Rental of a durable good[1]

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8 May 2005

[1]Cyril Hariton would like to acknowledge the Marie-Curie fellowship HPMF-CT-2002-02172 of the European Commission as well as the two following institutions: CORE (Université Catholique de Louvain, Belgium) and GREMAQ (University of Toulouse, France), for their hosting and their support.

Abstract

How should the owner of a durable good rent it to agents who desire to use it for different lengths of time? This question is important for many network industries: there are short run and long run users of gaz pipelines, and airports must choose between giving a particular slot to a regular airline or to keep it open for irregular charter flights. In order to examine this question, we build an infinite horizon stationary model where a monopoly seller rents a good. At each period, a number of potential buyers appear, with different lengths of demand. We study and compare the mechanisms that would be used by a profit maximizing and by a social welfare maximizing seller. We show that, in some precise sense of the term, a profit maximizing seller will favor long term renters.

JEL *codes*: C73, D44, D82, L96
Keywords: Auctions, Durable good, Dynamic games, Mechanism design, Reservation

Owners of infrastructure must often choose between providing access to consumers who desire to use it for different lengths of time. In this paper, we study this problem in a dynamic context and compare the strategies that would be used by a profit maximizing monopolist and a social welfare maximizing regulator.

We became interested in the problem while studying access to the Internet infrastructure. McKie-Mason and Varian have proposed to use Vickrey auctions, which they call "smart markets", to solve congestion problems in this network (McKie-Mason and Varian 1995, 1996, 1997). These auctions allocate resources efficiently and induce participants to reveal their true willingness to pay.[1] However, if some sellers want to use the network for an extended period of time, smart markets loose their efficiency inducing property, as we have shown in Crémer and Hariton (1999). Indeed, smart markets allocate access independently at each period of time: an agent who wishes to use the network for two consecutive periods must participate in two separate auctions. If he wins the first, he has no guarantee that he will have access during the second period. This problem is important for applications such as Internet video conferences where some users need a good quality connection over an extended period of time.

Similar problems arise in most network industries. Following the liberalization process in Europe,[2] vertically integrated industries such as railroad, electricity or gaz are progressively separated in independent undertakings: a network owner and service providers. The manager of the rail tracks must arbitrate between freight, which has irregular demand, and passenger trains, which have regular schedule and want a commitment that they will be allowed to use a specific time slot for several months. In the gaz and electricity industries, the manager of the transportation network must decide how much capacity to guarantee to a user, knowing that this might prevent him from accepting the request of a future user, who may have a higher willingness to pay.

Other transportation industries face similar concerns. For instance, an airport must arbitrate between leaving a landing slot available for (irregular) charter flights or committing it to an airline which wants to use it every day for regular flights. There exists a vast literature on congestion pricing in transportation in-

[1] In a few words, the proposed implementation of "smart markets" consists of allocating access freely in non-congested periods and, at each period of congestion, through standard Vickrey auctions. Customers attach a value to each packet of their messages. If the router faces congestion, packets with the highest valuations are selected up to the capacity. These packets are routed through the node of the network and their sender are required to pay the valuation attached to the first rejected packet.

[2] Several directives have modified the development of network industries in the European Community, such as directives 2001/12/EC, 2001/13/EC and 2001/14/EC for the rail sector [europa.eu.int/comm/transport/rail/legislation/legi_en.htm – see europa.eu.int/scadplus/leg/en/lvb/l24057.htm for a summary] or directive 96/92 for the electricity market [europa.eu.int/comm/energy/en/elec_single_market/index_en.html].

dustries (see Arnott, de Palma and Lindsey 1999 for an extensive bibliography). Following the seminal paper of Vickrey (1969), the attention has been mainly focused on agents with heterogeneous reservation prices; as far as we know, no paper has considered consumers with different lengths of use.

In order to study this problem, we consider a durable good which can be used by only one agent at the time. In each period when the good is not used, potential users vie for the right to use it; they are defined by two characteristics: their willingness to pay and the length of time during which they want to use the good (which we will call the *length of demand* or *horizon*). The owner of the asset must choose one of the users, or decide to leave the good unused in the hope of finding a "better" user in the future. We simplify the problem by assuming only the willingness to pay is private information; the lengths of demand are known by the seller. information.

We show that, at equilibrium, the auctioneer expects a per period benefit B. Each bidder with valuation v who competes to get the good for the i next periods has a net value for the auctioneer of $v - \left(1 + \delta + \ldots + \delta^{i-1}\right) B$ where δ is the discount rate. Introducing asymmetry of information with respect to the willingness to pay v adds the usual "informational rent" term that has to be deduced from v, which in turn modifies the equilibrium value of B, but leaves unchanged the qualitative nature of the allocation procedure. According to this procedure, the per period benefit B corresponds to a rental rate that the seller asks bidders to pay for each period of the rental of the good over the first one. This rental rate also plays an important role in the comparison by the auctioneer between short term and long term bidders, which we study in some detail. In particular, we show that a profit maximizing seller distorts the allocation in favor of the bidders with the longer demands (theorem 6).

Our framework has some relationship with the sequential auctions literature, where an auctionner sells sequentially different units of the same good, or different goods (Milgrom and Weber (1982), Weber (1983) and Maskin and Riley (1989)). Our framework introduces a major difference: if the auctioneer rents the good for more than one period, this prevents some futuer buyers from purchasing it. There is therefore competition between bidders that appear in different periods.

Our problem is also close to the multi-unit auctions studied by Branco (1995, 1996): In the multi-unit literature, an auctioneer with two goods to sell must compare separate offers for each of the two goods to offers for the bundle of both of them. There is a certain analogy to the model of the present paper, if we consider the rental of the good in different periods as different goods. Again, the type of competition is different as in our model the buyers of good 2 are not yet present when good 1 or the bundle are sold.

The paper is organized as follows. Section 2 describes the basic model and solves the problem when the bidders' willingness to pay is known to the auctioneer and section 3 studies the allocation procedure with asymmetric information. Section 4 compares the treatment of long run and short run buyers by a social welfare maximizing planner and a profit maximizing monopolist. Section 5 focuses on the distortion of the monopolist's allocation procedure auctioneer with respect to the social planner's policy. Section 6 concludes. All proofs are collected in the appendix.

1 The model

1.1 Demand

In each period $t = 0, \ldots, +\infty$, where the good is free, a set $\mathbb{I}$ of agents compete to obtain the right to use it. The "names" of these agents are the lengths of their demands, so that agent $i \in \mathbb{I}$ requires the good for i periods. Implicit in this notation is the assumption that the different bidders have different lengths of demand. This assumption lightens considerably the notation, and does not change our economically interesting results. We call I be the longest length of demand, so that $\mathbb{I} \subset \{1, \ldots, I\}$. For simplicity we will also denote by $\mathbb{I}$ the cardinality of the set $\mathbb{I}$, that is the number of bidders. For instance if $\mathbb{I} = \{2, 5, 6\}$, in each period where the good is free, there are bidders of lengths of demand 2, 5 and 6. We have $I = 6$ and, by abuse of notation, $\mathbb{I} = 3$.

The value that agent i attaches to the good, hereafter also called his *type* or his *valuation*, is $v_i \in [\underline{v}_i, \overline{v}_i]$ with $\underline{v}_i \geqslant 0$. Agent i wants the good for exactly i consecutive periods starting in the period in which he arrives on the market. In particular, his willingness to pay for less than i periods is zero, and if he arrives in period t his willingness to pay for the use of the good in periods $t+1$ to $t+i$ is also zero.

All agents, the buyers and the seller, have the same common knowledge discount factor δ. When selling the good to agent i in period t, with $i > 1$, the seller commits to sell it for i periods and not to renegotiate the allocation while the buyer is using it.[3]

1.2 Objective functions and informational constraints

A risk neutral seller allocates the use of the good, and we assume that there is no cost to doing so. We will distinguished two possible objective functions for the seller: maximizing profits or maximizing social welfare. We will also make

[3]We take this as an hypothesis, but it could easily be endogeneized as part of the optimal policies, by assuming that there is a high enough set up cost for each buyer when he begins to use the good.

two different informational assumptions: either the seller has full information and knows the types of the buyers, or he does not know them. This yields four models; three of them yield the same allocations, and it is only when the seller tries to maximize profits *and* does not know the types of the agents that we find distortions compared to the social optimum.

Denote π the infinite stream of expected benefits at any period t where the good is not committed. We will consider two different kinds of owners of the asset: a firm and a regulator. A "firm" maximizes profits and its "period utility" in a period where the good can be sold is equal to the total expected payment it receives

$$\pi = \sum_{i\in\mathbb{I}} \mathrm{E}_{v_i}\left[t_i\left(v_i\right)\right].$$

A "regulator", or "social planner", maximizes social welfare. Her period utility when the good can be sold is

$$\pi = \sum_{i\in\mathbb{I}} \mathrm{E}_{v_i}\left[v_i q_i\left(v_i\right)\right].$$

The mechanism used by the seller can be described by the functions $\{p_i, t_i\}_{i\in\mathbb{I}}$ where $p_i(v_i, v_{-i})$ is the probability that agent i of type v_i is given the object and $t_i\left(v_i\right)$ is his payment[4]. His expected probability of obtaining the good is $q_i\left(v_i\right) \equiv \mathrm{E}_{v_{-i}}\left[p_i\left(v\right)\right]$.

2 Maximizing social welfare with symmetric information

In this section, we begin by assuming that the seller can observe the types of the bidders when they appear, although, of course, she does not know the types of future bidders. We also assume that she maximizes social welfare.

The auctioneer cares about future expected welfare, and she takes into account the fact that by selling to agent i, the good will be available for use by another agent only in i periods. Therefore using Belman's principle of optimality, the maximal social welfare under symmetric information is the solution of the equation

$$\pi = V(\pi),$$

where $V(\pi)$ is the value of the problem $\mathcal{P}_1$

$$\max_{\{p_i(.),t_i(.)\}_{i\in\mathbb{I}}} \sum_{i\in\mathbb{I}} \left\{\mathrm{E}_{v_i}\left[t_i\left(v_i\right)\right] + \mathrm{E}_v\left[p_i\left(v\right)\delta^i\pi\right]\right\} + \left(1 - \sum_{i\in\mathbb{I}} \mathrm{E}_v\left[p_i\left(v\right)\right]\right)\delta\pi \tag{1}$$

[4]In principle, the payment could depend on the types or announced types of the other agents, but there is no loss of generality in assuming this possibility away.

subject to the constraints[5]

$$U_i(v_i) = q_i(v_i)\, v_i - t_i(v_i) \geqslant 0 \forall v_i, \qquad \text{for all } i, \qquad (IR_i)$$

$$p_i(v) \in [0,1] \qquad \text{for all } v, \text{ and all } i, \qquad (P_i)$$

$$\sum_{i \in \mathbb{I}} p_i(v) \leqslant 1 \qquad \text{for all } v, \qquad (P_0)$$

The individual rationality constraint (IR_i) translates the fact that the principal cannot force the agent to partic

2.1 Optimal allocations

Maximizing profits induces the auctioneer to extract as much as possible from agents, making the individual rationality constraint (IR_i) binding

$$\forall i, \forall v_i, \quad U_i(v_i) = 0 \quad \Leftrightarrow \quad t_i(v_i) = v_i q_i(v_i).$$

The objective function becomes

$$\pi = \max_{\{p_i(.)\}_{i \in \mathbb{I}}} \left[\sum_{i \in \mathbb{I}} \left\{ \mathrm{E}_v \left[\left[v_i - \delta \left(1 - \delta^{i-1}\right) \pi \right] p_i(v) \right] + \delta \pi \right\} \right].$$

A regulator has the same objective function. The solution of problem $\mathcal{P}_1$ is described by the following lemma, which is proved in the appendix.

Lemma 1. *$\{p_i^*(.), t_i^*(.)\}_i$ is a solution of the maximization problem $\mathcal{P}_1$ if and only if p_i^* is solution of the following problem*

$$W(\pi) \equiv \max_{\{p_i(.)\}_{i \in \mathbb{I}}} \left[\sum_{i \in \mathbb{I}} \mathrm{E}_v \left[\left[v_i - \delta \left(1 - \delta^{i-1}\right) \pi \right] p_i(v) \right] \right] + \delta \pi \tag{2}$$

subject to

$$\begin{cases} \forall i, \forall v, & p_i(v) \in [0,1], & (P_i) \\ \forall v, & \sum_{i \in \mathbb{I}} p_i(v) \leqslant 1 & (P_0), \end{cases}$$

and

$$\forall i, \forall v_i, \qquad t_i^*(v_i) = \mathrm{E}_{v_{-i}} \left[v_i p_i^*(v) \right].$$

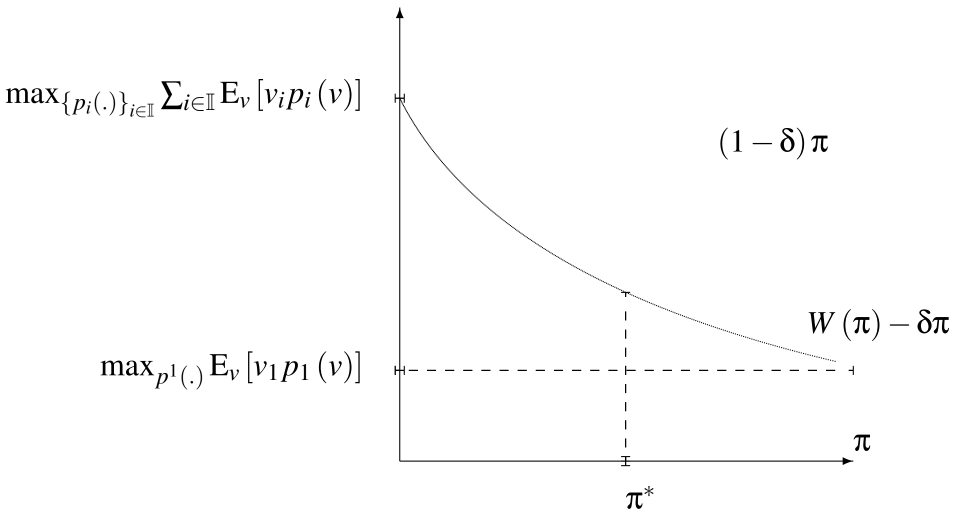

Figure 1: Optimal stationary profit with perfect information (when $1 \in \mathbb{I}$)

For a given π, the maximand is a simple sum of known coefficients multiplied by the probability that the mechanism designer has to choose. Therefore, for a given π, the buyer for whom $v_i - \delta\left(1-\delta^{i-1}\right)\pi$ is the highest obtains the good with probability 1. To determine π^*, we will use the following lemma, also proved in the appendix.

Lemma 2. *The expected per period benefit* $[W(\pi)-\delta\pi]$ *is decreasing and convex, with* $W(0)>0$.

As shown in figure 1, lemma 2 implies that $[W(\pi)-\delta\pi]$ crosses once the strictly increasing function $(1-\delta)\pi$, for $\pi=\pi^*$. We have proved the following proposition.

Proposition 1. *Under perfect information, the mechanism set by a profit maximizing seller or a social planner allocates the good to (one of) the agents with the highest positive* $[v_i - \delta\left(1-\delta^{i-1}\right)\pi^*]$. *If in any period* $[v_i - \delta\left(1-\delta^{i-1}\right)\pi^*]$ *is strictly negative for all i, then the good is not sold in this period.*

In the case $n_2 = 1$, $n_i = 0$ for all $i \neq 2$, valuation uniformly distributed on $[0,1]$, we find that the optimal mechanism leaves the good free if v_2 is less than a function which is well approximated by $.27 \times \delta$.

In the following subsection, we analyze some of the properties of these allocations. **implementation through second prize auctions.**

2.2 The rental rate

When there is no asymmetry of information, profit maximizing and efficient mechanisms are identical. Remember that π^* is the discounted social welfare associated with the infinite repeated allocation of the good, starting from a period in which the good is available. Then, when she has to choose one user, i.e. in a period when the good is available, proposition 1 tells the auctioneer either to allocate the good to (one of) the bidders with the highest

$$v_i + \delta^i \pi^*$$

if this quantity is greater than $\delta\pi^*$ and not to allocate the good otherwise.

Now, choosing the buyer with the greatest $v_i + \delta^i \pi^*$ is equivalent to choosing the one with the highest

$$\begin{aligned} \tilde{v}_i &\equiv v_i - [\pi^*] + \delta^i \pi^* \\ &= v_i - \left(1 + \delta + \ldots + \delta^{i-1}\right)(1-\delta)\pi^*. \end{aligned} \tag{3}$$

Thus, the seller rents the good at a price of $(1-\delta)\pi^*$, and chooses the bidder whose willingness to pay yields the highest benefit above the discounted sum of the *rental rate* over the whole duration of the agent's need. Note that it is not the profit per period which is taken into account by the allocation rule.

The condition

$$v_i + \delta^i \pi^* \geq \delta\pi^* \tag{4}$$

is equivalent to

$$\tilde{v}_i \geq -(1-\delta)\pi^*.$$

This expression can easily be interpreted. By not renting the good, the auctioneer incurs a cost (a loss of benefits) of $(1-\delta)\pi^*$, the one period rental rate. The surplus obtained by renting the good to agent i must be greater than this loss. If there is an agent 1 (i.e., if $1 \in \mathbb{I}$), for that agent (4) is equivalent to $v_1 \geq 0$, and the good is rented in every state of nature, which, of course is not true if $1 \notin \mathbb{I}$).

2.3 Comparative statics

We now show that if the distribution of agent types becomes unambiguously more favorable, then the optimal allocation mechanism changes in a way, to be made precise later, which favors short run buyers. Let us call $\tilde{F}_i(v_i)$ and $F_i(v_i)$ two cumulative distributions such that $\tilde{F}_i$ first degree stochastically dominates F_i: $\tilde{F}_i(v_i) \leqslant F_i(v_i)$ for all v_i with strict inequality for some values. Whatever π, $v_i - \delta\left(1-\delta^{i-1}\right)\pi$ is increasing in v_i, and therefore

$$\mathrm{E}_{v_i \sim \tilde{F}_i}\left[v_i - \delta\left(1-\delta^{i-1}\right)\pi\right] > \mathrm{E}_{v_i \sim F_i}\left[v_i - \delta\left(1-\delta^{i-1}\right)\pi\right].$$

Thus, the valuation the regulator attaches to bidder i is in average higher with the cumulative distribution $\tilde{F}_i$. Clearly, a shift from F_i to $\tilde{F}_i$ improves the welfare of the auctioneer, and we must have $\tilde{W}(\pi) > W(\pi)$ for all π, where we denote by a $\tilde{}$ the quantities corresponding to the distribution $\tilde{F}_i$. From figure 1, it is straightforward that $\tilde{\pi}^* > \pi^*$.[6]

We are now ready to prove the following proposition.

Proposition 2. *Assume that when the distribution function of the type of agent i buyers is F_i, and when the vector of types of other agents is v_{-k}, agent k of type v_k gets the good with probability zero, either because it is not allocated or because it is allocated to an agent in a* smaller *demand length. Then, when the distribution of the valuation of class i buyers is $\tilde{F}_i$, which first degree stochastically dominates F_i, in the same state of nature, agent k obtains the good with probability zero.*

Note that the proposition puts no constraint on the relationship between i and k. Its proof is straightforward. First, $v_k \leqslant \delta\left(1-\delta^{k-1}\right)\pi^*$ implies $v_k \leqslant \delta\left(1-\delta^{k-1}\right)\tilde{\pi}^*$, which implies that if the good is not allocated to agent k with F_i, it is not with $\tilde{F}_i$. Second,

$$v_k - \delta\left(1-\delta^{k-1}\right)\pi^* \leqslant v_r - \delta\left(1-\delta^{r-1}\right)\pi^*$$

for some elements v_r of v_{-k} with $r < k$ implies

$$v_k - \delta\left(1-\delta^{k-1}\right)\tilde{\pi}^* \leqslant v_r - \delta\left(1-\delta^{r-1}\right)\tilde{\pi}^*$$

which concludes the proof.

Corollary 1. *If in a state of nature v the good is not allocated when the distribution function of the type of agent i is F_i, then it is not when the distribution is $\tilde{F}_i$.*

For a given state of nature v, moving from F_i to $\tilde{F}_i$ lowers the overall probability to allocate de good because the seller, who expects a higher per period benefit, is more reluctant to rent the good to these same bidders v.

3 Asymmetric information

We now assume that there is asymmetry of information: the seller does not know the type of the bidders (howeverwe assume that he knows the lengths of demand[7]).

[6] Formally: $(1-\delta)\tilde{\pi}^* = \tilde{W}(\tilde{\pi}^*) - \delta\tilde{\pi}^* > W(\tilde{\pi}^*) - \delta\tilde{\pi}^*$, and because $(1-\delta)\pi$ is increasing in π while $W(\pi) - \delta\pi$ is decreasing, the result is straightforward.

[7] There are some circumstances under which the seller can obtain revelation of the length of demand as no cost. Consider for instance the case of a video conference. It may be possible to observe when it finishes, and the buyer might have no interest in renting for too short a time, if the conference has no value if it is cut short.

It is relatively straightforward to show that the presence of asymmetric information does not change the optimal mechanism for the regulator, who tries to maximize social welfare. We will therefore assume, unless stated explicitly, that the seller is a profit maximizing firm.

There are very few changes that need to be made to the model of section 2 in order to adapt the model. First, we interpret the v_is in the definition of the mechanism as the announced types of the sellers, as opposed to their types. Second, relying on the revelation principle, we add to the model an incentive compatibility constraint per bidder,

$$\forall v_i, \forall i, \qquad v_i = \arg\max_{\tilde{v}_i} \left\{ q_i(\tilde{v}_i)\, v_i - t_i(\tilde{v}_i) \right\} \qquad (IC_i)$$

Then, the optimal π is solution of

$$\pi = R(\pi)$$

where $R(\pi)$ is the value of the following problem $\mathcal{P}_2$, which is obtained by adding to problem $\mathcal{P}_1$ the incentive compatibility constraints.

$$U(\pi) = \max_{\{p_i(.),t_i(.)\}_{i\in\mathbb{I}}} \left[\sum_{i\in\mathbb{I}} \left\{ \mathrm{E}_{v_i}\left[X_i(v_i)\right] + \mathrm{E}_v\left[p_i(v)\,\delta^i\pi\right] \right\} + \left(1 - \sum_{i\in\mathbb{I}} \mathrm{E}_v\left[p_i(v)\right]\right) \delta\pi \right].$$

3.1 Allocation mechanisms

How will the presence of asymmetric information change the conclusions of the last section? Using Myerson's (1981) methodology, the set of individual rationality and incentive compatibility constraints of problem $\mathcal{P}_2$ can be rewritten as described in the following lemma.

Lemma 3. *A mechanism is incentive compatible and individually rational if and only if*

$$\forall i, \forall v_i, \forall s_i, \qquad (v_i - s_i)\left[q_i(v_i) - q_i(s_i)\right] \geqslant 0 \qquad (5)$$

$$\forall i, \forall v_i, \qquad U_i(v_i) = U_i(\underline{v}_i) + \int_{\underline{v}_i}^{v_i} q_i(x)\, dx \qquad (6)$$

$$\forall i, \qquad U_i(\underline{v}_i) \geqslant 0 \qquad (7)$$

(6)

Using lemma 3, problem $\mathcal{P}_2$ can be rewritten as described in the following lemma, whose proof can be found in the appendix.

Lemma 4. $\{p_i^{**}(.), t_i^{**}(.)\}_i$ *is a solution of the maximization program* $\mathcal{P}_2$ *if and only if* p_i^{**} *is the solution of the system of equations*

$$\begin{cases} V(\pi) \equiv \max_{\{p_i(.)\}_{i\in\mathbb{I}}} \left[\sum_{i\in\mathbb{I}} \mathrm{E}_v \left[\left[t_i(v_i) - \delta\left(1-\delta^{i-1}\right)\pi\right] p_i(v)\right]\right] + \delta\pi \\ V(\pi) = \pi \end{cases} \tag{8}$$

subject to

$$\begin{cases} \forall i, \forall v_i, \forall s_i, & (v_i - s_i)\left[q_i(v_i) - q_i(s_i)\right] \geqslant 0 & (IC_i') \\ \forall i, \forall v, & p_i(v) \in [0,1] & (P_i) \\ \forall v, & \sum_{i\in\mathbb{I}} p_i(v) \leqslant 1 & (P_0) \end{cases}$$

and

$$\forall i, \forall v_i, \qquad t_i^{**}(v_i) = \mathrm{E}_{v_{-i}}\left[v_i p_i^{**}(v) - \int_{\underline{v}_i}^{v_i} p_i^{**}(x, v_{-i})\, dx\right]. \tag{9}$$

As in the perfect information case, for a given π, the maximization problem ends up to be a simple sum of known coefficients multiplied by the probability that the mechanism designer has to choose. Thus, the object is allocated to agent with the highest positive coefficient and the following lemma proves the existence of a solution.

Lemma 5. *The expected per period benefit* $[V(\pi) - \delta\pi]$ *is decreasing and convex in* π*, with* $V(0) > 0$.

Thus the solution to the auctioneer allocation problem is given by the following rule.[8]

Proposition 3. *Under asymmetric information, a profit maximizing seller allocates the good to (one of) the agents within the highest positive* $J_i(v_i) - \delta(1 - \delta^{i-1})\pi^{**}$. *If in any period* $J_i(v_i) - \delta(1-\delta^{i-1})\pi^{**}$ *is strictly negative for all i, she does not sell the good in this period.*

In the case $n_2 = 1$, $n_i = 0$ for all $i \neq 2$, valuation uniformly distributed on $[0,1]$, we find that the optimal mechanism leaves the good free if v_2 is less than a function which is well approximated by $0.5 + 0.086\,\delta$.

3.2 Another rental rate

Proposition 3, the seller chooses the buyer i who maximizes

$$J_i(v_i) + \delta^i \pi^{**}$$

[8] If the problem is not regular, i.e. $J_i(v_i)$ is not increasing for all v_i and i, one can adapt this result by "making it regular", as done in Myerson (1981).

as long as this quantity is positive. This is equivalent to choosing agent i so as to maximize

$$\begin{aligned}\tilde{\tilde{v}}_i &\equiv J_i(v_i) - [\pi^{**}] + \delta^i \pi^{**} \\ &= J_i(v_i) - \left(1+\delta+\ldots+\delta^{i-1}\right)(1-\delta)\pi^{**}, \end{aligned} \tag{10}$$

as long as this quantity is at least equal to $-(1-\delta)\pi^{**}$. The interpretation parallels the interpretation of the efficient solution without asymmetry of information, with the valuation replaced by the virtual valuation, which includes the cost of extracting from the buyer information about its willingness to pay.

It may be worthwhile noting that, with asymmetry of information, the rental rate is not deducted from the willingness to pay of the agent, but from his virtual valuation. In particular, the formula is of the form $[J_i(v_i) - \text{Rental}]$ and not $[J_i(v_i - \text{Rental})]$. Indeed, the bidder's type, his willingness to pay, is not known to the auctioneer who induces its truthful revelation by releasing some informational rents to each agent: the magnitude of this rent depends on his type. On the contrary, the commitment cost induced by the all-or-nothing allocation rule is related to bidder's class, which is assumed to be publicly known. Thus, the virtual valuation reflects only the cost of truthful revelation of types, and not class.

As usual in the presence of asymmetry of information, agents with low valuations may never get the good. If there exists some $v_i^{\min}$ such that $J_i\left(v_i^{\min}\right) = \delta\left(1-\delta^{i-1}\right)\pi^{**} > 0$, then agents of classes i with $v_i < v_i^{\min}$ are excluded from the market. By excluding low valuation bidders, the auctioneer limits the rent of other bidders. This phenomenon is stronger than in standard static auctions, where virtual valuations are only constrained ro be positive.

4 Short vs Long term buyers

In this section, we examine the preferences of the planner between agents with different time horizons. We begin by the perfect information case and then turn to the asymmetric information case.

4.1 Perfect information

The *per period willingness to pay* of agent i of type v_i is

$$\frac{v_i}{1+\delta+\ldots+\delta^{i-1}}.$$

Consider two agents, i and k, $i < k$, with the same willingness to pay. From (3), the regulator is indifferent between renting the good to agents i and k if and only if

$$v_i - (1+\delta+\ldots+\delta^{i-1})(1-\delta)\pi^* = v_k - (1+\delta+\ldots+\delta^{k-1})(1-\delta)\pi^*$$
$$\Leftrightarrow \quad \delta^i(1+\delta+\ldots+\delta^{k-i-1})[v-(1-\delta)\pi^*] = 0.$$

This yields immediately the following proposition.

Proposition 4. *If two agents have the same per period willingness to pay v, the seller rents the good to the agent with the shortest length of demand if $v < v^*$, where $v^* = (1-\delta)\pi^*$ is the per period rental rate of the good at the optimum. If, on the contrary, $v > v^*$, the seller rents the good to the agent with the longest length of demand.*

It is possible to get a feeling for the critical value v^* by deriving a lower bound on the value of π^*. In order to do so, notice that the regulator could use the following suboptimal allocation policy: always pick the bidder i with the largest expected per period valuation, i.e. $i \in \arg\max_{k\in\mathbb{I}} \left[\mathrm{E}_{v_k}[v_k] / \left(1+\delta+\ldots+\delta^{k-1}\right)\right]$. By doing so, the seller secures an expected per period benefits of $\mathrm{E}_{v_i}[v_i] / \left(1+\delta+\ldots+\delta^{i-1}\right)$. Thus,

$$\pi^* \geqslant \max_{k\in\mathbb{I}} \left[\frac{\mathrm{E}_{v_k}[v_k]}{\left(1+\delta+\ldots+\delta^{k-1}\right)(1-\delta)}\right] = \max_{k\in\mathbb{I}} \left[\frac{\mathrm{E}_{v_k}[v_k]}{1-\delta^k}\right].$$

and the critical value v^* has the following lower bound

$$v^* \geqslant \max_{k\in\mathbb{I}} \left[\frac{\mathrm{E}_{v_k}[v_k]}{1+\delta+\ldots+\delta^{k-1}}\right]. \tag{11}$$

Thus, if two agents have the same per period valuation v, which is smaller than the maximum possible expected per period valuation of all types of agents, the seller allocates the good to the agent with the shortest horizon. Loosely speaking, this shows that the allocation mechanism tends to favor of short horizon agents, as in general we expect valuations to be smaller than the right hand side of (11).

Let now turn to a more general case than the one studied in proposition 4. Denote the per period willingness to pay of any agent i by

$$\hat{v}_i \equiv \frac{v_i}{1+\delta+\ldots+\delta^{i-1}}.$$

Then, following the previous analysis, the regulator compares agents i and k by comparing $\tilde{v}_i$ and $\tilde{v}_k$, that is

$$\tilde{v}_i \gtrless \tilde{v}_k \quad \text{iff} \quad [\hat{v}_i - v^*] \gtrless \frac{1-\delta^k}{1-\delta^i}[\hat{v}_k - v^*].$$

With $i < k$, the fraction in the right-hand side of the inequality is greater than 1. Whenever $\hat{v}_k > \hat{v}_i > v^*$, the auctioneer favors the long run buyer k and, if $\hat{v}_k < \hat{v}_i < v^*$ she prefers the short run buyer.

4.2 Asymmetric information

With asymmetric information, the comparison between two potential buyers depends not only on their horizon and their valuation, but also on the whole distribution of their types, which determines their virtual valuations. In order to provide a "clean" discussion of the preferences of the auctioneer, we will therefore assume not only that the agents have the same "per period valuation", but also the same "distribution of per period valuation".

Definition 1. *Two agents have the same distribution of per period valuation if there exists a distribution function $\tilde{\tilde{F}}$ and an associated density function $\tilde{\tilde{f}}$ such that*

$$\forall v_i \in [\underline{v}_i, \bar{v}_i], \quad F_i(v_i) = \tilde{\tilde{F}}\left(\frac{v_i}{1+\delta+\ldots+\delta^{i-1}}\right) \tag{12}$$

and

$$\forall v_k \in [\underline{v}_k, \bar{v}_k], \quad F_k(v_k) = \tilde{\tilde{F}}\left(\frac{v_k}{1+\delta+\ldots+\delta^{k-1}}\right).$$

From (12), we obtain

$$\begin{aligned} J_i(v_i) &= v_i - \left(1+\delta+\ldots+\delta^{i-1}\right) \frac{1-\tilde{\tilde{F}}\left(\frac{v_i}{1+\delta+\ldots+\delta^{i-1}}\right)}{\tilde{\tilde{f}}\left(\frac{v_i}{1+\delta+\ldots+\delta^{i-1}}\right)} \\ &= \left(1+\delta+\ldots+\delta^{i-1}\right) \tilde{\tilde{J}}\left(\frac{v_i}{1+\delta+\ldots+\delta^{i-1}}\right) \end{aligned} \tag{13}$$

where

$$\tilde{\tilde{J}}(v) \equiv v - \frac{1-\tilde{\tilde{F}}(v)}{\tilde{\tilde{f}}(v)}$$

is the virtual valuation computed from the distribution function $\tilde{\tilde{F}}$ and its associated density $\tilde{\tilde{f}}$. If the virtual valuations of bidders i and k are increasing, so is $\tilde{\tilde{J}}$.

Let $v_i = v_k = v$. The auctioneer is indifferent between allocating the good to agents i and k if

$$\begin{aligned} &\left(1+\delta+\ldots+\delta^{i-1}\right)\left[\tilde{\tilde{J}}(v) - (1-\delta)\pi^{**}\right] \\ &\qquad = \left(1+\delta+\ldots+\delta^{k-1}\right)\left[\tilde{\tilde{J}}(v) - (1-\delta)\pi^{**}\right], \end{aligned} \tag{14}$$

that is $v = v^{**}$ where v^{**} is defined[9] by $\tilde{\tilde{J}}(v^{**}) = (1-\delta)\,\pi^{**}$. The following proposition is then a straightforward consequence of equation (14).

Proposition 5. *If two agents with the same distribution of per period valuation have the same per period willingness to pay v, the seller prefers to rent the good to the agent with the shortest length of demand if $v < v^{**}$. If $v > v^{**}$, she would rather allocate it to the agent with the longest lenght of demand.*

The intuition is similar to that of the case without asymmetric information. The seller's average per period revenue is $\tilde{\tilde{J}}(v^{**})$. If the per period valuations of both bidders is larger than v^{**}, then the firm gets an "extra benefit" for each period of allocation, and allocates the good to the long term bidder. If, on the contrary, both bidders' per period willingness to pay is lower than v^{**}, then the regulator incurs a "loss". In order to limit this loss, the good is given to the shorter horizon bidder.

Turning to the more general case where agents have different per period willingness to pay, the seller prefers giving the good to buyer i than to buyer k if and only if

$$\left[\tilde{\tilde{J}}(\hat{v}_i) - \tilde{\tilde{J}}(v^{**})\right] > \frac{1-\delta^k}{1-\delta^i}\left[\tilde{\tilde{J}}(\hat{v}_k) - \tilde{\tilde{J}}(v^{**})\right].$$

Assume $k > i$. If $\hat{v}_k > \hat{v}_i > v^{**}$, the seller prefers the long run buyer k and, if $\hat{v}_k < \hat{v}_i < v^{**}$, the short run buyer i.

Thus, with asymmetry of information, the influence of the horizon of the agents on the optimal allocation has the same general structure than in the efficient case discussed earlier. For small per period valuations, short horizon buyers will be preferred. For larger per period valuations, long horizon buyers will be favored. However, it seems impossible to compare generally v^{**} and v^{*}, even in simple case where all the probability distribution functions satisfy properties such as the one described by definition 1.

The next section compares the allocation strategies of the profit maximizing seller and the social welfare maximizing regulator.

5 Efficient vs Profit maximizing allocation mechanisms

To compare both allocation procedures, in this section we compare the longest horizon bidders with a strictly positive probability of being allocated the good,

[9] For simplicity, we assume that such a v^{**} exists. It would not, for instance, if some other bidder had such high valuations with probability one that the optimal mechanism never allocated to good either to agent i or k.

depending on the objective of the seller. We show that, in this sense, the profit maximizing seller favors long run buyers – there is no "short termism" associated with profit maximizing in our model.

More precisely, we will say that a seller "never allocates the good to buyer of type i" if $p_i(v_i) = 0$ for all $v_i \in [\underline{v}_i, \overline{v}_i]$. The longest horizon buyer is the bidder j, such that $p_j(v_j) \neq 0$ for some v_j and such that the seller never allocates the good to a buyer of type $i > j$. Because the function p_i is increasing, if j is the longest horizon bidder of the firm, for any $i > j$, $p_i^{**}(\overline{v}_i) = 0$, which implies

$$J_i(\overline{v}_i) - \left(1 - \delta^i\right)\pi^{**} \leqslant J_k(\underline{v}_k) - \left(1 - \delta^k\right)\pi^{**}$$

for some $k < j$. As $J_i(\overline{v}_i) = \overline{v}_i$ and $J_k(\underline{v}_k) = \underline{v}_k - 1/f^k(\underline{v}_k) < \underline{v}_k$, this implies[10]

$$\begin{aligned} & \overline{v}_i - \left(1 - \delta^i\right)\pi^{**} < \underline{v}_k - \left(1 - \delta^k\right)\pi^{**} \\ \text{i.e.} \quad & \overline{v}_i - \underline{v}_k < \delta^k\left(1 - \delta^{i-k}\right)\pi^{**} \leqslant \delta^k\left(1 - \delta^{i-k}\right)\pi^{*}. \end{aligned}$$

Rewriting this last inequality yields $\tilde{\overline{v}}_i < \tilde{\underline{v}}_k$. Thus, bidder i is never allocated the good by the regulator. This proves the following proposition.

Proposition 6. *A profit maximizing firm distorts allocation decisions in favor of bidders with longer horizon, that is the firm's longest horizon buyer is larger than the regulator's one.*

The profit maximizing seller has incentives to increase the level of competition, and therefore invites more buyers to participate to the auction.

6 Conclusion

This paper develops an analysis of the repeated allocation of access by a seller who faces bidders with different lengths of demand, based on a new stationary approach to the pricing of capacity on durable assets.

There are five main insights derived from the analysis this paper. First, the problem cannot be avoided by simply comparing the rental rate that buyers with different lengths of demand are willing to pay, and one has to explicilty take these lenghts of demand in consideration. Typically, if two buyers have rather low valuations for the good, the optimal policies, chosen either by a firm or a regulator, will favor the short run user; if they have valuations on the high side, the optimal

[10]Recall that the profit maximizing firm gets an expected profit noted π^{**} and the social welfare maximizing regulator gets social welfare π^* such that $\pi^* > \pi^{**}$.

policies will favor the long run buyers. Second, one can use standard techniques to compute the optimal allocation mechanism under asymetry of information. Third, a profit maximizing firm will overall tend to favor long run bidders more than a social welfare maximizer, in the sense that the set of bidders that have a strictly positive probability of being awarded the good includes longer term bidders when the seller tries to maximize profits. Similarly, the profit maximizing firm will be more willing to reserve capacity for a buyer who is willing to wait.

We have constructed our model with a certain number of restrictive hypothesis that could easily be lifted. We have assumed that at most one bidder of any demand length appears at each period. It should be clear that this hypothesis could be lifted easily. Similarly, it would be easy to change the model to allow for a random number of bidders in each period.

Quite a number of interesting problems remain. In some circumstances there could be asymmetry of information about the length of demand. There could also be common shocks that affect all bidders. It would be interesting to compare how the regulator and the firm would react to periods of high or low demand. Finally, the presence of corrolated common value could create interesting signalling aspects, specially when several goods are for sale (or equivalently that the capacity of the asset can be divided among several users).

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Appendix

Proof of lemma 1

This follows directly by substitution in the objective function, once one has noticed that the individual rationality constraints are binding.

Note that while the firm must set the transfers $t_i(v_i)$ to $t_i^*(v_i)$, because the individual rationality is binding, there is no need for the social planner to do so. Indeed, the regulator has just to set transfer functions that satisfy individual rationality constraints. This is related to the linearity of the objective functions for the regulator where each transfer $t_i(v_i)$ appears both as a positive sum, collected by the auctioneer, and as a negative turn, extracted from bidder i, in the social welfare function. Both terms cancel. Eventually, $\{p_i^*(v), t_i^*(v_i)\}$ is just one solution to the problem of the auctioneer.

Proof of lemma 2

Let define

$$\tilde{W}(\pi, p) \equiv \sum_{i \in \mathbb{I}} \mathrm{E}_v \left[\left[v_i - \delta \left(1 - \delta^{i-1}\right) \pi \right] p_i(v) \right]$$

where p is the vector made of all p_i. This function is clearly decreasing in π. The convexity results from the fact that function $\tilde{W}$ is linear with respect to π. Simple computations show that $\tilde{W}(\alpha\pi_1 + (1-\alpha)\pi_2, p) = \alpha\tilde{W}(\pi_1, p) + (1-\alpha)\tilde{W}(\pi_2, p)$ for α between 0 and 1. Then, define $p(v|\pi)$ the vector made of probability functions that maximize $W(\pi)$ for a given π. It must be the case that $W(\pi) - \delta\pi = \tilde{W}(\pi, p(v|\pi))$ and that $W(\pi) - \delta\pi \geqslant \tilde{W}(\pi, p(v|\pi^*))$ for $\pi^* \neq \pi$. Combining the linearity of $\tilde{W}$ with respect to π and the relationship between $\tilde{W}(\pi, p)$, $W(\pi)$ and $p(v|\pi)$ yields

$$\begin{aligned}
&\tilde{W}(\alpha\pi_1 + (1-\alpha)\pi_2, p(v|\alpha\pi_1 + (1-\alpha)\pi_2)) \\
&\quad = \alpha\tilde{W}(\pi_1, p(v|\alpha\pi_1 + (1-\alpha)\pi_2)) + (1-\alpha)\tilde{W}(\pi_2, p(v|\alpha\pi_1 + (1-\alpha)\pi_2)) \\
&\quad \leqslant \alpha\tilde{W}(\pi_1, p(v|\pi_1)) + (1-\alpha)\tilde{W}(\pi_2, p(\pi_2)).
\end{aligned}$$

Thus,

$$W(\alpha\pi_1 + (1-\alpha)\pi_2) - \delta[\alpha\pi_1 + (1-\alpha)\pi_2] \leqslant \alpha[W(\pi_1) - \delta\pi_1] + (1-\alpha)[W(\pi_2) - \delta\pi_2]$$

and $W(\pi) - \delta\pi$ is convex.

Moreover, $W(0) - 0 = \max_{\{p_i(.)\}_i} [\sum_{i \in \mathbb{I}} \mathrm{E}_v [v_i p_i(v)]]$ must be positive if the overall problem has any economic sense: the right-hand side represents social welfare in a static optimal auction with perfect information.

Eventually, $\lim_{\pi \to +\infty} [W(\pi) - \delta\pi] = \max_{p^1(.)} \left[\mathrm{E}_v \left[v_1 p^1(v)\right]\right]$ which must also be positive and lower than $W(0) - 0$.

Proof of lemma 3

Constraints (IC_i) are equivalent, for any agent i, all v_i and $\tilde{v}_i$, to

$$U_i(v_i) \geqslant q_i(\tilde{v}_i)v_i - t_i(\tilde{v}_i)$$
$$U_i(\tilde{v}_i) \geqslant q_i(v_i)\tilde{v}_i - t_i(v_i)$$

or

$$q_i(v_i)(v_i - \tilde{v}_i) \geqslant U_i(v_i) - U_i(\tilde{v}_i) \geqslant q_i(\tilde{v}_i)(v_i - \tilde{v}_i) \tag{15}$$

Thus, equation (5) follows. This requires that the expected probability $q_i(v_i)$ must be non-decreasing as well as $U_i(v_i)$. Dividing (15) by $(v_i - \tilde{v}_i)$ and taking the limit as $\tilde{v}_i \to v_i$ yields, almost everywhere and for all i

$$\frac{dU^i(v_i)}{dv_i} = q_i(v_i) \geqslant 0. \tag{16}$$

Integrating (16) between $\underline{v}_i$ and v_i gives (6). Moreover, $U_i(v_i)$ being non-decreasing and (IR_i) induce (7).

Conversely, combining (6) and (5) yields (15) which is equivalent to (IC_i). Moreover, $q_i(v_i)$ is non-decreasing, so combining (6) and (7) implies (IR_i).

Proof of lemma 4

Lemma 3 sets that the maximization of both auctioneers can be written as maximizing (1) under constraints (5), (6), (7), (P_i) and (P_0). According to (6), the expected payment of a bidder can be written as

$$t_i(v_i) = q_i(v_i)v_i - U_i(v_i) = q_i(v_i)v_i - \int_{\underline{v}_i}^{v_i} q_i(x)\,dx - U_i(\underline{v}_i)$$

Moreover, standard manipulation of the integral yields

$$\begin{aligned}\mathrm{E}_{v_i}\left[\int_{\underline{v}_i}^{v_i} q_i(x)\,dx\right] &= \int_{\underline{v}_i}^{\overline{v}_i}\int_{\underline{v}_i}^{v_i} q_i(x) f_i(v_i)\,dx dv_i = \int_{\underline{v}_i}^{\overline{v}_i}\int_{x}^{\overline{v}_i} q_i(x) f_i(v_i)\,dv_i dx \\ &= \int_{\underline{v}_i}^{\overline{v}_i} q_i(x)\left[1 - F_i(x)\right]dx = \mathrm{E}_{v_i}\left[q_i(v_i)\frac{1 - F_i(v_i)}{f_i(v_i)}\right],\end{aligned}$$

For a profit maximizing seller, the objective is to maximize the bidders' expected payments while preserving individual rationality, so that the optimal mechanism is characterized by $U_i(\underline{v}_i) = 0$ which gives equation (9). Then, expected payment by agent i becomes

$$\mathrm{E}_{v_i}\left[t_i(v_i)\right] = \mathrm{E}_{v_i}\left[\left(v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}\right)q_i(v_i)\right]. \tag{17}$$

Finally, current expected benefits can be rewritten

$$\sum_i \mathrm{E}_{v_i}\left[\left(v_i - \frac{1-F_i(v_i)}{f_i(v_i)} - \delta\left(1-\delta^{i-1}\right)\pi\right) q_i(v_i)\right] + \delta\pi.$$

This is the expression given in (8) for a profit maximizing seller.

When dealing with efficiency, transfers are not taken into account in the objective function as far as they respect the incentive compatibility constraint. Thus, the objective function of the social planner can be rewritten

$$\sum_i \mathrm{E}_{v_i}\left[\left[v_i - \delta\left(1-\delta^{i-1}\right)\pi\right] q_i(v_i)\right] + \delta\pi.$$

which corresponds to (8) with $Y(v_i) = v_i$.

Thus, the auctioneer's problem is to maximize (8) under constraints (5), (6), (7), (P_i) and (P_0). What is left to be proved is that constraints (7) and (6) are all satisfied by the proposed transfer function $t_i^{**}(v_i)$. Simple computations show that this is the case. Eventually, the proposed solution of program (8) is a solution to program (1).

Note that, for the regulator, expression (17) about transfers holds but there is no obligation to set $U_i(\underline{v}_i) = 0$. Nevertheless, the particular transfer function $t_i^*(v_i)$ is a (non-unique) solution to the maximization problem (1).

Proof of lemma 5

Define the following function

$$\tilde{V}(\pi, p) \equiv \sum_{i\in\mathbb{I}} \mathrm{E}_v\left[\left[Y_i(v_i) - \delta\left(1-\delta^{i-1}\right)\pi\right] p_i(v)\right]$$

where $Y_i(v_i)$ corresponds to $J_i(v_i)$ when maximizing profits or v_i when taking care about efficiency. Then, function $\tilde{V}$ is clearly decreasing in π. The proof for convexity then follows the same line as the proof of lemma 2 because function $\tilde{V}$ is also linear in π. The result in $\pi = 0$ is also a consequence for the problem parameters to be economically meaningful in presence of asymmetric information.