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Abstract

This paper studies settings where a number of sellers of different reputations for honesty simultaneously offer sealed-bid, second-price, single-unit auctions for imperfect substitute goods to unit-demand buyers. Among other applications, these settings can serve as an abstraction of large-scale decentralized Internet auction marketplaces, such as eBay. I characterize the form of the bidding equilibria and derive expressions for the corresponding allocative efficiency and expected seller revenue. When bidders are restricted to submit at most one bid there exists a unique Bayes-Nash equilibrium that has the following form: Auctions are ranked according to their expected valuation, taking into account both the quality of the good and the seller's reputation. Buyers self-separate into a finite number of zones, according to their types. Buyers whose types fall in the k-th zone randomize between the top k auctions, assigning increasingly higher probability to selecting lower auctions. In such equilibria auction revenue is an increasing convex function of seller reputation. Allowing unit-demand bidders to place an arbitrary number of bids induces complex mixed strategy profiles where bidders place positive bids in all available auctions. The probabilistic nature of the bidding equilibria introduces allocative inefficiencies that are most severe when the number of bidders is roughly equal to the number of sellers. *JEL Classification: D44.*

1 Introduction

Internet auction sites started out as collections of separate auctions for unique collectibles, but evolved into markets where commodity goods like electronics or DVDs are traded. In many categories, several auctions for similar goods are ran quasi-simultaneously, and unit-demand bidders must adopt global multi-auction bidding strategies.[1] Their decision problem is further complicated by the fact that, even if they refer to the same product, most auction listings are not truly identical. Competing items are often sold in different conditions, have subtle differences in their features, etc. Last, but not least, the *reputation* of the various sellers, defined here as the probability of honest transaction fulfillment, is usually different.

Despite the increasing practical importance of such markets, there is relatively little theory on how unit-demand bidders should behave in simultaneous auctions of imperfect substitutes sold by sellers of different reputations. In addition to informing buyer behavior, such research will help sellers better understand how item condition and seller reputation affects expected revenue in the presence of competition. It will also assist market operators better understand the allocative efficiency of decentralized auction markets relative to centralized multi-unit auction mechanisms, such as the Vickrey-Clarke-Groves (VCG) mechanism, that have been shown to offer perfect efficiency (Clarke 1971; Groves 1973).

This paper contributes in this direction by offering a game-theoretic analysis of a setting where a number of sellers of different reputations simultaneously offer sealed-bid, second-price, single-unit auctions for imperfect substitute goods to unit-demand buyers. A seller's reputation is defined as the buyers' common subjective belief that the seller will fulfill the transaction. The buyers' decision problem is to select on which auctions to submit bids as well as how much to bid. The paper characterizes the form of the resulting Bayes-Nash bidding equilibria and derives expressions for the corresponding expected seller revenue, buyer surplus and allocative efficiency.

I develop the paper's core results in a baseline setting where sellers of different reputations offer simultaneous auctions of identical goods. When bidders are restricted to submit at most one bid, I show that there exists a unique Bayes-Nash equilibrium that has the following form: Sellers are ranked according to their reputation. Buyers self-select into a finite number of zones, according to their types (valuations). Buyers whose types fall in the highest zone always bid on the highest seller; buyers whose

[1] For example, a set of searches for "ipod nano 2GB" conducted on eBay during working hours on March 1, 2008 returned an average of 25 auctions ending per hour.

types fall in the 2nd highest zone randomize between the top two sellers, assigning higher probability to selecting the 2nd seller; more generally, buyers whose types fall in the k-th zone randomize between the top k sellers, assigning increasingly higher probability to selecting less reputable sellers. This bidding behavior is a form of probabilistic positive assortative matching, somewhat akin to Becker's (1973) famous marriage problem: bidders assess where they stand on the valuation scale and assign higher probability to bidding on the auction that "matches" their respective zone, while also occasionally bidding on "higher" auctions. The probabilistic nature of the matching between bidders and sellers creates allocative inefficiencies that are most severe when the number of bidders is roughly equal to the number of sellers. This important drawback of independent simultaneous auctions could, in theory at least, be remedied by combining single-unit auctions that end at roughly the same time into a single multi-item, provably efficient auction, such as a VCG auction.

In terms of seller revenue, the key property is that more reputable sellers attract higher valuation bidders with higher probability. Reputation, thus, has a double impact on a seller's expected revenue: First, it affects the amount that buyers are willing to bid (it is equal to the valuation of the good times the seller's reputation). Second, it affects the quality of buyers who are more likely to choose a seller's auction. In settings with competition a seller's expected revenue is, then, a convex function of his reputation.

I extend the analysis to the more general setting where buyers are allowed to bid on an arbitrary number of simultaneous auctions. Surprisingly, even though buyers have unit demand, I find that all optimal bidding strategies involve placing non-zero bids in all available auctions. The optimal bid amount in each auction is equal to the bidder's expected valuation of the respective auction (taking into account the seller's reputation) multiplied by the probability that the bidder will *not* receive the item from any of the other auctions, given her other bids. This relatively straightforward recursive formulation induces complex mixed strategy profiles. Equilibrium bid vectors of all except the lowest bidder types have one of two forms: I) a high bid (i.e. a bid that is close to the bidder's expected valuation) on one auction and low bids on all remaining auctions or II) intermediate bids on all auctions. Bidders randomize between using Type I and Type II bid vectors. When using Type I vectors they further randomize with respect to which auction they place their high bid on. Higher bidder types use Type I bid vectors more often and place their high bid on higher auctions with higher probability.

I conclude by showing that the one-bid results remain qualitatively identical in simultaneous auction settings where all sellers have the same reputation but sell imperfect substitute goods (e.g. used units of the same good in different conditions) as well as in settings where sellers sell imperfect substitute goods *and* have different reputations. I also identify some subtle differences in the ways that seller reputation and imperfect substitution impact bidding behavior in the multi-bid case. The paper, therefore, provides a fairly general analysis of bidding behavior in settings of simultaneous sealed-bid, second price auctions for imperfect substitutes.

This paper contributes to the growing literature on simultaneous auctions. In recent years, several disciplines have produced valuable contributions in this exciting area. In the economics literature, Krishna and Rosenthal (1996) study simultaneous sealed-bid, second-price auctions of identical objects to a combination of unit-demand and multi-unit demand buyers. Rosenthal and Wang (1996) extend the analysis in settings with common values. Peters and Severinov (1997) study simultaneous auctions of identical objects where sellers compete with each other by setting different reserve prices. Bikhchandani (1999) studies simultaneous auctions of heterogeneous objects to multi-unit demand buyers under complete information about buyer valuations. Szentes (2007) studies two-object, two-bidder simultaneous auctions in settings where information is complete and the objects are either complements or substitutes. Szentes and Rosenthal (2003) study majority auction games, that is, simultaneous sealed-bid auctions of identical objects among identical bidders who each want to win a specified fraction of the objects. Albano, Germano and Lovo (2006) study ascending simultaneous multi-unit auctions. Ellison, Fudenberg and Mobius (2004) study the conditions under which two competing and otherwise identical auction sites of different sizes can coexist in equilibrium. In the management science literature, Byde (2001) and Bertsimas, Hawkins and Perakis (2002) propose dynamic programming bidding algorithms for a single item in simultaneous or overlapping online auctions. Beil and Wein (2008) study a setting of two simultaneous auctions for identical goods where some of the unit-demand buyers are dedicated to each auction while others participate in both auctions. Gerding et al. (2007) study bidding equilibria of simultaneous second-price auctions of identical goods.

Simultaneous auctions have also generated substantial interest in the computer science/intelligent agents community. A lot of this interest has been motivated by the International Trading Agent Competition (Wellman et al. 2001), an annual event that challenges its entrants to design an automated trading agent capable of bidding in simultaneous online auctions for complementary and substitute goods. Most of the work in this area (for example, Greenwald and Boyan 2001; Anthony and Jennings

2003; Preist, Byde and Bartolini 2001; Stone et al. 2001) focuses on ascending auctions and proposes heuristic bidding algorithms that are tested by their authors experimentally or through simulation.

None of the above papers covers the topic of the current paper, namely a game-theoretic analysis of simultaneous second-price, sealed-bid, single-unit auctions of imperfect substitutes to unit-demand bidders under incomplete information about bidder valuations and uncertainty about the sellers' probability of transaction fulfillment.

The rest of the paper is organized as follows. Section 2 introduces the setting. Section 3 derives the form of the bidding equilibrium in the special case where goods are identical and bidders are restricted to submit at most one bid. Section 4 extends the analysis in the case where bidders can submit an arbitrary number of simultaneous bids. Section 5 looks at settings where goods are imperfect substitutes. Finally, Section 6 concludes and discusses possible directions of future work.

2 The setting

There are M sellers and N risk-neutral buyers trading in a market. Each seller offers a sealed-bid, second-price auction for one unit of an imperfect substitute good (e.g. a used unit of a given product in a variety of conditions), while each buyer has an inelastic demand for one unit of this good. All seller auctions begin and end simultaneously. Each buyer has a privately known type $t \in [0, 1]$ that determines her unit valuation. Buyer t's valuation of seller k's good is equal to $v_k(t) = tu_k$, where u_k represents the ratio of buyer t's valuation of seller k's good over her valuation of a hypothetical "ideal" good of the same class (e.g. a brand new unit of the product). All buyers are assumed to assign the same ratio u_k to seller k's good (i.e. we assume vertical differentiation). Buyer types are independently drawn from the same cumulative probability distribution $F(t)$ with associated density function $f(t) > 0$. Buyers know the total number of buyers N, their own type t, and the probability distribution of other buyers' types.

Each seller k, $1 \leq k \leq M$ has an associated reputation $0 \leq r_k \leq 1$, interpreted as the buyers' commonly held subjective belief that the seller will fulfill the transaction and will deliver the promised good. Reputations can arise from public information about a seller's past behavior provided by an online reputation mechanism (such as the one used by eBay) or from some other information that is available to all buyers.[2] A risk-neutral buyer of type t who wins seller k's auction at price p thus

[2] I do not make any assumptions with respect to whether such information is accurate or not. I simply assume that

expects to get surplus $tr_k u_k - p$.

The following two sections develop the basic results by focusing on a simplified setting where sellers have different reputations but sell a homogeneous good. This is equivalent to assuming that $u_k = 1$ for all k. I revisit the general case where goods are imperfect substitutes in Section 5.

In the rest of the paper I will assume that a seller's index k indicates his relative reputation rank within the set of competing sellers. Specifically, I assume that $r_1 \geq r_2 \geq ... \geq r_M$.

Trade is organized in the following way. All buyers simultaneously arrive in the market and discover their types. Sellers simultaneously announce their auctions. Buyers look up each seller's reputation, then submit bids to a subset of available auctions. Each auction is won by its highest bidder who pays an amount equal to the second-highest bid.

A buyer's decision problem has two distinct components: (i) select *on which* auctions to submit bids, (ii) decide *how much* to bid on each auction. We begin our characterization of bidding behavior in the special but important case where buyers are restricted to place at most one bid.

3 One-bid equilibria

Equilibrium bidding behavior has an elegant characterization in the special case where each unit-demand buyer is restricted to place at most one bid. When all simultaneous auctions are administered by the same auction house (e.g. eBay) such a restriction can be easily implemented by the auction intermediary.

3.1 Auction selection

Since all auctions are sealed-bid, once buyers have decided on which auction to place their bid, individual auctions proceed independently. According to the theory of second price auctions, buyer t's optimal bid on seller k's auction is independent of everybody else's choices and equal to her expected valuation $v_k(t) = tr_k$. This observation simplifies our problem considerably as it allows a buyer's bidding strategy to be uniquely determined by her auction selection strategy.

An *auction selection strategy* can be represented as a vector $\mathbf{s}(t) = (s_1(t), .., s_M(t)) \in [0, 1]^M$ where $s_k(t)$ denotes the probability that buyer t will bid (an amount equal to tr_k) on seller k's auction. An

it exists and that it is commonly interpreted by all buyers. Readers interested in questions that pertain to the design of reliable online reputation mechanisms are referred to Dellarocas (2003).

auction selection strategy is *pure* if and only if all components $s_k(t)$ are either 0 or 1. Throughout this section we restrict our attention to strategy vectors that are piecewise continuous in buyer type t.

If buyers are restricted to select at most one auction, they will choose the auction that maximizes their expected surplus, given their beliefs about every other buyer's strategy. Specifically, buyer t's expected surplus from bidding her expected valuation on seller k's auction is equal to:

$$\begin{aligned} V_k(t) = \textstyle\sum_{n=0}^{N-1} & Pr[n \text{ other bidders choose auction } k] \times Pr[\text{all } n \text{ other bidders have types} \leq t] \\ & \times r_k(t - E[\text{2nd highest type}|\text{highest type}{=}t \wedge \exists n \text{ other bidders}]) \end{aligned} \tag{1}$$

Observe that the expected surplus from winning an auction increases with a seller's reputation but also with the expected distance between the highest and second-highest bidders' types. Auctions by less reputable sellers that receive fewer bids may, thus, generate a higher expected surplus than auctions by more reputable sellers that receive more bids. Furthermore, the probability of winning an auction decreases with the number of bidders of similar or higher types. In selecting their strategy, buyers must, therefore, trade off the incentive to trade with reputable sellers (increases the probability that the buyer will receive the promised good) against the incentive to bid on less popular auctions (increases the probability of winning and decreases the expected payment). The tension between these two opposite forces gives rise to the resulting equilibria.

Given a selection strategy $\mathbf{s} : [0,1] \to [0,1]^M$, the following set of functions will play an important role in the analysis:

$$Q_k(t|\mathbf{s}) = \int_0^t s_k(u) f(u) du, \quad k = 1, ..., M \tag{2}$$

In the rest of the paper we will omit the dependence on $\mathbf{s}$ when it is implied. The quantity $Q_k(t)$ is equal to the probability that a buyer is of type t or lower *and* bids on seller k's auction. The quantity $Q_k(1)$ then represents the probability that a randomly chosen buyer bids on seller k's auction. The quantity $Q_k(1) - Q_k(t) = \int_t^1 s_k(u) f(u) du$ is equal to the probability that a buyer is of type t or higher *and* bids on seller k's auction. Finally, the quantity $1 - Q_k(1) + Q_k(t)$ is equal to the probability that a buyer is of type t or lower *or* does not bid on seller k's auction. This, in turn, is equal to the probability that a bidder of type t will win auction k if he is competing against exactly one other bidder.

The following Lemma is a generalization of standard auction theory results (McAfee and McMillan

1987; Riley and Samuelson 1981) in the context of simultaneous auctions:[3]

Lemma 1: *Let* $\mathbf{s}(t) = (s_1(t), .., s_M(t)) \in [0,1]^M$ *denote a buyer's beliefs about every other buyer's auction selection strategy and let* $Q_k(t|\mathbf{s})$ *be the functions defined by those beliefs and equation (2).*

1. *The number of bidders on seller* k*'s auction follows a binomial subjective probability distribution with mass function:*
$$P_k(m|\mathbf{s}) = \binom{N}{m} Q_k(1|\mathbf{s})^m \left(1 - Q_k(1|\mathbf{s})\right)^{N-m} \tag{3}$$

2. *The subjective probability that a buyer of type* t *who bids her expected valuation on seller* k*'s auction will win the auction is given by:*
$$W_k(t|\mathbf{s}) = \left(1 - Q_k(1|\mathbf{s}) + Q_k(t|\mathbf{s})\right)^{N-1} \tag{4}$$

3. *Buyer* t*'s expected surplus from bidding her expected valuation on seller* k*'s auction is given by:*
$$V_k(t|\mathbf{s}) = r_k \int_0^t \left(1 - Q_k(1|\mathbf{s}) + Q_k(x|\mathbf{s})\right)^{N-1} dx \tag{5}$$

In the above expressions, the quantity $\left(1 - Q_k(1|\mathbf{s}) + Q_k(t|\mathbf{s})\right)^{N-1}$ is equal to the probability that no other bidder has type higher than t *and* bids on seller k's auction.

A (*Bayes-Nash*) *auction selection equilibrium* is a strategy $\mathbf{s}^* : [0,1] \to [0,1]^M$ that maximizes the expected surplus of all buyer types subject to the assumption that all buyers believe that other buyers are also following strategy $\mathbf{s}^*$. Specifically, $\mathbf{s}^*$ satisfies the following incentive compatibility constraints:

$$\begin{array}{ll} s_k^*(t) = 0 \Rightarrow V_k(t|\mathbf{s}^*) \leq V_\ell(t|\mathbf{s}^*) & \\ 0 < s_k^*(t) < 1,\ 0 < s_\ell^*(t) < 1 \Rightarrow V_k(t|\mathbf{s}^*) = V_\ell(t|\mathbf{s}^*) & \text{for all } t \in [0,1] \text{ and } 1 \leq k, \ell \leq M \\ s_k^*(t) = 1 \Rightarrow V_k(t|\mathbf{s}^*) \geq V_\ell(t|\mathbf{s}^*) & \end{array} \tag{6}$$

The one-bid restriction requires that $\sum_{k=1}^M s_k^*(t) = 1 \quad \text{for all } t \in [0,1]$.

Intuition suggests that bidders whose types are near the top of the distribution (and who, therefore, have a good chance of winning whichever auction they decide to bid on) will select sellers with higher reputations, whereas bidders with low types will choose sellers with lower reputations in the hope of

[3]Full proofs of all propositions and lemmas are given in the Appendix.

maximizing their prospects of winning something. Interestingly, it turns out that no pure strategy auction selection equilibrium exists.

Proposition 1: *There exists no pure strategy one-bid auction selection equilibrium.*

An informal justification of the non-existence of a pure equilibrium can be based on the following argument. Suppose that a pure strategy equilibrium exists. Then, the assumption of piecewise continuous strategies implies that this equilibrium can be defined in terms of type zones $(\underline{t_k}, \overline{t_k}]$ such that, all types within a zone always bid on a specific seller's auction. Assume a pure strategy that prescribes that all buyers in zone $(0, t]$ bid on seller k's auction. Sellers whose types are near zero have the lowest probability of winning. If low types find it optimal to bid on seller k's auction then higher types will find it even more so. Therefore, any pure equilibrium must have $t = 1$: all buyers bid on seller k's auction. But then, any buyer who bids on another seller's auction, is guaranteed to win that auction. If there exists some other seller with positive reputation then buyers whose types are sufficiently close to the bottom of the distribution will prefer to deviate from the pure equilibrium. Thus, no pure strategy can be an equilibrium.

The following proposition sheds further light into the structure of the mixed auction selection equilibria:

Proposition 2: *All one-bid auction selection equilibria satisfy the following properties:*

1. $s_k^*(t_0) = 1$ *implies* $s_k^*(t) = 1$ *for all* $t \in (t_0, 1]$
2. $s_k^*(t_0) = 0$ *implies* $s_k^*(t) = 0$ *for all* $t \in (t_0, 1]$
3. $s_k^*(0) = 0$ *implies* $s_\ell^*(0) = 0$ *for all* $\ell > k$
4. *For all* $1 \leq k, \ell \leq M$, *if there exists* $t_0 \geq 0$ *such that* $s_k^*(t)$, $s_\ell^*(t)$ *both switch from positive values to zero at* t_0 *then* $r_k = r_\ell$.

Propositions 1 and 2 together with the assumption of piecewise continuous strategies allow us to provide a general characterization of auction selection equilibria:

1. All one-bid auction selection equilibria must be mixed everywhere, except for an interval at the top of the type space.

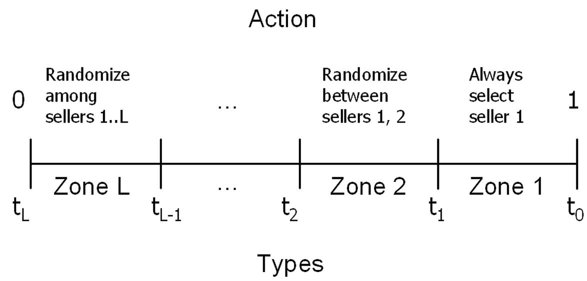


Figure 1: General form of one-bid auction selection equilibria.

2. Such equilibria can be characterized in terms of a sequence of type intervals:

$$(0, t_{L-1}], (t_{L-1}, t_{L-2}], .., (t_z, t_{z-1}], ..., (t_1, 1]$$

with the property that buyers whose types fall within interval $(t_z, t_{z-1}]$ randomize among the z sellers with the highest reputation.

3. The choice set (set of sellers among which buyers randomize) of a given type interval is equal to the choice set of the immediately preceding type interval minus the least reputable seller of that interval.

Figure 1 depicts the general form of these equilibria. Remarkably, it turns out that, for a given set of seller reputations, there exists a unique one-bid auction selection equilibrium with the above properties. Proposition 3 provides the details:

Proposition 3: *Consider a setting where $M \geq 2$ sellers with reputations $r_1 \geq r_2 \geq ... \geq r_M$ simultaneously offer sealed-bid, second-price auctions for one unit of a homogeneous good to $N \geq 2$ unit-demand buyers, independently drawn from the same type distribution $F(t)$. Define L as the lowest integer $2 \leq L \leq M-1$ for which $r_{L+1} < \left(\frac{L-1}{\sum_{i=1}^{L} \sqrt[N-1]{\frac{1}{r_i}}}\right)^{N-1}$. If no such integer exists then $L = M$. The following clauses describe the properties of the unique one-bid auction selection equilibrium.*

1. *Buyers are divided into L zones according to their type. Let t_z, $z = 0, 1, .., L$, $t_0 = 1$, $t_L = 0$ denote the zone delimiters. Buyers whose types satisfy $t_z < t \leq t_{z-1}$ belong to zone z.*

2. *Zone-z buyers randomly choose among sellers $k = 1, .., z$ with corresponding selection probabilities:*

$$s_{zk} = \frac{\sqrt[N-1]{\frac{1}{r_k}}}{\sum_{i=1}^{z} \sqrt[N-1]{\frac{1}{r_i}}}$$

3. *If $L < M$ then sellers $L+1, .., M$ are never chosen by any buyer.*

4. *Zone delimiters t_z, $z = 0, 1, .., L-1$ are solutions of the following equation:*

$$F(t_z) = \left(\sqrt[N-1]{r_{z+1}} \sum_{i=1}^{z} \sqrt[N-1]{\frac{1}{r_i}} \right) - (z-1)$$

5. *The expected number of bids on seller* k*'s auction is equal to:*

$$B_k = \begin{cases} N \left(1 - (L-1) \frac{\sqrt[N-1]{\frac{1}{r_k}}}{\sum_{i=1}^{L} \sqrt[N-1]{\frac{1}{r_i}}} \right) & \textit{if } k \leq L \\ 0 & \textit{if } k > L \end{cases}$$

Observe that each type zone's auction selection probabilities s_{zk} are inversely proportional to the reputations of the sellers among which buyers randomize. Thus, *less* reputable sellers are chosen more often by buyers of a given type zone. Nevertheless, because sellers of higher reputation are considered by buyers of more type zones, the total expected number of bids B_k on a seller's auction monotonically increases with his reputation.

The following is a high-level summary of the equilibrium described by Proposition 3: Sellers are ranked according to their reputation. Buyers self-select into a finite number of zones, according to their types (valuations). Buyers whose types fall in the highest zone always bid on the highest seller; buyers whose types fall in the 2nd highest zone randomize between the top two sellers, assigning higher probability to selecting the 2nd seller; more generally, buyers whose types fall in the k-th zone randomize between the top k sellers, assigning increasingly higher probability to selecting less reputable sellers. This bidding behavior is a form of probabilistic positive assortative matching, somewhat akin to Becker's (1973) famous marriage problem: bidders assess where they stand on the valuation scale and assign higher probability to bidding on the auction that "matches" their respective zone, while also occasionally taking chances on "higher" auctions.

The above auction selection equilibria *always* involve some mixing: irrespective of the distribution of seller reputations, buyer types $t \in (0, t_1]$, $t_1 = F^{-1}\left(\sqrt[N-1]{r_2/r_1}\right)$ (see Part 4 of Proposition 3) always randomize between *at least* the two most reputable sellers; only buyer types $t \in (t_1, 1]$ follow a pure bidding strategy.

An interesting aspect of the auction selection equilibrium is Part 3 of Proposition 3: that, if

$r_{L+1} < \left(\frac{L-1}{\sum_{i=1}^{L} \sqrt[N-1]{\frac{1}{r_i}}}\right)^{N-1}$, then sellers $L+1, ..., M$ do not receive any bids. The condition can be equivalently expressed as:

$$r_{L+1} < \left(\frac{L-1}{L}\right)^{N-1} \left(\frac{1}{L}\sum_{i=1}^{L}(r_i)^{-\frac{1}{N-1}}\right)^{-(N-1)}$$

The latter is a condition between seller-$(L+1)$'s reputation and the $\left(-\frac{1}{N-1}\right)$-power mean of all higher-ranked sellers' reputations. The condition is met as long as r_{L+1} is not substantially lower than the average reputation of higher ranked sellers. The intuitive interpretation of this condition, thus, is that if seller reputations are relatively uniformly spread out in the interval $[0,1]$ then all sellers will receive some bids (buyers whose types belong to the bottom zone randomize among all sellers). If, on the other hand, there is a cluster of sellers whose reputations are substantially higher than those of the rest of the sellers, then it is possible that no buyer will place any bids on the less reputable sellers' auctions.

Proposition 3 subsumes settings where all sellers have identical reputation as a special case. In such settings, $r_1 = ... = r_M = r$ implies $t_0 = t_1 = ... = t_{M-1} = 1$. Thus, $L = M$ and all buyers behave as zone-M buyers.

Corollary 1: *In settings where M sellers with identical reputations r simultaneously offer sealed-bid, second-price auctions for one unit of a homogeneous good to N unit-demand buyers, if buyers are restricted to place only one bid, they will randomly choose one of the M sellers with equal probability $1/M$.*

Of interest is also the behavior of the system as the number of buyers increases. Since $\partial^2\left(\sqrt[N-1]{r_k}\right)/\partial r_k \partial N < 0$, Proposition 3 implies that, as the number of bidders grows, the impact of reputations on auction selection becomes weaker. From $\lim_{N\to\infty} \sqrt[N-1]{r_k} = 1$, as the number of bidders becomes very large, buyers select auctions as if all sellers are identical.[4]

Corollary 2: *In settings where M sellers with different reputations simultaneously offer sealed-bid, second-price auctions for one unit of a homogeneous good to N unit-demand buyers, if the number of buyers grows to infinity and buyers are restricted to place only one bid, they will randomly choose one of the M sellers with equal probability $1/M$ irrespective of his reputation.*

[4]More formally, from Proposition 3, Part 4 the reader can verify that, for all $z = 0, 1, .., M-1$, it is $\lim_{N\to\infty} F(t_z) = 1$. Therefore, $t_0 = t_1 = ... = t_{M-1} = 1$ and all buyers behave as zone-M buyers.

The intuition behind this behavior is that, as the number of bidders increases, the expected difference between the first and second bids of any auction shrinks and, thus, the expected surplus from winning any auction goes to zero. Seller reputation then plays no role in a buyer's expected surplus. The reader should, of course, keep in mind that Corollary 2 refers to auction selection only. Even though, in settings with very large numbers of bidders, buyers ignore a seller's reputation when choosing *where* to place their bid, they take reputation into account when they select the bid amount. Bid amounts are always equal to a buyer's expected valuation tr_k, that includes the seller's reputation as a factor.

3.2 Allocative efficiency

The allocative efficiency of a system of auctions is an important market-level property since it characterizes the extent to which the market maximizes social welfare by allocating items to the buyers that value them most. In our setting an analysis of allocative efficiency has particularly important implications since it can help market operators assess the extent to which the, mostly uncoordinated, consumer-to-consumer simultaneous auction markets encountered on the Internet introduce allocative inefficiencies. The precise notion of efficiency that applies to our setting is given below:

Definition 1: *Consider a setting where M sellers with reputations $r_1 \geq r_2 \geq ... \geq r_M$ simultaneously offer sealed-bid, second-price auctions for one unit of a homogeneous good to N unit-demand buyers. Buyers are independently drawn from the same type distribution F, place at most one bid and follow a symmetric auction selection strategy $\mathbf{s}$. The expected allocative efficiency of the system of auctions under buyer type distribution F and strategy $\mathbf{s}$ is equal to:*

$$\eta(F, \mathbf{s}) = \frac{\sum_{k=1}^{M} r_k H_{1,k}(F, \mathbf{s})}{\sum_{k=1}^{\min(M,N)} r_k F_{N+1-k,N}}$$

where $H_{1,k}(F, \mathbf{s})$ is the highest bidder's expected type in seller k's auction and $F_{i,N}$ is the expected value of the ith order statistic of a sample of N values independently drawn from distribution F.[5]

The numerator of $\eta(F, \mathbf{s})$ is the expected social welfare resulting from the system of simultaneous auctions. The denominator is the expected maximum social welfare, attainable if, for any randomly drawn set of N buyer valuations, the highest seller is matched with the highest buyer, the second-

[5] We adopt the usual convention that the first order statistic is the minimum of the sample and the Nth order statistic is the maximum.

highest seller is matched with the second-highest buyer, etc.

The following lemma gives a closed form expression for the highest bidder's expected type.

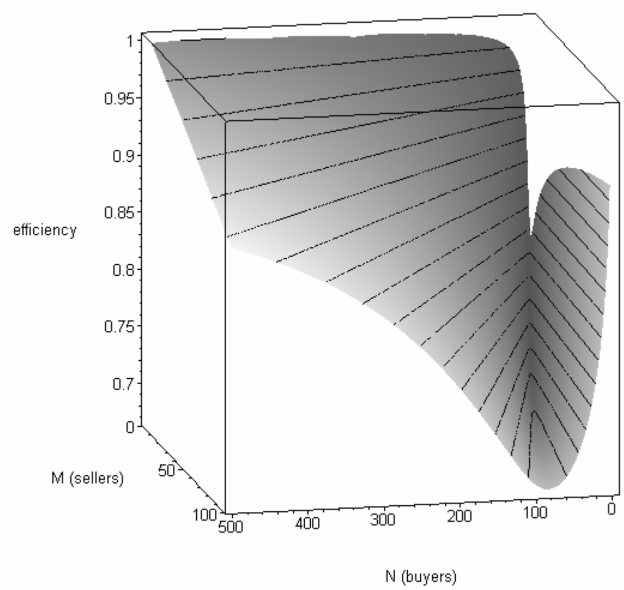
Lemma 2: *Consider a setting where M sellers with reputations $r_1 \geq r_2 \geq ... \geq r_M$ simultaneously offer sealed-bid, second-price auctions for one unit of a homogeneous good to N unit-demand buyers, independently drawn from the same type distribution $F(t)$. The highest bidder's expected type in seller k's auction is equal to:*

$$H_{1,k} = 1 - \int_0^1 (1 - Q_k(1) + Q_k(t))^N \, dt \tag{7}$$

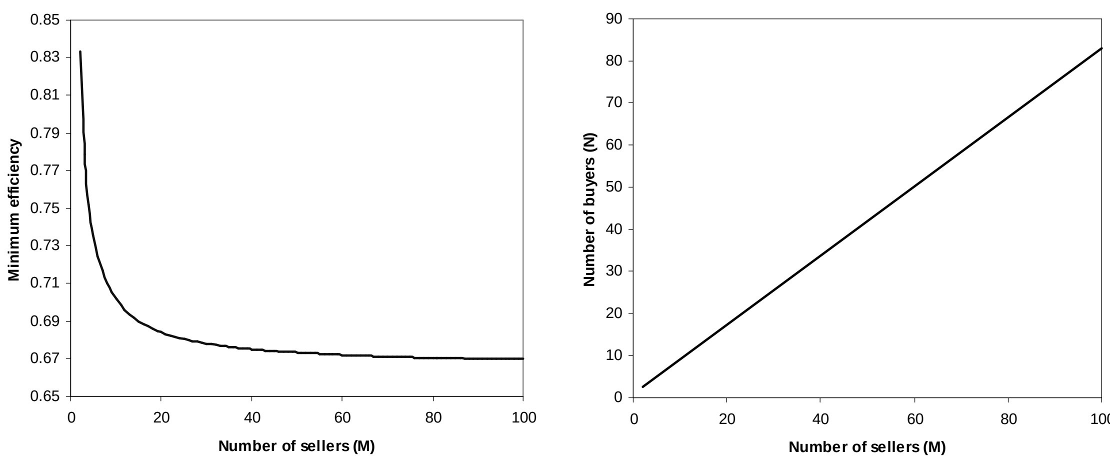
There is no closed-form expression for $\eta(F, \mathbf{s})$. However, given M, N, a set of seller reputations and a buyer type distribution F, it can be computed in a straightforward manner from (7) and the formulas of order statistics distributions (David and Nagaraja 2003).

Figure 2(a) depicts the allocative efficiency for $M = 2, ..., 100$, $N = 2, ..., 500$, uniformly spaced seller reputations $r_k = (M - k + 1)/M$ $(k = 1, ..., M)$ and iid uniform buyer valuations. When the ratio of bidders to sellers is very low or very high, efficiency is relatively high. On the other hand, when the number of bidders is roughly equal to the number of sellers, efficiency is low and can fall below 70%. Figure 2(b) depicts the minimum efficiency for $M = 2, ..., 100$ together with the corresponding number of bidders for which efficiency attains its minimum.

Allocative inefficiencies are due to imperfections in the matching between bidders and auctions that result from the probabilistic nature of the auction selection equilibrium of Proposition 3. Specifically, the lack of coordination between bidders makes it possible that two or more high bidder types will cluster on the same auction (in which case only one of them wins and the remaining auctions will be left to lower bidder types) whereas if these same bidders could coordinate and distribute their bids to different auctions they would all win, increasing social welfare. For example, consider a setting with two sellers and two buyers. There are three cases: (i) both buyer types fall in Zone 1, (ii) one type falls in Zone 1 and one in Zone 2, and (iii) both types fall in Zone 2. Proposition 3 predicts the following: In case (i) both buyers bid on the highest seller's auction. This is inefficient because one bidder will win nothing and one auction will receive no bids. In case (ii) the high buyer bids on the highest seller's auction; with probability s_{21} the second type also bids on the same auction, which leads to the drawbacks we have identified. Finally, in case (iii) with probability $(s_{21})^2$ both buyers will bid on the high seller's auction, whereas with probability $(s_{22})^2$ both buyers will bid on the low seller's



(a) Allocative efficiency as a function of the number of sellers (*M*) and buyers (*N*)



(b) Minimum allocative efficiency as a function of the number of sellers and corresponding number of buyers for which efficiency attains its minimum

Figure 2: Allocative efficiency of simultaneous auctions

auction. Both of these cases result in suboptimal efficiency.

As the number of bidders grows, the probability of such conflicts also increases. This leads to a rapid drop in allocative efficiency. At the same time, however, as the number of bidders grows even further, the number of high valuation bidders increases and the differences between the valuations of successive bidder types become very small. This, in turn, decreases the overall efficiency impact of the imperfect matching between bidders and auctions. It is straightforward to show the following:

Corollary 3: *As the number of bidders grows to infinity, the system of simultaneous auctions becomes asymptotically efficient.*

Ultimately, the problem boils down to the imperfect knowledge that each bidder has about every other bidder's valuation. The allocative inefficiencies identified in this section, thus, constitute the "price of anarchy" of uncoordinated auction markets. In theory, it is possible to solve this problem completely by centralizing all the simultaneously occurring single-item auctions into a single multi-item VCG auction and asking bidders to submit menus of bids for any subset of the available items. In settings where the number of bidders and the number of auctions are roughly equal this might be a sensible suggestion for auction marketplace operators to consider.

3.3 Impact of reputation on seller revenue

This section considers the impact of reputation on seller revenue in settings where each seller competes against sellers of different reputation. If the seller has reputation r, then his expected revenue from a second price auction is equal to $U = r \times E[\text{second highest bidder's type}]$. We begin by considering the case of a single monopolist seller. It is well known (see, for example, Riley and Samuelson 1981) that the second highest bidder's expected type is a function of the type distribution and equal to:

$$H_2(F) = \int_0^1 [tf(t) + F(t) - 1]F(t)^{N-1}dt \tag{8}$$

Integrating by parts and rearranging, equation (8) can be equivalently rewritten as:

$$H_2(F) = 1 + (N-1)\int_0^1 F(t)^N dt - N\int_0^1 F(t)^{N-1}dt \tag{9}$$

The following Lemma generalizes equation (9) in the case where there are M competing sellers offering simultaneous auctions.

Lemma 3: *Consider a setting where M sellers with reputations $r_1 \geq r_2 \geq ... \geq r_M$ simultaneously offer sealed-bid, second-price auctions for one unit of a homogeneous good to N unit-demand buyers, independently drawn from the same type distribution $F(t)$. The second highest bidder's expected type in seller k's auction is equal to:*

$$H_{2,k}^{\{r_1,...,r_M\}} = 1 + (N-1)\int_0^1 (1 - Q_k(1) + Q_k(t))^N \, dt - N\int_0^1 (1 - Q_k(1) + Q_k(t))^{N-1} \, dt \qquad (10)$$

The benchmark case is a setting where all sellers have the same reputation r. From Corollary 1, $Q_k(t) = F(t)/M$ for all k. The second highest bidder's expected type will then be identical for all auctions and equal to:

$$H_2^{\{M\}} = 1 + \frac{N-1}{M^N}\int_0^1 (M - 1 + F(t))^N \, dt - \frac{N}{M^{N-1}}\int_0^1 (M - 1 + F(t))^{N-1} \, dt$$

Proposition 4 characterizes the second highest bidder's expected type in a simultaneous auction setting where sellers have different reputations.

Proposition 4: *Consider a setting where a seller with reputation r_k (heretoforth referred to as* seller k*) competes with $M-1$ other sellers with reputations $r_1 \geq ... \geq r_{k-1} \geq r_{k+1} \geq ... \geq r_M$. All M sellers simultaneously offer sealed-bid, second-price auctions for one unit of a homogeneous good to N unit-demand buyers, independently drawn from the same type distribution $F(t)$. Let $H_{2,k}^{\{r_1,..,r_k,..,r_M\}}$ denote the expected value of the second highest bidder's type on seller* k*'s auction. The following statements are true:*

1. $H_{2,k}^{\{r_1,..,r_k,..,r_M\}}$ *is an increasing function of seller* k*'s own reputation* r.

2. $H_{2,k}^{\{r_1,..,r_k,..,r_M\}}$ *is a decreasing function of every other seller's reputation.*

The main corollary of Proposition 4 is that competition amplifies the impact of a reputation on a seller's expected auction revenue $U_k = r_k H_{2,k}^{\{r_1,..,r_M\}}$. When all sellers are identical, a seller's reputation affects the maximum valuation for that seller's good (it is scaled by r_k) but does not affect the second highest bidder's expected type (because all buyers randomize among all sellers). On the other hand, if sellers have different reputations, then sellers with higher (lower) reputation attract more (fewer) bidders of higher valuations and end up with higher (lower) expected second highest bidder types relative to the

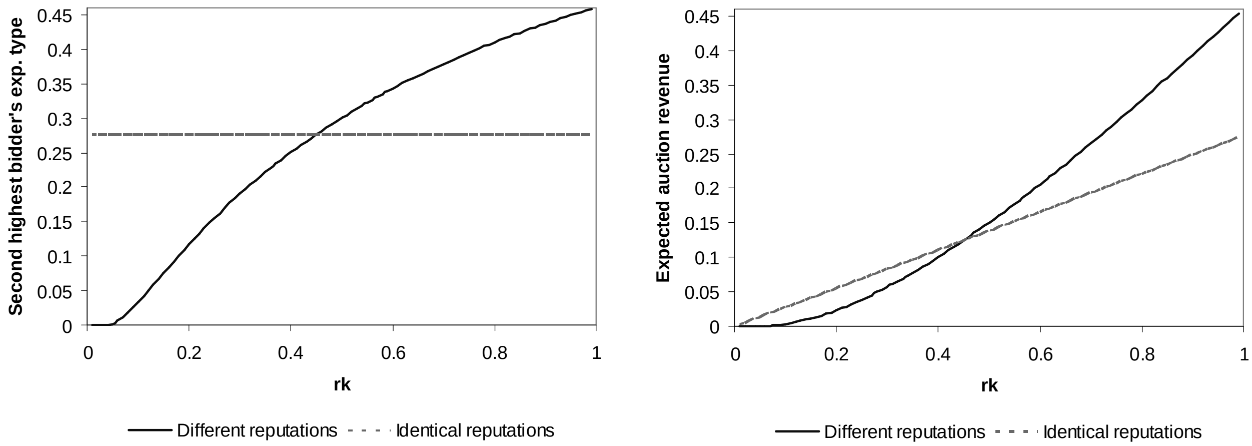


Figure 3: Second highest bidder's expected type $H_{2,k}^{\{r_1,...,r_M\}}$ and expected seller revenue $U_k = r_k H_{2,k}^{\{r_1,..,r_M\}}$ as a function of his reputation r_k when (a) all other $M-1$ sellers also have reputation r_k and (b) other sellers have fixed reputations i/M, $i = 1,..,M-1$. In both settings $M = 5$, $N = 10$ and buyer valuations are iid uniformly at $[0, 1]$.

baseline case of identical sellers. Reputation then has a double impact on expected seller revenue.

Figure 3 depicts this effect graphically by comparing the expected auction revenue of a seller of reputation r_k in the following two settings: (a) a setting where the other $M-1$ sellers also have reputation r_k, (b) a setting where the other $M-1$ sellers have reputations that are uniformly distributed in the interval $[0, 1]$. When all M sellers are identical, auction revenue is a linear function of seller reputation. However, when other sellers have different reputations, seller k's revenue is an increasing convex function of that seller's reputation. Furthermore, seller k's expected revenue surpasses that of the identical seller case when his reputation rises above a threshold that is roughly equal to the other $M-1$ sellers' average reputation. In settings with multiple competing sellers, each seller, then, has an additional incentive to maintain a good reputation.

4 Unrestricted number of bids

eBay currently does not limit the number of bids that buyers can simultaneously place on auctions for similar goods. Accordingly this section extends the preceding analysis to a setting where buyers are allowed to bid on an arbitrary number of simultaneous auctions. As before, I restrict the analysis to symmetric Bayes-Nash equilibria, that is, to equilibria where all bidders follow identical strategies. Even though all buyers have unit demand, it is plausible that some may then find it profitable to place several bids. There are two motivations for such behavior. First, by placing multiple bids, buyers increase their chances of winning at least one auction. Second, since sellers are unreliable, by winning more than one auction buyers increase their chances of *receiving* at least one item. On the other hand, if a buyer ends up receiving more than one items, she gets zero utility from additional units but still has to pay the respective auction prices.

When bidders are allowed to place any number of bids, individual bid amounts need no longer be equal to the bidder's respective expected valuations. The problem, thus, becomes one of finding a bid vector $\mathbf{b}(t) = (b_1(t), ..., b_M(t)) \in [0, t]^M$ that maximizes the bidder's expected global surplus from the system of auctions, given her beliefs about everybody else's bidding behavior. Let $G_k(b_k)$ denote a bidder's subjective probability of winning auction k conditional on placing bid b_k on that auction and also conditional on her beliefs about every other bidder's strategy. Let $g_k(b_k)$ denote the corresponding probability density function. A bidder's expected surplus from placing a vector of bids is given by:

$$\begin{aligned} V(t, \mathbf{b}(t)) = tPr[&\text{receive item from at least one seller}|\mathbf{b}(t)] \\ &- \textstyle\sum_{k=1}^{M} E[\text{payment to seller } k|b_k(t)] \end{aligned} \tag{11}$$

or, more formally, by:

$$V(t, \mathbf{b}(t)) = t\left[1 - \prod_{k=1}^{M}(1 - r_k G_k(b_k(t)))\right] - \sum_{k=1}^{M}\int_0^{b_k(t)} x g_k(x)dx \tag{12}$$

In the above expression $1 - r_k G_k(b_k(t))$ is the probability of *not receiving* an item from seller k. This includes the probability of not winning seller k's auction plus the probability of winning the auction but not receiving the item because seller k cheats. Accordingly, $\prod_{k=1}^{M}(1 - r_k G_k(b_k(t)))$ is the probability of not receiving the item from any seller and $1 - \prod_{k=1}^{M}(1 - r_k G_k(b_k(t)))$ the probability of receiving the item from at least one seller. The expression $\int_0^{b_k(t)} x g_k(x)dx$ is the well-known expression for the expected payment of a single second-price auction when bidding $b_k(t)$. Therefore, $\sum_{k=1}^{M}\int_0^{b_k(t)} x g_k(x)dx$ represents the buyer's total expected costs of participating to the system of auctions.

The following result shows that maximization of (12) implies non-zero bids in all available auctions for all bidder types in the interior of the type distribution:

Proposition 5: *For all $t \in (0, 1)$, any bid vector $\mathbf{b}(t)$ that maximizes (12) must have $b_k(t) > 0$ for all $k = 1, ..., M$.*

The form of the surplus-maximizing equilibrium bid vector can be obtained by setting the partial derivatives of (12) to zero:

$$\frac{\partial V(t,\cdot)}{\partial b_k} = g_k(b_k(t))\left[tr_k \prod_{j\in\{1,\ldots,M\}-\{k\}}(1-r_jG_j(b_j(t))) - b_k(t)\right] = 0 \tag{13}$$

If type t has positive probability density then at any symmetric equilibrium it must be $g_k(b_k(t)) > 0$. To see this, recall that $g_k(b_k(t))$ is equal to the probability that *some* bidder will post a bid equal to $b_k(t)$ on auction k. If $b_k(t)$ is part of one of type t's equilibrium bid vectors (there might be multiple such vectors in the case of mixed strategies) then it must be chosen by type t with positive probability, which implies that $g_k(b_k(t)) > 0$. The second part of (13) then yields:

$$b_k(t) = (tr_k)\prod_{j\in\{1,\ldots,M\}-\{k\}}(1-r_jG_j(b_j(t))) \tag{14}$$

In words, type t's optimal bid in auction k is equal to the bidder's expected valuation tr_k multiplied by the probability *of not receiving the item through any of the other auctions*, given the bidder's other bids $b_j(t)$ and her correct probability assessment $G_j(b_j(t))$ of winning each auction if every other bidder follows the same bidding strategy. The reader can verify that this expression is also equal to the bidder's marginal utility of winning auction k, given bids $b_j(t)$ in all other auctions. It is easy to show that the second partial derivative of (12) is negative, confirming that (14) is indeed the solution that maximizes (12).

The conceptual simplicity of recursive equation (14) is deceiving, since $G_j(b)$ depends on every other bidder's strategy over the entire type space, which (as it turns out) might be mixed and non-monotone. Observe that, for a given set of seller reputations r_k, each bid $b_k(t)$ can be equivalently characterized by its bid-to-valuation ratio:

$$\beta_k(t) = \frac{b_k(t)}{tr_k} = \prod_{j\in\{1,\ldots,M\}-\{k\}}(1-r_jG_j(b_j(t))) \tag{15}$$

This alternative characterization has the advantage of mapping the range of every auction's possible bids into the unit interval. From (15) it follows that a buyer's bid-to-valuation on each auction is negatively correlated with every one of her other bids. Therefore, the more a bidder focuses on winning a particular auction, the lower the bids she places on all other auctions. Based on this observation it is expected that bid vectors will have one of two general forms:

Type I a high bid (i.e. a bid whose bid-to-valuation ratio is close to one) on one of the auctions and low bids on all other auctions

Type II intermediate bids on all auctions

In the most general case, bidding strategies will be mixed and, therefore, defined by a probability density function $\sigma_k(t, \boldsymbol{\beta}) : [0,1] \times [0,1]^M \to [0,1]$, where $\boldsymbol{\beta} = (\beta_1, ..., \beta_M)$. Mixed strategy profiles that satisfy (15) can be computed using Monte Carlo methods. For example, Figures 4 and 5 depict aspects of the mixed bidding strategy profile in a setting where there are 3 sellers of reputations 0.9, 0.8 and 0.3 and 3 bidders whose valuations are uniformly distributed in $[0, 1]$.

Each of the three graphs of Figure 4 depicts the marginal probability distribution of the equilibrium bid-to-valuation ratio $\beta_k(t)$ for each of the three auctions (top graph=highest reputation seller, bottom graph=lowest reputation seller) and for all bidder types $t \in [0, 1]$. Darker shades of gray indicate higher probability of placing bids in the respective region.

Observe that low bidder types ($t < 0.2$) place bids that are close to their respective valuations on all three auctions. This is consistent with equation (15) since low types stand little chance of winning any auction. The bidding strategy distributions of higher types become increasingly diffuse. Close observation reveals the following patterns:

- The marginal probability distribution of $\beta_1(t)$ becomes bimodal for $t > 0.2$, indicating a probabilistic bifurcation of bid strategies: When using Type I bid vectors bidders randomize between placing a relatively high bid on auction 1 (accompanied by low bids on the other two auctions) and placing a low bid on auction 1 (accompanied by a relatively high bid on auction 2, see below). As bidder types increase the mean value of the high branch of $\beta_1(t)$ becomes increasingly close to 1 and acquires increasing probability mass (the latter fact is not clearly visible in the graphs), whereas the mean value of the, more diffuse, low branch declines. This implies that higher bidder types increasingly focus their attention on winning auction 1.

- The marginal probability distribution of $\beta_2(t)$ becomes multimodal for $t > 0.2$, suggesting that when bidders use Type I vectors they randomize between placing high and low bids on auction 2. This is consistent with the bimodality of $\beta_1(t)$'s distribution: when bidders place a higher bid on auction 1 they place a lower bid on auction 2 and vice versa. In contrast to the distribution of $\beta_1(t)$, as t increases the high branch of $\beta_2(t)$ neither monotonically increases nor attains increasing

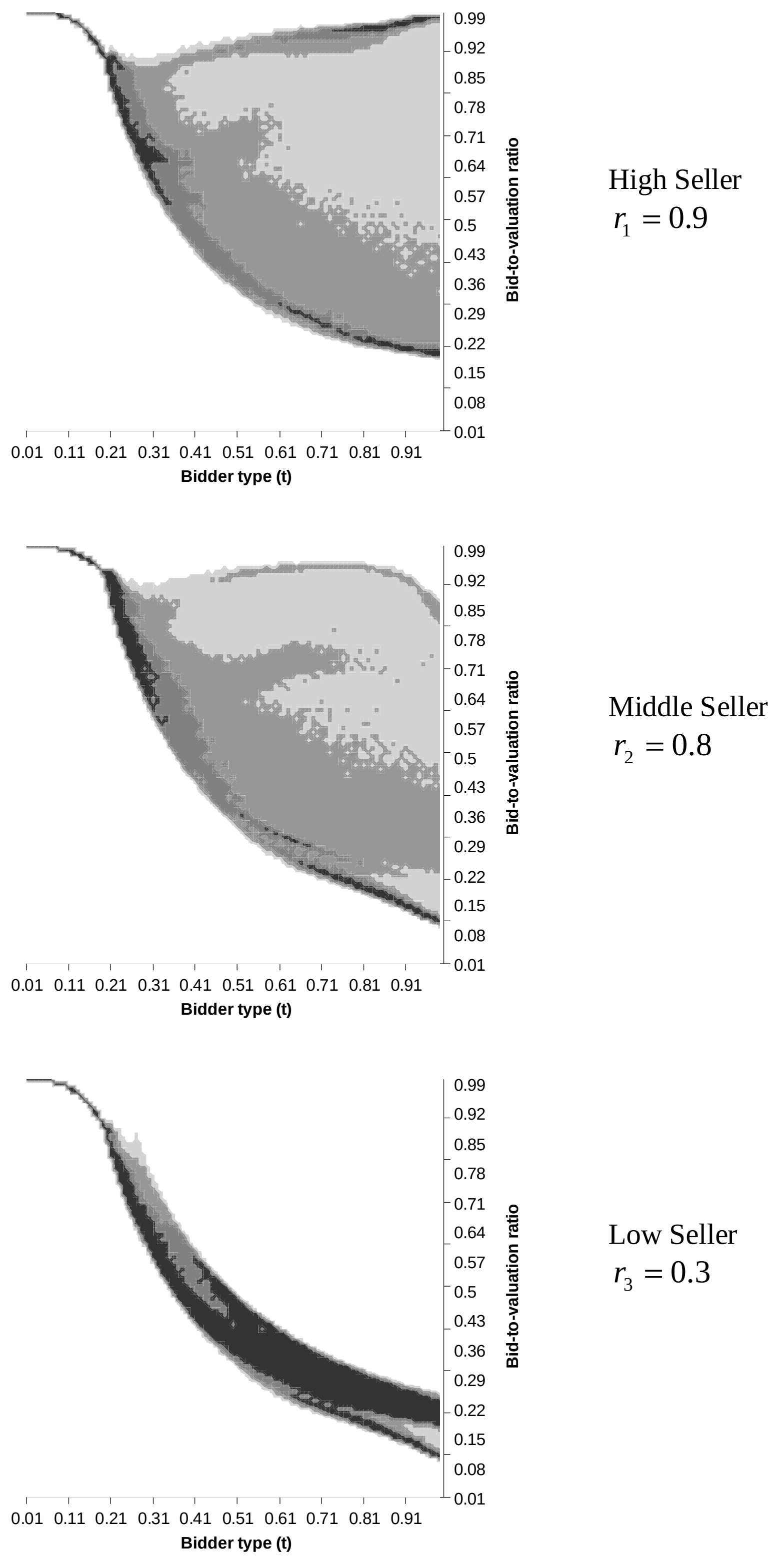


Figure 4: Marginal probability distributions of equilibrium bid-to-valuation ratios $\beta_k(t)$ in a simultaneous auction setting with 3 sellers of reputations 0.9, 0.8 and 0.3 and 3 bidders whose types are uniformly distributed in $[0, 1]$. Darker shades of gray indicate higher probability densities.

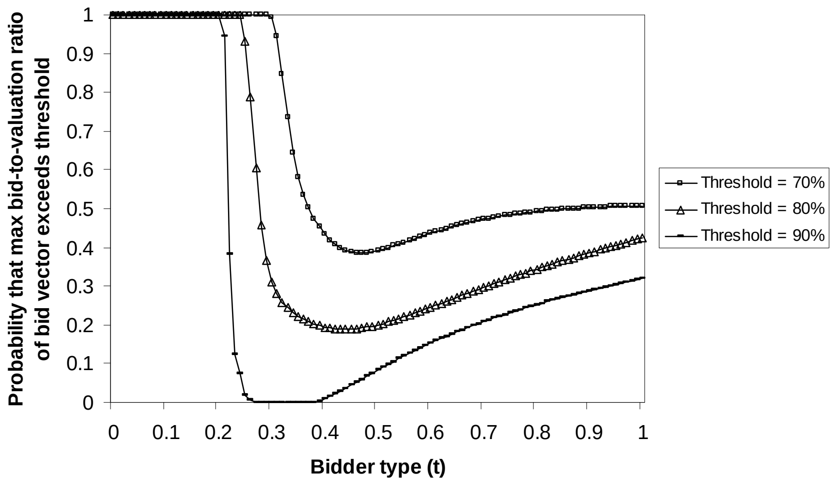


Figure 5: Probability of Type I bid vectors in a simultaneous auction setting with 3 sellers of reputations 0.9, 0.8 and 0.3 and 3 bidders whose types are uniformly distributed in $[0, 1]$.

probability mass. The probability mass of $\beta_2(t)$ for high t is concentrated at the bottom branch. Again, this is consistent with the patterns of bidding behavior on auction 1: since high bids on one auction imply low bids on all other auctions, the fact that high bidder types increasingly focus their attention on winning auction 1 implies that, with increasing probability, they lose their interest in seller 2 and thus place low bids on that seller's auction with increasing frequency.

- The marginal probability distribution of $\beta_3(t)$ is much less diffuse that the distributions of $\beta_1(t)$ and $\beta_2(t)$; its mean is a decreasing function of t. Because the reputation of seller 3 (0.3) is substantially lower than the reputations of the other two (0.9, 0.8) higher buyer types become increasingly uninterested in seriously bidding on that seller's auction, focusing instead on the top two auctions.

Figure 5 provides insight into bidders' relative use of Type I and Type II bid vectors. Specifically, the Figure plots the probability that different bidder types use bid vectors where the maximum bid-to-valuation ratio $\max_{k=1,..,M} \beta_k(t)$ is above 0.7 (70%), 0.8 (80%) and 0.9 (90%) respectively. We observe that, with the exception of very low types (who bid high on all auctions since they have small chance of winning any of them), higher bidder types use Type I vectors more frequently. This implies that higher bidder types increasingly focus their energies on winning one particular auction, placing low bids in all the rest.

In summary, even though its details are substantially more complex, the multi-bid equilibrium is somewhat similar, in spirit, to the one-bid equilibrium of Section 3. The lowest bidder types place bids close to their expected valuation on all auctions. Equilibrium bid vectors of higher bidder types have one of two forms: I) a high bid (i.e. a bid that is close to the bidder's expected valuation) on one auction and low bids on all remaining auctions or II) intermediate bids on all auctions. Bidders randomize between using Type I and Type II bid vectors. When using Type I vectors they further randomize with respect to which auction they place their high bid on. As their type increases, bidders use Type I bid vectors more often and place their high bid on higher auctions with higher probability. Just as in the one-bid case, the multi-bid equilibrium, thus, implements a form of probabilistic positive assortative matching between buyers and sellers.

5 Simultaneous auctions of imperfect substitute goods

So far our analysis has assumed that sellers promise to sell units of an identical good but differ in the reliability with which they are expected to fulfill their promises. This section shows how the preceding results generalize to settings where sellers of different reputations offer units of vertically differentiated imperfect substitute goods (e.g. used units of the same good in different conditions), a case that is, perhaps, even more pervasive in online auction settings.

To highlight the subtle differences between seller reputation and imperfect substitution on bidding strategies, we first consider the case where all sellers are reliable (i.e. their reputations are equal to one). We model imperfect substitution by assuming that buyer t's valuation of seller k's good is equal to $v_k(t) = tu_k$, where $u_k \in [0, 1]$ represents the ratio of buyer t's valuation of seller k's good over her valuation of a hypothetical "ideal" good of the same class (e.g. a brand new unit of the product). All buyers are assumed to assign the same ratio u_k to seller k's good. In the rest of the section we will refer to u_k as "item quality", keeping in mind that alternative interpretations (e.g. "item condition", "item age") are also plausible. We assume throughout that $u_1 \geq u_2 \geq ... \geq u_M$.

If we substitute u_k for r_k, buyer t's expected surplus from bidding her expected valuation on seller k's auction is given by expression (1). With this substitution, all results of Section 3 (one-bid equilibria) also apply to simultaneous auctions of imperfect substitute goods by reliable sellers.

The two settings have subtle differences in the case where buyers are allowed to submit an arbitrary number of bids. Assuming, as before, that each buyer has unit demand and does not receive any utility from additional units of the good, bidder t's expected surplus from placing a vector of bids $\mathbf{b}(t)$ in the new setting is equal to her expected valuation of the *maximum quality item she is likely to win* minus the sum of expected payments. If $u_1 \geq u_2 \geq ... \geq u_M$ the expected valuation of the maximum quality item that bidder t is likely to win by bidding $\mathbf{b}(t)$ is given by:

$$\begin{aligned} v(t,\mathbf{b}(t)) &= t\left[u_1 G_1(b_1(t)) + u_2(1-G_1(b_1(t)))G_2(b_2(t)) + ... + u_M\left(\prod_{j=1}^{M-1}(1-G_j(b_j(t)))\right)G_M(b_M(t))\right] \\ &= t\sum_{k=1}^{M} u_k\left(\prod_{j=1}^{k-1}(1-G_j(b_j(t)))\right)G_k(b_k(t)) \end{aligned}$$

The above expression is equal to the sum of each auction's expected valuation multiplied by the probability that the bidder will win that auction but will lose all higher auctions. The bidder's expected surplus from bidding $\mathbf{b}(t)$ is then equal to:

$$V(t,\mathbf{b}(t)) = t\sum_{k=1}^{M} u_k\left(\prod_{j=1}^{k-1}(1-G_j(b_j(t)))\right)G_k(b_k(t)) - \sum_{k=1}^{M}\int_0^{b_k(t)} x g_k(x)dx \tag{16}$$

Following a procedure analogous to the proof of Proposition 5 one can show that, for all $t \in (0,1)$, profit-maximizing bid vectors must have non-zero bids in all available auctions. The form of the surplus-maximizing equilibrium bid vector can be obtained, as before, by setting the partial derivatives of (16) to zero. The following expression obtains:

$$b_k(t) = t\left(\prod_{j=1}^{k-1}(1-G_j(b_j(t)))\right)\left[u_k - \sum_{\ell=k+1}^{M} u_\ell\left(\prod_{j=k+1}^{\ell-1}(1-G_j(b_j(t)))\right)G_\ell(b_\ell(t))\right] \tag{17}$$

In words, type t's optimal bid in auction k is equal to the probability that the bidder will not win any item whose quality is higher than k (i.e. any auction $j = 1, ..., k-1$) multiplied by the difference between her valuation of item k and her expected utility from auctions $j = k+1, ..., M$, given her other bids $b_j(t)$.[6] This is precisely the marginal utility from winning auction k given bids $b_j(t)$ in all other auctions: Winning item k only adds to the buyer's utility if she does not win any higher quality item; the amount it adds is the difference between that item's valuation and the expected utility from all lower auctions.

Equation (17) has similar properties to its counterpart (14) in Section 4. There is a similar negative relationship between a buyer's bid on each auction and her bids in every other auction. Numerical simulations (not reported here) reveal that bidding equilibria have qualitatively similar properties.

The most general case is one where sellers sell imperfect substitute goods *and* have different repu-

[6] If there exist more than one auctions whose quality is equal to u_k, the above formula still holds if we rearrange auctions so that index k is the lowest index among auctions whose quality is equal to u_k.

tations. Each auction is now characterized in terms of two parameters r_k, u_k, where r_k represents the seller's reputation and u_k represents the item's quality. A buyer of type t who wins seller k's auction at price p thus gets expected surplus $tr_k u_k - p$. If we substitute $(r_k u_k)$ for r_k and rearrange auctions so that $r_1 u_1 \geq r_2 u_2 \geq ... \geq r_M u_M$, buyer t's expected surplus from bidding her expected valuation on seller k's auction is given by expression (1). With these substitutions, all results of Section 3 (one-bid equilibria) also apply to simultaneous auctions of imperfect substitute goods by sellers of different reputations.

In the multi-bid setting, assuming that $u_1 \geq u_2 \geq ... \geq u_M$, the expression of a buyer's expected utility from placing a vector of bids must be modified to account for seller reputations as follows:

$$v(t, \mathbf{b}(t)) = t \sum_{k=1}^{M} u_k \left(\prod_{j=1}^{k-1} (1 - r_j G_j(b_j(t))) \right) r_k G_k(b_k(t))$$

The changes consist of multiplying the probability of winning an item by the corresponding seller's reputation and also replacing the probability $1 - G_j(b_j(t))$ of not winning auction j by the probability $1 - r_j G_j(b_j(t))$ of *not receiving an item* from auction j. The form of the surplus-maximizing equilibrium bid vector similarly changes to:

$$b_k(t) = tr_k \left(\prod_{j=1}^{k-1} (1 - r_j G_j(b_j(t))) \right) \left[u_k - \sum_{\ell=k+1}^{M} u_\ell \left(\prod_{j=k+1}^{\ell-1} (1 - r_j G_j(b_j(t))) \right) r_\ell G_\ell(b_\ell(t)) \right] \tag{18}$$

The reader can check that for $u_1 = ... = u_M = 1$, equation (18) reduces to (14).

6 Concluding Remarks

This paper derives the form of bidding equilibria in simultaneous sealed-bid, second price auction settings where sellers offer single units of imperfect substitute goods, buyers have unit demand, and sellers have different reputations for delivering on their promises. Among other possible applications, the model can serve as an abstraction of large-scale decentralized Internet auction marketplaces, such as eBay. The following points summarize the main results, integrating the impact of imperfect substitution and reputation on bidding behavior:

- If bidders are restricted to place at most one bid, the unique Bayes-Nash equilibrium corresponds to a form of probabilistic positive assortative matching: Bidders rank-order auctions with respect to their relative attractiveness, taking into account both the quality of the item and the seller's reputation. Bidders assess where they stand on the valuation type space relative to other bidders and assign higher probability to bidding on the auction that "matches" their respective type zone, while also occasionally bidding on "higher" auctions.

- The probabilistic nature of the equilibrium introduces allocative inefficiencies. Specifically, the lack of coordination between bidders makes it possible that two or more high bidder types will cluster on the same auction (in which case only one of them wins and the remaining auctions will be left to lower bidder types) whereas if these same bidders could coordinate and distribute their bids to different auctions they would all win, increasing social welfare. Inefficiencies can be substantial in settings where the number of bidders is roughly equal to the number of sellers. In theory it is possible to avoid such inefficiencies by replacing the set of simultaneous single-unit auctions with a, provably efficient, centralized multi-unit auction, such as a VCG auction.

- Competition amplifies the impact of a seller's reputation and item quality on his expected auction revenue. In the monopoly case, if bidders are risk-neutral, a seller's expected auction revenue is a linear function of his reputation and/or item quality. In simultaneous auction settings, seller revenue is an increasing convex function of reputation and/or item quality.

- Allowing unit-demand bidders to place an arbitrary number of bids induces complex mixed strategy profiles where bidders place positive bids in all available auctions. The lowest bidder types place bids close to their expected valuation on all auctions. Equilibrium bid vectors of higher bidder types have one of two forms: I) a high bid (i.e. a bid that is close to the bidder's expected valuation) on one auction and low bids on all remaining auctions or II) intermediate bids on all auctions. Bidders randomize between using Type I and Type II bid vectors. When using Type I vectors they further randomize with respect to which auction they place their high bid on. As their type increases, bidders use Type I bid vectors more often and place their high bid on higher auctions with higher probability.

The results of this paper can be extended in several directions. To retain tractability I have abstracted away from a number of aspects of real-life online auctions. First, most Internet auctions are ascending price English auctions. Second, they are not truly simultaneous but overlap in time. Third, unit-demand buyers who win multiple items have the opportunity to resell surplus items on the same electronic marketplace. It would be interesting to study to what extent these aspects affect the form of the bidding equilibria in actual practice. My analysis also assumes that buyers are fully rational and take the entire available set of auctions into consideration. The plausibility of this assumption is questionable in large-scale electronic markets, where the number of simultaneous auctions for imperfect substitutes is very large. A casual search on eBay in March 2008 revealed 668 simultaneous listings for a TiVo digital video recorder, 2367 listings for an Apple iPhone, and 6136 listings for a Blackberry smartphone. Most buyers do not perform an exhaustive search of all available listings before deciding on their bidding strategy. A particularly interesting research question then is to what extent the behavior of human bidders in experimental or real-life settings departs from the predictions of this paper and what aspect of bounded rationality can best explain such departures.

A Proofs

A.1 Proofs of Lemmas

Lemma 1

Part 1. Follows directly from the interpretation of the quantity $Q_k(1|\mathbf{s})$.

Part 2. The probability that a buyer of type t wins the auction of seller k is equal to:

$$W_k(t|\mathbf{s}) = \sum_{m=0}^{N-1} Pr[m \text{ other bidders choose } k|\mathbf{s}] \times Pr[\text{all } m \text{ other bidders have types} \leq t|\mathbf{s}] \tag{19}$$

From Part 1 it follows that $Pr[m \text{ other bidders choose } k|\mathbf{s}] = \binom{N}{m} Q_k(1|\mathbf{s})^m \left(1 - Q_k(1|\mathbf{s})\right)^{N-m}$. The subjective probability that a bidder who selects seller k has type t is given by:

$$g_k(t|\mathbf{s}) = Pr[t|k;\mathbf{s}] = Pr[k|t;\mathbf{s}]f(t)/Pr[k|\mathbf{s}] = s_k(t)f(t)/\int_0^1 s_k(t)f(t)dt = Q_k'(t|\mathbf{s})/Q_k(1|\mathbf{s})$$

The probability that all other m bidders of auction k have types less than or equal to t is equal to:

$$Pr[\text{all } m \text{ other bidders have types} \leq t] = \left(\int_0^t g_k(u|\mathbf{s})du\right)^m = \frac{Q_k(t|\mathbf{s})^m}{Q_k(1|\mathbf{s})^m}$$

Substituting into (19) and making use of the properties of binomial sums:

$$W_k(t|\mathbf{s}) = \sum_{m=0}^{N-1} \binom{N}{m} Q_k(1|\mathbf{s})^m \left(1 - Q_k(1|\mathbf{s})\right)^{N-m} \frac{Q_k(t|\mathbf{s})^m}{Q_k(1|\mathbf{s})^m} = \left(1 - Q_k(1|\mathbf{s}) + Q_k(t|\mathbf{s})\right)^{N-1}$$

Part 3. Buyer t's expected surplus from selecting seller k is equal to:

$$\begin{aligned} V_k(t|\mathbf{s}) = &\textstyle\sum_{m=0}^{N-1} Pr[m \text{ other bidders choose } k|\mathbf{s}] \times Pr[\text{all } m \text{ other bidders have types} \leq t|\mathbf{s}] \\ &\times r_k(t - E[\text{2nd highest type}|\text{highest type}{=}t; \exists m \text{ other bidders}, \mathbf{s}]) \end{aligned} \tag{20}$$

The conditional probability that the 2nd highest out of $m+1$ bidder types is equal to x if the highest type is equal to t is equal to:

$$Pr[\text{2nd type=x}|\text{1st type=}t; m+1 \text{ bidders}, \mathbf{s}] = \frac{Pr[\text{2nd type=x and 1st type=}t; m+1 \text{ bidders}, \mathbf{s}]}{Pr[\text{1st type=}t; m+1 \text{ bidders}, \mathbf{s}]}$$

where, from the theory of order statistics (see, for example, David and Nagaraja 2003):

$$\begin{aligned} Pr[\text{1st type=}t; m+1 \text{ bidders}, \mathbf{s}] &= G_k(t|\mathbf{s})^m G_k'(t|\mathbf{s})(m+1) \\ Pr[\text{2nd type=x and 1st type=}t; m+1 \text{ bidders}, \mathbf{s}] &= G_k(x|\mathbf{s})^{m-1} G_k'(x|\mathbf{s}) G_k'(t|\mathbf{s})(m+1)m \end{aligned}$$

Therefore:

$$\begin{aligned} &Pr[\text{2nd type=x}|\text{1st type=}t; m+1 \text{ bidders}, \mathbf{s}] \\ &\quad= \tfrac{G_k(x|\mathbf{s})^{m-1} G_k'(x|\mathbf{s}) G_k'(t|\mathbf{s})(m+1)m}{G_k(t|\mathbf{s})^m G_k'(t|\mathbf{s})(m+1)} = \tfrac{G_k(x|\mathbf{s})^{m-1} G_k'(x|\mathbf{s})m}{G_k(t|\mathbf{s})^m} = \tfrac{1}{G_k(t|\mathbf{s})^m} \tfrac{\partial G_k(x|\mathbf{s})^m}{\partial x} \end{aligned}$$

and:

$$\begin{aligned} &E[\text{2nd type=x}|\text{1st type=}t; m+1 \text{ bidders}, \mathbf{s}] \\ &\quad= \tfrac{\int_0^t x \frac{\partial G_k(x|\mathbf{s})^m}{\partial x} dx}{G_k(t|\mathbf{s})^m} = \tfrac{t G_k(t|\mathbf{s})^m - \int_0^t G_k(x|\mathbf{s})^m dx}{G_k(t|\mathbf{s})^m} = t - \tfrac{\int_0^t G_k(x|\mathbf{s})^m dx}{G_k(t|\mathbf{s})^m} \end{aligned}$$

Substituting into (20) we obtain:

$$\begin{aligned} V_k(t|\mathbf{s}) &= \textstyle\sum_{m=0}^{N-1} \binom{N-1}{m} Q_k(1|\mathbf{s})^m \left(1 - Q_k(1|\mathbf{s})\right)^{N-1-m} G_k(t|\mathbf{s})^m r_k \frac{\int_0^t G_k(x|\mathbf{s})^m dx}{G_k(t|\mathbf{s})^m} \\ &= r_k \textstyle\sum_{m=0}^{N-1} \binom{N-1}{m} \left(1 - Q_k(1)\right)^{N-1-m} \int_0^t Q_k(x|\mathbf{s})^m dx \\ &= r_k \textstyle\int_0^t \left(1 - Q_k(1|\mathbf{s}) + Q_k(x|\mathbf{s})\right)^{N-1} dx \end{aligned}$$

Lemma 2

From standard auction theory the highest bidder's expected type in a single auction setting with m bidders independently drawn from distribution $F(t)$ is $\int_0^1 t(m-1)f(t)F(t)^{m-1}dt = 1 - \int_0^1 F(t)^m dt$. The highest bidder's expected type in a setting with M simultaneous auctions and N sellers is equal to:

$$H_{1,k} = \sum_{m=0}^{N} Pr[m \text{ bidders choose } k] \times E[\text{highest bidder } |m \text{ bidders}]$$

where:

$$Pr[m \text{ other bidders choose } k] = \tbinom{N}{m} Q_k(1)^m \left(1 - Q_k(1)\right)^{N-m}$$
$$E[\text{highest bidder } |m \text{ bidders}] = 1 - \textstyle\int_0^1 G_k(t)^m dt,\ G_k(t) = Q_k(t)/Q_k(1)$$

Substituting and rearranging, taking into account the properties of binomial sums, produces the result.

Lemma 3

The second highest bidder's expected type in a setting with M simultaneous auctions and N sellers is equal to:

$$H_{2,k} = \sum_{m=0}^{N} Pr[m \text{ bidders choose } k] \times E[\text{second highest bidder } |m \text{ bidders}]$$

where:

$$Pr[m \text{ other bidders choose } k] = \tbinom{N}{m} Q_k(1)^m \left(1 - Q_k(1)\right)^{N-m}$$
$$E[\text{second highest bidder } |m \text{ bidders}] = 1 + (m-1)\textstyle\int_0^1 G_k(t)^m dt - m\int_0^1 G_k(t)^{m-1}dt,\ G_k(t) = Q_k(t)/Q_k(1)$$

Substituting and rearranging, taking into account the properties of binomial sums, produces the result.

A.2 Proofs of Propositions

Proposition 1

The proof is by contradiction. Suppose that there exists a pure strategy one-bid equilibrium in which types $t \in (0, t_0]$ always bid on seller k's auction. There are two possible cases:

$t_0 = 1$**.** Then $Q_k(t) = F(t)$, $Q_k(1) = 1$ and $Q_\ell(t) = 0$ for all $\ell \neq k$ and all $t \in (0, 1]$. From incentive compatibility constraint (6), since it is $V_k(0) = V_\ell(0) = 0$, the assumption $s_k(0) = 1$ implies that $\partial V_k(0)/\partial t \geq \partial V_\ell(0)/\partial t$ for all $\ell \neq k$. Substituting the above values of $Q_k(t)$ into (5) the inequality becomes:

$$r_k\left(1 - Q_k(1) + Q_k(0)\right)^{N-1} \geq r_\ell\left(1 - Q_\ell(1) + Q_\ell(0)\right)^{N-1}$$
$$r_k\left(1 - 1 + 0\right)^{N-1} \geq r_\ell\left(1 - 0 + 0\right)^{N-1}$$
$$0 \geq r_\ell$$

The above is a contradiction if there exists at least one other seller with nonzero reputation.

$t_0 < 1$. As before, the assumption $s_k(t) = 1$ for $t \in (0, t_0]$ requires that $\partial V_k(0)/\partial t \geq \partial V_\ell(0)/\partial t$. This is equivalent to:

$$r_k\left(1 - Q_k(1)\right)^{N-1} \geq r_\ell\left(1 - Q_\ell(1)\right)^{N-1} \tag{21}$$

The assumption of a pure equilibrium implies that, at type t_0, buyers will switch from seller k to another seller $\ell \neq k$. If strategies are piecewise continuous then expected buyer surpluses are continuous. The switch, therefore, implies that $V_k(t_0) = V_\ell(t_0)$ and $\partial V_k(t_0)/\partial t \leq \partial V_\ell(t_0)/\partial t$. The assumption $s_k(t) = 1$ for $t \in (0, t_0]$ implies $Q_k(t) = F(t)$ and $Q_\ell(t) = 0$ for all $t \in (0, t_0]$. The inequality of derivatives at $t = t_0$ is, then, equivalent to:

$$r_k\left(1 - Q_k(1) + F(t_0)\right)^{N-1} \leq r_\ell\left(1 - Q_\ell(1)\right)^{N-1} \tag{22}$$

Noting that $r_k\left(1 - Q_k(1) + F(t_0)\right)^{N-1} > r_k\left(1 - Q_k(1)\right)^{N-1}$, it is easy to see that requirements (21) and (22) lead to a contradiction.

Proposition 2

Part 1. Let $0 < t_0 < t_1 \leq 1$. From Proposition 1 I have established that all equilibria must be mixed at $t \in (0, t_0]$ for some positive t_0. Furthermore I have shown that sellers that are not chosen with positive probability by the lowest buyer types will not be chosen with positive probability by any buyer. I therefore restrict my attention on the subset S of sellers that are included in the choice set of the lowest-valuation bidders. Assume the existence of an equilibrium where $s_k^*(t) = 1$ for $t \in (t_0, t_1]$. For all $\ell \in S$, at $t = t_0$ it must be $V_k(t_0) = V_\ell(t_0)$ (equilibrium is mixed for all lower types) and $\partial V_k(t_0)/\partial t \geq \partial V_\ell(t_0)/\partial t$, or equivalently:

$$r_k\left(1 - Q_k(1) + Q_k(t_0)\right)^{N-1} \geq r_\ell\left(1 - Q_\ell(1) + Q_\ell(t_0)\right)^{N-1}$$

The assumption $s_k^*(t) = 1$ for $t \in (t_0, t_1]$ implies $Q_k(t) = Q_k(t_0) + F(t) - F(t_0) > Q_k(t_0)$ and $Q_\ell(t) = Q_\ell(t_0)$ for all $t \in (t_0, t_1]$. But this, in turn, implies that $\partial V_k(t)/\partial t > \partial V_k(t_0)/\partial t \geq \partial V_\ell(t_0)/\partial t$ and, consequently, that $V_k(t) > V_\ell(t)$: all buyer types higher than t_0 will then strictly prefer to bid on seller k's auction. Thus, it must be $t_1 = 1$ and $s_k^*(t) = 1$ for all $t \in (t_0, 1]$.

Part 2. Assume that there exists some $t_1 > t_0 \geq 0$ such that $s_k^*(t) > 0$ for $t \leq t_0$ and $s_k^*(t) = 0$ for $t \in (t_0, t_1]$. According to Part 1 of Proposition 2, for $t \leq t_0$ it cannot be $s_k^*(t) = 1$ (otherwise, it would have to be $s_k^*(t) = 1$ for all $t > t_0$). Thus, for $t \leq t_0$ the equilibrium has to be mixed. At $t = t_0$ it must then be $V_k(t_0) = V_\ell(t_0)$ (equilibrium is mixed) and $\partial V_k(t_0)/\partial t \leq \partial V_\ell(t_0)/\partial t$ for at least one $\ell \neq k$, or equivalently:

$$r_k \left(1 - Q_k(1) + Q_k(t_0)\right)^{N-1} \leq r_\ell \left(1 - Q_\ell(1) + Q_\ell(t_0)\right)^{N-1}$$

The assumption $s_k^*(t) = 0$ for $t \in (t_0, t_1]$ implies $Q_\ell(t) > Q_\ell(t_0)$ and $Q_k(t) = Q_k(t_0)$ for all $t \in (t_0, t_1]$. But this, in turn, implies that $\partial V_\ell(t)/\partial t > \partial V_\ell(t_0)/\partial t \geq \partial V_k(t_0)/\partial t$ and, consequently, that $V_\ell(t) > V_k(t)$: no buyer type higher than t_0 will then prefer to bid on seller k's auction. Thus, it must be $t_1 = 1$ and $s_k^*(t) = 0$ for all $t \in (t_0, 1]$.

Part 3. Suppose that there exists some $t_0 \geq 0$ such that $s_k^*(t) > 0$ for $t \leq t_0$ and $s_k^*(t) = 0$ for $t > t_0$. It must then be $Q_k(t_0) = Q_k(1)$. Assume now that there exists some $\ell > k$, such that $s_\ell^*(t) > 0$ for $t > t_0$. It must be $Q_\ell(t_0) < Q_\ell(1)$. Furthermore, our convention for ordering sellers implies that $r_\ell \leq r_k$. At $t = t_0$ it must be $V_k(t_0) = V_\ell(t_0)$ (by Part 2, $s_\ell^*(t) > 0$ for $t > t_0$ implies $s_\ell^*(t) > 0$ for $t \leq t_0$) and $\partial V_k(t_0)/\partial t \leq \partial V_\ell(t_0)/\partial t$, or equivalently:

$$r_k \left(1 - Q_k(1) + Q_k(t_0)\right)^{N-1} \leq r_\ell \left(1 - Q_\ell(1) + Q_\ell(t_0)\right)^{N-1}$$
$$r_k \left(1 - Q_k(1) + Q_k(1)\right)^{N-1} \leq r_\ell \left(1 - Q_\ell(1) + Q_\ell(t_0)\right)^{N-1}$$
$$r_k < r_\ell$$

The above contradicts the assumption $r_\ell \leq r_k$.

Part 4. At the point $t_0 \geq 0$ where $s_k^*(t), s_\ell^*(t)$ switch from positive values to zero it must be $\partial V_k(t_0)/\partial t = \partial V_\ell(t_0)/\partial t$. In addition, it must be $Q_k(t_0) = Q_k(1)$, and $Q_\ell(t_0) = Q_\ell(1)$. Substitution into (4) immediately produces the result.

Proposition 3

I will show that a given set of reputations $r_1 \leq ... \leq r_M$ defines a unique $L \leq M$, associated zone delimiters t_z, $z = 0, ..., L$, and selection probabilities $s_k(t)$ that satisfy the following incentive compatibility constraints:

IC1: $V_k(t) = V_z(t)$ for all $k = 1, .., z$ and $t \leq t_{z-1}$

IC2: $V_k(t) > V_z(t)$ for all $k = 1, .., z-1$ and $t > t_{z-1}$

Since $V_k(0) = 0$ for all k, at $t = 0$ (IC1) implies that $\partial V_k(0)/\partial t = \partial V_\ell(0)/\partial t$ for all $k = 1, ..., L$. If $L < M$ then it must also be $\partial V_k(0)/\partial t < \partial V_L(0)/\partial t$ for $k = L+1, .., M$. From $\frac{\partial V_k(t)}{\partial t} = r_k \left(1 - Q_k(1) + Q_k(t)\right)^{N-1}$ (IC1) then implies:

$$r_k \left(1 - Q_k(1)\right)^{N-1} = r_L \left(1 - Q_L(1)\right)^{N-1} \text{ for all } 1 \leq k \leq L \tag{23}$$

Equation (23) plus the one-bid constraint $\sum_{k=1}^{L} Q_k(1) = 1$ give:

$$Q_k(1) = 1 - (L-1) \frac{\sqrt[N-1]{\frac{1}{r_k}}}{\sum_{i=1}^{L} \sqrt[N-1]{\frac{1}{r_i}}} \tag{24}$$

Since, for $k = L+1, .., M$ it is $Q_k(t) = 0$ for all t, (IC2) is equivalent to:

$$r_k < r_L \left(1 - Q_L(1)\right)^{N-1} \text{ for all } L+1 \leq k \leq M$$

Substituting (24), the above expression becomes:

$$r_k < \left(\frac{L-1}{\sum_{i=1}^{L} \sqrt[N-1]{\frac{1}{r_i}}} \right)^{N-1} \text{ for all } L+1 \leq k \leq M$$

Since $r_{L+1} \geq r_{L+2} \geq ... \geq r_M$ the above inequality can be rewritten as:

$$r_{k+1} < \left(\frac{k-1}{\sum_{i=1}^{k} \sqrt[N-1]{\frac{1}{r_i}}} \right)^{N-1} \text{ for some } k < M \tag{25}$$

The number of equilibrium type zones L is equal to the lowest k that satisfies (25). If no $L < M$ satisfies (25) then $L = M$ and all sellers will receive bids with positive probability from at least some buyers.

Constraint (IC1) plus the fact that $V_k(0) = 0$ for all k implies that $\partial V_k(t)/\partial t = \partial V_z(t)/\partial t$ and $\partial^2 V_k(t)/\partial t^2 = \partial^2 V_z(t)/\partial t^2$ for all $k = 1, .., z$ and $t < t_{z-1}$. From $\frac{\partial V_k(t)}{\partial t} = r_k \left(1 - Q_k(1) + Q_k(t)\right)^{N-1}$, $\frac{\partial^2 V_k(t)}{\partial t^2} = r_k(N-1)\left(1 - Q_k(1) + Q_k(t)\right)^{N-2} s_k(t) f(t)$ we obtain:

$$s_k(t) = \sqrt[N-1]{\frac{r_z}{r_k}} s_z(t) \text{ for all } k = 1, .., z,\ t < t_{z-1}$$

Together with the one-bid condition $\sum_{k=1}^{z} s_k(t) = 1$ this implies that buyers whose types fall in zone z select seller k with constant probability:

$$s_{zk} = \frac{\sqrt[N-1]{\frac{1}{r_k}}}{\sum_{i=1}^{z} \sqrt[N-1]{\frac{1}{r_i}}} \tag{26}$$

I will now prove by induction that equality of derivatives $\partial V_k(t)/\partial t$ and the condition $Q_z(t_{z-1}) = Q_z(1)$ at type delimiters $t_1, .., t_{L-1}$ uniquely defines those delimiters as the solution of the equation:

$$F(t_z) = \left(\sqrt[N-1]{r_{z+1}} \sum_{i=1}^{z} \sqrt[N-1]{\frac{1}{r_i}} \right) - (z-1) \tag{27}$$

1. At $t = t_1$, the conditions $\partial V_1(t)/\partial t = \partial V_2(t)/\partial t$ and $Q_2(t_1) = Q_2(1)$ imply that:

$$r_1 \left(1 - (1 - F(t_1))\right)^{N-1} = r_2 \left(1\right)^{N-1}$$
$$F(t_1) = \sqrt[N-1]{\frac{r_2}{r_1}} = \left(\sqrt[N-1]{r_2}\ \sqrt[N-1]{\frac{1}{r_1}} \right) - (1-1)$$

2. Assume now that $F(t_y) = \left(\sqrt[N-1]{r_{y+1}} \sum_{i=1}^{y} \sqrt[N-1]{\frac{1}{r_i}} \right) - (y-1)$ for $y < z$. At $t = t_z$, the conditions $\partial V_z(t)/\partial t = \partial V_{z+1}(t)/\partial t$ and $Q_{z+1}(t_z) = Q_{z+1}(1)$ imply that:

$$r_z \left(1 - s_{zz}(F(t_{z-1}) - F(t_z))\right)^{N-1} = r_{z+1} \left(1\right)^{N-1}$$
$$r_z \left(1 - \frac{\sqrt[N-1]{\frac{1}{r_z}}}{\sum_{i=1}^{z} \sqrt[N-1]{\frac{1}{r_i}}} (\left(\sqrt[N-1]{r_z} \sum_{i=1}^{z-1} \sqrt[N-1]{\frac{1}{r_i}} \right) - (z-2) - F(t_z)) \right)^{N-1} = r_{z+1} \left(1\right)^{N-1}$$
$$F(t_z) = \left(\sqrt[N-1]{r_{z+1}} \sum_{i=1}^{z} \sqrt[N-1]{\frac{1}{r_i}} \right) - (z-1)$$

The requirement $F(t_z) > 0$ for $z = 1, ..., L-1$ implies that equilibria of the above form exist if and only if $\left(\sqrt[N-1]{r_{z+1}} \sum_{i=1}^{z} \sqrt[N-1]{\frac{1}{r_i}} \right) - (z-1) > 0$ or, equivalently, if $r_{z+1} > \left(\frac{z-1}{\sum_{i=1}^{z} \sqrt[N-1]{\frac{1}{r_i}}} \right)^{N-1}$ for all $1 < z \leq L-1$. This is consistent with the definition of L as the lowest integer k for which $r_{k+1} < \left(\frac{k-1}{\sum_{i=1}^{k} \sqrt[N-1]{\frac{1}{r_i}}} \right)^{N-1}$.

It is easy to show that the above strategy profile also satisfies constraint (IC2). From

$$\frac{\partial V_z(t)}{\partial t} = r_z \left(1 - Q_z(1) + Q_z(t)\right)^{N-1}$$

it follows that $\frac{\partial V_z(t)}{\partial t}$ increases with t for $t \leq t_{z-1}$ and then remains constant and equal to r_z for $t > t_{z-1}$. Let us now consider some $k < z$. The assumption of a mixed equilibrium implies $\frac{\partial V_z(t)}{\partial t} = \frac{\partial V_k(t)}{\partial t}$ for all $t \leq t_{z-1}$. At $t = t_{z-1}$, $\frac{\partial V_z(t)}{\partial t}$ turns into a constant whereas $\frac{\partial V_k(t)}{\partial t}$ keeps growing. Therefore, $\frac{\partial V_z(t)}{\partial t} < \frac{\partial V_k(t)}{\partial t} \Rightarrow V_z(t) < V_k(t)$ for all $k < z$ and all $t > t_{z-1}$.

From Lemma 1, the expected number of bids in each seller's auction is equal to $NQ_k(1)$. Substitution of equation (24) immediately proves Part 5 of the Theorem.

Proposition 4

Let $R_k(t) = 1 - Q_k(1) + Q_k(t)$. It is $0 \leq R_k(t) \leq 1$ and $R_k(1) = 1$ for all t. Equation (10) can then be rewritten as:

$$H_{2,k}^{\{r_1,...,r_M\}} = 1 + \int_0^1 \left[(N-1) \left(R_k(t)\right)^N - N \left(R_k(t)\right)^{N-1} \right] dt$$

Differentiating $(N-1)(f)^N - N(f)^{N-1}$ with respect to f we find that, for $0 \leq f < 1$, the derivative is negative. This means that the integrand $(N-1)\left(R_k(t)\right)^N - N\left(R_k(t)\right)^{N-1}$ is a monotonically declining function of $R_k(t)$. But this also means that, given two functions $R_k(t)$, $R_\ell(t)$:

(R): If $R_k(t) \leq R_\ell(t)$ for all $0 \leq t \leq 1$ and $R_k(t) < R_\ell(t)$ for at least some t then $H_{2,k}^{\{r_1,...,r_M\}} > H_{2,\ell}^{\{r_1,...,r_M\}}$

An equivalent way of stating result (R) is that if $R_k \succ_{FOSD} R_\ell$ (where $\succ_{FOSD}$ denotes strict first-order stochastic dominance ordering) then $H_{2,k}^{\{r_1,...,r_M\}} > H_{2,\ell}^{\{r_1,...,r_M\}}$.

I will apply the above result to show that, given $M-1$ other sellers with fixed reputations $r_1 \geq r_2 \geq ... \geq r_{M-1}$, the expected value of seller k's second highest bidder is an increasing function of his reputation r. Consider the function $R_k(t) = 1 - Q_k(1) + Q_k(t)$. From Proposition 3, fixing the reputations of all other sellers, $Q_k(1)$ is an increasing function of seller k's reputation (so $1 - Q_k(1)$ is a decreasing function of seller k's reputation). Furthermore, all auction selection probabilities s_{zk} are decreasing functions of seller k's reputation. Therefore, for all t for which seller k is considered by buyers with positive probability, $Q_k(t)$, and thus $R_k(t)$, are decreasing functions of seller k's reputation. When t surpasses the threshold buyer type t_k above which seller k is no longer considered, then $Q_k(t) = Q_k(1)$, and thus $R_k(t)$ attains its maximum value 1 and stays constant thereafter. From Proposition 3, threshold t_k is an increasing function of seller k's reputation. Taken together, the previous properties imply that, as seller k's reputation increases, function $R_k(t)$ decreases for small t and attains unity later. Thus, seller reputations generate a set of functions R_k that are monotonically ordered according to strict FOSD. According to result (R) this implies that the expected value of a seller's second highest bidder is a monotonically increasing function of that seller's reputation.

A similar approach can be used to prove Part 2 of this Proposition. The fundamental observation (again, from Proposition 3) is that if the reputation of seller k remains fixed but the reputation of some other seller increases then $Q_k(1)$ decreases and all auction selection probabilities s_{zk} increase.

Proposition 5

Consider a bid vector $\mathbf{b}(t)$ where $b_k(t) = 0$. I will show that there exists an $\epsilon > 0$, such that bidder $t \in (0, 1)$ can increase her expected surplus by setting $b_k(t) = \epsilon$. From (12) the probability of obtaining the item from at least one seller if one bids in all auctions except k is:

$$1 - \prod_{j \in \{1,\ldots,M\} - \{k\}} (1 - r_j G_j(b_j(t)))$$

Therefore, the marginal probability of obtaining the item from at least one seller if one places a non-zero bid in action k is equal to:

$$\left[1 - (1 - r_k G_k(b_k(t))) \prod_{j \in \{1,\ldots,M\} - \{k\}} (1 - r_j G_j(b_j(t)))\right] - \left[1 - \prod_{j \in \{1,\ldots,M\} - \{k\}} (1 - r_j G_j(b_j(t)))\right]$$
$$= r_k G_k(b_k(t)) \prod_{j \in \{1,\ldots,M\} - \{k\}} (1 - r_j G_j(b_j(t)))$$

The incremental cost of participating on auction k is $\int_0^{b_k(t)} x g_k(x) dx$. The marginal expected surplus from participating in auction k is, thus, equal to:

$$\Delta V = tr_k G_k(b_k(t)) \prod_{j \in \{1,\ldots,M\}-\{k\}} (1 - r_j G_j(b_j(t))) - \int_0^{b_k(t)} x g_k(x) dx \tag{28}$$

Integration by parts gives $\int_0^{b_k(t)} x g_k(x) dx = b_k(t) G_k(b_k(t)) - \int_0^{b_k(t)} G_k(x) dx$. Substituting into (28) we obtain:

$$\Delta V = G_k(b_k(t)) \left[tr_k \prod_{j \in \{1,\ldots,M\}-\{k\}} (1 - r_j G_j(b_j(t))) - b_k(t) \right] + \int_0^{b_k(t)} G_k(x) dx \tag{29}$$

If $r_j < 1$ it is $1 - r_j G_j(b_j(t)) > 0$. In the special case $r_j = 1$ the one-bid equilibrium (Proposition 3) shows that the top buyer type ($t = 1$) bids her expected valuation $b_j(1) = 1$ on auction j with probability 1 and also wins the auction with probability 1. This implies that $G_j(b_j(1)) = 1$. When $r_j = 1$ and $t = 1$ this behavior also holds in settings where the number of bids is unrestricted: since $1 - r_j G_j(b_j(1)) = 0$, maximization of equation (29) implies that the top type bids zero on all auctions $k \neq j$. But if type $t = 1$ bids $b_j(1) = 1$ on auction j, for any $t < 1$ it is $b_j(t) \leq t < 1$ and thus $G_j(b_j(t)) < G_j(1) = 1$. Hence, $1 - r_j G_j(b_j(t)) > 0$ even when $r_j = 1$.

If $1 - r_j G_j(b_j(t)) > 0$ for all j, for $t \in (0, 1)$ it is also $tr_k \prod_{j \in \{1,\ldots,M\}-\{k\}} (1 - r_j G_j(b_j(t))) > 0$. Then, for any $0 < \epsilon < tr_k \prod_{j \in \{1,\ldots,M\}-\{k\}} (1 - r_j G_j(b_j(t)))$ setting $b_k(t) = \epsilon$ makes both terms of (29) strictly positive and thus strictly increases the bidder's expected surplus.

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