker 1 with a price $p_1 \leq a^2$. Second, fix a price of $p_1 > a^2$. In this case, $c_1 > a$ along the boundary between M_2 and D_{12} (equation 2.5). This implies that matchmaker 2's duopoly revenue is decreasing in c_2 along the boundary, because condition (4.7) cannot be satisfied when $a/b \geq \frac{1}{2}$. For any fixed p_1, as c_2 increases c_1 decreases (equation 4.5) to the left of the dashed line in the D_{12} region (Figure 1), and by concavity matchmaker 2's duopoly revenue decreases with c_2. As c_2 further increases, c_1 may become a to the right of the dashed line (if p_1 is not too high). But in this case we have $\mathrm{d}c_1/\mathrm{d}c_2 = 0$ instead of $\mathrm{d}c_1/\mathrm{d}c_2 < 0$ (equation 4.5), and therefore the duopoly revenue continues to decrease to the right of the dashed line. Therefore, the maximum duopoly revenue is achieved at the boundary between M_2 and D_{12}. It follows that there is no equilibrium with a duopoly market structure if $a/b \geq \frac{1}{2}$.

As for Proposition 3, we say that the type distribution is sufficiently diffused if $a/b < \frac{1}{2}$. We have the following result:

Proposition 6. *Suppose that the type distribution is uniform. An equilibrium of the sequential-move game with a duopoly market structure exists if and only if the type distribution is sufficiently diffused. Further, in any such equilibrium the market coverage is at least as large as in the optimal structure of a two-price monopolist.*

An implication of Propositions 5 and 6 is that the comparison between competition and monopoly in terms of sorting efficiency can be ambiguous, because competition expands the coverage but reduces the quality difference. To give some sense of the comparison between competition and monopoly, we concentrate on a parameter range of the uniform type distribution where the equilibrium outcome can be explicitly calculated. This is the range where in equilibrium the first-mover serves all types and leaves some rents to the lowest type.

We assume that

$$\sqrt{19} - 4 < \frac{a}{b} < \frac{1}{2}.$$

Denote the equilibrium thresholds as $\tilde{c}_1$ and $\tilde{c}_2$, with corresponding prices $\tilde{p}_1$ and $\tilde{p}_2$. Since condition (4.7) is necessary for an equilibrium with $\tilde{c}_1 > a$, and since it cannot be satisfied when $a/b > \sqrt{19} - 4$, we have $\tilde{c}_1 = a$. Further, the argument leading to Proposition 6 implies $\tilde{p}_1 < a^2$. Since c_1 is no longer a variable, it is straightforward to solve for the equilibrium by working backwards. For any $p_1 < a^2$, the best response of matchmaker 2 in terms of c_2 can be found by revenue maximization. It is given by:

$$c_2 = \max\left\{a, \frac{1}{2}b - \frac{p_1}{b-a}\right\}. \tag{4.8}$$

Given the above best response, the optimal price p_1 for matchmaker 1 is given by

$$\tilde{p}_1 = \frac{1}{4}(b-2a)(b-a).$$

It can be easily verified that $\tilde{p}_1 < a^2$ because $a/b > \sqrt{19} - 4$. By equation (4.8), the equilibrium threshold for matchmaker 2 is therefore

$$\tilde{c}_2 = \frac{1}{4}(b+2a). \tag{4.9}$$

The corresponding equilibrium price is

$$\tilde{p}_2 = \frac{1}{8}(3b - 2a)(b - a).$$

We are ready to compare the total match value under competition and under the two-price monopolist. The total match value is given by (3.5). Under competition, the equilibrium thresholds are given by $\tilde{c}_1 = a$ and (4.9). The total match value is:

$$\tilde{R} = \frac{1}{16}(5b^2 + 6ba + 5a^2) - \frac{1}{64}b^2.$$

Under the monopolist, the optimal threshold $\hat{c}_1$ can be solved from (3.17) with $N = 2$, yielding $\hat{c}_1 = hb$, with $h = b(2\sqrt{6} + 3)/15$. The optimal threshold $\hat{c}_2$ can be then obtained from (3.15), as the market differentiation under monopoly is the same as under the planner. The corresponding total match value is:

$$\hat{R} = \frac{b - h}{16(b - a)}(5b^2 + 6bh + 5h^2).$$

For reference, we also compute the total match value under the planner. The efficient threshold c_1^* is given by (3.16) with $N = 2$, which implies $c_1^* = a$ in the parameter range under consideration. The optimal threshold c_2^* is given by (3.15). The total match value is therefore:

$$R^* = \frac{1}{16}(5b^2 + 6ba + 5a^2).$$

Evaluations of $\tilde{R}$ and $\hat{R}$ reveal that there is a critical level of a/b such that $\tilde{R} < \hat{R}$ if and only if the value of a/b is above the critical level.

The reason that the equilibrium outcome is less efficient in sorting than the monopoly outcome when a/b is close to $\frac{1}{2}$ can be understood as follows. We know from Proposition 1 that under the uniform type distribution assumption, the market differentiation is efficient under monopoly. The only source of sorting inefficiency under monopoly comes from insufficient market coverage compared with the planner's coverage. When a/b is close to $\frac{1}{2}$, the loss from insufficient coverage is small: $\hat{R} < R^*$ because $h > a$, but the difference shrinks when a/b increases towards $\frac{1}{2}$. In contrast, the equilibrium outcome has the efficient market coverage ($\tilde{c}_1 = c_1^* = a$) in the parameter range we are considering, but suffers from inefficiently small market differentiation (Proposition 5). Since the difference between $\tilde{R}$ and R^* is independent of a/b while the difference between $\hat{R}$ and R^* shrinks when a/b gets close to $\frac{1}{2}$, the equilibrium outcome has a lower total match value than the monopoly outcome.

5. Discussions

One important restriction in the present model of competing matchmaking is that each matchmaker is allowed to use only one price and create one search market. This restriction may be justified by the cost of setting up search markets in an actual matching environment, but we have made the assumption to simplify the analysis. What is critical in the restriction to one search market is not that matchmakers cannot create greater distinctions among search markets. Rather, the restriction to one search market is the simplest environment where price competition interferes with the sorting role of prices.

To see why our results on price competition and sorting efficiency are robust with respect to this restriction, we first argue that if sorting is perfectly achieved by prices, then the overtaking strategy becomes completely ineffective. Suppose that there is a continuum of search markets, one for each type, offered either by a single matchmaker, or by a continuum of matchmakers. The pricing schedule is such that each type maximizes its payoff by choosing the search market designed for the type. More precisely, if $q(x)$ is the price for the type-x search market, then perfect self-selection is guaranteed if $q'(x) = x$ for each type x in $[a, b]$. Imagine that an entrant to the market introduces a new search market with a price of $p \in [q(a), q(b)]$. The type that is most likely to deviate to the new search market is $q^{-1}(p)$, i.e. the type that is paying the price p in the present market structure, as this type has the largest set of "beliefs" about the match quality of the new search market that would induce deviation. However, type $q^{-1}(p)$ is exactly indifferent between staying in its present search market and deviation. The present market structure is stable by Definition 1, and in particular is not vulnerable to overtaking. Moreover, any deviating price $p > q(b)$ is unacceptable, and if $q(a) = 0$, there is no room for entry. Thus, when sorting is perfectly achieved by prices, price competition can only take the form of the usual Bertrand type.[16]

Next, we argue that as long as types are not perfectly sorted, overtaking is possible and price competition affects sorting. To see this, suppose that there is an active search market with a subscription fee p that has a non-degenerate type distribution. Then, this market can be overtaken by an entrant with a new search market with a price p' just above p. This follows because the type that is most likely to deviate is the highest type in the search market. Since the type distribution in the search market is non-degenerate, there is a price p' just above p such that the highest type strictly prefers the new search market. By Definition 1, the present market structure is unstable.

Appendix

In this appendix we complete the argument for case 4 in the proof of Proposition 4. Here, $p_1 \in [\hat{p}, g(\hat{p})]$, and we want to show that any price $p_2 \in (g^{-1}(p_1), s^{-1}(p_1))$ in the D_{21} region is dominated by either the best overtaking price (p_2 just above p_1) or the best undercutting price ($p_2 = g^{-1}(p_1)$). We refer to the revenue given by (4.2) as the "maximum overtaking revenue," and the revenue given by (4.3) as the "maximum undercutting revenue" for any p_1. Note that the maximum overtaking revenue decreases with p_1, while the maximum undercutting revenue increases in p_1. In addition, because for fixed $p_2 < p_1$ as p_1 increases c_2 either decreases or does not change and c_1 increases (see equations (2.2) and (2.3), with the roles of the two matchmakers reversed), matchmaker 2's maximum duopoly revenue in the D_{21} region is increasing in p_1.

The proof relies on two claims. The first claim is that there is a critical price $\overline{p}$ such that for any $p_1 \geq \overline{p}$ matchmaker 2's maximum duopoly revenue is achieved at the boundary between M_2 and D_{21}. Thus, the maximum duopoly revenue coincides with the maximum

[16] The same observation can be made in an alternative model of discrete types. Let x_i, $i = 1, \ldots, N$, be the types, with $x_i < x_j$ for $i < j$. Then the only stable market structure is a set of prices p_i, such that $p_1 = 0$ and $p_{i+1} = p_i + x_i(x_{i+1} - x_i)$ for $i = 1, \ldots, N-1$. It is not possible to overtake any search market because the prices are constructed so that for any deviating price p slightly above p_i for each i, the type most likely to deviate is exactly x_i but type x_i strictly prefer to stay in its current search market. It is not possible to undercut any search market either, because for any p slightly below p_i, the type most likely to deviate is x_{i-1}, which by construction strictly prefers to stay in its current search market.

undercutting revenue for any $p_1 \geq \overline{p}$. Note that matchmaker 1 makes zero revenue. The second claim is that at $p_1 = \overline{p}$ the maximum undercutting revenue, which equals the maximum duopoly revenue, is smaller than the maximum overtaking revenue. For any $p_1 < \overline{p}$, the maximum duopoly revenue is achieved in the interior of the D_{21} region, but it is smaller than when $p_1 = \overline{p}$ because as a function of p_1 it is increasing. Since the maximum overtaking revenue is decreasing in p_1, it follows from the second claim that the maximum duopoly revenue for any $p_1 < \overline{p}$ is smaller than the maximum overtaking revenue at the same p_1.

It will be helpful to refer to Figure 1 throughout the proof. The boundary between M_2 and D_{21}, defined by $p_2 = g^{-1}(p_1)$ with $c_1 = b$, has a kink at $p_1 = b(b - \mu(a,b)) + a\mu(a,b)$ (equation 2.6). Let $\tilde{p} = b(b - \mu(a,b)) + a\mu(a,b)$. Note that $\tilde{p}$ is the highest p_1 such that $c_2 = a$ on the boundary. If $p_1 \geq \tilde{p}$, then $c_2 > a$ throughout the D_{21} region. If instead $p_1 < \tilde{p}$, then we have $c_2 = a$ to the left of the dashed line (including the boundary between M_2 and D_{21}) and $c_2 > a$ to the right of the dashed line (see equation 2.7for the definition of the dashed line). We distinguish two cases. In the first case, we establish the two claims mentioned above by focusing on the section of the D_{21} region where $c_2 > a$. This allows us to characterize matchmaker 2's duopoly revenue as a function of c_2 (as opposed to p_2). It turns out that this is legitimate so long as the ratio a/b is small. In the second case, the two claims are established by analyzing matchmaker 2's duopoly revenue as a function of p_2 in the section where $c_2 = a$. This is legitimate so long as the ratio a/b is large. The two cases cover all parameter values of a and b.

Small a/b case. Consider the problem of choosing c_2 to maximize the revenue for matchmaker 2 in D_{21}, given by

$$(F(c_1) - F(c_2))c_2\mu(c_2, c_1),$$

where c_1 satisfies

$$p_1 = c_1(\mu(c_1, b) - \mu(c_2, c_1)) + c_2\mu(c_2, c_1).$$

Note that this maximization problem is valid only if $c_2 > a$. Under the uniform type distribution, the above relation becomes

$$p_1 = \frac{1}{2}(c_1 b + c_2^2), \tag{A.1}$$

from which we have

$$\frac{\mathrm{d}c_1}{\mathrm{d}c_2} = -\frac{2c_2}{b}. \tag{A.2}$$

It follows that the first order condition with respect to c_2 is

$$\frac{1}{2}c_1^2 - \left(\frac{2c_1}{b} + \frac{3}{2}\right) c_2^2 = 0. \tag{A.3}$$

Moreover, one can verify that equation (A.2) implies that the revenue is concave in c_2. Since $c_1 = b$ at the boundary between the two market structures M_2 and D_{21}, (A.1) and (A.3) imply that there is a unique $p_1 = \frac{4}{7}b^2$ such that equation (A.3) holds with equality at $p_2 = g^{-1}(p_1)$. Further, straightforward calculations reveal that at $p_1 = \frac{4}{7}b^2$, matchmaker 2's maximum undercutting revenue is smaller than its maximum overtaking revenue. Thus, we have established both claims mentioned above.

The above analysis is valid so long as the critical price $p_1 = \frac{4}{7}b^2$ is greater than $\tilde{p}$, which happens when $a/b < 1/\sqrt{7}$. If this does not hold, then for any $p_1 \geq \tilde{p}$, the maximum duopoly revenue is achieved at the boundary of M_2 and D_{21}, leaving matchmaker 1 with zero revenue. The same logic as before implies that charging a price p_2 in the D_{21} region is dominated by either overtaking or undercutting, if at $p_1 = \tilde{p}$ matchmaker 2's maximum overtaking revenue is greater than the maximum undercutting revenue. Using the assumption of uniform type distribution, we can verify that this is the case if the ratio a/b satisfies:

$$\sqrt{5 + 4(a/b)^2} + \frac{2(a/b) - 2(a/b)^3}{1 + (a/b)^2} < 3. \tag{A.4}$$

The above inequality holds for small values of a/b (about 0.4425).

Large a/b case. Now let us look at matchmaker's 2 revenue maximization problem when c_2 is fixed at a and c_1 is determined by the equation

$$c_1\mu(c_1, b) - p_1 = c_1\mu(a, c_1) - p_2. \tag{A.5}$$

Using the assumption of uniform type distribution, we can show that the revenue, given by $F(c_1)p_2$, is a concave function of p_2, and therefore the following first order condition is sufficient for revenue maximization:

$$p_2 = \frac{1}{2}p_1 - \frac{1}{4}a(b-a). \tag{A.6}$$

Since $c_1 = b$ at the boundary between the two market structures M_2 and D_{21}, the same analysis as in the small a/b case establishes the following two claims: (i) there is a unique $p_1 = \frac{1}{2}(2b-a)(b-a)$ such that equation (A.6) holds with equality at $p_2 = g^{-1}(p_1)$, and (ii) at $p_1 = \frac{1}{2}(2b-a)(b-a)$, for any value of a/b such that condition (A.4) is violated, matchmaker 2's maximum undercutting revenue is smaller than its maximum overtaking revenue.

The above analysis is valid, if for any value of a/b such that condition (A.4) is violated, the critical price $p_1 = \frac{1}{2}(2b-a)(b-a)$ is smaller than $\tilde{p}$, and if on the dashed line that separates the $c_2 = a$ section of D_{21} from the $c_2 > a$ section of D_{21}, matchmaker 2's revenue decreases with p_2 (or equivalently, c_2). The two conditions can be straightforwardly verified using the uniform type distribution assumption.

References

An, M. (1998), "Logconcavity versus Logconvexity: A Complete Characterization," *Journal of Economic Theory* 80, 350–369.

Armstrong, M. (2002), "Competition in Two-Sided Markets," Manuscript, Nuffield College, Oxford University.

Banks, J.; and Sobel, J. (1987), "Equilibrium Selection in Signaling Games," *Econometrica* 55, 647–662.

Damiano, E.; and Li, H. (2003), "Price Discrimination in Matching Markets," Manuscript, University of Toronto.

Ellison, G; Fudenberg, D.; and Mobius, M. (2002), "Competing Auctions," Manuscript, Havard University.

Ellison, G; and Fudenberg, D. (2002), "Knife-edge or Plateau: When Do Market Models Tip?" Manuscript, MIT.

Katz, M.; and Shapiro, K. (1985), "Network Externalities, Competition and Compatibility," *American Economic Review* 75, 424–440.

Caillaud, B.; and Jullien, B. (2000), "Competing Cybermediaries," *European Economic Review* 45, 797–808.

Caillaud, B.; and Jullien, B. (2001), "Chicken & Egg: Competing Matchmakers," Manuscript, Toulouse University.

Fujita, M. (1988), “A Monopolistic Competition Model of Spatial Agglomeration—Differentiated Product Approach,” *Regional Science and Urban Economics* 18, 87–124.

Gehrig, T. (1998), “Competing Markets,” *European Economic Review* 42, 277–310.

McAfee, R.P. (2002), “Coarse Matching,” *Econometrica* 70, 2025–2034.

Myerson, R. (1978), “Refinements of the Nash Equilibrium Concept,” *International Journal of Game Theory* 7: 73–80.

Rayo, L. (2002), “Monopolistic Signal Provision,” Manuscript, Stanford University.

Reny, P. (1999), “On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games,” *Econometrica* 67, 1029–1056.

Selten, R. (1975), “Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games,” *International Journal of Game Theory* 4: 25–55.