Network Markets and Consumers Coordination*

Attila Ambrus† and Rossella Argenziano‡

First Version: June 2003
This Version: March 2004

Abstract

This paper assumes that groups of consumers in network markets can coordinate their choices when it is in their best interest to do so, and when coordination does not require communication. It is shown that, despite this assumption, multiple asymmetric networks can coexist in equilibrium if consumers have heterogeneous reservation values. A monopolist provider might choose to operate multiple networks to price differentiate consumers on both sides of the market. Competing network providers might operate two networks such that one of them targets high reservation value consumers on one side of the market, while the other targets high reservation value consumers on the other side. Firms can obtain positive profits in price competition. In these asymmetric equilibria product differentiation is endogenized by the network choices of consumers. Heterogeneity of consumers is necessary for the existence of this type of equilibrium.

*We would like to thank Dirk Bergemann, Bernard Caillaud, Gergely Csorba, Erica Field, Drew Fudenberg, Dino Gerardi, Stephen Morris and seminar participants at Yale University, Harvard University, Olin School of Business, at the Conference on Two-Sided Markets organized by IDEI Toulouse and at the 31st TPRC for comments and suggestions

†Department of Economics, Harvard University, Cambridge, MA 02138, ambrus@fas.harvard.edu

‡Department of Economics, Yale University, New Haven, CT 06520-8268, rosa.argenziano@yale.edu

1 Introduction

A market has network externalities if consumers' utility from purchasing a product depends on which other consumers buy the same product. A highlighted special case of this is two-sided markets with network externalities. In these markets consumers are divided into two distinct subgroups. A consumers's utility on one side increases in the total number of consumers on the other side of the market who buy the same product (and possibly decreases in the number of consumers on the same side of the market). This applies to various situations in which two groups of agents need a common platform to interact and one or more firms own platforms and sell access to them. The higher the number of agents on one side who join a platform, the higher the utility of an agent on the other side of the platform because she has a higher number of potential partners with whom to trade or interact. One example of this is the market for on-line matchmaking services where the two sides are women and men. In other cases the two sides are buyers and sellers and the platforms are auction websites, directory services, classified advertisers and credit card networks (the sellers are merchants who accept the credit card and the buyers are credit card holders).

This paper investigates the decisions of firms regarding how many networks to operate and how to price them, and the network choices of consumers in two-sided markets with network externalities. We consider an extensive form game in which in the first stage firms announce registration fees for joining their networks. In the second stage consumers observe these fees and then simultaneously and independently choose networks or decide to stay out of the market.

The first contribution of this paper is methodological. Equilibrium analysis on network markets is an involved task. The coordination problems that arise among consumers result in a severe multiplicity of equilibria both in games in which there is a monopolist network provider and in games in which there are multiple providers. Consumers can have various self-fulfilling expectations regarding which networks other consumers join and whether they join any network at all. To address the issue of multiplicity of equilibria we use the concept of coalitional rationalizability, proposed by Ambrus ([02],[03]) to select equilibria and therefore derive qualitative predictions. Informally, this method corresponds to assuming that consumers can coordinate their network choices as long as it is in their joint interest and this coordination does not require explicit communication.

The second contribution of the paper is that this methodology allows us to analyze pricing games in which consumers on the market are heterogeneous. This makes it possible to ask a new set of questions. These include whether price discrimination among different type of consumers is possible without product differentiation, whether there can be multiple active networks operated by competing firms, attracting different types of consumers, and whether it can be in the interest of a monopolistic network provider to operate multiple networks, aimed at different sets of consumers.

Coordination failures and the resulting inefficiencies are relevant phenomena in network markets, since typically there are many anonymous consumers who cannot communicate with each other. Nevertheless, in some cases it is reasonable to expect consumers to be able to coordinate on a particular network. The simplest example is if there are two networks and one is cheaper on both sides of the market. Choosing this network is then a natural focal point on which consumers can coordinate. The central assumption of our paper is that consumers can coordinate their decisions to their advantage if their interests coincide and if coordination can be achieved without communication, as in the above case. In contrast, if there is no unique candidate network that consumers would agree to join, we do not assume successful coordination even if it is in the common interest of consumers. Our motivation for this is that if there are lot of small consumers on the market then it is practically impossible for them to get together and make explicit agreements on network choices.

Coalitional rationalizability allows us to incorporate the above assumption into the analysis. This noncooperative solution concept assumes that players can coordinate to restrict their play to a subset of the original strategy set if it is in the interest of every participant to do so. This defines a set of implicit agreements among players, which puts restrictions on beliefs that players can have at different stages of the game. These agreements are based on public information, the description of the game, and therefore do not require explicit communication. Furthermore, they are self-fulfilling in the sense that if participants expect each other to choose networks according to an agreement, then it is in their best interest to act according to the agreement. In the games that we analyze there are no such agreements among firms or between firms and consumers that are in the interest of every participant. But after certain price announcements there can be groups of consumers who can coordinate their network choices this way.

We investigate the subgame perfect equilibria of the above market games that are compatible with the additional assumption of coalitional rationalizability. We call these equilibria coalition perfect.

We analyze participation rates of consumers, prices charged by the firms, network sizes and firms' profits in coalition perfect equilibria. The analysis is carried out for the cases of both one and two firms operating on the market. In the former case we distinguish between the case that the firm can operate only one network and the case in which it can decide to operate multiple networks.

We show that it can be that in every coalition perfect equilibrium a monopolist network provider chooses to establish two networks if he is allowed to do so, and different type of consumers join these two networks. The intuition is that, if there are high reservation value consumers on both sides of the market, then the monopolist wants to extract surplus from both of these groups. However, if there are relatively few of these consumers and the monopolist operates only one network, then he can charge a high price on at most one side. This is because in order to charge a high price on one side, there have to be enough consumers on the other side of the network, which is only possible if the price charged on that side is low. On the other hand, if the monopolist establishes two networks such that one of them is cheap on one side of the market and the other network is cheap on the other side, then all consumers are willing to join some network and consumers with high reservation values are willing to join the network which is more expensive for them. The way price discrimination is achieved in these equilibria is through endogenous product differentiation. Networks are physically the same, but if one side of a network attracts a lot of consumers, then the other side of the network becomes more valuable for consumers.

In the case of two network providers competing for consumers we show that homogeneity of consumers on the same side of the market ensures that both firms' profits are zero in coalition perfect equilibrium, reestablishing the classic Bertrand result for markets without network externalities. This holds despite the fact that equilibrium prices do not have to be equal to the marginal cost. In fact, if the two sides are asymmetric in that consumers on one side care less about the network externality than consumers on the other side, then in every coalition perfect equilibrium the side that cares less gets strictly subsidized. Homogeneity of the consumers also implies that there cannot be multiple asymmetric networks in coalition perfect equilibrium. On the other hand, we show that if consumers are heterogeneous, then there can be equilibria in which there are two networks that attract different type of consumers and in which firms earn positive profits. The intuition is that although firms can steal each others' consumers by undercutting their rival's prices on both sides of the market, this move is not necessarily profitable. Consider the case in which one firm subsidizes consumers that join its network on one side and gets a positive price on the other side, while the other firm does the reverse - charging a positive price on the first side and subsidizing the second. Then undercutting might be unprofitable if it increases the number of consumers to be subsidized more than the number of consumers who pay a positive price.

In both the monopoly and the duopoly case the coalition perfect equilibria with two asymmetric networks have the feature that one network is larger and cheaper on one side of the market, while the other one is larger and cheaper on the other side. One example of this configuration is when a town has both a freely distributed newspaper with classified ads and one that is not freely distributed. The first newspaper is by definition cheaper and larger on the buyers' side. In order to compete with the freely distributed newspaper, typically the other newspaper has to have more ads posted on it, and therefore can only charge a smaller fee for posting ads. Therefore this newspaper is typically cheaper and larger on the sellers' side. Another example is on-line job search, where the two main platforms are Careerbuilder.com and Monster.com. Monster has a database of 25 million resumes versus Careerbuilder's 9 million, therefore larger on the job seekers' side. On the other hand, Careerbuilder has 45.2% of the job postings of the on-line job search market in the US, while Monster has only 37.5%.[1] Therefore Careerbuilder is larger on this side. And to post a job on Careerbuilder, a firm pays $269, while to post a job on Monster a firm pays $335.[2]

[1] 2004 February figures. The information about the size of the two databases is taken form www.careerbuilder.com and www.monster.com . The information about the number of job postings is obtained from Corzen.

[2] The base cost of posting a resume is zero on both sites, but job seekers pay extra fees for preferential treatment of their resumes (for example if they want them to come up at the top of search result lists obtained by firms). We do not have information on how many job seekers pay these extra fees, therefore we cannot make a correct price comparison on this side of the market.

2 Related literature

Recently a number of papers investigated the issue of optimal pricing and price competition on markets with two-sided network externalities. Below we mention the papers that are most closely related to ours. For a more extensive literature review, see for example Armstrong (2002).

Rochet and Tirole (2003) and Armstrong (2002) study monopolistic pricing and price competition between two firms on markets where the firms are platforms that try to attract two groups of agents. The models in these papers abstract away from coordination problems among consumers. First, they emphasize the case in which the networks' primary pricing instrument are transaction fees. Second, they assume differentiable demand functions for the networks, implicitly assuming differentiated networks (that consumers have heterogeneous inherent preferences between networks). Also, they focus on a particular symmetric equilibrium.

Jullien (2001) constructs a duopoly model that allows for more than two subgroups of consumers and for both inter-groups and intra-group network externalities. The setup of this paper differs from ours in that the intrinsic value of the good sold by each firm is assumed to be high compared to the network effect and also that one of the firms is highlighted in the sense that consumers always coordinate on the equilibrium which is the most favorable for this firm.

Ellison, Fudenberg and Mobius (2002) study competition between two auction sites. In their model, just like in ours, multiple asymmetric platforms can coexist in equilibrium, despite no product differentiation. In addition to this, just like us, they assume heterogeneous agents on both sides of the market. On the other hand, in their model consumers choose platforms ex ante, while in our paper they already know their types by the time they decide whether to join a network or not. Also the reason that multiple active networks can coexist in equilibrium is completely different in their model. They consider a finite number of buyers and sellers, therefore one of them switching from one platform to another adversely affects the market price on the latter platform.[3] In our model there is a continuum of consumers on both sides of the market, therefore this market-impact effect is absent.

[3] See Ellison and Fudenberg (2003) for a detailed analysis of this point.

The model in Damiano and Li (2003a and 2003b) is similar to ours in that consumers on a two-sided market are heterogeneous and that registration fees serve the role to separate different types of consumers. The main difference between our setup and theirs is that in the latter there is no network externality. Consumers care about the average quality of consumers on the other side of the network and not their number. On the other hand, in our model consumers are symmetric with respect to the external effect they generate on consumers on the other side.

The most similar model to ours is presented by Caillaud and Jullien (2001 and 2003). They analyze markets where firms are intermediaries offering matchmaking services to two groups of agents. The above papers assume that consumers on each side are homogeneous and their utility is linear in the number of consumers on the other side of the network. Like in our paper, both optimal pricing by a monopolist and price competition between two intermediaries are analyzed. Caillaud and Jullien select among equilibria by imposing monotonicity on the demand function of consumers and by assuming full market coverage in equilibrium.[4][5] As opposed to this, we impose the assumption of coalitional rationalizability on expectations of individual players, not an assumption on the aggregate demand function. We consider this type of restriction more plausible. Furthermore it allows us to drop the restrictive assumptions of homogeneity and linear utility functions without making the model intractable.

In our model different types of consumers might select to join different networks. In this aspect, our analysis is connected to the literature on price discrimination (for an overview see Varian (1987)) multiproduct pricing (see Baumol et al. (1982)) and the theory of screening (for an overview see Salanie (1997)).

3 The Model

There are three types of games that we consider in the paper. In the first there is a monopolist firm and it can only establish one network. In the second there is again only one firm, but it can decide whether to establish

[4] For a comparison between our results and the ones in Caillaud and Jullien (2003), see Subsection 7.1.

[5] There is a similar distinction with respect to this point between Damiano and Li (2003b) and our paper, too.

one or two networks. In the third there are two competing firms and each can establish only one network. In all three type$s$ of games the set of players consists of the firm(s) and two distinct sets of consumers, side 1 and side 2 ones. We assume that there is a continuum of consumers of mass 1 on both sides of the market. We denote the set of these games Γ^{M1}, Γ^{M2} and Γ^{D} respectively. Let $\Gamma^{M} = \Gamma^{M1} \cup \Gamma^{M2}$ and let $\Gamma = \Gamma^{M} \cup \Gamma^{D}$.

First we provide an informal description of games in Γ, then we define them formally.

Every game in Γ is a multi-stage game with observable actions, consisting of three stages. In the first stage the monopolist or the two duopolists simultaneously choose how many networks to establish. In the second the firms simultaneously set prices (registration fees) on their networks. Firms can charge different registration fees on the two sides of a network. Furthermore, they can charge negative prices on either side of networks (subsidizing consumers on that side). Finally consumers, having observed the previous choices, choose which network to join, if any. With the exception of Section 10 we assume that consumers can join at most one network (exclusivity of networks).

Firms maximize profits. It is assumed throughout the paper that firms are symmetric. Furthermore, with the exception of Section 9 we assume that the cost of operating a network is zero, independently of the number of consumers joining the network. Then the payoff of the firm is the sum of the revenues collected from the firm's networks, where the revenue collected from a network is sum of the revenue collected on side 1 and the revenue collected on side 2.

A consumer's utility if not joining any network is zero. A consumer's utility if joining a network decreases in the registration fee she has to pay, increases in the number of people joining the network from the other side of the market and weakly decreases in the number of people joining from her side of the market. Consumers have individual specific utility functions (we do not assume that they are homogeneous), but we assume that consumers' utilities are quasilinear in money. Also, we assume that consumers do not have any inherent preference for joining one network or another, they only care about the number of people joining the networks and the price they have to pay.

We start the formal construction by defining the set of players, I.

For games in Γ^M let $I = I^M \equiv \{A, (C_i^1)_{i\in[0,1]}, (C_i^2)_{i\in[0,1]}\}$ and for games in Γ^D let $I = I^D \equiv \{A, B, (C_i^1)_{i\in[0,1]}, (C_i^2)_{i\in[0,1]}\}$, where C_i^1 ($i \in [0,1]$) denotes consumer i on side 1, C_i^2 ($i \in [0,1]$) denotes consumer i on side 2, while A and B denote the firms. For $j = 1,2$ let $C^j = \underset{i\in[0,1]}{\cup} C_i^j$. Let $C = C^1 \cup C^2$.

Next we define the action sets of players at different stages, and then the set of strategies S in the whole game.

In stage 1 and in stage 2 player A moves if $G \in \Gamma^M$, and players A and B move simultaneously if $G \in \Gamma^D$. In stage 3 players in C choose actions simultaneously.

The first stage action set is $\{1, ..., \overline{n}\}$ for each player that moves in that stage, where $\overline{n} = 1$ if $G \in \Gamma^{M1} \cup \Gamma^D$, and $\overline{n} = 2$ if $G \in \Gamma^{M2}$. If $G \in \Gamma^{M1} \cup \Gamma^D$, then second stage action set is R^2 for each player that moves in that stage. If $G \in \Gamma^{M2}$, then the second stage action set is R^2 for A at nodes that follow action choice 1 in the first stage, and the second stage action set is R^4 for A at nodes that follow action choice 2 in the first stage. For every $C_i^j \in C$ the third stage action set is $\{\emptyset, 1\}$ if $G \in \Gamma^{M1}$, $\{\emptyset, A, B\}$ if $G \in \Gamma^D$, $\{\emptyset, 1\}$ at nodes following a first stage action choice of 1 if $G \in \Gamma^{M2}$, and $\{\emptyset, 1, 2\}$ at nodes following a first stage action choice of 2 if $G \in \Gamma^{M2}$.

For any $G \in \Gamma$ let S_A denote the set of strategies of player A, S_B the set of strategies of player B (if $G \in \Gamma^D$) and S_i^j the set of strategies of C_i^j for $j = 1,2$ and $i \in [0,1]$. Note that strategies in S_i^j specify actions for C_i^j after any history of length 2.

Let the set of strategy profiles be $S = S^M \equiv S_A \underset{i\in[0,1]}{\times} S_i^1 \underset{i\in[0,1]}{\times} S_i^2$ for $G \in \Gamma^M$ and $S = S^D = S_A \times S_B \underset{i\in[0,1]}{\times} S_i^1 \underset{i\in[0,1]}{\times} S_i^2$ for $G \in \Gamma^D$.

Let $k \in \{A, B\}$, $j \in \{1,2\}$, $i \in [0,1]$ and $s \in S$. For any $k \in \{A, B\}$ let $n_k(s)$ denote the first stage action choice that s_k specifies. If $G \in \Gamma^M$ and $n_k(s) = 1$, then let $(p_1^1(s), p_1^2(s))$ denote the second stage action choice that s_k specifies. Similarly, for any $j \in \{1,2\}$ and $i \in [0,1]$ let $c_i^j(s)$ denote the action choice that s_i^j specifies after first stage action $n_k(s)$ and second stage action $(p_1^1(s), p_1^2(s))$. If $G \in \Gamma^{M2}$ and $n_k(s) = 2$, then let $(p_1^1(s), p_1^2(s), p_2^1(s), p_2^2(s))$ denote the second stage action choice that s_k specifies. Similarly, let $c_i^j(s)$ denote the action choice that s_i^j specifies after first stage action $n_k(s)$ and second stage action $(p_1^1(s), p_1^2(s), p_2^1(s), p_2^2(s))$. If $G \in \Gamma^D$, then let $(p_k^1(s), p_k^2(s))$ denote the second stage action choice that s_k specifies. Similarly, let $c_i^j(s)$ denote the action choice that s_i^j specifies after second stage actions $(p_A^1(s), p_A^2(s))$ and $(p_B^1(s), p_B^2(s))$.

Lastly, we define the payoff functions of players: $\pi_k(s)$ for $k = A, B$ and $U_i^j(s)$ for $C_i^j \in C$.

Let $s \in S$. If $G \in \Gamma^M$ and $k \in \{1, 2\}$ is such that $\{c_i^j(s) = k\}$ is measurable, then let $N_k^j(s) = \int\limits_{i:c_i^j(s)=1} di$. Similarly, if $G \in \Gamma^M$ and $k \in \{A, B\}$ is such that $\{c_i^j(s) = k\}$ is measurable, then let $N_k^j(s) = \int\limits_{i:c_i^j(s)=1} di$.

If $G \in \Gamma^M$ and $n_A(s_A) = 1$ then $\pi_A(s) = \sum\limits_{j=1,2} p_1^j(s) N_1^j(s)$. If $G \in \Gamma^{M2}$ and $n_A(s_A) = 2$ then $\pi_A(s) = \sum\limits_{k=1,2} \sum\limits_{j=1,2} p_k^j(s) N_k^j(s)$. If $G \in \Gamma^D$ then $\pi_k(s) = \sum\limits_{j=1,2} p_k^j(s) N_k^j(s)$ for $k \in \{A, B\}$.

Let $U_i^j(s) = 0$ if $c_i^j(s) = \emptyset$. Let $U_i^j(s) = g_i^j(N_{c_i(s)}^j, N_{c_i(s)}^{-j}) - p_{c_i(s)}^j(s)$ if $c_i^j(s) \neq \emptyset$. We assume that for every $j = 1, 2$ and $i \in [0, 1]$ g_i^j weakly decreases in its first argument and strictly increases in the second, and that $g_i^j(N, 0) = 0 \; \forall \; N \in [0, 1]$. Furthermore, we assume that $g_i^j(N, 1) = g_i^j(N', 1)$ $\forall \; N, N' \in [0, 1]$ and $j = 1, 2$.

The assumption that consumers care positively about the number of consumers on the other side of the market is the main feature of two-sided markets with network externalities. The assumption that they care negatively about the number of consumers from the same side of the market applies to a wide variety of situations in which consumers on the same side are competitors of each other. For example if the networks are matchmaking services, then people on the same side of the market compete with each other for matches with people on the other side. Therefore, everything else being equal, when joining a network a consumer has higher expected utility the less people from her side join the same network. Despite it is hard to come up with examples in which participants on both sides of the market do not care about the number of consumers from the same side, most of the existing literature makes this assumption, presumably for analytical purposes. Our specification allows for caring negatively about the number of consumers from the same side, unless all consumers from the other side are present on the same network. If the latter case we assume that consumers on the same side are no longer competitors.[6]

For every $j = 1, 2$ and $i \in [0, 1]$ let $u_i^j = g_i^j(0, 1)$.[7] We call u_i^j the reservation value of consumer i on side j.

If $g_i^1 = g_{i'}^1$ and $g_i^2 = g_{i'}^2$ $\forall\ i, i' \in [0, 1]$, then we say that consumers on both sides are homogeneous.

A special case of the above specification, that received highlighted attention in the existing literature, is when for every $j = 1, 2$ and $i \in [0, 1]$ it holds that $g_i^j(N_{c_i(s)}^j, N_{c_i(s)}^{-j}) = u_i^j N_{c_i(s)}^{-j}$. This assumes both a linear functional form and that there is no competition among consumers on the same side.

4 Coalitional rationalizability and coalition perfect equilibrium

The central assumption of our paper is that at every stage of the game players can coordinate their actions whenever it is unambiguously in their interest and it does not require communication. The formal concept we use is coalitional rationalizability (Ambrus[02] and [03]).

Coalitional rationalizability is a reasoning procedure that iteratively eliminates certain strategies from the strategy space. At each step of this elimination procedure groups of players (coalitions) consider implicit agreements

[6] A scenario involving matchmaking services which validates this assumption is if there are ideal matches in the population (there is an ideal match for every person on the other side of the market, whose ideal match is exactly this person). In this case if someone's ideal match is present on the same network, she can be sure of being matched with her perfect match. Therefore once every consumer is present from one side on a network, consumers on the other side of the network are not competitors of each other. On the other hand if someone's perfect match is not present on the same network, which happens with positive probability if not everyone from the other side is present, she does care about how much competition she faces from the same side of the market.

[7] which by the assumption above is equal to $g_i^j(x, 1)$ for every $x \in [0, 1]$

on not playing certain strategies, or alternatively on restricting their play to a certain subset of the original strategy space. A restriction is called supported if it is in the interest of every player in the coalition to make it, in the sense that it increases the player's expected payoff. The concept assumes that every supported restriction by every coalition is made at every step of the procedure. This defines an iterative procedure of eliminating strategies. Strategies surviving this procedure are called coalitionally rationalizable. For the formal construction see Ambrus[02].

In our analysis of network market games we restrict attention to pure strategy subgame perfect equilibria in which players play coalitionally rationalizable strategies in every subgame. We call these outcomes coalition perfect equilibria.[8]

Definition: $s \in S$ is a coalition perfect equilibrium if it is a subgame perfect Nash equilibrium and in every subgame every player plays some coalitionally rationalizable strategy.

Intuitively, coalition perfect equilibrium requires that supported restrictions can be made not only at the beginning of the game, but after any publicly observed history, and that players at any stage of the game foresee restrictions that are made at later stages.

The sequential structure of the game implies that there are no supported restrictions in the game that involve both firms and consumers. By the time consumers move, firms already made their choices and those choices were observed by the consumers. And since consumers cannot commit themselves to make choices that are not in their interest, there are no credible implicit agreements between firms and consumers. A consequence of this is that in games with only one firm coalition perfect equilibria is equivalent to Nash equilibria in which consumers play some coalitionally rationalizable Nash equilibrium in every consumer subgame.

[8] It is possible to show that in the games that we consider the requirement that players play coalitionally rationalizable strategies in every subgame is outcome equivalent to the concept of extensive form coalitional rationalizability (see Ambrus[03]) and therefore coalition perfect equilibria are equivalent to subgame perfect equilibria that are extensive form coalitionally rationalizable.

Theorem 1 *If $G \in \Gamma^M$ then $s \in S$ is a coalition perfect equilibrium iff s is a Nash equilibrium and s_{-A} specifies a coalitionally rationalizable Nash equilibrium in every $G^c \in \Gamma^M(G)$.*

In games with two firms we could not establish a general equivalence result like this, although we did not find examples in which coalitional rationalizability puts extra restrictions on firms' action choices. It is possible to show that if consumers on both sides of the market are homogeneous, then Theorem 1 holds for games with two firms as well. We suspect that the result holds more generally though. The two firms are competitors and they only move once in the game, therefore their possibilities to make credible and mutually advantageous agreements are very limited in our model.

Next we establish the existence of pure strategy coalition perfect equilibria in games with one firm. The sufficient condition that we provide is that the consumers' utility functions are differentiable with respect to the measure of other consumers on the same network and there is a uniform upper bound on these derivatives. In games with two firms existence in pure strategies is not guaranteed. In all concrete examples and restricted settings considered in this paper, like the case of homogeneous consumers, there exists a pure strategy coalition perfect equilibrium. Furthermore, it is possible to show that every game with two firms has a coalition perfect equilibrium in mixed strategies in which every consumer plays a pure strategy (only firms mix). We only restrict attention to pure strategy equilibria to keep the analysis tractable.

Theorem 2 *Let $G \in \Gamma^M$. Let g_i^j be differentiable for every $j = 1, 2$, $i \in [0, 1]$ and assume $\exists\ U > 0$ such that $\frac{\partial g_i^j(N^{-j}, N^j)}{\partial N^{-j}} \leq U$ and $-U \leq \frac{\partial g_i^j(N^{-j}, N^j)}{\partial N^j}$ $\forall\ j = 1, 2$, $i \in [0, 1]$, $N^{-j}, N^j \in [0, 1]$ (there is a uniform bound on the partial derivatives of the utility functions). Then $\exists\ s \in S$ such that s is a coalition perfect equilibrium of G.*

Our interpretation for coalition perfect equilibrium is that coalitional rationalizability puts restrictions on players beliefs concerning other players' choices in certain subgames, and we look at subgame perfect equilibria that are compatible with these restrictions. Another interpretation would be that players use coalitional rationalizability to simplify the game, by ruling out the possibility of certain strategies being played, and we investigate the subgame perfect Nash equilibria of the simplified game.

To provide some intuition on how the concept works, we provide a formal definition of supported restriction in consumer subgames, and three concrete examples of it.

Let $G^c = (C, S^c, u^c) \in \Gamma^c$. Let $A = \underset{j=1,2}{\times} \underset{i=[0,1]}{\times} A_i^j \subset S^c$ and $B = \underset{j=1,2}{\times} \underset{i=[0,1]}{\times} B_i^j \subset A$. Let $J \subset C$. For every $j = 1,2$ and $i \in [0,1]$ let $\Omega_{-j,i}$ be the set of probability distributions over $S^c_{-j,i}$. For every $A' \subset S^c$ let $\Omega_{-j,i}(A')$ $= \{\omega_{-j,i} : \int\limits_{a_{-j,i} \in A'_{-j,i}} \omega_{-j,i}(a_{-j,i}) = 1\}$.

Definition: B is a supported restriction by J given A if$(S^c)_i^j$

1) $B_i^j = A_i^j$, $\forall\ C_i^j \notin J$, and

2) $\forall\ C_i^j \in J$, and $\omega_{-j,i} \in \Omega_{-j,i}(A)$ for which $\exists\ s_i^j \in A_i^j / B_i^j$ such that $s_i^j \in BR_j(\omega_{-j,i})$ it is the case that

$u_i^j(s_i^j, \omega_{-j,i}) < \max\limits_{t_i^j \in (S^c)_i^j} u_i^j(t_i^j, \tau_{-j,i})\ \forall\ \tau_{-j,i}$ such that $\tau_{-j,i} \in \Omega_{-j,i}(A)$ and $\tau_{C/J} = \omega_{C/J}$.

The first condition above requires that only the strategies of those consumers who are members of the given coalition are restricted. The second condition requires that for any player in the coalition it holds that no matter what her beliefs are concerning the choices of consumers outside the coalition, her expected payoff is always strictly higher if the restriction is made (if every player in the coalition only chooses from among strategies in B) than if the restriction is not made and it is rational for her to play some strategy outside the restriction. To put it in simpler terms, it is required that for every player in the coalition making the restriction is strictly preferred to playing some strategy outside the restriction.

For the first example suppose that $G \in \Gamma^D$ and that for every $j = 1,2$ and $i \in [0,1]$ it holds that $g_i^j(N_{c(i)}^{-j}, N_{c(i)}^j) = uN_{c(i)}^{-j}$, where $u > 0$ (consumers have linear utility functions and there is no conflict of interest among consumers on the same side). Consider the consumer subgame that follows price announcements $p_k^j = 0\ \forall\ j = 1,2$ and $k = A, B$. Then the restriction to join either A's network or B's network (or agreeing upon not to stay out of the market) is a supported restriction for the coalition of all consumers, because if the restriction is made, then any possible conjecture that is compatible with it is such that a best response to it yields expected payoff of at least $u/2$ (the conjecture should allocate an expected size of at least $1/2$ to one of the two networks), while staying out of the market yields zero payoff. Because prices charged by the two networks are the same, no more strategies are eliminated by coalitional rationalizability in this subgame. Both joining A's network and joining B's network are coalitionally rationalizable for every consumer and therefore this subgame has three coalitionally rationalizable Nash equilibria. Either every consumer joins A or every consumer joins B, or one half of the consumers on both sides joins each network.

Next, consider the subgame in the above game that follows price announcements $p_A^j = u/4$ and $p_A^j = u/2 \ \forall \ j = 1, 2$. In this subgame A is a supported restriction for the coalition of all players, since it yields payoff $3u/4$ to all consumers, while joining B's network can yield a payoff of at most $u/2$ and staying out yields 0. Coalitional rationalizability pins down a unique strategy profile in this subgame.

In the examples above consumers on the same side of the market were homogeneous. If consumers are heterogeneous, then the set of coalitionally rationalizable outcomes in a subgame might only be reached after multiple rounds of agreements. The next example is the simplest possible demonstration of this. Consider $G \in \Gamma^D$ in which for every $j = 1, 2$ and $i \in [0, 1]$ it holds that $g_i^j(N_{c(i)}^{-j}, N_{c(i)}^j) = u_i^j N_{c(i)}^{-j}$ and that $u_i^1 = 1$ $i \in [0, 1/2]$, $u_i^1 = 1/2$ $i \in (1/2, 1]$ and $u_i^2 = 1$ $i \in [0, 1]$. In words, consumers on side 2 are homogeneous, while half of the consumers on side 1 have relatively low reservation values. Consider the subgame following price announcements $p_A^1 = .4, p_A^2 = .8$ and $p_2^1 = .8, p_2^2 = .4$. Initially, there is no supported restriction for the coalition of all consumers. Consumers would prefer to coordinate their network choices, but coordinating on A is better for side 1 consumers (since the registration fee is lower), while coordinating on B is better for side 2 consumers. However, note that joining B is not rationalizable for any consumer C_i^1 for $i \in (1/2, 1]$ and therefore not joining B is a supported restriction for these consumers. Given this, the maximum utility that other consumers can get by joining B is smaller than the utility they get if all consumers join A. Once it is established that players C_i^1 for $i \in (1/2, 1]$ only consider strategies $\emptyset$ or A, it is a supported restriction for the coalition of all consumers to join A. Therefore coalitional rationalizability pins down a unique outcome in this subgame as well.

We conclude by noting that the solution concept that we use is not equivalent to coalition-proof Nash equilibrium (Bernheim, Peleg and Whinston (1987)) or Pareto efficiency in the consumer subgames, even if there is no conflict of interest among consumers on the same side. In particular, consider again the first example, with the 0 price announcements. As shown, the profile in which half of the consumers on both sides join A's network and the other half of them join B's is a coalitionally rationalizable Nash equilibrium of the subgame. It is not a coalition proof Nash equilibrium though and it is not Pareto efficient. For example, the players who join A in this proposed profile could jointly deviate to B and be all better off, without any subgroup of them wanting to deviate further. In fact it would be a Pareto improvement, since players already on B's network would be better off as well. Our point though is that players who join B in the proposed profile have the same incentives to switch and without explicit coordination it is not obvious whether a player should switch or stay. Coalition-proof Nash equilibrium takes the position that players can successfully coordinate even in those subgames where explicit coordination is needed, while we only require it in cases in which explicit coordination is not necessary, keeping in mind that most applications of our model are such that there is a large number of consumers who do not regularly interact with each other. In subgames in which there is a unique Pareto efficient outcome for the consumers coalition perfect equilibrium implies that the Pareto efficient outcome is played. But in subgames with multiple Pareto efficient outcomes it is consistent with coalition perfect equilibrium that a Pareto inefficient outcome is played. We emphasize that these typically include not only subgames in which two networks charge exactly the same prices, but a much larger set of subgames in which one network is cheaper on one side of the market, while the other is cheaper on the other side.

In the above examples we examined properties of coalitionally rationalizable Nash equilibria in particular consumer subgames. Therefore we only demonstrated that consumers' network choices might be different in some coalition perfect equilibria than in coalition proof Nash equilibria or in Pareto undominated Nash equilibria. For an example showing that in addition to this the range of equilibrium prices and profits can be different in coalition perfect equilibria, see Section 8.

5 Monopolist with one network

In this subsection we consider games in $\Gamma^M(1)$. There is a monopolist network provider A, who can only establish one network.

On a market without network externalities, subgame perfection, which is implicitly assumed when the demand function of consumers is derived, guarantees that a monopolist can achieve the maximum profit compatible with Nash equilibrium. To give a simple example, if all consumer have a reservation value $u > 0$ for some indivisible good and the firm charges a price strictly below u, then subgame perfection implies that all consumers buy the good. Then in any subgame perfect Nash equilibrium (from now on: SPNE) the firm gets a profit of u times the number of consumers.

On markets with network externalities the above result does not hold. Typically there are many different equilibria of the pricing game with one network provider. The consumers face a coordination problem when deciding on whether to join a network or not and they can have multiple self-fulfilling expectations after a given price announcement[9]. This results *in* a wide range of subgame perfect equilibrium prices, consumer participation measures and profit levels[10].

In sharp contrast, Theorem 1 below establishes that if there is no conflict of interest among consumers on the same side of the market, then in every coalition perfect equilibrium the monopolist gets the maximum profit compatible with Nash equilibrium. The intuition behind this result is the following. In any Nash equilibrium consumers who join the network have to get nonnegative utility. Then the set of consumers who join the network in some Nash equilibrium get strictly positive utility if prices are strictly

[9] The multiplicity is somewhat less severe than in the case of one-sided markets with network externalities, where typically there are always equilibria in which all consumers stay out of the market. A monopolist network provider is guaranteed to have positive profit in SPNE. The intuition is that the firm now has the option of preventing the most pessimistic expectations that would keep consumers out of the market by charging a slightly negative price on one side of the market, which guarantees that all consumers of that side join the network. This makes it possible to charge a positive price on the other side of the market and still guaranteed to have somel consumers joining the network on that side.

[10] For a complete characterization of the SPNE for the case where the two sides are symmetric see an earlier version of this paper on the authors' websites.

smaller than the above equilibrium prices and they all join the network (independently whether others join as well or not). Therefore it is a supported restriction for these consumers to join the network at any price smaller than the equilibrium price. Also note that if in the above equilibrium the price on one side is negative, then all consumers from that side have to join the network in the equilibrium, therefore charging slightly smaller prices cannot increase the number of consumers that have to be subsidized. All this establishes that the firm is guaranteed to get a profit that is arbitrarily close to the above equilibrium profit, which implies that there is no coalition perfect equilibrium in which the firm's profit is strictly below this level.

The proofs of all theorems that are stated in the main section of the paper are in the Appendix.

Theorem 3 *Assume there is no conflict of interest among consumers on the same side. Let $s \in S$ be any Nash equilibrium profile and let $s' \in S$ be any coalition perfect equilibrium profile. Then $\pi_A(s') \geqslant \pi_A(s)$.*

The next theorem establishes a similar result for the case when there is potential conflict of interest among consumers on the same side, but they have equal reservation values. Note that in this case the Nash equilibrium with the highest profit implies that on each side the monopolist charges a price equal to the reservation value at that side, and that all consumers join the network. Since coalitional rationalizability guarantees that for any pair of prices that is below the pair of reservation values every consumer joins the network, the firm can extract the whole aggregate consumer surplus. The assumption that consumers can implicitly coordinate their choices in this case ultimately hurts them because the firm can extract all the potential consumer surplus on the market. SPNE requires that it is not credible that after an off equilibrium price announcement consumers do not choose individually optimal strategies. On top of that, coalition perfect equilibrium implies that it is not credible that after an off equilibrium price announcement consumers play strategies that are not coalitionally rationalizable which, as in the case of homogeneous consumers, can adversely effect their payoffs in equilibrium.

Theorem 4 *Assume* $u_i^k = u^k \ \forall \ k = 1, 2$ *and* $i \in [0, 1]$ *and let* s *be a coalition perfect equilibrium. Then* $p_A^k(s) = u^k$ *and* $N_A^k(s) = 1 \ \forall \ k = 1, 2$.

A corollary of the previous theorem is that in the case of equal reservation values for consumers on the same side it is never in the interest of the firm to establish more than one network. Theorem 2 shows that by providing one network the firm can extract the maximum possible gross consumer surplus on the market. If there are two active networks then gross consumer surplus is smaller than in the above case, and since no consumer can get negative utility in any Nash equilibrium, the profit of the firm is strictly smaller than in the one network case.

We conclude this subsection by providing a necessary condition for a network size (the aggregate number of consumers joining the network on side 1 and 2) to be part of a coalition perfect equilibrium if consumers have linear utility functions and the consumers' reservation values are distributed according to a continuously differentiable c.d.f. The condition reveals the trade-off that the monopolist network provider faces when determining how many consumers to target on each side of the market.

Theorem 5 *Assume* $g_i^j(N^j, N^{-j}) = u_i^j N^{-j} \ \forall \ j = 1, 2$ *and* $i \in [0, 1]$. *Assume that for* $j = 1, 2$ *it holds that* $F^j(u)$ *is continuously differentiable and that it is strictly increasing for* u *such that* $F^j(u) \in (0, 1)$. *Then for any* $x^1, x^2 \in (0, 1)$ *it holds that* $\exists \ s \in S$ *such that* s *is a coalition perfect equilibrium,* $N_A^1(s) = x^1$ *and* $N_A^2(s) = x^2$ *only if the following conditions hold:*

$$(1) \qquad F_1^{-1}(1 - x_1) + F_2^{-1}(1 - x_2) - f_1^{-1}(1 - x_1)x_1 = 0$$

and

$$(2) \qquad F_2^{-1}(1 - x_2) + F_1^{-1}(1 - x_1) - f_2^{-1}(1 - x_2)x_2 = 0$$

The first equality corresponds to x_1 being the optimal network size on side 1, given that the size of the network on side 2 is x_2. The left hand side of the equality shows that increasing x_1 has a threefold effect on the profit of the firm. The first term corresponds to the increase in revenue on side 1 that comes from getting registration fee from more people on that side. The second term corresponds to the increase in the revenue on side 2 that comes from the fact that an increase in the number of people on side 1 makes it possible to increase price on side 2 and have the same number of people joining the network (the network externality effect). And the third term corresponds to the decrease in revenue on side 1 coming from the fact that the registration fee has to be lowered in order to attract more people on side 1 to the network. The other equality is symmetric and corresponds to x_2 being optimal given x_1.

Combining the two equalities in the above necessary condition yields:

$$\frac{x_1}{x_2} = \frac{f_1(1-x_1)}{f_2(1-x_2)}$$

This condition corresponds to the equilibrium requirement that a unit increase in the size of the network on side 1 has to have the same effect on profits as a unit increase in the size on side 2.

6 Monopolist who can operate multiple networks

This section investigates a monopolist network provider's decision on how many networks to operate and how to price them. Since the analysis of the model with more than two networks and heterogeneous consumers is involved, in this paper we restrict attention to the case when the monopolist can operate at most two networks.

First we show in a simple setting that it can indeed be better for the monopolist to operate two networks, even if there is no conflict of interest among consumers on the same side (so that it would be socially optimal to have all consumers on the same network). The example also demonstrates that on network markets the monopolist can effectively price discriminate among consumers through registration fees, even without product differentiation. Next we show that if the monopolist creates two networks, multiple coalition perfect equilibria might exist, with different network price choices and profit levels. Third, we provide a necessary condition on the distribution of consumer types for the monopolist to operate two networks in equilibrium. Finally, we provide a necessary condition for a pair of network sizes to be part of the most profitable coalition perfect equilibrium for a particular class of games with a continuum of consumer types.

In this subsection we consider games in $\Gamma^M(2)$. There is a monopolist network provider A, who can choose to operate either one or two networks.

6.1 Two types of consumers on each side

Here we examine the monopolist's decision on the number of networks in the following simple context. The two sides of the market are symmetric, there are only two types of consumers on each side, there is no competition among consumers on the same side and consumers have linear utility functions. Our main goal is to illustrate that it might be optimal for the monopolist to operate multiple networks to price differentiate among different types of consumers, and to develop some intuition on when that might be the case and how the monopolist can achieve price discrimination.

Assume that $g_i^k(N_{c(k,i)}^k, N_{c(k,i)}^{-k}) = u_i^k \cdot N_{c(k,i)}^{-k}$ $\forall$ $k = 1,2$ and $i \in [0,1]$. Also assume that for every $k \in \{1,2\}$ $u_i^k = h$ for $i \in [0,a]$ and $u_i^k = l$ for $i \in (a,1]$, where $l < h$. For every $k \in \{1,2\}$ let $H^k = \underset{i\in[0,a]}{\cup} C_i^k$ and $L^k = \underset{i\in(a,1]}{\cup} C_i^k$.

In words, there are two types of consumers on both sides. In each side a fraction a of the consumers have a reservation value h, which is higher than l, the reservation value of the rest of the consumers. For ease of exposition we refer to consumers with reservation value h as high types, and consumers with reservation value l as low types[11].

As a first step we characterize the set of coalition perfect equilibria when the firm can only establish one network. Note that by Theorem 2 a coalition perfect equilibrium exists for all parameter values a, h and l. The coalition perfect equilibrium is almost always unique, but depends on the values of parameters a, h and l. If l is relatively low and a is high, then the monopolist targets only the high type consumers and charges a high price on both sides.

[11] note that high and low type only refers to the reservation value of consumers and not to their quality in terms of how desireable a consumer's presence is on the network for consumers on the other side. In our model all consumers are ex ante identical in terms of this external effect.

If l is relatively high and a is small, then the monopolist targets all consumers and charges a low price on both sides. In cases in between the monopolist might target all consumers on one side and only the high types on the other side, by charging a low and a high price. These results are both intuitive and in accordance with classic results from the literature on multi product pricing with heterogeneous consumers.

Define the following cutoff points:

$$t_1 \equiv 2a - 1 \tag{1}$$
$$t_2 \equiv \frac{a}{2-a} \tag{2}$$

Notice that if $a \geq \frac{1}{2}$ then $0 \leq t_1 \leq t_2 \leq \frac{1}{2}$, while if $a \leq \frac{1}{2}$ then $t_1 \leq 0 \leq t_2 \leq \frac{1}{2}$. Also, notice that both t_1 and t_2 are strictly increasing in a.

Theorem 6 *If $s \in S$ is a coalition perfect equilibrium, then*:

Theorem 1 *1.*

2. *(targeting all consumers)*
   If $\frac{l}{h} < \max\{0, t_1\}$, then $p^1(s) = p^2(s) = ah$, $c_i^1(s) = c_i^2(s) = A$ $\forall$ $i \in [0,a]$ and $c_i^1(s) = c_i^2(s) = \emptyset$ $\forall$ $i \in (a,1]$

3. *(targeting all consumers on one side, only high types on the other)*
   If $\frac{l}{h} \in (\max\{0,t_1\}, t_2)$ then either $p^1(s) = h$, $p^2(s) = al$, $c_i^1(s) = A$ $\forall$ $i \in [0,a]$, $c_i^1(s) = A$ $\forall$ $i \in (a,1]$ and $c_i^2(s) = A$ $\forall$ $i \in [0,1]$ or $p^1(s) = al$, $p^2(s) = h$, $c_i^1(s) = A$ $\forall$ $i \in [0,1]$, $c_i^1(s) = A$ $\forall$ $i \in [0,a]$ and $c_i^1(s) = \emptyset$ $\forall$ $i \in (a,1]$

4. *(targeting only high types)*
   If $\frac{l}{h} \in (t_2, 1)$ then $p^1(s) = p^2(s) = l$, $c_i^1(s) = c_i^2(s) = A$ $\forall i \in [0,1]$

   Finally, if $\frac{l}{h} = t_1$ both strategy profiles of type 1 and type 2 are coalition perfect equilibria and similarly if $\frac{l}{h} = t_2$ then both strategy profiles of type 2 and 3 are coalition perfect equilibria.

Note that if $a < 1/2$, then there is no coalition perfect equilibrium in which the monopolist charges a high price on both sides of the market, targeting only high types. Charging a high price on one side of the market has to be accompanied by charging a low price on the other side. The reason is that there have to be enough consumers on the other side on the network for high types on the first side to be willing to pay the high price. The monopolist therefore cannot extract a high level of consumer surplus from both sides of the market simultaneously.

Consider now $G \in \Gamma^{M}(2)$, the case when the monopolist can decide to operate two networks. The next theorem shows that for some combinations of the parameters in every coalition perfect equilibrium the monopolist chooses to operate two networks and high and low type consumers on the same side of the market choose different networks.

Define the following cutoff points:

$$z_1 \equiv 4a - 1 \tag{3}$$

$$z_2 \equiv \frac{a(1-2a)}{1-a} \tag{4}$$

Notice that if $a \in \left[0, 1 - \frac{\sqrt{2}}{2}\right]$ then $t_1 \leq z_1 \leq t_2 \leq z_2$. and both z_1 and z_2 are strictly increasing in a.

Theorem 7 *If $s \in S$ is a coalition perfect equilibrium, $a \in \left(0, 1 - \frac{\sqrt{2}}{2}\right)$ and $\frac{l}{h} \in (\max\{0, z_1\}, z_2)$ then $n(s_A) = 2$. Furthermore, $p_A^1(s_A) = q_A^2(s_A) = h(1-2a) + al$, $p_A^2(s_A) = q_A^1(s_A) = al$ and $c_i^1(s) = 1\ \forall\ i \in [0, a]$, $c_l^2(s) = 1\ \forall\ i \in (a, 1]$, $c_i^1(s) = 2\ \forall\ i \in (a, 1]$ and $c_i^2(s) = 2\ \forall\ i \in [0, a]$*

Note that this range of parameter values cuts into the region in which a monopolist with one network would target all consumers on both sides and into the region in which it targets only high type consumers on one side and all consumers on the other.

The reason that operating two networks can increase the monopolist's profits for some parameter values is the following. By establishing two networks and pricing them differently the monopolist implements a form of second degree price discrimination. In particular, if the proportion of high types is sufficiently low, then the monopolist can separate the low types and the high types on each side even if he cannot observe their reservation values, by charging a high price on side 1 and a low price on side 2 in one network, and doing the opposite on the other network[12]. An appropriate choice of prices results in low type consumers choosing networks that are relatively cheap for them, while high type consumers choosing the ones that are relatively expensive for them. In equilibrium the two networks, despite being physically equivalent, end up being of different quality. In our framework the quality of a network for a consumer is determined by how many consumers join the network on the other side of the market. If the majority of consumers on each side of the market are low types, then when all low type consumers on side 1 join one network, that network becomes higher quality for side 2 consumers. Similarly when all low type consumers on side 2 join one network, that network becomes higher quality for side 1 consumers. Since in the above equilibria the low type consumers join different networks, one network ends up being high quality for side 1 consumers, while the other one for side 2 consumers. High type consumers have a higher willingness to pay for quality and therefore are willing to join the networks that are more expensive for them.

The result that in equilibrium the monopolist separates consumers on the same side by offering them two products that have different prices and qualities is standard in the adverse selection literature[13]. What is special to this model is that the two networks are ex-ante identical and product differentiation is endogenous. The quality of a network is determined in equilibrium by the network choices of the consumers, which are driven by the prices of the networks.

The reason the monopolist might be better off by the price discrimination is that it can extract a large consumer surplus from high type consumers simultaneously on both sides of the market, something that it cannot achieve by operating only one network (see Figure 1 for an illustration).

[12] note that coalitional rationalizability implies that in order to have two active networks in equilibrium they either have to have exactly the same prices, or one network has to be relatively cheaper on one side, while the other one on the other side. Therefore there cannot be coalition perfect equilibria with two networks with one being large and expensive on both sides, while the other small and cheap on both sides.

[13] See Mussa-Rosen (1978) and Maskin-Riley (1984).

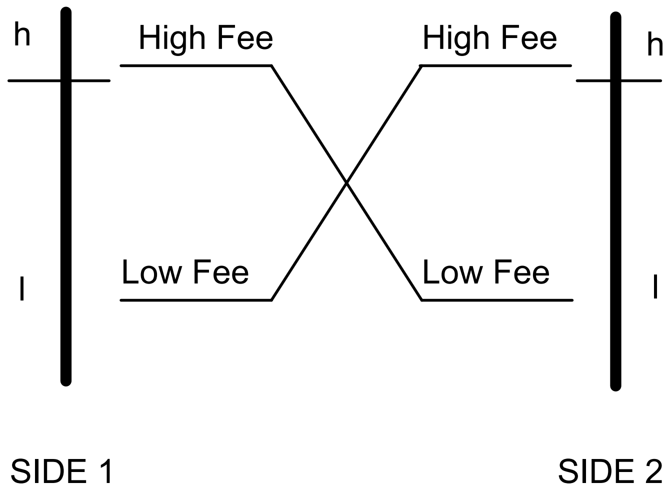

Figure 1

Notice that in the above equilibrium the firm sacrifices some gross consumer surplus in order to be able to extract a high share of the surplus from consumers with high reservation values on both sides of the market.

Since there is no conflict of interest among consumers on the same side of the market and therefore it is always socially efficient if all participating consumers are on the same network.

Despite this, the aggregate social welfare in the situation in which the monopolist is not allowed to operate multiple networks can be both higher or lower than in the situation in which it can only operate one. If $\frac{l}{h} \in (t_2, z_2)$ then a monopolist operating only one network charges prices (l, l) and all consumers join the network. This generates a higher aggregate surplus than if the monopolist can operate two networks, because the same set of consumers participate in the market both cases, but more surplus is generated if they all join same network. As far as consumer surplus is concerned, high types are better off if the monopolist can only run one network and low types are indifferent (they get zero utility in both cases). On the other hand if $\frac{l}{h} \in (z_1, t_2)$, then being restricted to operate one network the monopolist sets a price of h on one side and la on the other. Only high types join the network on the first side and all consumers on the other side. In this case high type consumers are better off if the monopolist can operate two networks and low types are again indifferent. Furthermore, it is straightforward to establish that aggregate social surplus is higher in the case of two networks. In case of two networks aggregate surplus is $2(ah\,(1-2a)+al)+2a^2\,(h-l)$, while in case of one network it is $ah+al+a^2\,(h-l)$. The difference of the two surpluses is $ah+al-3a^2h-a^2l$, which is positive given that $\frac{l}{h}<1$ and $\frac{l}{h}\geq z_1=4a-1$ implies $a<\frac{1}{4}$.

Finally, note that equilibrium prices and quantities have to satisfy the "incentive compatibility constraints" that a high type consumer should prefer the more expensive network, while a low type consumer should prefer the cheaper network. Furthermore, since staying out of the market is an option to every consumer, consumers have to get nonnegative utility in equilibrium - a "participation constraint". One feature of the above result, which is consistent with the literature on adverse selection, is that the incentive compatibility constraints for the high types and the participation constraints for the low types are binding in equilibrium.

6.2 More heterogeneity

Theorems 6 and 7 in the previous subsection imply that if there are two types of consumers on each side of the market, then for almost all parameter values the number of networks and the prices are uniquely determined in coalition perfect equilibrium. Furthermore, for *all* parameter values the profit of the firm is uniquely pinned down. The next example shows that these results do not generalize for the case of more than two consumer types. If there is a coalition perfect equilibrium in which the monopolist establishes two networks, then there might be a range of equilibrium prices and profit levels in coalition perfect equilibrium. Therefore in these cases coalitional rationalizability does not guarantee that the monopolist can get the highest possible profit compatible with Nash equilibrium.

Consider $G\in\Gamma^{M2}$ such that $g_i^j(N_{c_i(s)}^j,N_{c_i(s)}^{-j})=u_i^jN_{c_i(s)}^{-j}\ \forall\ j=1,2$ and $i\in[0,1]$. For every $j=1,2$ and $i\in[0,0.16]$ let $u_i^j=1$, for every $j=1,2$ and $i\in(0.16,0.2]$ let $u_i^j=0.82$, and for every $j=1,2$ and $i\in(0.2,1]$ let $u_i^j=0.02$.

In words, consumers have linear utility functions, with no conflict of interest among consumers on the same side. On both sides 0.16 of consumers with reservation value 1 (we refer to them as type H) 0.04 of consumers have reservation value 0.82 (type M) and 0.8 of consumers with reservation value 0.02 (type L).

Let S^* denote the set of coalition perfect equilibria of G.

Claim 1 establishes that in coalition perfect equilibrium the monopolist always establishes two networks. There is a range of coalition perfect equilibria in which H type consumers join the network that is more expensive on their side of the market, while M and L type consumers join the network that is cheaper on their side. Also there is an equilibrium in which one network attracts H and M type consumers on one side and M and L type consumers on the other, while the second network attracts L type consumers on the first side and H type consumers on the other. The range of profits in coalition perfect equilibria is $[0.214\,72, 0.224]$.

Claim 1 *If $s \in S^*$, then $n(s) = 2$. For every $q_1^1, q_1^2, q_2^1, q_2^2$ such that $\exists$ $k = 1, 2$ for which $q_k^{3-k}, q_{3-k}^k \le 0.0032$, $q_{3-k}^k + 0.557\,6 \le q_k^k(s) \le q_{3-k}^k + 0.68$, $q_k^{3-k} + 0.557\,6 \le q_{3-k}^{3-k}(s) \le q_k^{3-k} + 0.68$ and $0.84(q_k^{3-k} + q_{3-k}^k) + 0.16(q_k^k(s) + q_{3-k}^{3-k}(s)) \ge 0.214\,72$ $\exists$ $s \in S^*$ such that*

(1) $p_i^j(s) = q_i^j$ $\forall$ $j = 1, 2$ and $i \in [0, 1]$

(2) $c_i^k(s) = k$, $c_i^{3-k}(s) = 3 - k$ $\forall$ $i \in [0, 0.16]$ and $c_i^k(s) = 3 - k$, $q_i^{3-k}(s) = k$ $\forall$ $i \in [0.16, 1]$

Furthermore, for any $k \in \{1, 2\}$ $\exists$ $s \in S^*$ such that $p_k^1(s) = 0.004$, $p_k^2(s) = 0.560\,8$, $p_{3-k}^1(s) = 0.604$, $p_{3-k}^2(s) = 0.0032$, $c_i^1(s) = k$ $\forall$ $i \in [0, 0.2]$, $c_i^1(s) = 3 - k$ $\forall$ $i \in (0.2, 1]$, $c_i^2(s) = 3 - k$ $\forall$ $i \in [0, 0.16]$ and $c_i^2(s) = k$ $\forall$ $i \in (0.16, 1]$.

There is no $s \in S^*$ such that $\pi_A(s) < 0.214\,72$.

The intuition behind the multiplicity result is that in the consumer subgames following the price announcements stated in Claim 1 there are multiple coalitionally rationalizable Nash equilibria. In one of them only H types join the expensive sides of the networks and M and L type consumers join the cheap sides. This gives a higher profit to the monopolist than 0.214 72 and it is compatible with coalition perfect equilibrium. But there are also coalitionally rationalizable Nash equilibria of these subgames in which all consumers on one side and H and M type consumers on the other join the same network (L type consumers on this side either stay out, or join the other network if the registration fee they have to pay for that is nonpositive). In this case the monopolist's profit is no more than the maximum profit attainable by operating only one network. Therefore if consumers' choices are such that in all of these subgames the latter type of equilibrium applies, then the monopolist chooses prices according to the second type of equilibrium in the Claim. In the subgames following these price announcements there is a unique coalitionally rationalizable Nash equilibrium, resulting in a profit of $0.214\,72$. This is then the lower bound of the firm's profits in coalition perfect equilibrium.[14]

For further analysis of this game that assumes a stronger form of consumer coordination see Section 8.

We point out that if there is no conflict of interest among consumers on the same side, then multiplicity of equilibrium profit levels is only possible if there is at least one coalition perfect equilibrium in which the monopolist establishes two networks. By Theorem 3 the monopolist's profit is uniquely determined in coalition perfect equilibrium if he establishes only one network.

6.3 General specification with no conflict of interest on the same side

The question of when it is in the interest for the monopolist to operate multiple networks is difficult to answer in general. Subsection 6.1 gives a precise answer for the special case of two consumer types on both sides of the market. One result in that setting is that the reservation values of high and low type consumers have to be sufficiently different for the monopolist wanting to establish two networks. Here we show that this result can be extended to a more general setting. In particular Theorem 8 establishes that if there is no conflict of interest among consumers on the same side of the market and on both sides of the market the ratio of the highest and lowest consumer reservation value is less than $3+2\sqrt{2}\approx 5.828\,4$, then in any coalition perfect equilibrium there is only one active network.

Theorem 8 *Consider any $G\in\Gamma^{M2}$ in which $g_i^j(N^j,N^{-j})=g_i^j(M^j,N^{-j})$ $\forall\ N^j,N^{-j},M^{-j}\in[0,1]$. If for every $j=1,2$ and $i,i'\in[0,1]$ it holds*

[14] more precisely, in games where prices are arbitrarily close to these equilibrium prices there is a unique coalitionally rationalizable Nash equilibrium.

that $u_i^j / u_{i'}^j < 3 + 2\sqrt{2}$ then for every $s \in S$ such that s is a coalition perfect equilibrium either $n_A(s) = 1$, or $n_A(s) = 2$ and $\exists\ k, j \in \{1, 2\}$ such that $N_k^j(s) = 0$ and $p_k^j(s) = 0$.

The proof of Theorem 8 shows that if the condition of the theorem holds, then establishing one network and charging prices equal to the lowest reservation values is always more profitable than establishing two active networks. Recall that establishing two active networks implies sacrificing some consumer surplus, which limits the price that the monopolist can charge on the networks. Therefore it can only be profitable if there are some consumers on both sides of the market with sufficiently high reservation values relative to the rest of the consumers.

We note that the ratio in Theorem 8 is not tight. We leave it to future research to provide tighter conditions for the monopolist not wanting to establish multiple networks.

7 Duopoly

In this section we consider games in G^D. Just like in the case of one firm operating on the market, typically there are many different types of SPNE, including one in which no consumers participate in the market. There are also equilibria in which firms get positive profits.[15]

We investigate whether the assumption of coalitional rationalizability reestablishes the result on markets without network externalities that firms' profits are zero in Bertrand competition. Furthermore we ask whether there exist coalition perfect equilibria in which asymmetric networks coexist on the market, with different types of consumers choosing different networks. First we address these issues in the special case that all consumers on the same side

[15] The presence of so called "divide and conquer" strategies (introduced by Innes and Sexton[93], and then analyzed in the context of network markets by Jullien[01] and Caillaud and Jullien[01]) restricts the set of SPNE. If a firm charges a sufficiently low (negative) price compared to its rival on one side of the market, it can make its network a dominant choice for consumers on that side. Then it can charge a high price on the other side of the market and still make sure that consumers join its network on that side. Despite this there is typically a severe multiplicity of equilibria. For a complete characterization of the SPNE for the case where the two sides are symmetric and consumers have linear utility functions, see an earlier version of this paper, available on the authors' websites.

of the market have the same reservation values, where we can characterize the set of coalition perfect equilibria. Then we investigate general utility functions, where we provide an example that shows that not all the results of the special case can be generalized, and obtain a partial characterization result.

7.1 Homogeneous consumers

In this subsection we assume that consumers on the same side have the same reservation value: $u_i^k = u_j^k \equiv u^k \; \forall \; k = 1, 2$ and $i, j \in [0, 1]$. A particular case of this is when consumers on the same side are homogeneous: $g_i^k() = g_j^k() \; \forall \; k = 1, 2$ and $i, j \in [0, 1]$.

Theorem 9 establishes that in this case coalition perfect equilibria of the duopoly game have similar properties to subgame perfect equilibria of duopoly pricing games with no network externalities. Namely in every coalition perfect equilibrium both firms' profits are zero, if both of them are active then they charge the same prices and have the same size, and no consumer stays out of the market. The difference is that in this two-sided market environment prices do not have to be zero, despite the assumption that marginal cost is zero on both sides. It is compatible with coalition perfect equilibrium that consumers on one side of the market pay a positive price for joining a network, while consumers on the other side are subsidized to join. In fact Theorem 10 below shows that if the two sides are asymmetric (the reservation value of consumers on side 1 is different than that of consumers on side 2), then in every coalition perfect equilibrium the side with the smaller reservation value gets strictly subsidized, and consumers on the other side pay a strictly positive price.

Theorem 9 *If $s \in S$ is a coalition perfect equilibrium, then either*

1. $\exists \; k \in \{A, B\}$ *such that* $N_k^j(s) = 1 \; \forall \; j = 1, 2$ *and* $i \in [0, 1]$, $p_k^1(s) = -p_k^2(s)$ *and* $p_k^j(s) \leq u^j \; \forall \; j = 1, 2$

or

2. $N_A^j(s) + N_B^j(s) = 1 \; \forall \; j = 1, 2$, $p_A^1(s) = p_B^1(s) = -p_A^2(s) = -p_B^2(s)$ *and* $p_A^j(s) \leq u^j \; \forall \; j = 1, 2$

Furthermore, if $g_i^j(N^j, N^{-j}) = g_i^j(\widehat{N^j}, N^{-j}) \ \forall\ j = 1, 2,\ i \in [0,1]$ *and* $N^{-j}, N^j, \widehat{N^j} \in [0,1]$ *then* $N_k^1(s) + N_k^2(s) > 0$ *for both* $k = 1, 2$ *implies that* $N_k^1(s) = N_k^2(s) = 1/2 \ \forall\ k = 1, 2.$

Theorem 9 states that in any coalition perfect equilibrium either all consumers join the same network, or the two networks charge the same prices. In all equilibria no consumer stays out of the market and all consumers on the same side of the market pay the same price. This market price on one side of the market is just the negative of the price on the other side. Therefore either both prices are zero, or consumers on one side pay a positive price while consumers on the other side receive an equivalent subsidy. In all equilibria both firms get zero profit. Finally, if there is no conflict of interest among consumers on the same side of the market, then if both firms are active not only their prices but also their sizes are the same (on both sides exactly the same number of consumers join A's and B's network).

The intuition behind the zero profit result is that coalitional rationalizability implies that slightly undercutting the competitor's price on both sides of the market results in stealing the whole market. Then if in some profile at least one firm's profit is positive, then at least one firm could profitably deviate by slightly undercutting the other firm's prices. Furthermore, even in profiles in which firms get zero profit but active firms do not charge the same prices at least one firm could profitably deviate by undercutting.

The result of full consumer participation comes from the fact that in all the above equilibria either the market price (the price charged by the active firm(s)) is negative on one side, or the market price is zero on both sides. The first case implies that consumers on the side with the negative market price do not stay out of the market, and then the zero profit result can be used to show that there has to be full consumer participation on the other side as well. The intuition behind the result that every consumer joins some network is that if prices are zero, then it is a supported restriction for the coalition of all consumers to agree upon joining some network. This is because even the most pessimistic expectation compatible with the agreement (namely that other consumers are equally dispersed between the two networks) yields a positive expected payoff, while staying out of the market gives a zero payoff.

Theorem 10 *Let* $s \in S$ *be a coalition perfect equilibrium. If* $u^k < u^{-k}$ *for* $k \in \{1, 2\}$*, then* $N_f^1(s) + N_f^2(s) > 0$ *for some* $f \in \{A, B\}$ *implies that* $p_f^k(s) \in [-u^{-k}, u^k - u^{-k}]$

Theorem 10 states that if consumers on one side have a higher reservation value than consumers on the other side, then in a coalition perfect equilibrium the price charged on the side which has the smaller reservation value has to be in an interval that is strictly below zero. The minimal amount of subsidy that consumers on this side get is the difference between the reservation values. This result is not a consequence of coalitional rationalizability, but comes from the restrictions that "divide and conquer" strategies put on any SPNE[16]. If the market price is positive on the side with the low reservation value, then it is relatively cheap to steal consumers on that side, and then a higher price can be charged on the side with the high reservation value.

Next we characterize the set of coalition perfect equilibria for the case of linear utility functions and no competition among consumers on the same side of the market.

Theorem 11 *Let* $g_i^k(N_{c(k,i)}^k, N_{c(k,i)}^{-k}) = N_{c(k,i)}^{-k} u^k \; \forall \; k = 1,2$ *and* $i, j \in [0,1]$.

-If $u^1 = u^2 \equiv u$*, then two types of coalition perfect equilibria* $s \in S$ *exist:*

1. *(monopoly equilibria with zero profits)* $\exists \; f \in \{A, B\}$ *such that* $c_i^k(s) = f \; \forall \; k = 1,2$ *and* $i \in [0,1]$, $p_f^1(s) = -p_f^2(s)$ *and* $p_f^k(s) \leq u \; \forall \; k = 1,2$
2. *(symmetric equilibria with zero profits)* $N_A^1(s) = N_A^2(s) = N_B^1(s) = N_B^2(s) = 1/2$, $p_A^1(s) = p_B^1(s) = -p_A^2(s) = -p_B^2(s)$ *and* $p_A^k(s) \leq u/2 \; \forall \; k = 1,2$

-If $u^l < u^{-l}$ *for some* $l \in \{1,2\}$ *and* $u^{-l} \leq 2u^l$*, then two types of coalition perfect equilibria* $s \in S$ *exist:*

1. *(monopoly equilibria with zero profits)* $\exists \; f \in \{A, B\}$ *such that* $c_i^k(s) = f \; \forall \; k = 1,2$ *and* $i \in [0,1]$, $p_f^1(s) = -p_f^2(s)$ *and* $p_f^l(s) \in [-u^{-l}, u^l - u^{-l}]$
2. *(symmetric equilibria with zero profits)* $N_A^1(s) = N_A^2(s) = N_B^1(s) = N_B^2(s) = 1/2$, $p_A^1(s) = p_B^1(s) = -p_A^2(s) = -p_B^2(s)$ *and* $p_f^l(s) \in [\frac{-u^{-l}}{2}, u^l - u^{-l}]$

[16] this is why Caillaud and Jullien[01] obtain a similar result for equilibria in which both firms are active

-If $u^l < u^{-l}$ for some $l \in \{1,2\}$ and $u^{-l} > 2u^l$, then only one type of coalition perfect equilibrium $s \in S$ exists:

(monopoly equilibria with zero profits) $\exists\ f \in \{A, B\}$ such that $c_i^k(s) = f$ $\forall\ k = 1, 2$ and $i \in [0, 1]$, $p_f^1(s) = -p_f^2(s)$ and $p_f^l(s) \in [-u^{-l}, u^l - u^{-l}]$.

Theorem 11 establishes that if firms are symmetric, then there exist zero profit equilibria with both one and two active firms, and that either side can be subsidized in equilibrium (or prices can be zero). If the larger reservation value is less than twice the smaller one, it is still true that there are equilibria with both one and two active firms, but the side with the smaller reservation value is always subsidized. If the larger reservation value is more than twice the smaller one, then there are only monopoly equilibria. This is because if the side with the smaller reservation value gets a subsidy that is less than the difference of the reservation values, then there is a "divide and conquer" deviation that is profitable. But if there are two active networks, then the price that can be charged on consumers in the other side is at most half of their reservation value (otherwise they would get negative utility). Then since firms' profits have to be nonnegative in equilibrium, there cannot be coalition perfect equilibria in the above case.

The above characterization result for linear utility functions, no competition among consumers on the same side of the market gives an opportunity for a direct comparison with the predictions of Caillaud and Jullien[01a] in the case of asymmetric sides, since that is the context of their investigation. By assuming monotonicity of the demand function, they obtain the same set of equilibria with two active firms. On the other hand, their refinement selects a larger set of equilibria with one active firm, including equilibria in which the active firm gets positive profits. Furthermore, full participation is an extra assumption in their model, while it is a result in our paper.

7.2 The general case

Example 1 shows that the strong result that we obtained in the previous, namely that firms make zero profits, does not hold in general if consumers are heterogeneous. The example also points out that there can be equilibria in which consumers with different reservation values pay different prices for joining a network.

Example 1 Let $\widehat{G} \in \Gamma^2$ be such that $U_i^j = N_{c(i,k)}^{-j} \cdot u_i^j - p_{c(i,k)}^j \; \forall \; j = 1, 2$ and $i \in [0, 1]$. For $j = 1, 2$ define $T_I^j = \{C_i^k : i \in [0, 0.4]\}$, $T_{II}^j = \{C_i^j : i \in (0.4, 0.55]\}$, $T_{III}^j = \{C_i^j : i \in (0.55, 0.65]\}$ and $T_{IV}^j = \{C_i^j : i \in (0.65, 1]\}$. Let $u_i^j = 2.55 \; \forall \; C_i^j \in T_I^k$, $u_i^j = 0.51 \; \forall \; C_i^j \in T_{II}^j$, $u_i^j = 0.46 \; \forall \; C_i^j \in T_{III}^j$ and $u_i^j = 0.15 \; \forall \; C_i^j \in T_{IV}^j \; \forall \; j = 1, 2$.

In words, consumers have linear utility functions, with no competition among consumers on the same side. On both sides of the market, a mass of consumers with measure 0.4 have reservation value 2.55 ('I' types), a mass of consumers with measure 0.15 have reservation value 0.51 ('II' types), a mass of consumers with measure 0.1 have reservation value 0.46 ('III' types), while a mass of consumers with measure 0.35 have reservation value 0.15 ('IV' types).

Claim 2 establishes that this game has a coalition perfect equilibrium in which one firm charges a price of 0.31 on side 1 and -0.2 on side 2, while the other firm charges -0.2 on side 1 and 0.31 on side 2. All type 'I' consumers on side 1 and type 'II'-'IV' consumers on side 2 join the first firm, while all type 'I' consumers on side 2 and type 'II'-'IV' consumers on side 1 join the second firm. Notice that in this profile both firms get a profit of $0.31 \times 0.4 - 0.2 \times 0.6 = 0.04$, which is strictly positive.

Claim 2 *There exists $s \in S$ such that s is a coalition perfect equilibrium of $\widehat{G}$ and the following hold:*

$$p_A^1(s) = p_B^2(s) = 0.31,\; p_A^2(s) = p_B^1(s) = -0.2,$$
$$c_i^1(s) = A \; \forall \; C_i^1 \in T_I^1,\; c_i^1(s) = B \; \forall \; C_i^1 \in T_{II}^1 \cup T_{III}^1 \cup T_{IV}^1,$$
$$c_i^2(s) = B \; \forall \; C_i^2 \in T_I^2,\; c_i^1(s) = A \; \forall \; C_i^2 \in T_{II}^2 \cup T_{III}^2 \cup T_{IV}^2.$$

The example is especially striking because the utility functions of consumers are such that there is no conflict of interest among consumers on the same side. Furthermore, in the described equilibrium the firms charge different prices, and consumers on the same side of the market with different reservation values end up paying different prices for the market good, despite the fact that reservation prices are private information of the consumers.

Notice that every consumer on both sides of the market joins some network. Type 'I' consumers on both sides of the market pay a registration fee of 0.31 for joining a network, and in equilibrium they face a measure of 0.6

consumers from the other side of the market. All other consumers on both sides of the market are subsidized, they pay a registration fee of -0.2. In the equilibrium they face only a measure of 0.4 consumers from the other side of the market.

This equilibrium structure is similar to the equilibria in the previous section, in which the monopolist achieved price discrimination by operating two networks. In particular, one network is cheaper on one side of the market, while the other one is cheaper on the other side. A larger fraction of consumers, those having relatively low reservation values, join the cheap network sides, which makes it worthwhile for the remaining, high reservation value, consumers to join the expensive network sides. These similarities are consequences of assuming that after every price announcement consumers play some coalitionally rationalizable Nash equilibrium. In the monopoly case it is never in the interest of the firm to establish two networks that are priced equally, since that would just split consumers into two networks, generating less consumer surplus and therefore less profit. In the duopoly case there cannot be a coalition perfect equilibrium with positive profits and equally priced networks, because of the usual Bertrand competition undercutting argument. Coalitional rationalizability ensures that a slight undercutting by a firm is enough to "steal the whole market" and that it increases the firm's profit. Therefore in both cases, for different reasons, the two networks have to be priced differently. But two active firms that are priced differently are only compatible with coalitional rationalizability if neither of them is cheaper on both sides of the market (then consumers would all switch to that network). Then it is the consequence of Nash equilibrium that if both networks are active, then on both sides more consumers join the relatively cheaper network, since only that can make it attractive for some consumers to join the other, expensive side.

The main intuition on why competition does not drive profits down to zero in the above example is that with heterogeneous consumers deviation strategies based on undercutting, which are always effective due to the assumption of coalitional rationalizability, are not necessarily profitable. In particular, since the suggested equilibrium is such that each firm subsidizes one side of the market, undercutting requires that the deviating firm pays a subsidy to a large share of consumers. Consider for example an undercutting deviation by firm B. If B announces prices $(0.31 - \varepsilon, -0.2 - \varepsilon)$ for some small $\varepsilon > 0$, then coalitional rationalizability implies that all consumers who do not stay out of the market join B's network. These consumers will be types 'I'-'III' from side 1, and every consumer from side 2. If ε is close to zero, this gives a profit to B that is close to $0.31 \times 0.65 - 0.2 = .00\,15$. This upper limit is strictly smaller though than the equilibrium profit of B. The reason is that heterogeneity of consumers implies that undercutting increases the number of consumers joining the network by a larger amount on the side where the price is negative.

The same intuition applies to "divide and conquer" type strategies. A firm can lower its price so that it makes it a dominant choice for some type of consumers to join its network and then it can charge a high price on the other side of the market and still make sure that some consumers join its network on that side as well. But if consumers are heterogeneous, then the proportion of consumers who are willing to pay the increased price on the latter side might be too low to compensate for the costs associated with lowering the price (increasing the subsidy) on the first side. In the example above none of these deviations are profitable.

Theorem 12 establishes that the basic features of the above example hold for every equilibrium in which some firm's profit is positive in every game where there is no conflict of interest among consumers on the same side.

Theorem 12 *Let $g_i^j(N^j, N^{-j}) = g_i^j(\widehat{N^j}, N^{-j})$ $\forall$ $j = 1, 2$, $i \in [0, 1]$ and $N^{-j}, N^j, \widehat{N^j} \in [0, 1]$ and let $s \in S$ be a coalition perfect equilibrium such that $\pi_k(s) > 0$ for some $k \in \{A, B\}$. Then (1) $N_k^j(s) > 0$ $\forall$ $j \in \{1, 2\}$ and $k \in \{A, B\}$; (2) $\exists$ $j \in \{1, 2\}$ such that $p_A^j(s) > p_B^j(s)$, $p_A^{3-j}(s) \leq p_B^{3-j}(s)$, $N_A^j(s) \leq N_B^j(s)$ and $N_A^{3-j}(s) > N_B^{3-j}(s)$.*

If in a coalition perfect equilibrium some firm has positive profit then in that equilibrium both firms have to be active and firm A's network has to be (weakly) more expensive and smaller on one side of the market and (weakly) cheaper and larger on the other than firm B's network. Furthermore, the two networks have to be asymmetric in the sense that on at least one side the networks charge different prices and on at least one side a different fraction of consumers choose A's network than B's. The intuition behind these results is that if there is only one active network on the market getting positive profit, then the other firm could improve its profit by slightly undercutting its competitor on both sides of the market (in which case coalitional rationalizability guarantees that it steals the whole market).

The same would be true if the two networks charged the same prices and an equal number of consumers chose A's and B's networks. Finally, coalitional rationalizability excludes the possibility that one firm is more expensive on both sides of the market and still has some consumers joining its network.

8 More effective consumer coordination

Coalition perfect equilibrium corresponds to the assumption that consumers can coordinate their network choices as long as it does not require explicit communication. This section shows that the range of equilibrium prices is not necessarily the same if consumers can coordinate their network choices more effectively (presumably by explicit coordination). More precisely, we show by an example that the set of equilibrium prices can be different in coalition perfect equilibrium and in extensive form coalition-proof Nash equilibrium (Bernheim et al.(1987)).

Consider again game $G \in \Gamma^{M2}$ introduced in Subsection 6.2.

From now on we refer to extensive form coalition-proof Nash equilibrium as coalition-proof equilibrium. Let C^{**} denote the set of coalition-proof equilibria of G.

Recall from Subsection 6.2 that there is a range of prices and profit levels that are compatible with coalition perfect equilibrium. Claim 3 establishes that with respect to network prices and equilibrium network choices there is a unique coalition-proof equilibrium, and it is the coalition perfect equilibrium that gives the lowest possible profit to the firm. In this equilibrium the firm establishes two networks. Type H and M consumers on side 1 and type L consumers on side 2 join one network. Type H and M consumers on side 2 and type L consumers on side 1 join the other one.

Claim 3 *$C^{**} \neq \emptyset$. If $s \in C^{**}$, then*

(i) $n(s) = 2$

(ii) $\exists\ k \in \{1, 2\}$ such that $p_k^1(s) = 0.004$, $p_k^2(s) = 0.560\,8$, $p_{3-k}^1(s) = 0.604$, $p_{3-k}^2(s) = 0.0032$, $c_i^1(s) = k\ \forall\ i \in [0, 0.2]$, $c_i^1(s) = 3 - k\ \forall\ i \in (0.2, 1]$, $c_i^2(s) = 3 - k\ \forall\ i \in [0, 0.16]$ and $c_i^2(s) = k\ \forall\ i \in (0.16, 1]$

To provide an intuition why the coalition perfect equilibria in which only H type consumers join the expensive network sides and both type M and L consumers the cheap sides, consider the coalition perfect equilibria with the maximum possible profit. In this equilibrium the firm charges prices $(0.683\,2, 0.0032)$ on one network and $(0.0032, 0.683\,2,)$ on the other. Type H consumers on side 1 and type M and L consumers on side 2 join the first network, while type H consumers on side 2 and type M and L consumers on side 1 join the second network. In the subgame following the above price announcements this network choice profile is indeed a coalitionally rationalizable Nash equilibrium. But it is not a coalition-proof Nash equilibrium. Given the above profile of network choices, the deviation in which type M consumers on side 2 and type H consumers on side 1 switch to the first network is profitable and credible (there is no further profitable deviation by any subcoalition of the deviators). But note that the deviation in which type M consumers on side 1 and type 2 consumers on side 2 switch to the second network is also profitable and credible. Type H and M consumers have an incentive to coordinate on choosing the same network, but it is not clear which one. Type H consumers on side 1 and type M consumers on side 2 prefer coordinating on network 1, while type M consumers on side 1 and type H consumers on side 2 prefer coordinating on network 2. Since it is not obvious which network these consumers should coordinate on, it is consistent with coalitional rationalizability that type H and type M consumers on the same side choose different networks. Furthermore, it is easy to check that these choices constitute a Nash equilibrium (there are profitable coalitional deviations, but not individual ones). This example highlights the conceptual differences between coalitional rationalizability and coalition-proof Nash equilibrium. In the former groups of players can only coordinate their moves from an ex ante perspective, while in the latter groups of players can contemplate making agreements given a candidate equilibrium profile. This in our view can only correspond to scenarios in which consumers can get together before making their network choices and can explicitly negotiate over them.

Recall from Subsection 6.2 that in the subgames where prices are arbitrarily close to the one following the price announcements in Claim 3 there is a unique coalitionally rationalizable Nash equilibrium. This gives a lower bound on the firm's profit in coalition perfect equilibrium. These coalitionally rationalizable Nash equilibria are also the unique coalition proof Nash equilibria of the corresponding subgames, establishing that the profit of the firm in any coalition proof equilibrium is equal to the minimum profit that the firm can get in coalition perfect equilibrium.

By changing the parameter values of the above game one can provide an example in which in coalition-proof equilibrium the firm establishes only one network, while there are coalition-proof equilibria with two networks and higher profit.

It is also possible to provide examples in which there are two competing firms and the range of prices is different in coalition perfect and coalition-proof equilibria.

In the example presented above the set of coalition-proof equilibria are strictly contained in the set of coalition perfect equilibria. Theorem 10 establishes that this generalizes to all games in which there is no conflict of interest among consumers on the same side of the market, both in the case of one or two network providers.

Theorem 13 *Consider any* $G \in \Gamma$ *in which* $g_i^j(N^j, N^{-j}) = g_i^j(M^j, N^{-j})$ $\forall\ N^j, N^{-j}, M^{-j} \in [0,1]$*. The set of coalition-proof equilibria of any* $G \in \Gamma$ *in which* $g_i^j(N^j, N^{-j}) = g_i^j(N^j, M^{-j})$ $\forall\ N^j, N^{-j}, M^{-j} \in [0,1]$ *is a subset of the set of coalition perfect equilibria of* G*.*

In general though, if consumers' utilities depend negatively on the number of consumers on the same network from the same side of the market, there is no containment relationship between the set of coalition-proof and the set of coalition perfect equilibria.

We conclude this section by pointing out that the set of equilibria in which consumers choose some Pareto efficient equilibrium in every subgame following price announcements (from now on Pareto perfect equilibria) is different from both the set of coalition perfect and coalition-proof equilibria. For example, it is easy to establish that in G there are Pareto perfect equilibria in which the firm establishes only one network. The one which yields the lowest possible profit to the firm corresponds to a divide and conquer strategy. On one side the firm sets price 0, while on the other it sets price 0.82. All consumers join the network on the first side, and type H and M consumers on the other. The firm's profit in this equilibrium is 0.164, strictly smaller than its profit in coalition-proof equilibrium. In general, if consumers are heterogeneous, then the Pareto efficiency condition in subgames is different than the conditions implied by coalition-proofness or coalitional rationalizability.

9 Positive Marginal Cost

In the previous sections we assumed zero marginal cost for providing the network good. In this section we briefly discuss how the results extend to the case where firms face constant but possibly positive marginal costs.

Assume the same model framework as described in Section 3, with the following modification in the firms' payoff functions. Let $m \geq 0$. If $G \in \Gamma^M$ and $n_A(s_A) = 1$ then $\pi_A(s) = \sum\limits_{j=1,2} (p_1^j(s) - m)N_1^j(s)$. If $G \in \Gamma^{M2}$ and $n_A(s_A) = 2$ then $\pi_A(s) = \sum\limits_{k=1,2}\sum\limits_{j=1,2} (p_k^j(s)N_k^j(s) - m)$. If $G \in \Gamma^D$ then $\pi_k(s) = \sum\limits_{j=1,2} (p_k^j(s) - m)N_k^j(s)$ for $k \in \{A, B\}$.

The results of Section 5 (there is a monopolist network provider who can operate only one network) apply to this context with minimal modification. Theorem 3, which states that if there is no conflict of interest among consumers on the same side then the monopolist's profit in every coalition perfect equilibrium is equal to the maximum profit it can get in any Nash equilibrium, still holds. Theorem 4 continues to hold as well, provided that the common reservation values on both sides of the market are larger than the marginal cost. Finally, the necessary conditions provided in Theorem 5 for a network size to be part of a coalition perfect equilibrium are only slightly different:

(1′) $\quad F_1^{-1}(1 - x_1) + F_2^{-1}(1 - x_2) - f_1^{-1}(1 - x_1)x_1 = m$

and

(2′) $\quad F_2^{-1}(1 - x_2) + F_1^{-1}(1 - x_1) - f_2^{-1}(1 - x_2)x_2 = m$

In particular it still holds that $\frac{x_1}{x_2} = \frac{f_1(1-x_1)}{f_2(1-x_2)}$ is a necessary condition for a coalition perfect equilibrium in which not all consumers join the network. The proofs of the above claims, since they are similar to the proofs of the corresponding theorems in Section 5, are omitted.

The qualitative conclusions of Section 6 and 7 hold as well, with one exception. If m is higher than a certain threshold, then coalitional rationalizability does not exclude the possibility of market shutdown (that no consumer joins any network) in equilibrium. To make this point clear, below we state the version of Theorem 11 that holds for the present context, for the case of symmetric sides (consumers' reservation values on the two sides are the same).[17]

Theorem 14 *Assume that there are two network providers, A and B. Let $g_i^k(N_{c(k,i)}^k, N_{c(k,i)}^{-k}) = N_{c(k,i)}^{-k} u \; \forall \; k = 1,2$ and $i, j \in [0,1]$.*

-If $m < u/4$, then two types of coalition perfect equilibria $s \in S$ exist:

1. *(monopoly equilibria with zero profits) $\exists \; k \in \{A, B\}$ such that $c_i^j(s) = k \; \forall \; j = 1,2$ and $i \in [0,1]$, $p_k^1(s) = 2m - p_f^2(s)$ and $p_k^j(s) \leq u \; \forall \; j = 1,2$*
2. *(symmetric equilibria with zero profits) $N_A^1(s) = N_A^2(s) = N_B^1(s) = N_B^2(s) = 1/2$, $p_A^1(s) = p_B^1(s) = 2m - p_A^2(s) = 2m - p_B^2(s)$ and $p_A^k(s) \leq u/2 \; \forall \; k = 1,2$*

-If $m \in [u/4, u/3)$, then three types of coalition perfect equilibria $s \in S$ exist. The two types stated above, plus:

3. (market shutdown) $N_A^1(s) = N_A^2(s) = N_B^1(s) = N_B^2(s) = 0$, $p_k^j(s) \geq 0 \; \forall \; j \in \{1,2\}$ and $k \in \{A, B\}$ and $u \leq p_A^j(s) + p_B^j(s)$ for some $j \in \{1,2\}$

-If $m \in [u/3, u)$, then three types of coalition perfect equilibria $s \in S$ exist. The three types stated above, plus:

4. (price=MC, partial participation) $\exists \; k \in \{A, B\}$ such that $p_k^1(s) = p_k^2(s) = m$, $N_k^1(s) = N_k^2(s) = m/u$

The theorem states that if the marginal cost is low enough, then the same type of coalition perfect equilibria exist as in the zero marginal cost case. In particular in every coalition perfect equilibrium firms' profits are zero and every consumer joins some network. If the marginal cost is larger than one quarter of the reservation value of consumers then in addition to the above market shutdown can happen in coalition perfect equilibrium. Finally, if the marginal cost exceeds half of the reservation value, there is a coalition

[17] for the case of asymmetric sides the theorem can be modified analogously.

perfect equilibrium with partial participation. The result that both firms get zero profit in every coalition perfect equilibrium holds for all marginal cost values.

The reason that market shutdown can happen in equilibrium if $m \geq u/4$ is that if all prices are nonnegative and on one side of the market the sum of the prices charged by the two networks is at least u, then it might no longer be a supported restriction for the coalition of all consumers to agree upon not staying out of the market.[18] If marginal costs are high, then the average price charged on consumers is high and then if consumers face two equally priced networks, then ex ante coordination to join some network becomes harder. The same intuition applies for the result that for even higher marginal costs there exist equilibria with partial consumer participation.

10 Multi-homing

On some two-sided markets network choices are naturally mutually exclusive, at least over a given period of time. For example people looking for partners for a date can only be in one entertainment facility at a time even if they would be willing to pay extra money to be able to look around for matches at different places. In other contexts consumers joining multiple platforms - we follow the existing literature and call it multi-homing - is not impossible, but uncommon. On the other hand there are contexts in which multi-homing is natural and widespread. People looking for used cars or looking for jobs sign up for multiple platforms, for example get multiple newspapers with classified ads.

It is straightforward to show that the main conclusions of the paper remain valid in the context in which consumers are allowed to multi-home. In particular if there is no conflict of interest among consumers on the same side of the market then there cannot be a coalition perfect equilibrium in which there are two active networks and one is more expensive on both sides of the market than the other one. If there are multiple active networks in

[18] For example if both firms charge a fee of $u/2$ on side 1 and 0 on side 2, which can be shown to be part of a coalition perfect equilibrium, then agreeing upon not staying out of the market does not guarantee positive utility to side 1 consumers, since if they expect $1/2$ of side 2 consumers to join A's network and $1/2$ of them to join B's network, then joining either network would give 0 expected utility for them.

coalition perfect equilibrium that are not equally priced, then it has to be that one network is cheaper and larger on one side of the market and the other network is cheaper and larger on the other side. And if consumers are homogeneous, then it still holds that in every coalition perfect equilibrium both firms get zero profit.

Multi-homing does not change the result that there are games in Γ^{M2} such that in every coalition perfect equilibrium the monopolist establishes two networks. In fact it can be shown that the possibility of multi-homing does not change the constraints on prices that the monopolist faces if wanting to operate two active networks, but how the high reservation value consumers join multiple networks, increasing the monopolist's revenue.[19] In short, multi-homing makes operating two networks more attractive and therefore there is a larger set of games in which the monopolist runs two networks in coalition perfect equilibrium.

It is also easier to construct an example in Γ^D in which firms' profits are positive in coalition perfect equilibrium. Consider $G \in \Gamma^D$ such that $g_i^j(N^j_{c(j,i)}, N^{-j}_{c(j,i)}) = N^{-j}_{c(j,i)} u_i^j \ \forall \ i, j \in [0,1]$. Assume that a measure 0.2 of consumers on each side of the market have reservation value 1 ('H' types) and a measure 0.8 of the consumers have reservation value 0.2 ('L' types). Now modify the game allowing consumers to multi-home. Formally assume that consumers' action sets are $\{A, B, AB, \emptyset\}$ instead of just $\{A, B, \emptyset\}$ and define consumers' payoffs if choosing AB the following way: $U_i^j(s) = N_A^{-j}(s)u_i^j + N_B^{-j}(s)u_i^j - p_A^j(s) - p_B^j(s)$ if $s_i^j = AB$. Then it can be shown that there is a coalition perfect equilibrium s such that $p_A^1(s) = p_B^2(s) = 0.8$, $p_A^2(s) = p_B^1(s) = 0.04$, $c_i^j(s) = AB$ for 'H' types, $c_i^2(s) = A$ for 'L' types on side 2 and $c_i^1(s) = B$ for 'L' types on side 1. High reservation value consumers multi-home, while low reservation value consumers join only the cheaper network.

11 Conclusions and possible extensions

This paper analyzed pricing decisions of firms and platform choices of consumers on two-sided markets with network externalities, assuming that con-

[19] It is possible to show that in the context of 6.1 if with no multi-homing the monopolist runs one network and targets high reservation value consumers on one side of the market and every consumer on the other, then with multi-homing in coalition perfect equilibrium it always runs two networks and the high reservation value consumers join both networks.

sumers are coalitionally rational and therefore have some ability to coordinate their decisions. Consumers on the market are allowed to have heterogeneous preferences. To keep the analysis tractable several simplifying assumptions were made in other dimensions. Sections 9 and 10 discuss relaxing some of these, but it is easy to come up with other extensions that would make the model more realistic in most settings. One is allowing the firms to use more complicated pricing instruments. In some contexts it is reasonable to assume that firms can only charge registration fees or entrance fees on their consumers. But on other markets on top of this (or instead of this) they can charge usage fees, or fees for successful matches. In addition to this firms might use contingent offers, give special offers to randomly selected consumers or employ other strategies to influence consumers' choices and induce them coordinate on joining their networks. Another extension would be dropping the assumption that serving every consumer induces the same marginal cost on the firms or more generally that there is no correlation between consumers' reservation values and the marginal cost they induce on the firm. This would introduce adverse selection into our model, which is a key component in a lot of two-sided markets, including the market for health insurance. Finally, there are contexts in which our implicit assumption that all consumers are symmetric with respect to the network externality they generate (their "attractiveness" to consumers on the other side) is unrealistic. Allowing a positive correlation between consumers' reservation values and the network externality they generate would change the qualitative predictions of our model. Some of these directions are addressed in the existing literature, others are left for future research.

12 Appendix

Some extra notation for the Appendix.

For every $D \subset C$ such that $\{i : C^i \in D\}$ is measurable with respect to the Lebesgue measure, let $N(D) = \int\limits_{C^i \in D} di$.

Let $S^c = \underset{i\in[0,1]}{\times} S_i^1 \underset{i\in[0,1]}{\times} S_i^2$ for $G \in \Gamma$.

For every $G \in \Gamma$ let $\Gamma^c(G)$ be the set of all subgames of G that start in the third stage. We refer to elements of $\Gamma^c(G)$ as consumer subgames.

Proof of Theorem 1 By definition a coalition perfect equilibrium of $G \in$

Γ is a subgame perfect Nash equilibrium of G and specifies a coalitionally rationalizable profile in every $G^c \in \Gamma^c(G)$.

Let $\widehat{T}$,$\widehat{S} \subset S$ and $J \subset I$ be such that $\widehat{T}$ is a supported restriction by J given $\widehat{S}$ in G. For every $G^c \in \Gamma^c(G)$ and $s \in S$ let $s(G^c)$ denote strategy that s specifies in G^c. For every $T \subset S$ let $T(G^c) = \{s(G^c) : s \in T\}$. Then it is straightforward to establish that the definition of supported restriction implies that $\widehat{T}(G^c)$ is a supported restriction by $J/\{A\}$ given $\widehat{S}(G^c)$ in G^c.

Furthermore, if for some $G^c = (C, S^c, U^c) \in \Gamma^c(G)$, $\widehat{T}^c, \widehat{S}^c \subset S^c$ it holds that $\widehat{T}^c$ is not a supported restriction by any $J \subset C$ given $\widehat{S}^c$ in G^c, then there are no $\widehat{T}, \widehat{S} \subset S$ and $\widehat{J} \subset C \cup \{A\}$ such that $\widehat{T}$ is a supported restriction by $\widehat{J}$ given $\widehat{S}$, $\widehat{S}(G^c) = \widehat{S}^c$ and $\widehat{T}(G^c) = \widehat{T}^c$. This implies that $s \in S$ is a perfect coalitionally rationalizable profile iff for every $C_i^j \in C$ it holds that s_i^j is coalitionally rationalizable in every $G^c \in \Gamma^c(G)$ and s^A is a best response to some conjecture that is concentrated on strategies of players in C that are coalitionally rationalizable in every $G^c \in \Gamma^c(G)$. This implies the claim. QED

Lemma 1 Let $G \in \Gamma$ and let $G^c = (C, S^c, u^c) \in \Gamma^c(G)$. Then $\exists\ s \in S^c$ such that s is a coalitionally rationalizable Nash equilibrium of G^c.

Proof of Lemma 1 For any $A \subset S^c$ such that $A = \underset{j=1,2}{\times} \underset{i\in[0,1]}{\times} A_i^j \neq \emptyset$ let $F(A)$ be the collection of supported restrictions in S^c given A. Let $A^0, A^1, ...$ be such that $A^0 = S^c$ and $A^k = \underset{B \in F(A^{k-1})}{\cap} B\ \forall\ k \geq 1$. Note that $A^* = \lim\limits_{k\to\infty} A^k$ is the set of coalitionally rationalizable strategies in G^c. Let $k \geq 0$ and assume $A^k \neq \emptyset$. For every $i \in [0,1]$ and $j = 1,2$ let $\overline{u}_i^j(k) = \sup\limits_{\omega_{-i,j} \in \Omega_{-i,j}(A^k), a_{i,j} \in A_{i,j}^k} u_{i,j}^c(a_{i,j}, \omega_{-i,j})$. Let $(a_{i,j}^m, \omega_{-i,j}^m)_{m=1,2,...}$ be such that $a_{i,j}^m \in A_{i,j}^k$ and $\omega_{-i,j}^m \in \Omega_{-i,j}(A^k)\ \forall\ m \geq 1$, and $u_{i,j}^c(a_{i,j}^m, \omega_{-i,j}^m) \to \overline{u}_i^j(k)$ as $m \to \infty$. Since $A_{i,j}^k$ is finite, $(a_{i,j}^m)$ has a subsequence $(a_{i,j}^{m_n})$ such that $\lim\limits_{m_n\to\infty} a_{i,j}^{m_n} = a \in A_{i,j}^k$. Then by the definition of supported restriction $a \in B_{i,j}$ for every $B \subset A^k$ such that B is a supported restriction in G^c given A^k. Since i and j were arbitrary, this implies $A^{k+1} \neq \emptyset$. Then by induction $A^0 = S^c \neq \emptyset$ implies $A^* \neq \emptyset$.

Note that $(S^c)_i^j = (S^c)_{i'}^{j'} \equiv \overline{S}^c\ \forall\ i, i' \in [0,1]$ and $j, j' \in \{1,2\}$. Order pure strategies in $\overline{S}^c$ any way, such that $\overline{S}^c = \{a_1, ..., a_n\}$. Define $x(s)$:

$\overline{S}^c \to R^{2n}$ such that $x_k(s) = N(C_i^1 \in C^1 : c_i^1(s) = a_k)\ \forall\ i \in 1,...,n$ and $x_k(s) = N(C_i^2 \in C^2 : c_i^2(s) = a_{k-n})\ \forall\ i \in n+1,...,2n$.

Let $\Theta = \{\theta \in R^{2n} : \exists\ s \in A^*$ s.t. $x(s) = \theta\}$. It is easy to establish that Θ is a nonempty, compact and convex subset of R^{2n}. For every $\theta \in \Theta$, $i \in [0,1]$ and $j = 1,2$ let $BR_i^j(\theta) = \{a \in \overline{S}^c : \exists\ s \in S$ s.t. $x(s) = \theta$ and $a \in BR_i^j(s_{-j,i})\}$. Now define the correspondence $F : \Theta \to R^{2n}$ such that $F(\theta) = \{y : \exists\ s \in S^c$ s.t. $s_i^j \in BR_i^j(\theta)$, and $y = x(s)\}$. It is straightforward to establish that $BR_i^j(s_{-j,i}) \subset (A^*)_i^j\ \forall\ i \in [0,1]$ and $j = 1,2$. This implies that $F(\theta) \subset \Theta\ \forall\ \theta \in \Theta$, therefore F is a correspondence from Θ to itself. Note that F is nonempty valued, since $\overline{S}^c$ is finite, so the best response correspondence is nonempty valued. Furthermore, if $s, t \in \overline{S}^c$ are such that $s_i^j, t_i^j \in BR_i^j(\theta)$ for some $\theta \in \Theta$, then $z_i^j \in BR_i^j(\theta)$ for every $z \in \overline{S}^c$ for which it holds that $\forall\ i \in [0,1]$ and $j = 1,2$ either $z_i^j = s_i^j$ or $z_i^j = t_i^j$. This implies that F is convex valued. Finally, since for every $i \in [0,1]$ and $j = 1,2$ g_i^j is continuous, it holds that for every $i \in [0,1]$ and $j = 1,2$ $BR_i^j(\theta)$ is upper hemicontinuous, which implies that F is upper hemicontinuous.

The above imply that all the conditions for Kakutani's fixed point theorem hold for F , therefore it has a fixed point θ^*. That implies $\exists\ s^* \in A^*$ such that $x(s^*) = \theta^*$ and $(s^*)_i^j \in BR_i^j(\theta^*)\ \forall\ \theta \in \Theta$, $i \in [0,1]$ and $j = 1,2$. This implies $(s^*)_i^j \in BR_i^j(s^*_{-j,i})\ \forall\ \theta \in \Theta$, $i \in [0,1]$ and $j = 1,2$, which establishes that s^* is a coalitionally rationalizable Nash equilibrium. QED

Proof of Theorem 2 Let $s_{-A} \in S_{-A}$ be such that s_{-A} specifies a coalitionally rationalizable Nash equilibrium in every $G^c \in \Gamma^c(G)$, and that for every $n = 1,2$ and $(p_k^j)_{j=1,2\ k=1,...,n} \in R^{2n}$ it holds that $s_{-A}(n, (p_k^j)_{j=1,2\ k=1,...,n}) = s_{-A}(n, (\widehat{p}_k^j)_{j=1,2\ k=1,...,n})$ where $\widehat{p}_k^j = \max(p_k^j, M)$. Lemma 1 and the assumption that $g_i^j(N^{-j}, N^j) < M$ $\forall\ i \in [0,1]$ and $j = 1,2$ together imply that there exists a profile s like that. Now let $\pi^* = \sup\limits_{s_A \in S_A} \pi_A(s_A, s_{-A})$. If $\pi^* = 0$, then for any $s_A \in S_A$ such that $n(s_A) = 1$ and $p_1^1(s_A), p_1^2(s_A) \geq 0$ it holds that (s_A, s_{-A}) is a coalition perfect equilibrium. Let $\pi^* > 0$. Then $\exists\ s_A^1, s_A^2, ...$ such that $\pi_A(s_A^k, s_{-A}) \to \pi^*$ as $k \to \infty$, and that also satisfies $p_m^j(s_A^k) \leq M\ \forall\ j = 1,2$ and $m = 1,...,n_A(s_A^k)$. W.l.o.g. let $n(s_A^k) = n(s_A^{k'}) \equiv \overline{n}\ \forall\ k, k' \geq 1$ (taking such a subsequence is always possible). Then $s_A^1, s_A^2, ...$ has a subsequence $s_A^{n_1}, s_A^{n_2}, ...$ such that $s_A^{n_k} \to s_A^* \in S_A$. Furthermore, $s_A^{n_1}, s_A^{n_2}, ...$ has a subsequence

$s_A^{m_1}, s_A^{m_2}, ...$ such that $s_{-A}(\overline{n}, p_k^j(s_A^{m_n})_{j=1,2\ k=1,...,\overline{n}}) \equiv s_{-A}^n \to s_{-A}^{c*} \in \{\emptyset, 1, ..., \overline{n}\}^{[0,1]\times[0,1]}$.

Let $G^{c*} = (C, S^{c*}, u^{c*})$ denote the subgame following first stage action $\overline{n}$ and second stage action $(p_k^j(s_A^n)_{j=1,2\ k=1,...,\overline{n}})$. It is straightforward to establish that continuity of g_i^j for every $j = 1, 2$, $i \in [0, 1]$, and that s_{-A}^n is a Nash equilibrium in the subgame following $(\overline{n}, p_k^j(s_A^n)_{j=1,2\ k=1,...,\overline{n}})$ together imply that s_{-A}^{c*} is a Nash equilibrium in G^{c*}. Now, assume $A \subset S^{c*}$ and $B \subset S^{c*}$ are such that A is a supported restriction by $J \subset C$ given B in G^{c*}. Then the starting assumption on the utility functions implies that $\exists\ (A^n, B^n)_{n=1,2,...}$ such that $A^n, B^n \subset S^{c*}\ \forall\ n = 1, 2, ...,$ A^n is a supported restriction by $J \subset C$ given B^n in the subgame following $(\overline{n}, p_k^j(s_A^n)_{j=1,2\ k=1,...,\overline{n}})$, and $(A^n, B^n) \to (A, B)$ as $n \to \infty$. Then since s_{-A}^n is coalitionally rationalizable in the subgame following $(\overline{n}, p_k^j(s_A^n)_{j=1,2\ k=1,...,\overline{n}})$, s_{-A}^{c*} is coalitionally rationalizable in G^{c*}. Also note that for any $t \in S$ such that $t_A = s_A^*$ and t_{-A} specifies s_{-A}^{c*} in G^{c*} it holds that $\pi_A(t) = \sup\limits_{s_A \in S_A} \pi_A(s_A, s_{-A})$.

Let $s'_{-A} \in S_{-A}$ be such that s'_{-A} specifies s_{-A}^{c*} in G^{c*}, and in every other subgame G^c s'_{-A} specifies the same strategy as s_{-A}. Then $\pi_A(s_A^*, s'_{-A}) \geq \pi_A(s'_A, s'_{-A})\ \forall\ s'_A \in S_A$ by construction. Furthermore, s'_{-A} specifies a coalitionally rationalizable Nash equilibrium in every $G^c \in \Gamma^c$, therefore (s_A^*, s'_{-A}) is a coalition perfect equilibrium. QED

Proof of Theorem 3 Let $\pi_A(s) = \pi$. If $\pi = 0$ the claim is trivial, since in every Nash equilibrium and therefore in every coalition perfect Nash equilibrium the firm has to get nonnegative profit (since announcing prices above 0 guarantees that).

Suppose now that $\pi > 0$. That implies that at least on one side of the market the monopolist charges a strictly positive profit and has a strictly positive market share.

Let $\widehat{C^j}(s) = \{C_i^j : c_i^j(s) = A\}$ for $j = 1, 2$. Then for $C_i^j \in \widehat{C^j}(s)$ and $N^j \in [0, 1]$ it holds that $g_i^j(N^j, N_A^{-j}(s)) - p_A^j(s) \geqslant 0$ (note that $g_i^j(N^j, N_A^{-j}(s))$ is constant in N^j). Then for every $\varepsilon > 0$ and $C_i^j \in \widehat{C^j}(s)$ it holds that $g_i^j(N_A^{-j}(s)) - p_A^j(s) + \varepsilon > 0$, which implies that after a price announcement of $\left(p_A^1(s) - \varepsilon, p_A^2(s) - \varepsilon\right)$ joining the network is a supported restriction for $\widehat{C^1}(s)$ and $\widehat{C^2}(s)$.

If $p_A^j(s) > 0$ for $j = 1, 2$, then it has to be the case that $N_A^j(s) \geq 0$ and $Max\left\{N_A^j(s), N_A^{-j}(s) > 0\right\}$. Then the firm can guarantee a profit arbitrarily close to π by charging prices $\left(p_A^1(s) - \varepsilon, p_A^2(s) - \varepsilon\right)$. If $p_A^j(s) > 0$ and $p_A^{-j}(s) = 0$ then it has to be the case that $N_A^j(s) > 0$ and $N_A^{-j}(s) \geq 0$. In this case, by charging $\left(p_A^1(s) - \varepsilon, p_A^2(s) - \varepsilon\right)$ the monopolist gets market shares $N_A^j \geq N_A^j(s)$ and $N_A^{-j} = 1$ and profits arbitrarily close to π. Finally, if $p_A^j(s) > 0$ and $p_A^{-j}(s) = 0$ it has to be the case that $N_A^j(s)$ and $N_A^{-j}(s) = 1$. Then, by charging $\left(p_A^1(s) - \varepsilon, p_A^2(s) - \varepsilon\right)$ the monopolist gets market shares $N_A^j \geq N_A^j(s)$ and $N_A^{-j} = 1$ and again profits arbitrarily close to π.

That means that if consumers play only coalitionally rationalizable strategies, then the firm can guarantee a profit arbitrarily close to π by charging prices $\left(p_A^1(s) - \varepsilon, p_A^2(s) - \varepsilon\right)$ where $\varepsilon > 0$ is small enough. This implies that it cannot be that $\pi(s') < \pi$. QED

Proof of Theorem 4 Let $p_1 = u_1 - \varepsilon$ and $p_2 = u_2 - \varepsilon$, where $\varepsilon > 0$. Consider the restriction $D \equiv \underset{i \in [0,1]}{\times} \{A\} \underset{i \in [0,1]}{\times} \{A\}$ given $\underset{i \in [0,1]}{\times} \{A, \emptyset\} \underset{i \in [0,1]}{\times} \{A, \emptyset\}$ by $C^1 \cup C^2$ in the subgame following price announcements (p_1, p_2). For any $k = 1, 2$, $i \in [0, 1]$ and $\omega_{-(k,i)} \in \Omega_{-(k,i)}(D)$ it holds that $U_i^k(A, \omega_{-(k,i)}) = \varepsilon > 0$ and therefore the above restriction is supported. But then in every coalition perfect equilibrium s it holds that $c_i^j(s \mid p_1, p_2) = A \ \forall\ j = 1, 2$ and $i \in [0, 1]$. Then $\pi_A(s) \geq u_1 + u_2 - 2\varepsilon$ for any $\varepsilon > 0$, which in turns means that $\pi_A(s) \geq u_1 + u_2$. Since $u_1 + u_2$ is the maximum possible consumer surplus and consumers cannot get negative utility in equilibrium, $\pi_A(s) = u_1 + u_2$. Then it has to be that $p_k(s) = u_k$ and $N_A^k(s) = 1 \ \forall\ k = 1, 2$, since $p^k(s) > u_k$ $(k = 1, 2)$ would imply $N_A^k(s) = 0$. QED

Proof of Theorem 5 First note that for any $p_1, p_2 \geq 0$ and for any $\overline{s}$ such that $\overline{s}$ is a Nash equilibrium of $G^c(p_1, p_2)$ it holds that $\overline{s}_i^k = A$ implies $\overline{s}_j^k = A \ \forall\ k = 1, 2$ and $i, j \in [0, 1]$ such that $u_i^k \geq u_j^k$. Consider now $p_1 = F_1^{-1}(1 - x_1)x_2$ and $p_2 = F_2^{-1}(1 - x_2)x_1$ (since F_j is strictly increasing and continuous in u if $F_j(u) \in (0, 1)$ for any $j = 1, 2$, $F_j^{-1}(x)$ is well defined for $x \in (0, 1)$). Then for any $\varepsilon > 0$ and $s \in S$ coalition perfect equilibrium it holds that $c_i^j(s \mid p_1 - \varepsilon, p_2 - \varepsilon) = A \ \forall\ C_i^j \in C^1 \cup C^2$ such that $F_j(u_i^j) \geq 1 - x_j$. Then $\pi_A(s \mid p_1 - \varepsilon, p_2 - \varepsilon) \geq (p_1 - \varepsilon)x_1 + (p_2 - \varepsilon)x_2$. Furthermore, note that $N_A^1(s), N_A^2(s) \in (0, 1)$ implies that $p_A^1(s) = F_1^{-1}(1 - N_A^1(s))N_A^2(s)$ and $p_A^2(s) = F_2^{-1}(1 - N_A^2(s))N_A^1(s)$. Therefore for $x_1 = N_A^1(s)$ and $x_2 = N_A^2(s)$ it has to hold that $F_1^{-1}(1 - x_1) +$

$F_2^{-1}(1-x_2))x_1x_2 \geq F_1^{-1}(1-\widetilde{x}_1)+F_2^{-1}(1-\widetilde{x}_2))\widetilde{x}_1\widetilde{x}_2 \;\forall\; \widetilde{x}_1,\widetilde{x}_2 \in (0,1)$. Since F_1 and F_2 are continuously differentiable, so are F_1^{-1} and F_2^{-1}. Then x_1 and x_2 have to satisfy the following first order necessary conditions: $\frac{\partial((F_1^{-1}(1-x_1)+F_2^{-1}(1-x_2))x_1x_2)}{\partial x_1} = (F_1^{-1}(1-x_1)+F_2^{-1}(1-x_2))x_2 - f_1^{-1}(1-x_1)x_1x_2 = 0$ and $\frac{\partial((F_1^{-1}(1-x_1)+F_2^{-1}(1-x_2))x_1x_2)}{\partial x_2} = (F_1^{-1}(1-x_1)+F_2^{-1}(1-x_2))x_1 - f_2^{-1}(1-x_2)x_1x_2 = 0$

These imply the claim. QED

Proof of Theorem 6 Let $s \in S$ be a coalition perfect equilibrium. If $p^1(s), p^2(s) < l$ then joining the network is a supported restriction for the coalition of all consumers. The supremum of the profit the firm in this price range is $2l$ and the firm can get a profit arbitrarily close to it by charging $\left(p^1(s), p^2(s)\right) = (l-\varepsilon, l-\varepsilon)$ for small enough $\varepsilon > 0$. If $p^1(s), p^2(s) < ah$ then joining the network is a supported restriction for the coalition of consumers that involve the high types from both sides of the market. Therefore the monopolist can guarantee a profit arbitrarily close to $2a^2h$ by charging prices $\left(p^1(s), p^2(s)\right) = (ah-\varepsilon, ah-\varepsilon)$ for small enough $\varepsilon > 0$. If $p^j(s) < h$ and $p^{-j}(s) < al$ for some $j \in \{1,2\}$ then joining the network is a supported restriction for $C^{-j} \cup \{C_i^j : i \in [0,a]\}$. Therefore the monopolist can guarantee a profit arbitrarily close to $a(h+l)$ by charging prices $\left(p^1(s), p^2(s)\right) = (h-\varepsilon, al-\varepsilon)$ for small enough $\varepsilon > 0$.

The above establish that $\pi_A(s) \geq \max(2l, 2a^2h, a(h+l))$.

If $p^j(s) > h$ for some $j \in \{1,2\}$, then $N^j(s) = 0$. Then $N^{-j}(s) > 0$ only if $p^{-j}(s) \leq 0$. In any case $\pi_A(s) < \max(2l, 2a^2h, a(h+l))$. Therefore $p^j(s) \leq h$ for $j = 1,2$.

It cannot be that $p^j(s) \leq l \;\forall\; j \in \{1,2\}$ and $p^j(s) < l$ for some $j \in \{1,2\}$, since then $\pi_A(s) < 2l$.

If $p^j(s) > l \;\forall\; j \in \{1,2\}$ then $c_i^j(s) = \emptyset \;\forall\; j = 1,2$ and $i \in (a,1]$. Then it cannot be that $p^j(s) > ah$ for some $j \in \{1,2\}$, otherwise $c_i^j(s) = \emptyset \;\forall\; j = 1,2$ and $i \in [0,1]$ and therefore $\pi_A(s) = 0$. Furthermore, it cannot be that $p^j(s) < ah$ for some $j \in \{1,2\}$, otherwise $\pi_A(s) < 2a^2h$.

Suppose now that $p^j(s) > l$ and $p^{-j}(s) \leq l$ for some $j \in \{1,2\}$. It cannot be that $p^{-j}(s) < al$ since then $\pi_A(s) < a(l+h)$. If $p^{-j}(s) > al$, then $p^j(s) > ah$ or $p^{-j}(s) > ah$ implies $\pi_A(s) = 0$. Then $p^j(s) < ah$ or $p^{-j}(s) < ah$ implies $\pi_A(s) < 2a^2h$. Finally, $p^j(s) = ah$ and $p^{-j}(s) = ah$

contradict that $p^j(s) > l$ and $p^{-j}(s) \leq l$. This concludes that $p^{-j}(s) = al$. Then $p^j(s) = h$, otherwise $\pi_A(s) < a(l+h)$.

Consider first the case that $ah \geqslant l$. If $p^j(s) \geq ah$ then only high types can join the network in equilibrium. Furthermore, consumers on side j only join in equilibrium if at least some low types join the network from the other side. For that to be possible in equilibrium, it has to be the case that $p^{-j}(s) \leq al$. The above imply that if ${}^j(s) > h$ for $j = 1$ or $j = 2$ then $\pi(s) \leq ah + al$ But note that the firm can get a profit arbitrarily close to this amount by charging $(h - \varepsilon, al - \varepsilon)$ for small enough $\varepsilon > 0$, since then joining the network is a supported restriction for the coalition of consumers involving all high types on side 1 and all consumers on side 2.

Consider now the case that $l > ah$. If ${}^j(s) > l$ for $j = 1$ or $j = 2$ the same arguments as above establish that $\pi(s) \leq ah + al$, but the monopolist can get a profit arbitrarily close to $ah + al$ by charging $(h - \varepsilon, al - \varepsilon)$ for small enough $\varepsilon > 0$.

This concludes that $\pi(s) \leq \max(2l, 2a^2h, ah + al)$, but if consumers play only coalitionally rationalizable strategies then the monopolist can always guarantee a profit arbitrarily close to $\max(2l, 2a^2h, ah+al)$. But then it has to be the case that in every coalition perfect equilibrium $\pi(s) = \max(2l, 2a^2h, ah + al)$. This establishes that the prices charged by the monopolist are either (l, l) or (ah, ah) or (h, al) or (al, h) in any coalition perfect equilibrium, and in the first case $c_i^1(s) = c_i^2(s) = A \; \forall i \in [0, 1]$, in the second case $c_i^1(s) = c_i^2(s) = A \; \forall \; i \in [0, a]$ and $c_i^1(s) = c_i^2(s) = \emptyset \; \forall \; i \in (a, 1]$ and in the third case either $c_i^1 = A \; \forall \; i \in [0, a]$, $c_i^1 = A \; \forall \; i \in (a, 1]$ and $c_i^2 = A \; \forall \; i \in [0, 1]$ or $p^1(s) = al$, $p^2(s) = h$, $c_i^1 = A \; \forall \; i \in [0, 1]$, $c_i^1 = A \; \forall \; i \in [0, a]$ and $c_i^1 = \emptyset \; \forall \; i \in (a, 1]$.

The above imply that if $2l > \max(2a^2h, ah + al)$ then $p^j(s) = l$ and $N^j(s) = 1 \; \forall \; j \in \{1, 2\}$. If $2a^2h > \max(2l, ah + al)$ then $p^j(s) = ah$ and $N^j(s) = a \; \forall \; j \in \{1, 2\}$. And if $ah + al > \max(2l, 2a^2h)$ then $p^j(s) = h$, $p^{-j}(s) = al$, $N^j(s) = a$ and $N^{-j}(s) = 1$ for some $j \in \{1, 2\}$.

Suppose $2a^2h > al + ah$. Since $a > 0$, this is equivalent to $(2a-1)h > l$. The latter implies $a^2h > l$ since $a^2h - (2a-1)h = (a-1)^2h > 0$. Therefore $(2a-1)h > l$ implies that $2a^2h > \max(2l, ah + al)$.

Suppose now that $ah + al < 2l$. It is equivalent to $l > \frac{a}{2-a}h$. The latter implies $l > a^2h$ since $\frac{a}{2-a}h - a^2h = ah\frac{1-2a+a^2}{2-a} > 0$. Therefore $2a^2h < 2l$. This establishes that if $l > \frac{a}{2-a}h$ then $2l > \max(2a^2h, ah + al)$.

Note that $(2a-1)h < \frac{a}{2-a}h$. If $l \in ((2a-1)h, \frac{a}{2-a}h)$ then $2a^2h < al + ah$ and $ah + al > 2l$ and therefore $ah + al > \max(2l, 2a^2h)$.

Let $G^1 = (I, S^1, u^1) \in \Gamma^c$ be the subgame following price announcements (ah, ah) and let $s^1 \in S^1$ be such that $(s^1)^1_i = (s^1)^2_i = A \; \forall \; i \in [0, a]$ and $(s^1)^1_i = (s^1)^2_i = \emptyset \; \forall \; i \in (a, 1]$. Let $G^2 = (I, S^2, u^2) \in \Gamma^c$ be the subgame following price announcements (h, al) and let $s^2 \in S^2$ be such that $(s^1)^1_i = A$ $\forall \; i \in [0, a]$, $(s^1)^1_i = \emptyset \; \forall \; i \in (a, 1]$ and $(s^1)^2_i = A \; \forall \; i \in [0, 1]$. Let $G^3 = (I, S^3, u^3) \in \Gamma^c$ be the subgame following price announcements (l, l) and let $s^3 \in S^3$ be such that $(s^3)^1_i = (s^3)^2_i = A \; \forall \; i \in [0, 1]$. It is straightforward to establish that for $k = 1, 2, 3$ s^k is a coalitionally rationalizable Nash equilibrium in G^k. Let now $s_{-A} \in S_{-A}$ be any profile that specifies s^k in G^k for every $k = 1, 2, 3$ and specifies some arbitrary coalitionally rationalizable Nash equilibrium in every other $G^c \in \Gamma^c$. By Lemma 1 there exists a profile like that. Let s_A be such that $p_A(s_A) = (ah, ah)$. Let s'_A be such that $p_A(s'_A) = (h, al)$. And let s''_A be such that $p_A(s''_A) = (l, l)$. If $\frac{l}{h} = 2a - 1$

then $2a^2h = ah + al = \max(2l, 2a^2h, ah + al)$. The above then establish that both (s_A, s_{-A}) and (s'_A, s_{-A}) are coalition perfect equilibria. If $\frac{l}{h} = \frac{a}{2-a}$ then $2l = ah + al = \max(2l, 2a^2h, ah + al)$. The above then establish that both (s'_A, s_{-A}) and (s''_A, s_{-A}) are coalition perfect equilibria. QED

Proof of Theorem 7 If $s_{-A} \in S_{-A}$ is such that consumers play a coalitionally rationalizable Nash equilibrium in every consumer subgame, then in the subgame following $n_A(s_A, s_{-A}) = 2$ and $p^1_1 = p^2_2 = la - \varepsilon$, $p^2_1 = p^1_2 = la + (1-a)h - 2\varepsilon$ ($\varepsilon > 0$), it has to hold that $n^1_i = 2$, $n^2_i = 1 \; \forall \; i \in [0, a]$ and $n^1_i = 1$, $n^2_i = 2 \; \forall \; i \in (a, 1]$. To see this, define $A \subset S$ such that $A = \underset{i \in [0,1], j=1,2}{\times} A^j_i$ and $A^j_i \equiv \{\emptyset, 1, 2\} \; \forall \; i \in [0, a]$, $j = 1, 2$, $A^1_i \equiv \{\emptyset, 1\} \; \forall \; i \in (a, 1]$ and $A^2_i \equiv \{\emptyset, 2\} \; \forall \; i \in (a, 1]$. Also define $B \subset S$ such that $B = \underset{i \in [0,1], j=1,2}{\times} B^j_i$ and $B^1_i \equiv \{2\} \; \forall \; i \in [0, a]$, $B^2_i \equiv \{1\} \; \forall \; i \in [0, a]$, $j = 1, 2$, $B^1_i \equiv \{1\} \; \forall \; i \in (a, 1]$ and $A^2_i \equiv \{2\} \; \forall$ $i \in (a, 1]$. First note that $A^j_i \times S_{-1,i}$ is a supported restriction by C^j_i given $S \; \forall \; i \in [0, 1]$ and $j = 1, 2$ (since strategies in S^j_i / A^j_i are never best responses for C^j_i). Next, B is a supported restriction given A by $C^1 \cup C^2$ since it gives the best possible payoff to every consumer in this subgame, given A. Therefore $n^1_i = 2$, $n^2_i = 1 \; \forall \; i \in [0, a]$ and $n^1_i = 1$, $n^2_i = 2 \; \forall \; i \in (a, 1]$ is the only coalitionally rationalizable strategy in the above subgame.

Since ε can be arbitrarily small positive, the above establishes that if $s \in S$ is a coalition perfect equilibrium, then $\pi_A(s) \geq 2(la + (1-a)ah) \equiv \pi^*$.

Suppose $s_{-A} \in S_{-A}$ is such that consumers play a coalitionally rationalizable Nash equilibrium in every consumer subgame and $\exists\, \widehat{s}_A \in S_A$ such that $\pi_A(\widehat{s}_A, s_{-A}) \geq \pi^*$ and it is not the case that $n_A(\widehat{s}_A, s_{-A}) = 2$, and $p^1_j(\widehat{s}_A, s_{-A}) = p^2_{-j}(\widehat{s}_A, s_{-A}) = la$, $p^2_j(\widehat{s}_A, s_{-A}) = p^1_{-j}(\widehat{s}_A, s_{-A}) = la + (1-a)h$ for some $j = 1, 2$.

Let $s = (\widehat{s}_A, s_{-A})$.

First suppose $n_A(s) = 1$. If $l \in \left((4a-1)h, \frac{a}{2-a}h\right)$, then by Theorem 4 $\pi_A(s) \leq (l+h)a$. But for $l > (4a-1)h$ it holds that $(l+h)a < 2(la + (1-a)ah) \equiv \pi^*$, a contradiction. If $l \in \left((\frac{a}{2-a}h, \frac{a(1-2a)}{1-a}h\right)$, then by Theorem 4 $\pi_A(s) \leq 2l$. But $l < \frac{a(1-2a)}{1-a}h$ implies $2l < 2(la + (1-a)ah) \equiv \pi^*$, a contradiction.

Therefore $n_A(s) = 2$.

It cannot be that $N^j_k = 0$ for some $j = 1, 2$ and $k = 1, 2$ since then either $N^{-j}_k(s) = 0$ or $p^{-j}_k(s) \leq 0$ (otherwise consumers choosing network k in s would get negative utility, contradicting the assumption on s_{-A}). In either case $\pi_A(s)$ is smaller or equal to the supremum of profits attainable by a strategy in which A operates only one network. Then, as established above, $\pi_A(s) < \pi^*$. Therefore $N^j_k > 0\ \forall\ j = 1, 2$ and $k = 1, 2$.

Let $H^j = \{C^j_i : i \in [0, a]\}$ and $L^j = \{C^j_i : i \in (a, 1]\}\ \forall\ j = 1, 2$.

Let $X^j_k = \{C^j_i : n^j_i(s) = k\}\ \forall\ j = 1, 2$ and $k = 1, 2$.

First we establish that it cannot be that for some $j = 1, 2$ both $X^j_1 \cap L^j = \emptyset$ and $X^j_2 \cap L^j = \emptyset$. If $X^j_1 \cap L^j = \emptyset$ and $X^j_2 \cap L^j = \emptyset\ \forall\ j = 1, 2$ then $\pi_A(s) < 2a^2h < \pi^*$. Otherwise, w.l.o.g. assume $X^2_1 \cap L^2 = \emptyset$ and $X^2_2 \cap L^2 = \emptyset$ and $X^1_2 \cap L^1 \neq \emptyset$. Then consider a deviation s' by the firm such that $n_A(s') = 1$, $p^1(s') = \max(0, p^1_1(s)\frac{N^2_1+N^2_2}{N^2_2} - \varepsilon)$ and $p^2(s') = \max(0, p^2_2(s) - \varepsilon)$. In the subgame following the above prices it is a supported restriction for $X^1_1 \cup X^1_2 \cup X^2_1 \cup X^2_2$ (note that $X^1_2 \cap L^1 \neq \emptyset$ and therefore $p^1_1(s) \leq lN^2_2$) to choose 1, which guarantees a profit of at least $\pi' \equiv p^1(s')(N^1_1 + N^1_2) + p^2(s')(N^2_1 + N^2_2)$. Consider now deviation s'' by the firm such that $n_A(s'') = 1$, $p^1(s'') = \max(0, p^2_1(s)\frac{N^2_1+N^2_2}{N^2_1} - \varepsilon)$ and $p^2(s'') = \max(0, p^2_1(s) - \varepsilon)$. In the subgame following the above prices it is a supported restriction for $X^1_2 \cup X^2_1 \cup X^2_2$ to choose 1, which guarantees a profit of at least $\pi'' \equiv p^1(s'')N^1_2 + p^2(s'')(N^2_1 + N^2_2)$. It is straightforward to verify that both $\pi' \leq \pi_A(s)$ and $\pi'' \leq \pi_A(s)$, and therefore at least one of the above deviations yields higher profit than $\pi_A(s)$. And since $n_A(s') = 1$ and $n_A(s'') = 1$, it holds that $\pi' < \pi^*$ and $\pi'' < \pi^*$.

Next we establish that it cannot be that for some $j = 1,2$ both $X_1^j \cap L^j \neq \emptyset$ and $X_2^j \cap L^j \neq \emptyset$. If $X_1^j \cap L^j \neq \emptyset$ and $X_2^j \cap L^j \neq \emptyset \ \forall \ j = 1,2$ then $p_k^j(s) \leq lN_k^{-j}(s) \ \forall \ k = 1,2$ and $j = 1,2$. Then $\pi_A(s) < 2l < \pi^*$. Otherwise w.l.o.g. assume $X_1^1 \cap L^1 \neq \emptyset$, $X_2^1 \cap L^1 \neq \emptyset$ and $X_1^2 \cap L^2 \neq \emptyset$. Then $\pi_A(s) < (h+l)N_1^1(s)N_2^2(s) + 2lN_1^2(s)N_2^1(s)$, since $p_1^1(s) \leq lN_2^2(s)$, $p_2^1(s) \leq lN_2^1(s)$, $p_1^2(s) \leq lN_1^1(s)$ and $p_2^2(s) < hN_2^1(s)$. Note that $(h+l)N_2^2(s) < (h+l)a < \pi^*$ and therefore $(h+l)N_2^2(s) < (h+l)N_1^1(s)N_2^2(s) + 2lN_1^2(s)N_2^1(s)$. This

implies $(h+l)N_2^2(s) < 2lN_1^2(s) < 2l$. Furthermore, $2l < \pi^*$ and therefore $2l < (h+l)N_1^1(s)N_2^2(s) + 2lN_1^2(s)N_2^1(s)$. This implies $2l(1 - N_1^2(s)N_2^1(s)) < (h+l)N_1^1(s)N_2^2(s)$ which implies $2lN_1^1(s) < (h+l)N_1^1(s)N_2^2(s)$ which implies $2l < (h+l)N_2^2(s)$, a contradiction.

Next we establish that it cannot be that for some $k = 1,2$ both $X_k^1 \cap L^1 \neq \emptyset$ and $X_k^2 \cap L^2 \neq \emptyset$. Suppose otherwise. Then, as established above, $X_{-k}^1 \cap L^1 = \emptyset$ and $X_{-k}^2 \cap L^2 = \emptyset$, but $X_{-k}^1 \neq \emptyset$ and $X_{-k}^2 \neq \emptyset$. This implies $N_k^{-j}(s)l - p_k^j(s) \geq N_{-k}^{-j}(s)l - p_{-k}^j(s) \ \forall \ j = 1,2$ and $N_k^{-j}(s)h - p_k^j(s) \leq N_{-k}^{-j}(s)h - p_{-k}^j(s) \ \forall \ j = 1,2$, which implies $p_k^j(s) \leq p_{-k}^j(s) \ \forall \ j = 1,2$. Then by Lemma 1 it has to be that $p_k^j(s) = p_{-k}^j(s) \ \forall \ j = 1,2$. But since $p_k^j(s) \leq N_k^{-j}(s)l < l \ \forall \ j = 1,2$, this implies $\pi_A(s) < 2l < \pi^*$.

Therefore $\exists \ k \in \{1,2\}$ such that $X_k^1 \cap H^1 = X_k^1$ and $X_{-k}^2 \cap H^2 = X_k^2$. W.l.o.g. let $k = 1$.

Note that $p_1^2(s) \leq lN_1^1$ and $p_2^1(s) \leq lN_2^2$ since $X_1^2 \cap L^2 \neq \emptyset$, $X_2^1 \cap L^1 \neq \emptyset$ and by definition no consumer can get negative utility in any subgame if s is played. Then $hN_1^2 - p_1^1(s) \geq hN_2^2 - p_2^1(s)$ implies $p_1^1(s) \leq lN_2^2 + h(1 - N_2^2)$ and $hN_2^1 - p_2^2(s) \geq hN_1^1 - p_1^2(s)$ implies $p_2^2(s) \leq lN_1^1 + h(1 - N_1^1)$. This establishes that $\pi_A(s) \leq l(N_1^1N_1^2 + N_2^1N_2^2) + (lN_2^2 + h(1 - N_2^2))N_1^1 + (lN_1^1 + h(1 - N_1^1))N_2^2 \leq l(N_1^1 + N_2^2) + h(N_1^1 + N_2^2 - 2N_1^1N_2^2)$.

Note that $\frac{\partial(l(N_1^1+N_2^2)+h(N_1^1+N_2^2-2N_1^1N_2^2))}{\partial N_1^1} = h + l - 2hN_2^2 \geq h + l - 2ha > 0$ (since the starting assumptions imply $a < 1/2$). Similarly it holds that $\frac{\partial(l(N_1^1+N_2^2)+h(N_1^1+N_2^2-2N_1^1N_2^2))}{\partial N_2^2} = h + l - 2hN_1^1 \geq h + l - 2ha > 0$. Therefore $\pi_A(s) < l2a + h(2a - 2a^2) = \pi^*$ unless $p_1^2(s) = p_2^1(s) = al$, $N_1^2(s) = N_2^1(s) = 1 - a$, $p_1^1(s) = p_2^2(s) = la + h(1-a)$ and $N_1^1(s) = N_2^2(s) = a$.

This concludes the claim. QED

Proof of Claim 1 Let $\pi_A^1 = \max \pi_A(s)$ st $s \in S$, $n_A(s_A) = 1$ and s_{-A} specifies a coalitionally rationalizable Nash equilibrium in every $G^c \in \Gamma^c$. It is easy to show that $\pi_A^1 = 0.168$.

Let s be a coalition perfect equilibrium.

It is easy to show that if $N_k^j(s) = 0$ for some $j, k \in \{1, 2\}$ then $\pi_A(s) \leq \pi_A^1$.

If it does not hold that $p_k^1(s), p_{3-k}^2(s) \leq 0.02 \times 0.8 = 0.016$ for some $k = 1, 2$, $\pi_A(s) \leq \pi_A^1$. Fix this k. Since s is a Nash equilibrium, if it does not hold that $p_k^1(s) \leq 0.02N_k^2(s)$ and $p_{3-k}^2(s) \leq 0.02N_{3-k}^1(s)$ then $\pi_A(s) \leq \pi_A^1$.

Suppose $c_i^1(s) = 3 - k$ for some $i \in (0.16, 0.2]$ and $c_i^2(s) = k$ for some $i \in (0.16, 0.2]$. Then $c_i^1(s) = 3 - k \; \forall \; i \in [0, 0.16]$ and $c_i^2(s) = k \; \forall \; i \in [0, 0.16]$. From the starting assumption $p_k^2(s) \leq p_{3-k}^2(s) + 0.82 \times (N_k^1(s) - N_{3-k}^1(s))$ and $p_{3-k}^1(s) \leq p_k^1(s) + 0.82 \times (N_{3-k}^2(s) - N_k^2(s))$. Then it is straightforward to establish that $\pi_A(s) \leq (0.2 \times 0.02 + 0.6 \times 0.82 \times 0.2) \times 2 = 0.204\,8$.

Suppose $c_i^1(s) = 3 - k$ for some $i \in (0.16, 0.2]$ but $-\exists \; i \in (0.16, 0.2]$ such that $c_i^2(s) = k$. Then $N_k^2(s) \leq 0.16$ and $N_{3-k}^1(s) \leq 0.2$. From the starting assumption $p_{3-k}^1(s) \leq p_k^1(s) + 0.82 \times (N_{3-k}^2(s) - N_k^2(s))$. In order to have $N_k^2(s) > 0$ it has to be that $p_k^2(s) \leq p_{3-k}^2(s) + N_k^1(s) - N_{3-k}^1(s)$. Then it is straightforward to establish that $\pi_A(s) \leq 0.02 \times 0.16 + 0.68 \times 0.82 \times 0.2 + 0.02 \times 0.2 + 0.6 \times 0.16 = 0.214\,72$.

Consider the subgame following price announcements $(0.003\,2 - \varepsilon, 0.560\,8 - 2\varepsilon, 0.604 - 2\varepsilon, 0.004 - \varepsilon)$ where $\varepsilon > 0$. It can be shown that the only coalitionally rationalizable outcome in this subgame is s^c such that $(s^c)_i^1 = 2 \; \forall \; i \in [0, 0.16]$, $(s^c)_i^1 = 1 \; \forall \; i \in (0.16, 1]$, $(s^c)_i^2 = 1 \; \forall \; i \in [0, 0.2]$ and $(s^c)_i^2 = 2 \; \forall \; i \in (0.2, 1]$. This establishes that $\pi_A(s) \geq 0.214\,72$. Since $0.214\,72 > \pi_A^1$, it also establishes that $n_A(s_A) = 2$.

Suppose $-\exists \; i \in (0.16, 0.2]$ such that $c_i^1(s) = 3 - k$ and $-\exists \; i \in (0.16, 0.2]$ such that $c_i^2(s) = k$. If $p_k^2(s) > p_{3-k}^2(s) + 0.68$ or $p_{3-k}^1(s) > p_k^1(s) + 0.68$ then it can be shown that $\pi_A(s) \leq \pi_A^1$. If $p_k^2(s) < p_{3-k}^2(s) + 0.557\,6$ or $p_{3-k}^1(s) < p_k^1(s) + 0.557\,6$ then it can be shown that the starting assumption $-\exists \; i \in (0.16, 0.2]$ such that $c_i^1(s) = 3 - k$ and $-\exists \; i \in (0.16, 0.2]$ such that $c_i^2(s) = k$ contradicts that s is a coalition perfect equilibrium. If $p_k^j(s) + 0.6 \geq p_{3-k}^j(s)$ and $p_{3-k}^{3-j}(s) + 0.6 \geq p_k^{3-j}(s)$ then $\pi_A(s) \leq (0.02 \times 0.16 + 0.6 \times 0.16) \times 2 = 0.198\,4$.

Let now $(p_1^1, p_1^2, p_2^1, p_2^2)$ such that for some $j \in \{1, 2\}$, $p_k^j + 0.68 \geq p_{3-k}^j > p_k^j + 0.6$ and $p_{3-k}^{3-j} + 0.68 \geq p_k^{3-j} \geq p_{3-k}^{3-j} + 0.557\,6$. Consider subgame $G^c = (C, S^c, u^c)$ following price announcements $(p_1^1, p_1^2, p_2^1, p_2^2)$. Let $s^c \in S^c$ such that $(s^c)_i^j = 3 - k \; \forall \; i \in [0, 0.16]$, $(s^c)_i^j = k \; \forall \; i \in (0.16, 1]$, $(s^c)_i^{3-j} = k \; \forall \; i \in [0, 0.16]$ and $(s^c)_i^{3-j} = 3 - k \; \forall \; i \in (0.16, 1]$. Let $t^c \in S^c$ such that $(t^c)_i^j = k \; \forall \; i \in [0, 1]$, $(t^c)_i^{3-j} = k \; \forall \; i \in [0, 0.2]$ and for $i \in (0.2, 1]$ it holds that $(t^c)_i^{3-j} = \emptyset$ if $p_{3-k}^{3-j} \geq 0$ and $(t^c)_i^{3-j} = 3 - k$ if $p_{3-k}^2 < 0$. Then it can be established that both s^c and t^c are coalitionally rationalizable equilibria in G^c. Note that if $p_l^h(s) = p_l^h \; \forall \; h, l \in \{1, 2\}$ and s_{-A} specifies t^c in G^c, then $\pi_A(s) \leq \pi_A^1$. Also note that $\exists\ p_1^1, p_1^2, p_2^1, p_2^2$ such that $p_k^j + 0.68 \geq p_{3-k}^j > p_k^j + 0.6$ and $p_{3-k}^{3-j} + 0.68 \geq p_k^{3-j} \geq p_{3-k}^{3-j} + 0.557\,6$ for some $j \in \{1,2\}$ and such that if $p_l^h(s) = p_l^h$ $\forall\ h, l \in \{1,2\}$ and s_{-A} specifies s^c in G^c, then $\pi_A(s) > 0.214\,72$. It is easy to establish that $\pi_A(s)$ is maximized for prices $(0.003\,2, 0.683\,2, 0.683\,2, 0.003\,2)$ and for $(0.683\,2, 0.003\,2, 0.003\,2, 0.683\,2)$. The maximal profit is $(0.02 \times 0.16 + 0.68 \times 0.16) \times 2 = 0.224$.

Let now $r_k^j, r_{3-k}^{3-j} \geq 0$, $r_k^j + 0.68 \geq r_{3-k}^j > r_k^j + 0.6$ and $r_{3-k}^{3-j} + 0.68 \geq r_k^{3-j} \geq r_{3-k}^{3-j} + 0.557\,6$ for some $j \in \{1,2\}$ and $0.16(r_{3-k}^j + r_k^{3-j}) + 0.84(r_k^j + r_{3-k}^{3-j}) \geq 0.214\,72$. Consider $s \in S$ such that $p_l^h(s) = r_l^h\ \forall\ h, l \in \{1,2\}$ and s_{-A} is as follows. In the subgame following $(r_1^1, r_1^2, r_2^1, r_2^2)$ let s_{-A} specify s^c such that $(s^c)_i^j = 3-k\ \forall\ i \in [0, 0.16]$, $(s^c)_i^j = k\ \forall\ i \in (0.16, 1]$, $(s^c)_i^{3-j} = k\ \forall\ i \in [0, 0.16]$ and $(s^c)_i^{3-j} = 3-k\ \forall\ i \in (0.16, 1]$. In any subgame following $(p_1^1, p_1^2, p_2^1, p_2^2) \neq (r_1^1, r_1^2, r_2^1, r_2^2)$ let s_{-A} specify t^c such that $(t^c)_i^j = k\ \forall\ i \in [0,1]$, $(t^c)_i^{3-j} = k\ \forall\ i \in [0, 0.2]$ and for $i \in (0.2, 1]$ it holds that $(t^c)_i^{3-j} = \emptyset$. In any other $G^c \in \Gamma^c$ let s_{-A} specify any coalitionally rationalizable Nash equilibrium. By Lemma 1 there exists s like this. By construction s_{-A} specifies a coalitionally rationalizable Nash equilibrium in every $G^c \in \Gamma^c$. By the above considerations s_A is a best response to s_{-A} which concludes that s is a coalition perfect equilibrium.

It is similarly straightforward to construct an equilibrium $s \in S$ such that for some $j, k \in \{1,2\}$ $p_k^j(s_A) = 0.003\,2$, $p_k^{3-j}(s_A) = 0.560\,8$, $p_{3-k}^{3-j}(s_A) = 0.004$, $p_{3-k}^j(s_A) = 0.604$ and in the subgame following $p(s)$ s_{-A} specifies s^c such that $(s^c)_i^j = 3-k\ \forall\ i \in [0, 0.16]$, $(s^c)_i^j = k\ \forall\ i \in (0.16, 1]$, $(s^c)_i^j = k\ \forall\ i \in [0, 0.2]$ and $(s^c)_i^{3-j} = 3-k\ \forall\ i \in (0.2, 1]$. QED

Proof of Theorem 8 Suppose $\exists\ s \in S$ such that s is a coalition perfect equilibrium, $n_A(s) = 2$ and $N_k^j(s) > 0\ \forall\ j, k \in \{1,2\}$. It is straightforward to establish that $p_k^j(s) \geq 0$, otherwise $\exists\ s'_A \in S_A$ such that $n_A(s'_A) = 1$ and $\pi_A(s'_A, s_{-A}) > \pi_A(s_A)$.

By Lemma 9.1 (see below) $\exists\ k \in \{1,2\}$ such that $p_k^1(s) \leq p_{3-k}^1(s)$ and $p_k^2(s) \geq p_{3-k}^2(s)$, otherwise . Let $l^1 = (\inf_{i \in [0,1]: c_i^j(s) = k} u_i^1)(N_k^2(s) + N_{3-k}^2(s))$, $h^1 = (\inf_{i \in [0,1]: c_i^j(s) = 3-k} u_i^1)(N_k^2(s) + N_{3-k}^2(s))$, $l^2 = (\inf_{i \in [0,1]: c_i^j(s) = 3-k} u_i^2)(N_k^1(s) + N_{3-k}^1(s))$ and $h^2 = (\inf_{i \in [0,1]: c_i^j(s) = k} u_i^2)(N_k^1(s) + N_{3-k}^1(s))$. Then Nash equilibrium implies $p_k^1(s) \leq N_k^2(s) l^1 / (N_k^2(s) + N_{3-k}^2(s))$, $p_{3-k}^2(s) \leq N_k^1(s) l^2 / (N_k^1(s) +$

$N^1_{3-k}(s))$, $p^1_{3-k}(s) \leq (N^2_k(s) - N^2_k(s))h^1/(N^2_k(s) + N^2_{3-k}(s)) + p^1_k(s)$ and $p^2_k(s) \leq (N^1_{3-k}(s) - N^1_k(s))h^2/(N^1_k(s) + N^1_{3-k}(s)) + p^2_{3-k}(s)$. Let $x^1 = N^1_k(s)/(N^1_k(s) + N^1_{3-k}(s))$ and $x^2 = N^2_{3-k}(s)/(N^2_{3-k}(s) + N^2_k(s))$. Then $\pi_A(s) \leq x^2 l^1(1 - x^1) + x^1((1 - 2x^2)h^1 + x^2 l^1) + x^1 l^2(1 - x^2) + x^2((1 - 2x^1)h^2 + x^1 h^2) = h^1 x^1 + h^2 x^2 + l^1 x^2 + l^2 x^1 - 2h^1 x^1 x^2 - 2h^2 x^1 x^2$. Taking first order conditions it is easy to verify that the latter expression is maximized at $x_1 = \frac{h^2+l^1}{2h^1+2h^2}$, $x_2 = \frac{h^1+l^1}{2h^1+2h^2}$. Substituting these values into the expression yields $\pi_A(s) \leq \frac{(h^1+l^2)(h^2+l^1)}{2(h^1+h^2)}$.

Let $\widehat{C}(s) = \{C^j_i \in C : c^j_i(s) \neq \emptyset\}$. Notice that if $s'_A \in S_A$ is such that $n_A(s'_A) = 1$ and $p^j_A(s'_A) = l^j/(N^{3-j}_k(s) + N^{3-j}_{3-k}(s)) - \varepsilon \; \forall \, j \in \{1, 2\}$, where $\varepsilon > 0$, then $\pi_A(s'_A, s_A) \geq l^1 + l^2 - \varepsilon(\sum_{j=1,2}\sum_{k=1,2} N^j_k(s))$, since the assumptions that there is no conflict of interest among consumers on the same side and that s is a Nash equilibrium together guarantee that in the consumer subgame following the above price announcements joining the network is a supported restriction for all players in $\widehat{C}(s)$. Since s is a Nash equilibrium, this implies $\pi_A(s) \geq l^1 + l^2$. Therefore $\frac{(h^1+l^2)(h^2+l^1)}{2(h^1+h^2)} \geq l^1 + l^2$.

It is straightforward to verify that for any $h_1 + h_2 = h > 0$ and $l_1 + l_2 = l > 0$ the expression $\frac{(h_1+l_2)(h_2+l_1)}{2(h_1+h_2)} - l_1 - l_2$ is maximized at $h_1 = h_2 = h/2$, $l_1 = l_2 = l/2$. In that case $\frac{(h_1+l_2)(h_2+l_1)}{2(h_1+h_2)} - l_1 - l_2 = h^2 - 6hl + l^2$. Then s being a Nash equilibrium implies $h^2 - 6hl + l^2 \geq 0$. Since $h > l$, this implies $h \geq (3 + 2\sqrt{2})l$. Therefore if $(\max_{i\in[0,1]} u^j_i)/(\min_{i\in[0,1]} u^j_i) < 3 + 2\sqrt{2} \; \forall \, j \in \{1, 2\}$, then s cannot be a Nash equilibrium, a contradiction. QED

Lemma 9.1 Let $G^c = (C, S^c, u^c)$ be the subgame following price announcements $(p^1_A, p^2_A, p^1_B, p^2_B)$. If $p^j_A < p^j_B \; \forall \, j = 1, 2$ then B is not coalitionally rationalizable in G^c for any $C^j_i \in C$. If also $p^j_A < u^j \; \forall \, j \in \{1, 2\}$ then A is the unique coalitionally rationalizable strategy in G^c for every $C^j_i \in C$. Similarly if $p^j_B < p^j_A \; \forall \, j = 1, 2$ then A is not coalitionally rationalizable in G^c for any $C^j_i \in C$. If also $p^j_B < u^j \; \forall \, j \in \{1, 2\}$ then B is the unique coalitionally rationalizable strategy in G^c for every $C^j_i \in C$.

Proof of Lemma 9.1 Consider $p^j_A < p^j_B \; \forall \, j = 1, 2$. If $p^j_B > u^j$ for some $j \in \{1, 2\}$ then in G^c choosing B is not rationalizable for any $C^j_i \in C$. Observe that $p^j_A \leq u^j \; \forall \, j \in \{1, 2\}$ implies $p^j_A < u^j \; \forall \, j \in \{1, 2\}$. For every $C^j_i \in C$ the maximum utility C^j_i can expect when joining B is

$u^j - p_B^j$. The minimum utility that C_i^j can expect if every consumer in C joins A is $u^j - p_A^j(s) > u^j - p_B^j$. If $p_A^j(s) < u^j \; \forall \; j \in \{1,2\}$ then this implies that joining A is a supported restriction for consumers in C, which implies the claim.

The other case is perfectly symmetric. QED

Lemma 9.2 For every $k = 1,2$ and $i \in [0,1]$ it holds that $\emptyset$ is not coalitionally rationalizable for C_i^k in the subgame following price announcements $(0,0,0,0)$.

Proof of Lemma 9.2 Let $\widehat{S}_i^k = \{A, B\} \; \forall \; k = 1,2$ and $i \in [0,1]$. Consider the restriction $\widehat{S} \equiv \underset{i\in[0,1]}{\times} \widehat{S}_i^1 \underset{i\in[0,1]}{\times} \widehat{S}_i^2$ given S^c by C in the subgame following price announcements $(0,0,0,0)$. Let $k \in \{1,2\}$, $i \in [0,1]$ and $\omega_{-k,i} \in \Omega_{-k,i}(\widehat{S})$. Let $n_A^{-k}(\omega_{-k,i}) = \int\limits_{t_{-i}\in S_{-i}} g_j(C_j^{-k} \in C^{-k}/C_i^{-k} : t_j^k(0,0,0,0) = A) d\omega_{-k,i}$ and let $n_B^{-k}(\omega_{-k,i}) = \int\limits_{t_{-i}\in S_{-i}} g_j(C_j^{-k} \in C^{-k}/C_i^{-k} : t_j^k(0,0,0,0) = B) d\omega_{-k,i}$. Since $\omega_{-k,i} \in \Omega_{-k,i}(\widehat{S})$, $\min(n_A^{-k}(\omega_{-i}), n_B^{-k}(\omega_{-i})) > 0$. Then playing a best response strategy to $\omega_{-k,i}$ yields a positive expected payoff to C_i^k. Since $U_i^k(\emptyset) = 0 \; \forall \; i \in [0,1]$, this implies that the above restriction is supported for C and therefore $\emptyset$ is not a coalitionally rationalizable strategy for any $k = 1,2$ and $i \in [0,1]$. QED

Lemma 9.3 Let s be a coalition perfect equilibrium. If $N_A^k(s) > 0$ for some $k = 1,2$ and $N_B^1(s) = N_B^2(s) = 0$ then (i) $p_A^1(s) = -p_A^2(s)$, (ii) $p_A^k(s) \leq u^k \; \forall \; k = 1,2$ and (iii) $N_A^1(s) = N_A^2(s) = 1$. Similarly if $N_B^k(s) > 0$ for some $k = 1,2$ and $N_A^1(s) = N_A^2(s) = 0$ then (i) $p_B^1(s) = -p_B^2(s)$, (ii) $p_B^k(s) \leq u^k \; \forall \; k = 1,2$ and (iii) $N_B^1(s) = N_B^2(s) = 1$.

Proof of Lemma 9.3 Note that $N_B^1(s) = N_B^2(s) = 0$ implies $\pi_B(s) = 0$. Suppose $p_A^k(s) > u^k$ for some $k = 1,2$. Then $N_A^k(s) = 0$ since consumers cannot get negative utility in s. Then $N_A^{-k}(s) > 0$ implies that $p_A^{-k}(s) \leq 0$, again because consumers cannot get negative utility in s. Since A cannot have negative profit in s, this implies $p_A^{-k}(s) = 0$. Consider the deviation $(u - \varepsilon, -\varepsilon)$ by B, where $\varepsilon > 0$. In the subgame following this deviation it is a supported restriction for $C^1 \cup C^2$ to play B, because that profile yields the highest possible payoff in this subgame for every $C_i^k \in C$, and choosing A or $\emptyset$ yields a strictly smaller payoff than this maximum no matter what strategies other consumers play. Therefore $s_i^k(p_A^1(s), p_A^2(s), u-\varepsilon, -\varepsilon) = B \ \forall\ k=1,2$ and $i \in [0,1]$. Then B's profit after this deviation is $u - 2\varepsilon$, which is positive for small enough profits, a contradiction. This concludes that $p_A^k(s) \leq u^k \ \forall\ k = 1,2$.

Suppose now that $p_A^1(s) + p_A^2(s) > 0$. Consider the deviation $(p_A^1(s) - \varepsilon, p_A^2(s) - \varepsilon)$ by B, where $\varepsilon > 0$. By lemma 9.1 $s_i^k(p_A^1(s), p_A^2(s), p_A^1(s) - \varepsilon, p_A^2(s) - \varepsilon) = B \ \forall\ k = 1,2$ and $i \in [0,1]$. Then B's profit after this deviation is $p_A^1(s) + p_A^2(s) - 2\varepsilon$, which is positive for small enough ε, a contradiction. This concludes that $p_A^1(s) + p_A^2(s) \leq 0$.

Suppose now that $p_A^1(s) + p_A^2(s) < 0$. This implies $p_A^k(s) < 0$ for some $k = 1,2$. Then $N_B^k(s) = 0$ implies $N_A^k(s) = 1$, since A strictly dominates $\emptyset$ for side 1 consumers. But then $p_A^1(s) + p_A^2(s)$ implies $\pi_A(s) = p_A^1(s)N_A^1(s) + p_A^2(s)N_A^2(s) < 0$, a contradiction. This concludes that $p_A^1(s) + p_A^2(s) \leq 0$.

If $p_A^k(s) < 0$ for some $k = 1,2$, then $N_B^k(s) = 0$ implies $N_A^k(s) = 1$. Then $\pi_A(s) \geq 0$ implies that $N_A^k(s) = 1 \ \forall\ k = 1,2$.

Consider now $p_A^1(s) = p_A^2(s) = 0$. Then $\pi_A(s) = 0$. If $p_B^k(s) < 0$ for some $k = 1,2$, then $\emptyset$ is a strictly dominated strategy for side k consumers, and therefore $N_B^k(s) = 0$ implies $N_A^k(s) = 1$. Then choosing A yields utility $u_{-k} > 0$ for side $-k$ consumers, and therefore $N_B^{-k}(s) = 0$ implies $N_A^{-k}(s) = 1$. Suppose now that $p_B^1(s) > 0$ and $p_B^2(s) = 0$. Then by lemma 9.1 a deviation $\min(u - \varepsilon, p_B^1(s) - \varepsilon), -\varepsilon$ by A for $\varepsilon > 0$ guarantees that all consumers join A, which for small enough ε yields positive profit for A, contradicting that s is an equilibrium. A symmetric argument rules out that $p_B^1(s) = 0$ and $p_B^2(s) > 0$. If $p_B^1(s) = p_B^2(s) = 0$, then lemma 9.2 implies that $N_A^k(s) + N_B^k(s) = 1 \ \forall\ k = 1,2$, and then $N_B^1(s) = N_B^2(s) = 0$ implies $N_A^1(s) = N_A^2(s) = 1$. QED

Lemma 9.4 Let s be a coalition perfect equilibrium such that $N_A^k(s) > 0$ for some $k = 1,2$ and $N_B^k(s) > 0$ for some $k = 1,2$. Then $p_A^1(s) = p_B^1(s) = -p_A^2(s) = -p_B^2(s)$ and $p_A^k(s) \leq u^k \ \forall\ k = 1,2$. If $g_i^j(N^j, N^{-j}) = g_i^j(\widehat{N^j}, N^{-j}) \ \forall\ j = 1,2$, $i \in [0,1]$ and $N^{-j}, N^j, \widehat{N^j} \in [0,1]$ then $N_A^1(s) = N_A^2(s) = N_B^1(s) = N_B^2(s) = 1/2$.

Proof of Lemma 9.4 Suppose $p_f^k(s) > u^k$ for some $k = 1,2$ and $f \in \{A, B\}$. W.l.o.g. assume $p_A^1(s) > u^1$. Then $N_A^1(s) = 0$ and therefore $N_A^2(s) > 0$. This is only compatible with consumers choosing A in s playing a best response and A not getting negative profits if $p_A^2(s) = 0$. Then by lemma 9.1 a price announcement $(u^1 - \varepsilon, -\varepsilon)$ by B for $\varepsilon > 0$ guarantees that all consumers join B, which for small enough

ε yields positive profit for B. Therefore $\pi_A(s) = 0$ and $\pi_B(s) > 0$. The latter can only be if both $N_B^1(s) > 0$ and $N_B^2(s) > 0$, which imply that $p_B^k(s) \leq u^k \; \forall \; k = 1, 2$. Then by Lemma 9.1 a deviation $p_B^1(s) - \varepsilon, p_B^2(s) - \varepsilon$ by A for $\varepsilon > 0$ guarantees that all consumers join A. For small enough ε this deviation profit is close to $p_B^1(s) + p_B^2(s)$. If $p_B^k(s) \geq 0 \; \forall \; k = 1, 2$, then $\pi_B(s) > 0$ implies $p_B^1(s) + p_B^2(s) > 0$, which implies that the above deviation is profitable for small enough ε. If $p_B^2(s) \leq 0$, then $\pi_B(s) > 0$ implies $N_B^1(s) > 0$, but then $N_A^2(s) > 0$ contradicts that every consumer plays a best response in s. Therefore $p_B^2(s) > 0$. If $p_B^1(s) < 0$ and $p_B^2(s) > 0$, then $N_B^1(s) = 1$ since B is the unique best response in s after the equilibrium price announcements for side 1 consumers, and therefore $p_B^1(s) + p_B^2(s) \geq p_B^1(s) N_B^1(s) + p_B^2(s) N_B^2(s) = \pi_B(s) > 0$. This again implies that the above deviation for A is profitable for small enough ε, contradicting that s is a Nash equilibrium. This concludes that $p_f^k(s) \leq u^k \; \forall \; k = 1, 2$ and $f \in \{A, B\}$. Suppose $p_A^k(s) \neq p_B^k(s)$ for some $k = 1, 2$. W.l.o.g. assume $p_A^1(s) > p_B^1(s)$. Then $p_A^2(s) \leq p_B^2(s)$, otherwise lemma 9.1 implies $N_A^1(s) = N_A^2(s) = 0$. Suppose first that $N_A^1(s) = N_B^1(s) = 0$. Then $N_A^2(s) > 0$ and $N_B^2(s) > 0$. This is only compatible with consumers being in equilibrium and firms not getting negative profit if $p_A^2(s) = p_B^2(s) = 0$. Then $\pi_B(s) = 0$. Then by lemma 9.1 a deviation $\min(u - \varepsilon, p_A^1(s) - \varepsilon), -\varepsilon$ by B for $\varepsilon > 0$ guarantees that all consumers join A, which for small enough ε yields positive profit for B, contradicting that s is an equilibrium. Suppose next that $N_A^2(s) = N_B^2(s) = 0$. Then $N_A^1(s) > 0$ and $N_B^1(s) > 0$, which contradicts that s is a Nash equilibrium, since $N_A^2(s) = N_B^2(s) = 0$ and $p_A^1(s) > p_B^1(s)$ implies that given $s_{-1,i}$ B is a better response than A in the subgame following the equilibrium price announcements for every $C_i^1 \in C^1$. This concludes that $N_k^1(s) > 0$ for some $k = A, B$ and $N_k^1(s) > 0$ for some $k = A, B$. But then $N_A^1(s) \leq N_B^1(s)$ and $N_A^2(s) > N_B^2(s)$, otherwise $p_A^1(s) > p_B^1(s)$ and $p_A^2(s) \leq p_B^2(s)$ imply that some consumers are not playing a best response in s. Consider now following two deviations. The first is $(p_B^1(s) - \varepsilon, p_B^2(s) - \varepsilon)$ by A, and the second is $(p_A^1(s) - \varepsilon, p_A^2(s) - \varepsilon)$ by B. Since $p_f^k(s) \leq u^k \; \forall \; k = 1, 2$ and $f \in \{A, B\}$, lemma 9.1 implies that $s_i^k(p_B^1(s) - \varepsilon, p_B^2(s) - \varepsilon, p_B^1(s), p_B^2(s)) = A$ and $s_i^k(p_A^1(s), p_A^2(s), p_B^1(s) - \varepsilon, p_B^2(s) - \varepsilon) = B \; \forall \; k = 1, 2$ and $i \in [0, 1]$. Then the first deviation yields a profit $p_B^1(s) + p_B^2(s) - 2\varepsilon$ to A, while the second yields $p_A^1(s) + p_A^2(s) - 2\varepsilon$ to B. The sum of these deviation profits is $p_A^1(s) + p_A^2(s) + p_B^1(s) + p_B^2(s) - 4\varepsilon$. The sum of the two firms' equilibrium profits is

$N_A^1(s)p_A^1(s)+N_A^2(s)p_A^2(s)+N_B^1(s)p_B^1(s)+N_B^2(s)p_B^2(s) \equiv \pi^*$. Note that $p_B^1(s) < 0$ implies that $N_A^1(s)+N_A^2(s) = 1$, since then $\emptyset$ is never a best response for any $C_i^1 \in C^1$. Similarly, $p_A^2(s) < 0$ implies that $N_B^1(s) + N_B^2(s) = 1$. Then by $N_A^1(s) \leq N_B^1(s)$, $N_A^2(s) > N_B^2(s)$, $p_A^1(s) > p_B^1(s)$ and $p_A^2(s) \leq p_B^2(s)$ it has to hold that $N_A^1(s)p_A^1(s) + N_A^2(s)p_A^2(s) + N_B^1(s)p_B^1(s)+N_B^2(s)p_B^2(s) < \frac{1}{2}(p_A^1(s)+p_A^2(s)+p_B^1(s)+p_B^2(s))$. The left hand side of this inequality is nonnegative (it is the sum of equilibrium profits), therefore the right hand side is positive, which implies that also $N_A^1(s)p_A^1(s) + N_A^2(s)p_A^2(s) + N_B^1(s)p_B^1(s) + N_B^2(s)p_B^2(s) < p_A^1(s) + p_A^2(s) + p_B^1(s) + p_B^2(s)$. But that implies that for small enough ε the sum of the two deviation profits above is larger than the sum of the two equilibrium profits, implying that at least one of the deviations is profitable, a contradiction. This concludes that $p_A^k(s) = p_B^k(s)\ \forall\ k = 1, 2$. Suppose that $\pi_A(s)+\pi_B(s) > 0$. W.l.o.g. assume $\pi_A(s) \geq \pi_B(s)$. Then $\pi_B(s) < p_A^1(s)+p_A^2(s) \leq \pi_A(s)+\pi_B(s)$ (note that $p_A^k(s) = p_B^k(s)$ $\forall\ k = 1, 2$, and that $p_A^k(s) < 0$ implies that $C_i^k(s) \neq \emptyset\ \forall\ C_i^k \in C^k$). By lemma 9.1 a deviation $p_A^1(s) - \varepsilon, p_A^2(s) - \varepsilon$ by B for $\varepsilon > 0$ guarantees that all consumers join B, which yields a profit of $p_A^1(s)+p_A^2(s)-2\varepsilon$ to B. This implies that the above deviation is profitable for small enough ε, a contradiction. Then $\pi_A(s) + \pi_B(s) \leq 0$ and since equilibrium profits have to be nonnegative, $\pi_A(s) = \pi_B(s) = 0$. Suppose $p_A^1(s) + p_A^2(s) > 0$. By lemma 9.1 a deviation $p_A^1(s) - \varepsilon, p_A^2(s) - \varepsilon$ by B for $\varepsilon > 0$ guarantees that all consumers join B, which yields a profit of $p_A^1(s)+p_A^2(s)-2\varepsilon$ to B. But for small enough ε this profit is positive, which contradicts that $\pi_B(s) = 0$ and that s is an equilibrium. This concludes that $p_A^1(s) + p_A^2(s) \leq 0$. Suppose $p_A^1(s) + p_A^2(s) < 0$. Then $p_A^k(s) = p_B^k(s) < 0$ for some $k = 1, 2$. Then $N_A^k(s) + N_B^k(s) = 1$, since $\emptyset$ is never a best response for any $C_i^k \in C^k$ in the subgame after the equilibrium price announcements. But then $\min(\pi_A(s), \pi_B(s)) < 0$, contradicting that s is a Nash equilibrium. This concludes that $p_A^1(s) + p_A^2(s) = 0$. If $p_A^k(s) = p_B^k(s) < 0$ for some $k = 1, 2$, then $N_A^k(s)+N_B^k(s) = 1$. Then nonnegativity of equilibrium profits implies that also $N_A^{-k}(s) + N_B^{-k}(s) = 1$ and that $N_A^1(s) = N_A^2(s)$, $N_B^1(s) = N_B^2(s)$. If $p_A^1(s) = p_B^1(s) = p_A^2(s) = p_B^2(s) = 0$, then by lemma 9.2 $N_A^k(s) + N_B^k(s) = 1\ \forall\ k = 1, 2$. Suppose now that $g_i^j(N^j, N^{-j}) = g_i^j(\widehat{N^j}, N^{-j})\ \forall\ j = 1, 2,\ i \in [0, 1]$ and $N^{-j}, N^j, \widehat{N^j} \in [0, 1]$. As shown above, $N_A^k(s) + N_B^k(s) > 0\ \forall\ k = 1, 2$. Then $p_A^k(s) = p_B^k(s)\ \forall\ k = 1, 2$ implies $N_A^k(s) = N_B^k(s)\ \forall\ k = 1, 2$. This implies $\pi_A(s) = \pi_B(s)$. If $p_A^k(s) = p_B^k(s) < 0$ for some $k = 1, 2$, then the above implies that

$N^k_A(s) = N^k_B(s) = 1/2$. Then $p^1_A(s) + p^2_A(s) = 0$ and nonnegativity of equilibrium profits together imply that also $N^{-k}_A(s) = N^{-k}_B(s) = 1/2$. If $p^1_A(s) = p^1_B(s) = p^2_A(s) = p^2_B(s) = 0$, then $N^j_A(s) + N^j_B(s) = 1$ $\forall$ $j = 1, 2$ and the fact that s is a Nash equilibrium imply that $N^j_A(s) = N^j_B(s) = 1/2$ $\forall$ $j = 1, 2$. QED

Proof of the Theorem: Lemma 9.3 and lemma 9.4 establish that there are no other coalition perfect equilibria with one or two active firms than those stated in the claim. All that remains to be shown is that there is no coalition perfect equilibrium with no active firm.

Suppose $N^k_A(s) + N^k_B(s) = 0$ $\forall$ $k = 1, 2$. Then $\pi_A(s) = \pi_B(s) = 0$. If $p^k_f(s) < 0$ for some $k = 1, 2$ and $f = A, B$, then $N^k_A(s) + N^k_B(s) = 1$, since $\emptyset$ is a never best response strategy for any $C^k_i \in C^i$, a contradiction. Suppose now that $\exists$ $k \in \{A, B\}$ such that $p^j_k(s) \geq 0$ $\forall$ $j = 1, 2$ and $p^l_j(s) > 0$ for some $l \in \{1, 2\}$. W.l.o.g. assume $p^1_A(s) > 0$ (and $p^2_A(s) \geq 0$). By lemma 9.1 the deviation $\min(u^1 - \varepsilon, p^1_A(s) - \varepsilon)$, $\min(u^2 - \varepsilon, p^2_A(s) - \varepsilon)$ by B for $\varepsilon > 0$ guarantees that every consumer joins B, and it yields strictly positive profit for small enough ε, a contradiction. If $p^j_k(s) = 0$ $\forall$ $j = 1, 2$ and $k = A, B$, then $N^j_A(s) + N^j_B(s) = 1$ $\forall$ $j = 1, 2$ by lemma 9.3. This concludes that if s is a coalition perfect equilibrium, then it cannot be that $N^j_A(s) + N^j_B(s) = 0$ $\forall$ $j = 1, 2$. QED

Proof of Theorem 10 W.l.o.g. assume that $k = 1$ (the other case is perfectly symmetric), so $u^1 < u^2$.

By Theorem 9 if $N^1_k(s) + N^2_k(s) > 0$ for some $k \in \{A, B\}$, then $p^1_k(s) = -p^2_k(s)$ and $p^l_k(s) \leq u^l$ $\forall$ $l = 1, 2$. Furthermore, $\pi_A(s) = \pi_B(s) = 0$.

Assume $N^1_A(s) + N^2_A(s) > 0$ and suppose $p^1_A(s) > u^1 - u^2$. Consider the deviation $p^1_A(s) - u^1 - \varepsilon, u^2 - \varepsilon$ by B for $\varepsilon > 0$. In the subgame following the deviation B is a strictly dominant strategy for every $C^1_i \in C^1$, therefore it is the only rationalizable strategy. But then B is the only rationalizable strategy in the subgame for every $C^2_i \in C^2$ too. Therefore after the above deviation B's profit in s is $p^1_A(s) - u^1 + u^2 - 2\varepsilon$. Since $p^1_A(s) > u^1 - u^2$, this profit is strictly positive for small enough ε, contradicting that s is an equilibrium. A perfectly symmetric argument shows that it cannot be that $N^1_B(s) + N^2_B(s) > 0$ and $p^1_B(s) > u^1 - u^2$. QED

Proof of Theorem 11

Lemma 11.1 Let $G^c = (C, S^c, u^c)$ be the subgame following price announcements $(p_A^1, p_A^2, p_B^1, p_B^2)$. Let $\widehat{s} \in S^c$ be such that $\widehat{s}_i^j = A$ $\forall$ $j = 1, 2$ and $i \in [0, 1]$. Let $\widehat{t} \in S^c$ be such that $\widehat{t}_i^j = B$ $\forall$ $j = 1, 2$ and $i \in [0, 1]$. If $p_A^j \leq p_B^j$ for some $j = 1, 2$, $p_A^j \leq p_B^j + u^j$ $\forall$ $j = 1, 2$, and $p_A^j \leq u^j$ $\forall$ $k = 1, 2$ then $\widehat{s}$ is a coalitionally rationalizable Nash equilibrium of G^c. If $p_B^j \leq p_A^j$ for some $j = 1, 2$, $p_A^j \leq p_A^j + u^j$ $\forall$ $j = 1, 2$, and $p_B^j \leq u^j$ $\forall$ $k = 1, 2$ then $\widehat{t}$ is a coalitionally rationalizable Nash equilibrium of G^c.

Proof of Lemma 11.1 Consider the case that $p_A^1 \leq p_B^1$, $p_A^k \leq p_B^k + u^k$ $\forall$ $k = 1, 2$, and $p_A^k \leq u^k$ $\forall$ $k = 1, 2$. Let D be such that $D = \underset{i \in [0,1]}{\times} D_i^1 \underset{i \in [0,1]}{\times} D_i^2$, $D_i^k \subset (S^c)_i^k$ and $A \in D_i^k$ $\forall$ $k = 1, 2$ and $i \in [0, 1]$. Note that for every $C_i^1 \in C^1$ it holds that the outcome in which all consumers on both sides of the market choose A yields the highest possible payoff to C_i^1 in the subgame following price announcements $(p_A^1, p_A^2, p_B^1, p_B^2)$. Therefore there cannot be any $\widehat{C} \subset C^1 \cup C^2$ and $\widehat{D} \subset D$ such that $\widehat{D}$ is a supported restriction given D by $\widehat{C}$ in the above subgame, and $C_i^1 \in \widehat{C}$ for some $i \in [0, 1]$. Furthermore, if some $C_i^2 \in C^2$ expects every $C_i^1 \in \widehat{C}$ to chooses A with probability 1 in the above subgame, then A is a best a best response for her, no matter what her conjecture is concerning other players in C^2. Therefore there cannot be any $\widehat{C} \subset C^1 \cup C^2$ and $\widehat{D} \subset D$ such that $\widehat{D}$ is a supported restriction given D by $\widehat{C}$ in the above subgame, and $\widehat{C} \subset C^2$. This concludes that $\widetilde{D} \equiv \underset{\widehat{D} \in \mathcal{F}(D)}{\cap} \widehat{D}$ is such that $A \in \widetilde{D}_i^k$ $\forall$ $k = 1, 2$ and $i \in [0, 1]$. Since $A \in (S^c)_i^k$ $\forall$ $k = 1, 2$ and $i \in [0, 1]$, an iterative argument establishes that A is a coalitionally rationalizable strategy for C_i^k in the subgame following price announcements $(p_A^1, p_A^2, p_B^1, p_B^2)$, for every $k = 1, 2$ and $i \in [0, 1]$. It is straightforward to check that $\widehat{s}$ is a Nash equilibrium in the subgame. Combining the previous two statements establishes the claim that $\widehat{s}$ is a coalitionally rationalizable Nash equilibrium of the subgame.

The other cases are perfectly symmetric. QED

Proof of the Theorem: W.l.o.g. let $u^1 \leq u^2$.

Theorems 9 and 10 imply that if s is a coalition perfect equilibrium and $N_k^1(s) + N_k^2(s) > 0$ $\forall$ $k \in \{A, B\}$ then $p_A^1(s) = p_B^1(s) \leq u^1 - u^2$, $p_A^2(s) = p_B^2(s) = -p_A^1(s)$ and $N_k^1(s) = N_k^2(s) = 1/2$ $\forall$ $k \in \{A, B\}$. But then $u^2 > 2u^1$ implies $p_A^2(s) > u^2/2$, contradicting that s is a Nash equilibrium.

Furthermore $p_A^k(s) > u^k$ for some $k \in \{1,2\}$ contradicts that s is a Nash equilibrium. These observations and theorems 9 and 10 establish that there cannot be any other coalition perfect equilibria than those in the claim. All that remains to show is that all the outcomes in the claim of are coalition perfect equilibrium outcomes.

Consider $s \in S$ such that $p_A^1(s) = p_B^1(s) = -p_A^2(s) = -p_B^2(s)$, $p_A^j(s) \leq u^j$ $\forall\, j \in \{1,2\}$, if $u^1 < u^2$ then $p_A^1(s) \leq u^1 - u^2$,

and consumers' strategies are the following:

-$c_i^j(s) = A$ $\forall\, j = 1,2$ and $i \in [0,1]$

-all consumers join A in subgames following price announcements $(p_A^1(s), p_A^2(s), q_1', q_2')$ such that $q_1' \geqslant p_A^1(s) - u^1$, $q_2' \geqslant p_2(s) - u^2$ and it is not the case that both $q_1' < p_A^1(s)$ and $q_2' < p_A^2(s)$

-all consumers join B in subgames following price announcements $(p_1', p_2', p_B^1(s), p_B^2(s))$ such that $(p_1', p_2') \neq (p_A^1(s), p_A^2(s))$, $p_k' \geqslant p_A^k(s) - u^k$ $\forall\, k \in \{1,2\}$ and it is not the case that $p_1' < p_A^1(s)$ and $p_2' < p_A^2(s)$

-consumers play some arbitrary coalitionally rationalizable Nash equilibrium in every other consumer subgame.

By lemmas 9.1 and 11.1 the consumers play a coalitionally rationalizable Nash equilibrium in every subgame in the above profile. Consider deviations (q_1', q_2') by B. If $(q_1', q_2') << (p_A^1(s), p_A^2(s))$ then all consumers join B and B's profit is negative. If $q_1' < p_A^1(s) - u^1$ and $p_A^1(s) < 0$ then since no consumer on side 2 joins him if $q_2' > u^2$, he ends up with negative profit. If $q_1' < p_A^1(s) - u^1$ and $p_A^1(s) \geqslant 0$ then the starting assumption implies $u^1 = u^2$. Then since no consumer on side 2 joins B if $q_2' > p_A^2(s) + u^2$, B's profit again has to be negative. A similar argument establishes that setting $q_2' < p_A^2(s) - u^2$ yields a negative profit. After every other deviation B gets a zero market share, which concludes that neither firm has a profitable deviation and therefore s is a coalition perfect equilibrium. Similar arguments establish that A does not have a profitable deviation, which concludes that s is a coalition perfect equilibrium.

Consider now $s' \in S$ such that $p_A^1(s') = p_B^1(s') = -p_A^2(s') = -p_B^2(s')$, $p_A^j(s') \leq u^j$ $\forall\, j \in \{1,2\}$, if $u^1 < u^2$ then $p_A^1(s') \leq u^1 - u^2$,

and consumers' strategies are the following:

-$c_i^j(s') = B$ $\forall\, j = 1,2$ and $i \in [0,1]$

-all consumers join A in subgames following price announcements $(p_A^1(s'), p_A^2(s'), q_1', q_2')$ such that $(q_1', q_2') \neq (p_B^1(s'), p_B^2(s'))$, $q_1' \geqslant p_A^1(s') - u^1$, $q_2' \geqslant p_2(s') - u^2$ and it is not the case that both $q_1' < p_1(s')$ and $q_2' < p_2(s')$

-all consumers join B in subgames following price announcements $(p_1', p_2', p_B^1(s'), p_B^2(s'))$ such that $p_k' \geqslant p_A^k(s') - u^k$ $\forall\, k \in \{1,2\}$ and it is not the case that $p'_1 < p^1_A(s')$ and $p'_2 < p^2_A(s')$

-consumers play some arbitrary coalitionally rationalizable Nash equilibrium in every other consumer subgame.

Similar arguments as above establish that s' is a coalition perfect equilibrium.

Assume now that $u^2 \leq 2u^1$ and consider $s \in S$ such that $p^1_A(s) = p^1_B(s) = -p^2_A(s) = -p^2_B(s)$, $p^j_A(s) \leq u^j/2\ \forall\, j \in \{1,2\}$, if $u^1 < u^2$ then $p^1_A(s) \leq u^1 - u^2$ and:

-$c^1_i(s) = c^2_i(s) = A\ \forall\ i \in [0, 1/2]$ and $c^1_i(s) = c^2_i(s) = A\ \forall\ i \in (1/2, 1]$

-all consumers join A in subgames following price announcements $(p^1_A(s), p^2_A(s), q'_1, q'_2)$ such that $(q'_1, q'_2) \neq (p^1_A(s), p^2_A(s))$, $q'_1 \geqslant p^1_A(s) - u^1$, $q'_2 \geqslant p^2_A(s) - u^2$ and it is not the case that both $q'_1 < p^1_A(s)$ and $q'_2 < p^1_A(s)$

-all consumers join B in subgames following price announcements $(p'_1, p'_2, p^1_B(s), p^2_B(s))$ such that $(p'_1, p'_2) \neq (p^1_A(s), p^2_A(s))$, $p'_1 \geqslant p^1_B(s) - u$, $p'_2 \geqslant p^2_B(s) - u$ and it is not the case that both $p'_1 < p^1_B(s)$ and $p'_2 < p^2_B(s)$

-consumers play some arbitrary coalitionally rationalizable Nash equilibrium in every other consumer subgame.

Again Lemma 9.1 and Lemma 11.1 establish that s specifies a coalitionally rationalizable Nash equilibrium in every subgame. Furthermore, it is straightforward to check that no firm has a profitable deviation, establishing that s is a coalition perfect equilibrium. QED

Proof of Claim 2 Define s the following way.

For $p^1_A, p^2_A, p^1_B, p^2_B$ such that neither $(p^1_A, p^2_A) = (0.31, -0.2)$, nor $(p^1_B, p^2_B) = (-0.2, 0.31)$, let s specify any coalitionally rationalizable Nash equilibrium in the subgame following price announcements $(p^1_A, p^2_A, p^1_B, p^2_B)$. By lemma 1 there exists a strategy like that.

Let $p^1_A, p^2_A, p^1_B, p^2_B$ be such that $(p^1_A, p^2_A) = (0.31, -0.2)$ and $(p^1_B, p^2_B) \neq (-0.2, 0.31)$. Let $G^c = (C, S^c, u^c)$ be the subgame following price announcements $(p^1_A, p^2_A, p^1_B, p^2_B)$. Let $\widehat{S}^c$ denote the set of coalitionally rationalizable strategies of G^c. By lemma 1 $\widehat{S}^c \neq \emptyset$. Then let $s^c \in \widehat{S}^c$ be such that $(s^c)^j_i = A$ if $A \in (\widehat{S}^c)^j_i$ and $(s^c)^j_i = B$ only if $B = (\widehat{S}^c)^j_i$. It is easy to establish that s^c is a coalitionally rationalizable Nash equilibrium of G^c. Let s specify s^c in G^c.

Let $p^1_A, p^2_A, p^1_B, p^2_B$ be such that $(p^1_A, p^2_A) \neq (0.31, -0.2)$ and $(p^1_B, p^2_B) = (-0.2, 0.31)$. Let $G^c = (C, S^c, u^c)$ be the subgame following price announcements $(p^1_A, p^2_A, p^1_B, p^2_B)$. Let $\widehat{S}^c$ denote the set of coalitionally rationalizable strategies of G^c. By lemma 1 $\widehat{S}^c \neq \emptyset$. Then let $s^c \in \widehat{S}^c$ be such that

$(s^c)_i^j = B$ if $B \in (\widehat{S}^c)_i^j$ and $(s^c)_i^j = A$ only if $A = (\widehat{S}^c)_i^j$. It is easy to establish that s^c is a coalitionally rationalizable Nash equilibrium of G^c. Let s specify s^c in G^c.

In the subgame following the (candidate equilibrium) price announcements $(0.31, -0.2, -0.2, 0.31)$, the only rationalizable and therefore only coalitionally rationalizable strategy is A for every $C_i^1 \in T_M^1 \cup T_L^1$. Similarly, the only rationalizable and therefore only coalitionally rationalizable strategy is B for every $C_i^2 \in T_M^2 \cup T_L^2$. But for every $C_i^1 \in T_H^1$, and every conjecture $\theta_{-1,i} \in \Theta_{-1,i}$ such that $\theta_{2,j}(A) = 1 \ \forall\ C_j^1 \in T_M^1 \cup T_L^1$ it holds that $A \in BR_{1,i}(\theta_{-1,i})$, which establishes that A is a coalitionally rationalizable strategy for C_i^1. A similar argument establishes that for every $C_i^2 \in T_H^2$ it holds that B is a coalitionally rationalizable strategy for C_i^2. This concludes that s specifies a coalitionally rationalizable profile in this subgame. It is straightforward to check that this profile is also a Nash equilibrium of the subgame.

Since by construction s specifies a coalitionally rationalizable Nash equilibrium after any other vector of price announcements, we conclude that in every consumer subgame s specifies a coalitionally rationalizable Nash equilibrium.

Note that $\pi_A(s) = \pi_B(s) = 0.4 \times 0.31 - 0.2 \times 0.6 = .00\,4$.

Consider deviations for B.

1. pricing above $(0.31, -0.2)$

If $s'_B \in S_B$ is such that $p_B^1(s'_B) \geq 0.31$ and $p_B^2(s'_B) \geq -0.2$, then by construction $\pi_B(s'_B, s_{-B}) = 0$, since in the subgame following the above price announcements whenever B is a coalitionally rationalizable strategy for $C_i^j \in C$, so is A.

2. undercutting on both sides

2.a. Consider $s'_B \in S_B$ such that $0.31 > p_B^1(s'_B) > 0.15$ and $p_B^2(s'_B) < -0.2$. Then $c_i^1(s'_B, s_{-B}) = B \ \forall\ C_i^1 \in T_I^1 \cup T_{II}^1 \cup T_{III}^1$, $c_i^1(s'_B, s_{-B}) = \emptyset \ \forall\ C_i^1 \in T_{IV}^1$, and $c_i^2(s'_B, s_{-B}) = B \ \forall\ C_i^2 \in C^2$. The supremum of $\pi_B(s'_B, s_{-B})$ is $0.31 \times 0.65 - 0.2 = .00\,15$ and therefore these deviations are unprofitable.

2.b. Consider $s'_B \in S_B$ such that $0.31 = p_B^1(s'_B)$ and $p_B^2(s'_B) < -0.2$ or $0.31 > p_B^1(s'_B)$ and $p_B^2(s'_B) = -0.2$. Then it is straightforward to show that $\pi_B(s'_B, s_{-B}) < .00\,15$.

2.c. Consider $s'_B \in S_B$ such that $p_B^1(s'_B) \leq 0.15$ and $p_B^2(s'_B) < -0.2$. Then $c_i^1(s'_B, s_{-B}) = B \ \forall\ C_i^1 \in T_I^1 \cup T_{II}^1 \cup T_{III}^1$, $c_i^1(s'_B, s_{-B}) \in \{B, \emptyset\} \ \forall\ C_i^1 \in T_{IV}^1$, and $s_i^2(0.31, -0.2, p^1, p^2) = B \ \forall\ C_i^2 \in C^2$. Then $\pi_B(s'_B, s_{-B}) < 0.15 - 0.2 = -0.05$.

3. undercutting on side 1 and increasing price on side 2

3.a. Consider $s_B' \in S_B$ such that $0.31 > p_B^1(s_B') > 0.15$ and $p_B^2(s_B') > -0.2$. Then $c_i^1(s_B', s_{-B}) = A \; \forall \; C_i^1 \in T_I^1 \cup T_{II}^1 \cup T_{III}^1$, $c_i^1(s_B', s_{-B}) = \emptyset \; \forall \; C_i^1 \in T_{IV}^1$, and $c_i^2(s_B', s_{-B}) = A \; \forall \; C_i^2 \in C^2$. Therefore $\pi_B(s_B', s_{-B}) = 0$.

3.b. Consider $s_B' \in S_B$ such that $p_B^1(s_B') \leq 0.15$ and $-0.1475 > p_B^2(s_B') \geq -0.2$. Then $c_i^1(s_B', s_{-B}) = B \; \forall \; C_i^1 \in C^1$ and $c_i^2(s_B', s_{-B}) = B \; \forall \; C_i^2 \in C^2$. Then $\pi_B(s_B', s_{-B}) < 0.15 - 0.1475 = .00\,25$.

3.c. Consider $s_B' \in S_B$ such that $0 < p_B^1(s_B') \leq 0.15$ and $-0.1475 \leq p_B^2(s_B')$. Then $c_i^1(s_B', s_{-B}) = A \; \forall \; C_i^1 \in T_I^1 \cup T_{II}^1 \cup T_{III}^1$, $c_i^1(s_B', s_{-B}) = \emptyset \; \forall \; C_i^1 \in T_{IV}^1$ and $c_i^2(s_B', s_{-B}) = \emptyset, \forall \; C_i^2 \in C^2$. Therefore $\pi_B(s_B', s_{-B}) = 0$.

3.d. Consider $s_B' \in S_B$ such that $-0.15 \leq p_B^1(s_B') \leq 0$ and $-0.1475 \leq p_B^2(s_B')$. Then $c_i^1(s_B', s_{-B}) = A \; \forall \; C_i^1 \in T_I^1 \cup T_{II}^1 \cup T_{III}^1$, $c_i^1(s_B', s_{-B}) = \emptyset \; \forall \; C_i^1 \in T_{IV}^1$ and $c_i^2(s_B', s_{-B}) = \emptyset, \forall \; C_i^2 \in C^2$. The $\pi_B(s_B', s_{-B}) \leq 0$.

3.e. Consider $s_B' \in S_B$ such that $-.2 \leq p_B^1(s_B') < -0.15 = 0.31 - 0.46$ and $p_B^2(s_B') < -.132\,5 = -0.2 + 0.15 \times 0.45$. Then $c_i^1(s_B', s_{-B}) = B \; \forall \; C_i^1 \in C^1$ and $c_i^2(s_B', s_{-B}) = B \; \forall \; C_i^2 \in C^2$. Then $\pi_B(s_B', s_{-B}) < -0.15 - 0.132\,5 < 0$.

3.f. Consider $s_B' \in S_B$ such that $-.2 \leq p_B^1(s_B') < -0.15$ and $p_B^2(s_B') \geq -.132\,5$. Then $c_i^2(s_B', s_{-B}) \neq B \; \forall \; C_i^2 \in C^2$ and $\pi_B(s_B', s_{-B}) < 0$.

3.g. Consider $s_B' \in S_B$ such that $0.31 - 0.35 \times 2.55 = -.582\,5 \leq p_B^1(s_B') < -0.2$ and $p_B^2(s_B') < -0.098 = -0.2 + 0.51 \times 0.2$. Then $\pi_B(s_B', s_{-B}) < 0$.

3.h. Consider $s_B' \in S_B$ such that $-.582\,5 \leq p_B^1(s_B') < -0.2$ and $-0.098 \leq p_B^2(s_B') < 0.31 = -0.2 + 2.55 \times 0.2$. Then $c_i^1(s_B', s_{-B}) = A \; \forall \; C_i^1 \in T_I^1$, $c_i^1(s_B', s_{-B}) = B \; \forall \; C_i^1 \in T_{II}^1 \cup T_{III}^1 \cup T_{IV}^1$, $c_i^2(s_B', s_{-B}) = B \; \forall \; C_i^2 \in T_I^2$ and $c_i^2(s_B', s_{-B}) = B \; \forall \; C_i^2 \in T_{II}^2 \cup T_{III}^2 \cup T_{IV}^2$. Then $\pi_B(s_B', s_{-B}) < 0$.

3.i. Consider $s_B' \in S_B$ such that $-.582\,5 \leq p_B^1(s_B') < -0.2$ and $p_B^2(s_B') \geq 0.31$. Then $c_i^2(s_B', s_{-B}) \neq B \; \forall \; C_i^2 \in C^2$ and $\pi_B(s_B', s_{-B}) < 0$.

3.j. Consider $s_B' \in S_B$ such that $0.31 - 0.45 \times 2.55 = -.837\,5 \leq p_B^1(s_B') < -0.582\,5$ and $p_B^2(s_B') < 0.076 = -0.2 + 0.6 \times 0.46$. Then $c_i^1(s_B', s_{-B}) = B \; \forall \; C_i^1 \in C^1$, $c_i^2(s_B', s_{-B}) = B \; \forall \; C_i^2 \in T_I^2 \cup T_{II}^2 \cup T_{III}^2$ and $c_i^2(s_B', s_{-B}) = A \; \forall \; C_i^2 \in T_{IV}^2$. Then $\pi_B(s_B', s_{-B}) < -0.582\,5 + 0.076 \times 0.65 = -.533\,1$.

3.k. Consider $s_B' \in S_B$ such that $-.837\,5 \leq p_B^1(s_B') < -0.582\,5$ and $p_B^2(s_B') \geq 0.076$. Then $c_i^2(s_B', s_{-B}) \neq B \; \forall \; C_i^2 \in C^2$ and $\pi_B(s_B', s_{-B}) < 0$.

3.l. Consider $s_B' \in S_B$ such that $0.31 - 0.6 \times 2.55 = -1.22 \leq p_B^1(s_B') < -.837\,5$ and $p_B^2(s_B') < .106 = -0.2 + 0.51 \times 0.6$. Then $c_i^1(s_B', s_{-B}) = B \; \forall \; C_i^1 \in C^1$, $c_i^2(s_B', s_{-B}) = B \; \forall \; C_i^2 \in T_I^2 \cup T_{II}^2$ and $c_i^2(s_B', s_{-B}) = A \; \forall \; C_i^2 \in T_{III}^2 \cup T_{IV}^2$. Then $\pi_B(s_B', s_{-B}) < -0.8375 + 0.106 \times 0.55 = -.779\,2$.

3.m. Consider $s_B' \in S_B$ such that $-1.22 \leq p_B^1(s_B') < -.837\,5$ and $0.31 > p_B^2(s_B') \geq .106$. Then $c_i^1(s_B', s_{-B}) = A \; \forall \; C_i^1 \in T_I^1$, $c_i^1(s_B', s_{-B}) = B \; \forall \; C_i^1 \in T_{II}^1 \cup T_{III}^1 \cup T_{IV}^1$, $c_i^2(s_B', s_{-B}) = B \; \forall \; C_i^2 \in T_I^2$ and $c_i^2(s_B', s_{-B}) = B$

$\forall\ C_i^2 \in T_{II}^2 \cup T_{III}^2 \cup T_{IV}^2$. Then $\pi_B(s_B', s_{-B}) < -0.8375 \times 0.6 + 0.31 \times 0.4 = -.378\,5$.

3.n. Consider $s_B' \in S_B$ such that $-1.22 \leq p_B^1(s_B') < -.837\,5$ and $0.31 \leq p_B^2(s_B')$. Then $c_i^2(s_B', s_{-B}) \neq B\ \forall\ C_i^2 \in C^2$ and $\pi_B(s_B', s_{-B}) < 0$.

3.o. Consider $s_B' \in S_B$ such that $0.31 - 2.55 = -2.\,24 \leq p_B^1(s_B') < -1.22$ and $-0.2 + 2.55 \times 0.6 = 1.33 > p_B^2(s_B')$. Then $c_i^1(s_B', s_{-B}) = A\ \forall\ C_i^1 \in T_I^1$, $c_i^1(s_B', s_{-B}) = B\ \forall\ C_i^1 \in T_{II}^1 \cup T_{III}^1 \cup T_{IV}^1$, $c_i^2(s_B', s_{-B}) = B\ \forall\ C_i^2 \in T_I^2$ and $c_i^2(s_B', s_{-B}) = A\ \forall\ C_i^2 \in T_{II}^2 \cup T_{III}^2 \cup T_{IV}^2$. Then $\pi_B(s_B', s_{-B}) < -1.22 + 1.33 \times 0.4 = -.\,688$.

3.p. Consider $s_B' \in S_B$ such that $-2.\,24 \leq p_B^1(s_B') < -1.22$ and $1.33 \leq p_B^2(s_B')$. Then $c_i^2(s_B', s_{-B}) \neq B\ \forall\ C_i^2 \in C^2$ and $\pi_B(s_B', s_{-B}) < 0$.

3.q. Consider $s_B' \in S_B$ such that $p_B^1(s_B') < -2.24$ and $2.24 > p_B^2(s_B')$. Then $c_i^1(s_B', s_{-B}) = B\ \forall\ C_i^1 \in C^1$, therefore $\pi_B(s_B', s_{-B}) < 0$.

3.r. Consider $s_B' \in S_B$ such that $p_B^1(s_B') < -2.24$ and $2.24 \leq p_B^2(s_B') < 2.\,35 = -.2 + 2.55$. Then $c_i^1(s_B', s_{-B}) = B\ \forall\ C_i^1 \in C^1$, $c_i^2(s_B', s_{-B}) = B\ \forall\ C_i^2 \in T_I^2$ and $c_i^2(s_B', s_{-B}) = A\ \forall\ C_i^2 \in T_{II}^2 \cup T_{III}^2 \cup T_{IV}^2$. Then $\pi_B(s_B', s_{-B}) < -2.24 + 2.35 \times 0.4 = -1.\,3$.

3.s. Consider $s_B' \in S_B$ such that $p_B^1(s_B') < -2.24$ and $2.35 \leq p_B^2(s_B')$. Then $c_i^2(s_B', s_{-B}) \neq B\ \forall\ C_i^2 \in C^2$ and $\pi_B(s_B', s_{-B}) < 0$.

4. undercutting on side 2 and increasing price on side 1

4.a. Consider $s_B' \in S_B$ such that $0.31 \leq p_B^1(s_B')$ and $-0.2 - 0.65 \times 0.15 = -.\,297\,5 \leq p_B^2(s_B') < -0.2$. Then $c_i^1(s_B', s_{-B}) \neq B\ \forall\ C_i^1 \in C^1$, $c_i^2(s_B', s_{-B}) \neq B\ \forall\ C_i^2 \in C^2$. Therefore $\pi_B(s_B', s_{-B}) = 0$.

4.b. Consider $s_B' \in S_B$ such that $0.31 \leq p_B^1(s_B') < 0.46$ and $p_B^2(s_B') < -0.297\,5$. Then $c_i^1(s_B', s_{-B}) = B\ \forall\ C_i^1 \in T_I^1 \cup T_{II}^1 \cup T_{III}^1$, $c_i^1(s_B', s_{-B}) = \emptyset\ \forall\ C_i^1 \in T_{IV}^1$, $c_i^2(s_B', s_{-B}) = B\ \forall\ C_i^2 \in C^2$. Then $\pi_B(s_B', s_{-B}) < 0.46 \times 0.65 - 0.297\,5 = .00\,15$.

4.c. Consider $s_B' \in S_B$ such that $0.31 \leq p_B^1(s_B') < 0.46$ and $p_B^2(s_B') \geq -0.297\,5$. Then $c_i^1(s_B', s_{-B}) \neq B\ \forall\ C_i^1 \in C^1$, $c_i^2(s_B', s_{-B}) \neq B\ \forall\ C_i^2 \in C^2$. Therefore $\pi_B(s_B', s_{-B}) = 0$.

4.d. Consider $s_B' \in S_B$ such that $0.46 \leq p_B^1(s_B') < .\,488\,5 = 0.31 + 0.51 \times 0.35$ and $p_B^2(s_B') < -.\,455 = -0.2 - 0.1 \times 2.55$. Then $c_i^1(s_B', s_{-B}) = B\ \forall\ C_i^1 \in T_I^1 \cup T_{II}^1$, $s_i^1(0.31, -0.2, p^1, p^2) = \emptyset\ \forall\ C_i^1 \in T_{III}^1 \cup T_{IV}^1$, $c_i^2(s_B', s_{-B}) = B\ \forall\ C_i^2 \in C^2$. Then $\pi_B(s_B', s_{-B}) < 0.488\,5 \times 0.55 - .\,455 = -0.186\,33$.

4.e. Consider $s_B' \in S_B$ such that $0.46 \leq p_B^1(s_B') < .\,488\,5$ and $p_B^2(s_B') \geq -.\,455$. Then $c_i^1(s_B', s_{-B}) \neq B\ \forall\ C_i^1 \in C^1$, $c_i^2(s_B', s_{-B}) \neq B\ \forall\ C_i^2 \in C^2$. Therefore $\pi_B(s_B', s_{-B}) = 0$.

4.f. Consider $s_B' \in S_B$ such that $0.\,488\,5 \leq p_B^1(s_B') < 0.82 = 2.55 \times 0.2 + 0.31$ and $-.\,531\,5 > p_B^2(s_B')$. Then $c_i^1(s_B', s_{-B}) = B\ \forall\ C_i^1 \in T_I^1$,

$c_i^1(s'_B, s_{-B}) = \emptyset\ \forall\ C_i^1 \in T_{II}^1 \cup T_{III}^1 \cup T_{IV}^1$, $c_i^2(s'_B, s_{-B}) = B\ \forall\ C_i^2 \in C^2$. Then $\pi_B(s'_B, s_{-B}) < 0.82 \times 0.4 - .531\,5 = -0.203\,5$.

4.g. Consider $s'_B \in S_B$ such that $0.488\,5 \leq p_B^1(s'_B) < 0.82 = 2.55 \times 0.2 + 0.31$ and $-.531\,5 > p_B^2(s'_B)$. Then $c_i^1(s'_B, s_{-B}) \neq B\ \forall\ C_i^1 \in C^1$ and if $p_B^2(s'_B) > 0$ then $c_i^2(s'_B, s_{-B}) \neq B\ \forall\ C_i^2 \in C^2$. Therefore $\pi_B(s'_B, s_{-B}) \leq 0$.

4.h. Consider $s'_B \in S_B$ such that $0.82 \leq p_B^1(s'_B) < 1.202\,5 = 0.31 + 2.55 \times 0.35$ and $p_B^2(s'_B) < -.837\,5 = -0.2 - 0.25 \times 2.55$. Then $c_i^1(s'_B, s_{-B}) = B\ \forall\ C_i^1 \in T_I^1$, $c_i^1(s'_B, s_{-B}) = \emptyset\ \forall\ C_i^1 \in T_{II}^1 \cup T_{III}^1 \cup T_{IV}^1$, $c_i^2(s'_B, s_{-B}) = B\ \forall\ C_i^2 \in C^2$. Then $\pi_B(s'_B, s_{-B}) < 1.202\,5 \times 0.4 - 0.837\,5 = -0.356\,5$.

4.i. Consider $s'_B \in S_B$ such that $0.82 \leq p_B^1(s'_B) < 1.202\,5 = 0.31 + 2.55 \times 0.35$ and $p_B^2(s'_B) \geq -.837\,5$. Then $c_i^1(s'_B, s_{-B}) \neq B\ \forall\ C_i^1 \in C^1$ and if $p_B^2(s'_B) > 0$ then $c_i^2(s'_B, s_{-B}) \neq B\ \forall\ C_i^2 \in C^2$. Therefore $\pi_B(s'_B, s_{-B}) \leq 0$.

4.j. Consider $s'_B \in S_B$ such that $1.202\,5 \leq p_B^1(s'_B) < 2.55$ and $p_B^2(s'_B) < -1.857\,5 = -.2 - 2.55 \times 0.65$. Then $c_i^1(s'_B, s_{-B}) = B\,\forall\,C_i^1 \in T_I^1$, $c_i^1(s'_B, s_{-B}) = \emptyset\,\forall\,C_i^1 \in T_{II}^1 \cup T_{III}^1 \cup T_{IV}^1$, $c_i^2(s'_B, s_{-B}) = B\,\forall\,C_i^2 \in C^2$. Then $\pi_B(s'_B, s_{-B}) < 2.55 \times 0.4 - 1.857\,5 = -0.837\,5$.

4.k. Consider $s'_B \in S_B$ such that $1.202\,5 \leq p_B^1(s'_B) < 2.55$ and $p_B^2(s'_B) \geq -1.857\,5$. Then $c_i^1(s'_B, s_{-B}) \neq B\ \forall\ C_i^1 \in C^1$ and if $p_B^2(s'_B) > 0$ then $c_i^2(s'_B, s_{-B}) \neq B\ \forall\ C_i^2 \in C^2$. Therefore $\pi_B(s'_B, s_{-B}) \leq 0$.

4.l. Consider $s'_B \in S_B$ such that $2.55 \leq p_B^1(s'_B)$. Then $c_i^1(s'_B, s_{-B}) \neq B\ \forall\ C_i^1 \in C^1$ and if $p_B^2(s'_B) > 0$ then $c_i^2(s'_B, s_{-B}) \neq B\ \forall\ C_i^2 \in C^2$. Therefore $\pi_B(s'_B, s_{-B}) \leq 0$.

This establishes that there is no profitable deviation for B. Perfectly symmetric arguments establish that there is no profitable deviation for A.

This concludes that s is a coalition perfect equilibrium. QED

Proof of Theorem 12 Let s be a coalition perfect equilibrium.

Suppose first that $N_k^j(s) = 0$ for some $j \in \{1,2\}$ and $k \in \{A,B\}$. W.l.o.g. assume $k = A$ and $j = 1$. Then either $N_A^1(s) = N_A^2(s) = 0$ or $N_A^2(s) > 0$ and $p_A^1(s) = 0$. In either case $\pi_A = 0$ and then by the starting assumption $\pi_B(s) > 0$. Let $\widehat{C} = \{C_i^j : c_i^j(s) = B\}$. Note that $p_B^j(s) < 0$ for some $j \in \{1,2\}$ implies that $C^j \subset \widehat{C}$. Consider now deviation $(p_B^1(s) - \varepsilon, p_B^2(s) - \varepsilon)$ for $\varepsilon > 0$ by A. Similar arguments as in Lemma 9.1 establish that in the subgame after this deviation A is the unique coalitionally rationalizable strategy for every $C_i^j \in \widehat{C}$. But then for small enough ε the deviation is profitable, a contradiction. Therefore $N_k^j(s) > 0\ \forall\ j \in \{1,2\}$ and $k \in \{A,B\}$.

If $p_A^j(s) > p_B^j(s)\ \forall\ j = 1, 2$ then similar arguments as in 9.1 establish that $N_A^1(s) + N_A^2(s) = 0$, contradicting the above result. Similarly it cannot be that $p_A^j(s) < p_B^j(s)\ \forall\ j = 1, 2$.

Consider now $p_A^j(s) = p_B^j(s)\ \forall\ j = 1, 2$. Let $\widehat{C} = \{C_i^j : c_i^j(s) \neq \emptyset\}$. There exists $k \in \{A, B\}$ such that $\pi_k(s) \leq (\pi_1(s) + \pi_2(s))/2 > 0$. W.l.o.g. assume $k = A$. Consider deviation $(p_B^1(s) - \varepsilon, p_B^2(s) - \varepsilon)$ by A. Similar arguments as in Lemma 9.1 establish that in the subgame after this deviation A is the unique coalitionally rationalizable strategy for every $C_i^j \in \widehat{C}$. Therefore if ε is small enough then after this deviation A's profit is larger than $(\pi_1(s) + \pi_2(s))/2$ (note that $p_B^j(s) < 0$ for some $j = 1, 2$ implies that $C^j \subset \widehat{C}$), a contradiction.

Finally, notice that if $p_A^j(s) \leq p_B^j(s)$ for some $j \in \{1, 2\}$, then $N_B^j(s) > 0$ and the assumption that s is a Nash equilibrium imply that $N_B^{3-j}(s) \geq N_A^{3-j}(s)$. If $p_A^j(s) < p_B^j(s)$ for some $j \in \{1, 2\}$, then $N_B^j(s) > 0$ and the assumption that s is a Nash equilibrium imply that $N_B^{3-j}(s) > N_A^{3-j}(s)$. Similarly if $p_B^j(s) \leq p_A^j(s)$ (correspondingly $p_B^j(s) < p_A^j(s)$) for some $j \in \{1, 2\}$, then $N_B^{3-j}(s) \leq N_A^{3-j}(s)$ (correspondingly $N_B^{3-j}(s) < N_A^{3-j}(s)$). QED

Lemma 2 Consider any $G \in \Gamma$ in which $g_i^j(N^j, N^{-j}) = g_i^j(M^j, N^{-j})\ \forall\ N^j, N^{-j}, M^{-j} \in [0, 1]$. Then for every $G^c = (C, S^c, U^c) \in \Gamma^c$ it holds that $\exists\ s \in S^c$ such that s is a coalition proof Nash equilibrium of G^c. Furthermore, if $s \in S^c$ is a coalition proof Nash equilibrium of G^c, then s is also a coalitionally rationalizable Nash equilibrium.

Proof of Lemma 2 Let $G = (I, S, U) \in \Gamma$. Let $G^c = (C, S^c, U^c) \in \Gamma^c(G)$. Analogously to Ambrus(02), define game $\widehat{G} = (\widehat{C}, \widehat{S}, \widehat{U})$ to be a restriction of G^c iff

(i) $\widehat{C} = C$
(ii) $\widehat{S}_i^j \subset S_i^{cj}\ \forall\ i \in I,\ j = 1, 2$
(iii) $\widehat{U}_i^j$ is a restriction of U_i^{cj} on $\widehat{S}_i^j$

First we establish that for every $G \in \Gamma$ and every $G^c \in \Gamma^c(G)$ it holds that every restriction of G^c has a coalition-proof Nash equilibrium on pure strategies.

Consider first $G \in \Gamma^M$. Let G^c be a consumer subgame which follows $n(s_A) = 1$. Let $\widehat{G} = (\widehat{C}, \widehat{S}, \widehat{U})$ be a restriction of G^c.

Let $\widehat{R}(1) \subset C$ be the set of players for whom choosing 1 is rationalizable in $\widehat{G}$. Define $s \in \widehat{S}$ such that $s_i^j = 1$ iff $C_i^j \in \widehat{R}(1)$. Note that for every $C_i^j \in \widehat{R}(1)$ either $\widehat{S}_i^j = \{1\}$ or $\widehat{U}_i^j(s) \geq 0$, otherwise choosing 1 cannot be rationalizable for C_i^j (given that the starting assumption on g_i^j implies that C_i^j cannot have a rationalizable expectation in $\widehat{G}$ that gives him higher expected payoff in case of choosing 1 than the one that allocates probability 1 that every other player in $R^s(1)$ chooses 1). But then there cannot be any coalitional deviation t from s by any $B \subset C$ for which $s_i^j \neq t_i^j$ for some $C_i^j \in \widehat{R}(1)$. Furthermore, there cannot be any coalitional deviation t from s by any $B \subset C$ for which $s_i^j \neq t_i^j$ for some $C_i^j \notin \widehat{R}(1)$, because then 1 would be a rationalizable choice for C_i^j, a contradiction.

Next, let G^c be a consumer subgame of G which follows $n(s_A) = 2$ and let $\widehat{G} = (\widehat{C}, \widehat{S}, \widehat{U})$ be a restriction of G^c. Let $\widehat{R}$ be the set of pure strategy rationalizable profiles in $\widehat{G}$.

Let $\mathcal{X}_1$ denote the collection of sets $X \subset C$ for which it holds that for every $Y \subset X, Y \neq X$ it holds that $\exists\ Z \subset X/Y, Z \neq \emptyset$ such that for every $C_i^j \in Z$ it holds that (i) $1 \in \widehat{S}_i^j$, (ii) $\widehat{U}_i^j(s) > \widehat{U}_i^j(t)\ \forall\ s,t \in \widehat{R}$ such that $s_{i'}^{j'} = 1$ $\forall\ C_{i'}^{j'} \in Y \cup Z$, $t_i^j \neq 1$ and $\widehat{U}_{i'}^{j'}(t) \geq 0\ \forall\ C_{i'}^{j'} \in C$ such that $s_{i'}^{j'} = 2$. It is straightforward to establish that $X, X' \in \mathcal{X}_1$ implies $X, X' \in \mathcal{X}_1$. Therefore $X_1^* = \underset{X \in \mathcal{X}_1}{\cup} X \in \mathcal{X}_1$. Let $\mathcal{Y}_2$ be the collection of sets $Y \subset C/X_1^*$ such that for every $C_i^j \in Y$ it holds that (i) $2 \in \widehat{S}_i^j$ (ii) either $\widehat{S}_i^j = \{2\}$ or $\widehat{U}_i^j(s) \geq 0\ \forall$ $s \in \widehat{S}$ such that $s_{i'}^{j'} = 2\ \forall\ C_{i'}^{j'} \in Y$. Let $Y_2^* = \underset{Y \in \mathcal{Y}_2}{\cup} Y$. It is easy to establish that $Y_2^* \in \mathcal{Y}_2$. Now define $\widehat{s} \in \widehat{S}$ such that $s_i^j = 1$ if $C_i^j \in X_1^*$, $s_i^j = 2$ if $C_i^j \in Y_2^*$ and $s_i^j = \emptyset$ if $C_i^j \in C/(Y_2^* \cup X_1^*)$. Suppose there is a profitable coalitional deviation $s \in \widehat{S}$ from $\widehat{s}$ by $\widetilde{C} \subset C$. But $X_1^* \cap \widetilde{C} \neq \emptyset$ contradicts the definition of X_1^*. That implies that $C/X_1^* \cap \widetilde{C} \neq \emptyset$ contradicts the definition of Y_2^*, a contradiction. Since there is no profitable coalitional deviation from $\widehat{s}$, it is a coalition-proof Nash equilibrium. Note that since G^c is itself a restriction of G^c, it has a coalition proof Nash equilibrium.

Proving the claim for $G^c \in \Gamma^c(G)$ if $G \in \Gamma^D$ can be done exactly the same way as for the previous set of subgames in the case of $G \in \Gamma^M$.

It is straightforward to generalize Claim 8 in Ambrus(02) to normal form games with infinite number of players (the proof is essentially the same). Then the claim established above implies that in every consumer subgame G^c of every $G \in \Gamma$ it holds that every coalition proof Nash equilibrium is a coalitionally rationalizable Nash equilibrium. QED

Proof of Claim 3 The proof of Claim 1 establishes that the subgame following price announcements $(0.0032-\varepsilon, 0.5608-2\varepsilon, 0.604-2\varepsilon, 0.004-\varepsilon)$ for some $\varepsilon > 0$ has a unique coalitionally rationalizable profile s^c

such that $(s^c)^1_i = 2 \ \forall\ i \in [0, 0.16]$, $(s^c)^1_i = 1 \ \forall\ i \in (0.16, 1]$, $(s^c)^2_i = 1 \ \forall\ i \in [0, 0.2]$ and $(s^c)^2_i = 2 \ \forall\ i \in (0.2, 1]$. By Lemma 2 this is the unique coalition proof Nash equilibrium of the subgame. This establishes that in any coalition proof equilibrium $s \in S$ it holds that $\pi_A(s) \geq 0.214\,72$.

Let now $(p^1_1, p^2_1, p^1_2, p^2_2)$ such that for some $j, k \in \{1, 2\}$ $p^j_k, p^{3-j}_{3-k} < 0.02$, $p^j_k + 0.68 \geq p^j_{3-k} > p^j_k + 0.6$ and $p^{3-j}_{3-k} + 0.68 \geq p^{3-j}_k \geq p^{3-j}_{3-k} + 0.557\,6$ and $(p^j_k, p^{3-j}_k, p^j_{3-k}, p^{3-j}_{3-k}) \neq (0.003\,2, 0.560\,8, 0.604, 0.004)$. Consider subgame $G^c = (C, S^c, u^c)$ following price announcements $(p^1_1, p^2_1, p^1_2, p^2_2)$. Let $s^c \in S^c$ be a Nash equilibrium of G^c. As shown in the proof of Claim 1, if s^c is such that $(s^c)^j_i = k$ for some $i \in (0.16, 0.2]$ or $(s^c)^{3-j}_i = 3 - k$ for some $i \in (0.16, 0.2]$ then for any $s \in S$ such that $p_A(s_A) = (p^1_1, p^2_1, p^1_2, p^2_2)$ and s_{-A} specifies s^c in G^c it holds that $\pi_A(s_A) < 0.214\,72$. Suppose s^c is such that $(s^c)^j_i = k$ implies $i \in [0, 0.16]$ and $(s^c)^{3-j}_i = 3 - k$ implies $i \in [0, 0.16]$. Let $\widehat{C} = \{C^j_i \in C : C^j_i \neq k\} \cup \{C^{3-j}_i \in C : i \in [0, 0.2],\ C^{3-j}_i \neq k\}$. Let $t^c \in S^c$ such that $(t^c)^l_i = k$ if $C^l_i \in \widehat{C}$ and $(t^c)^l_i = (s^c)^l_i$ otherwise. It is straightforward to establish that t^c is a profitable deviation for $\widehat{C}$ from s^c and that there is no further profitable deviation from it by a subcoalition of $\widehat{C}$. This concludes that if $s \in S$ is such that $p^j_k(s) = p^j_k \ \forall\ j, k \in \{1, 2\}$ and s_{-A} specifies a coalitionally rationalizable Nash equilibrium in every $G^c \in \Gamma^c$ then $\pi_A(s_A) < 0.214\,72$.

Following the same arguments as in the proof of Claim 3 it is straightforward to establish that if s is such that: (1) there are no $j, k \in \{1, 2\}$ such that $p^j_k, p^{3-j}_{3-k} < 0.02$, $p^j_k + 0.68 \geq p^j_{3-k} > p^j_k + 0.6$ and $p^{3-j}_{3-k} + 0.68 \geq p^{3-j}_k \geq p^{3-j}_{3-k} + 0.557\,6$, and (2) s_{-A} specifies a coalitionally rationalizable Nash equilibrium in every $G^c \in \Gamma^c$, then $\pi_A(s_A) < 0.214\,72$. Lemma 2 then implies that if s is a coalition proof equilibrium and there are no $j, k \in \{1, 2\}$ such that $p^j_k, p^{3-j}_{3-k} < 0.02$, $p^j_k + 0.68 \geq p^j_{3-k} > p^j_k + 0.6$ and $p^{3-j}_{3-k} + 0.68 \geq p^{3-j}_k \geq p^{3-j}_{3-k} + 0.557\,6$, then $\pi_A(s_A) < 0.214\,72$, a contradiction.

The above imply that if s is a coalition proof equilibrium, then $\exists\ j, k \in \{1, 2\}$ such that $(p^j_k(s), p^{3-j}_k(s), p^j_{3-k}(s), p^{3-j}_{3-k}(s)) = (0.003\,2, 0.560\,8, 0.604, 0.004)$. Furthermore, if $s_A \in S_A$ is such that $(p^j_k(s_A), p^{3-j}_k(s_A), p^j_{3-k}(s_A), p^{3-j}_{3-k}(s_A)) = (0.003\,2, 0.560\,8, 0.604, 0.004)$ then it is easy to construct $s_{-A} \in S_{-A}$ such that (s_A, s_{-A}) is a coalition proof equilibrium. In the subgame following $p_A(s_A)$ let s_{-A} specify s^c such that $(s^c)^j_i = k \ \forall\ i \in (0.16, 1]$, $(s^c)^j_i = 3 - k \ \forall\ i \in [0, 0.16]$, $(s^c)^{3-j}_i = k \ \forall\ i \in [0, 0.2]$ and $(s^c)^{3-j}_i = 3 - k \ \forall\ i \in (0.2, 1]$. For every other $G^c \in \Gamma^c(G)$ let s_{-A} specify some coalition proof Nash equilibrium. By Lemma 2 there exists a profile like that. QED

Proof of Theorem 13 Straightforward from Lemma 2, since in every coalition-proof equilibrium s of every $G \in \Gamma$ it holds that in every $G^c \in \Gamma^c(G)$ consumers play some coalition-proof Nash equilibrium, which is also a coalitionally rationalizable Nash equilibrium. Furthermore, s is a Nash equilibrium of G and therefore s_A is a best response to s_{-A}. This concludes the claim. QED

Proof of Theorem 14 The same arguments used in the proof of Theorems 9-11 establish that for every coalition perfect equilibria $s \in S$ the following hold. First, if $N_k^1(s) + N_k^2(s) > 0$ for some $k \in \{A, B\}$ then $p_k^1(s) + p_k^2(s) = 2m$. Second, if $N_k^1(s) + N_k^2(s) > 0 \ \forall\ k \in \{A, B\}$ then $p_A^j(s) = p_B^j(s)$ and $N_A^j(s) = N_B^j(s) = 1/2 \ \forall\ j \in \{1, 2\}$. Then the same arguments used in the proof of Lemma 9.2 establish that for $m < u/4$ it has to be that $N_A^j(s) = N_B^j(s) = 1 \ \forall\ j \in \{1, 2\}$ and the same arguments used in the proof of Theorem 11 establish that there exist equilibria of type (1) and (2) in the claim.

It follows from the above that if $N_k^1(s) + N_k^2(s) > 0$ and $N_{-k}^1(s) + N_{-k}^2(s) = 0$ for some $k \in \{A, B\}$ then either $N_k^1(s) = N_k^2(s) = 1$, or $p_k^1(s) = p_k^2(s) = m$ and $N_k^1(s) = N_k^2(s) = m/u$ (otherwise either the consumers or firm k are not in equilibrium). Assume now that $m < u/3$, $p_k^1(s) = p_k^2(s) = m$, $N_k^1(s) = N_k^2(s) = m/u$ and $N_{-k}^1(s) + N_{-k}^2(s) = 0$. It cannot be that $p_A^j(s) + p_B^j(s) < u \ \forall\ j = 1, 2$ since the same arguments as in the proof of Lemma 9.2 establish that in any subgame following price announcement $(p_A^1, p_A^2, p_B^1, p_B^2)$ such that $p_A^j + p_B^j < u \ \forall\ j = 1, 2$ it holds that $\emptyset$ is not a coalitionally rationalizable strategy for any $C_i^j \in C$. It cannot be that $p_B^j(s) < 0$ because then $\emptyset$ is not a coalitionally rationalizable strategy for any $C_i^j \in C^j$. Finally, it cannot be that $p_B^1(s)+p_B^2(s) > 2m$, because then for small enough $\varepsilon > 0$ the deviation $(\min(u-\varepsilon, p_B^1(s)-\varepsilon), \min(u-\varepsilon, p_B^1(s)-\varepsilon))$ by A would yield positive profit and therefore be profitable. But it is easy to establish that the three requirements above cannot hold simultaneously if $m < u/3$, a contradiction.

All that remains to show is that type (3) equilibria exist if $m \geq u/4$ and that type (4) equilibria exist if $m \geq u/3$.

For the first, define $s \in S$ $s \in S$ such that $p_A^1(s) = p_B^1(s) = 2m$, $p_A^2(s) = p_B^2(s) = 0$ and consumers' strategies are the following:

-$c_i^j(s) = \emptyset \ \forall\ j = 1, 2$ and $i \in [0, 1]$

-all consumers join A in subgames following price announcements ($p_A^1(s)$, $p_A^2(s)$, q_1', q_2') such that $q_1' \geqslant p_A^1(s) - u$, $q_2' \geqslant p_2(s) - u$ and it is not the case that both $q_1' < p_A^1(s)$ and $q_2' < p_A^2(s)$

-all consumers join B in subgames following price announcements $(p_1', p_2', p_B^1(s), p_B^2(s))$ such that $(p_1', p_2') \neq (p_A^1(s), p_A^2(s))$, $p_k' \geqslant p_A^k(s) - u \; \forall \; k \in \{1, 2\}$ and it is not the case that $p_1' < p_A^1(s)$ and $p_2' < p_A^2(s)$

-consumers play some arbitrary coalitionally rationalizable Nash equilibrium in every other consumer subgame.

The same arguments as in Lemma 11.1 establish that s specifies a coalitionally rationalizable Nash equilibrium after the equilibrium price announcements. The same arguments then in the proof of Theorem 11 then establish that s specifies a coalitionally rationalizable Nash equilibrium in every other consumer subgame and that there is no profitable deviation for the firms.

For the second, define $t \in S$ such that $p_A^1(t) = p_A^2(t) = m$, $p_B^1(t) = 2m$, $p_B^2(t) = 0$ and consumers' strategies are the following:

-$c_i^j(t) = A \; \forall \; j = 1, 2$ and $i \in [0, m/u]$, $c_i^j(t) = \emptyset \; \forall \; j = 1, 2$ and $i \in (m/u, 1]$

-all consumers join A in subgames following price announcements $(p_A^1(t), p_A^2(t), q_1', q_2')$ such that $q_1' \geqslant p_A^1(t) - u$, $q_2' \geqslant p_2(t) - u$ and it is not the case that both $q_1' < p_A^1(t)$ and $q_2' < p_A^2(t)$

-all consumers join B in subgames following price announcements $(p_1', p_2', p_B^1(t), p_B^2(t))$ such that $(p_1', p_2') \neq (p_A^1(t), p_A^2(t))$, $p_k' \geqslant p_A^k(t) - u \; \forall \; k \in \{1, 2\}$ and it is not the case that $p_1' < p_A^1(t)$ and $p_2' < p_A^2(t)$

-consumers play some arbitrary coalitionally rationalizable Nash equilibrium in every other consumer subgame.

Similar considerations that used above to show that s is a coalition perfect equilibrium establish that t is a coalition perfect equilibrium. QED

References

[1] AMBRUS, A. (2002): "Coalitional rationality," *mimeo Princeton University*

[2] AMBRUS, A. (2003): "Dynamic coalitional agreements in multi-stage games," *mimeo Princeton University and Yale University*

[3] ARMSTRONG, M. (2002): "Competition in Two-Sided Markets," *mimeo UCL.*

[4] BAUMOL, W.J., J. PANZAR and R. WILLIG (1982): "*Contestable Markets and the Theory of Industry Structure",* Harcourt Brace, Jovanovich, New York, NY.

[5] BENTAL, B and M. SPIEGEL (1995): "Network Competition, Product Quality, and Market Coverage in the Presence of Network Externalities," *Journal of Industrial Economics,* 43, 197-208.

[6] BERNHEIM, B.D., B. PELEG and M.D.WHINSTON (1987): "Coalition-Proof Nash Equilibria I. Concepts," *Journal of Economic Theory,* 42, 1-12.

[7] CAILLAUD, B. and B. JULLIEN (2003): "Chicken & Egg: Competition Among Intermediation Service Providers," *Rand Journal of Economics,* 34(2), 309-328.

[8] CAILLAUD, B. and B. JULLIEN (2001): "Competing Cybermediaries," *European Economic Review,* 45, 797-808.

[9] DAMIANO, E. and H. LI (2003a): "Price discrimination in matching markets," *mimeo University of Toronto*

[10] DAMIANO, E. and H. LI (2003b): "Competing Matchmaking," *mimeo University of Toronto.*

[11] ECONOMIDES, N. (1996): "The Economics of Networks," *International Journal of Industrial Organization*, 14, 673-699

[12] ECONOMIDES, N. and SKRZYPACZ, A. (2003) : "Standards Coalitions Formation and Market Structure in Network Industries," *Working Paper no. EC-03-08, Stern School of Business, N.Y.U.*

[13] ELLISON, G., D. FUDENBERG and M. MOBIUS (2002): "Competing Auctions," *mimeo MIT and Harvard University.*

[14] ELLISON, G. and D. FUDENBERG (2003) : "Knife-Edge or Plateau: When Do Market Models Tip?," *mimeo Harvard University and MIT.*

[15] FARRELL, J. and P. KLEMPERER (2001): "Coordination and Lock-In: Competition with Switching Costs and Network Effects," *mimeo University of California at Berkeley and Nuffield College, Oxford University.*

[16] INNES R. and R. SEXTON (1993): "Customer Coalitions, Monopoly Price Discrimination, and Generic Entry Deterrrence," *European Economic Review,* 37, 1569-1597.

[17] JULLIEN, B. (2001): "Competing in Network Industries: Divide and Conquer," *mimeo IDEI Tolouse.*

[18] KATZ, M. and C. SHAPIRO (1994): "System Competition and Network Effects", *Journal of Economic Perspectives,* 8, 93-115.

[19] MASKIN, E and J. RILEY (1984): "Monopoly and Incomplete Information," *RAND Journal of Economics,* 15, 171-196.

[20] MUSSA, M and S. ROSEN (1978): "Monopoly and Product Quality," *Journal of Economic Theory,* 18, 301-317.

[21] ROCHET, J.C. and J. TIROLE (2002): "Cooperation among Competitors: Some Economics of Payment Card Associations," *RAND Journal of Economics,* 33, 549-570.

[22] ROCHET, J.C. and J. TIROLE (2003): "Platform Competition in Two-Sided Markets," *Journal of the European Economic Association,* 1(4), 990-1029.

[23] SALANIE, B. (1997): *"The economics of contracts: A primer,"* MIT Press, Cambridge, MA.

[24] SEGAL, I. (2003): "Coordination and Discrimination in Contracting with Externalities: Divide and Conquer?" *Journal of Economic Theory* 113(2), pp. 147-327.

[25] VARIAN, H. (1988): "Price Discrimination," in Schmalensee, R. and R. Willig (editors) *Handbook of Industrial Organization,* Elsevier, Amsterdam.