

Tail Risk Premia and Predictability

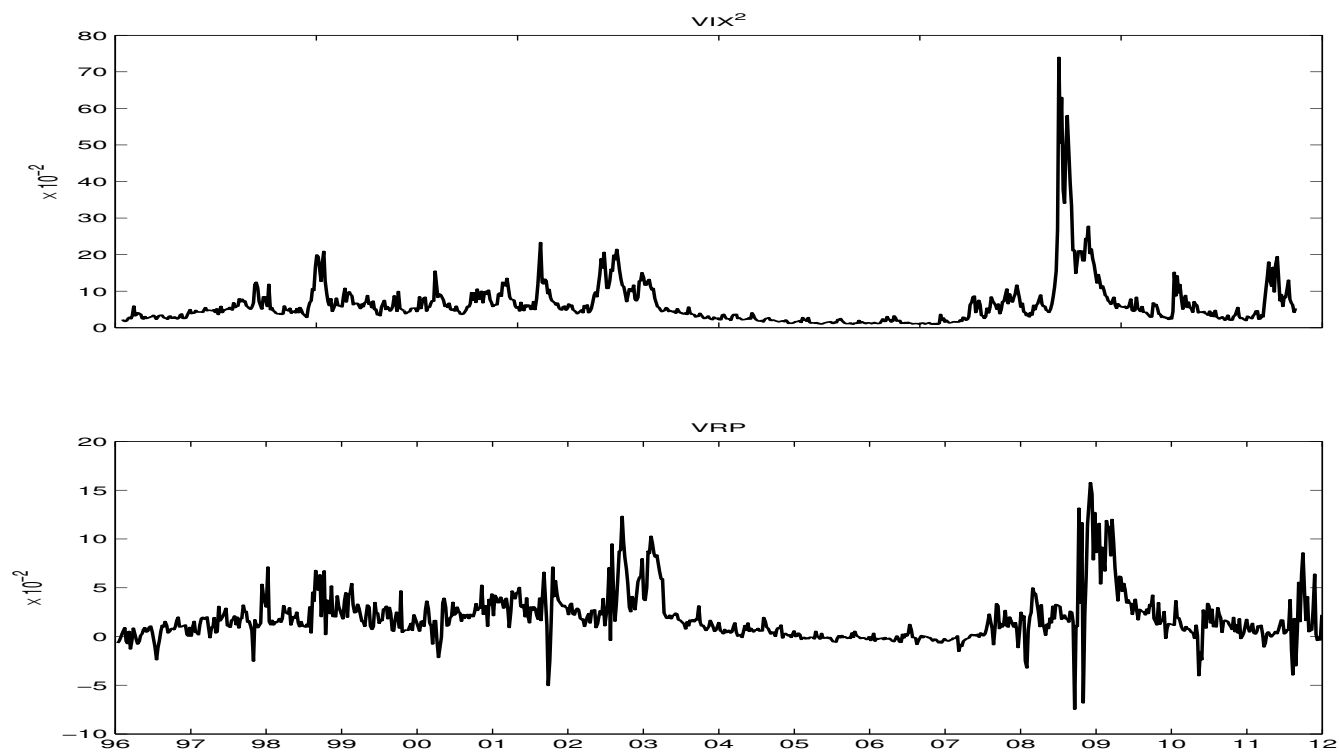
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Motivation

- Market Volatility changes over time
- this risk is significantly rewarded by investors
 - At-the-money Black-Scholes implied Volatility $>$ Historic Volatility
- Compensation for Variance Risk Called Variance Risk Premium
- Variance Risk Premium Varies a lot



Motivation

Traditional Models with Time-Varying Risk Premium \propto to volatility and its factors, but

Quarterly Predictive regressions									
Constant	-2.08 (-0.56)	0.24 (0.06)	6.60 (1.60)	92.41 (2.17)	73.35 (1.81)	20.63 (1.32)	7.39 (1.24)	6.92 (2.18)	5.53 (1.54)
VRP	0.47 (2.86)								
VIX^2		0.19 (1.41)							
RV			0.00 (0.00)						
$\log(P/E)$				-2.28 (-1.97)					
$\log(P/D)$					-1.42 (-1.62)				
DFSP						-1.39 (-0.90)			
TMSP							-0.46 (-0.17)		
RREL								3.27 (0.625)	
CAY									3.23 (1.78)
Adj. R^2	6.82	2.49	-0.47	6.55	4.19	1.18	-0.43	0.43	4.13

Bollerslev, Tauchen and Zhou (2009, RFS)

Motivation

- Therefore Variance Risk Premium contains different source of information from market volatility
- Variance Risk Premium is compensation for two different types of risk: Jump Risk and Time-Varying Volatility
- Big Part of Variance Risk Premium due to Jump Tail Events [Bollerslev and Todorov (2011, JF)]
- Where is the predictive ability of the Variance Risk Premium coming from?
- Can we improve the predictive ability by separating the Tail Risk Premium from the Variance Risk Premium?
- What are the underlying economic sources behind the predictability: time-varying risk aversion and/or time-varying economic uncertainty?

Outline

- Formal Setup and Notation
- Variance Risk Premium and its Decomposition
- Estimation of Tail Risk from Options
- Market Return Predictability Results
- Portfolio Returns Predictability Results

Formal setup and notation

- The S&P futures price has the following dynamics under the actual \mathbb{P} distribution:

$$\frac{dX_t}{X_{t-}} = a_t dt + \sigma_t dW_t + \int_{\mathbb{R}} (e^x - 1) \tilde{\mu}^{\mathbb{P}}(dt, dx)$$

- a_t drift
- σ_t (arbitrary) stochastic process
- $\mu(dt, dx)$ counting measure for jumps
- $\tilde{\mu}^{\mathbb{P}}(dt, dx) = \mu(dt, dx) - \nu_t^{\mathbb{P}}(dx)dt$ compensated (“demeaned”) jump measure
- $\nu_t^{\mathbb{P}}(dx)$ jump intensity process

Formal setup and notation

- Corresponding risk-neutral, or \mathbb{Q} distribution, dynamics:

$$\frac{dX_t}{X_{t-}} = (r_{f,t} - \delta_t)dt + \sigma_t dW_t^{\mathbb{Q}} + \int_{\mathbb{R}} (e^x - 1) \tilde{\mu}^{\mathbb{Q}}(dt, dx)$$

- $r_{f,t}$ is risk-free rate
- δ_t is dividend yield
- σ_t stochastic volatility **same** under \mathbb{P} and \mathbb{Q}
- $\tilde{\mu}^{\mathbb{Q}}(dt, dx) = \mu(dt, dx) - \nu_t^{\mathbb{Q}}(dx)dt$ compensated (“demeaned”) jump measure
- $\nu_t^{\mathbb{Q}}(dx)$ jump intensity process generally **differs** from $\nu_t^{\mathbb{P}}(dx)$ (for jumps away from zero)

Formal setup and notation

- Variance risk premium

$$VRP_{t,\tau} = \mathbb{E}_t^{\mathbb{P}} \left(IV_{[t,t+\tau]} + JV_{[t,t+\tau]}^{\mathbb{P}} \right) - \mathbb{E}_t^{\mathbb{Q}} \left(IV_{[t,t+\tau]} + JV_{[t,t+\tau]}^{\mathbb{Q}} \right)$$

- Continuous integrated variation:

$$IV_{[t,t+\tau]} = \int_t^{t+\tau} \sigma_s^2 ds$$

- Jump Variation:

$$JV_{[t,t+\tau]}^{\mathbb{P}} = \int_t^{t+\tau} \int_{\mathbb{R}} x^2 \nu_s^{\mathbb{P}}(dx) ds$$
$$JV_{[t,t+\tau]}^{\mathbb{Q}} = \int_t^{t+\tau} \int_{\mathbb{R}} x^2 \nu_s^{\mathbb{Q}}(dx) ds$$

Variance Risk Premium Decomposition

$$\begin{aligned} VRP_{t,\tau} = & \left(\mathbb{E}_t^{\mathbb{P}}(IV_{[t,t+\tau]}) - \mathbb{E}_t^{\mathbb{Q}}(IV_{[t,t+\tau]}) \right) + \left(\mathbb{E}_t^{\mathbb{P}}(JV_{[t,t+\tau]}^{\mathbb{P}}) - \mathbb{E}_t^{\mathbb{Q}}(JV_{[t,t+\tau]}^{\mathbb{P}}) \right) \\ & + \left(\mathbb{E}_t^{\mathbb{Q}}(JV_{[t,t+\tau]}^{\mathbb{P}}) - \mathbb{E}_t^{\mathbb{Q}}(JV_{[t,t+\tau]}^{\mathbb{Q}}) \right) \end{aligned}$$

- First two terms involve the difference between the \mathbb{P} and \mathbb{Q} expectations of the **same** variation measures
 - Naturally associated with investors willingness to hedge against changes in the investment opportunity set
- Last terms involves the difference between the \mathbb{P} and \mathbb{Q} jump variation under the **same** probability measure \mathbb{Q}
 - Purged from the compensation for time-varying jump intensity risk
 - Reflects the “special” treatment of jump risk

Variance Risk Premium Decomposition

- Impossible to separately identify the different terms in the decomposition of the Variance Risk Premium without additional (strong) parametric assumptions
- Focussing on the jump “tails” a measure that parallels the last term may be estimated non-parametrically from out-of-the-money options
- Naturally interpreted as a measure for investor fears
- Much of the predictability inherent in the variance risk premium “sits” in this new fear measure

More formally....

Variance Risk Premium Decomposition

$$\lim_{\tau \downarrow 0} \frac{1}{\tau} V R P_{t, \tau} = \int_{\mathbb{R}} x^2 (\nu_t^{\mathbb{P}}(dx) - \nu_t^{\mathbb{Q}}(dx))$$

- The variance risk premium at the very short maturity is solely due to compensation for jump risk
- At longer maturities the compensation for the changes in the investment opportunities starts contributing
- Suggests way to isolate the jump tail component of the Variance Risk Premium

Jump Tail Risk Premia

- Left and right jump tail risk premia

$$LJP_{t,\tau} = \mathbb{E}_t^{\mathbb{P}}(LJV_{[t,t+\tau]}^{\mathbb{P}}) - \mathbb{E}_t^{\mathbb{Q}}(LJV_{[t,t+\tau]}^{\mathbb{Q}})$$

$$RJP_{t,\tau} = \mathbb{E}_t^{\mathbb{P}}(RJV_{[t,t+\tau]}^{\mathbb{P}}) - \mathbb{E}_t^{\mathbb{Q}}(RJV_{[t,t+\tau]}^{\mathbb{Q}})$$

$$LJV_{[t,t+\tau]}^{\mathbb{P}} = \int_t^{t+\tau} \int_{x < -k_t} x^2 \nu_s^{\mathbb{P}}(dx) ds \quad RJV_{[t,t+\tau]}^{\mathbb{P}} = \int_t^{t+\tau} \int_{x > k_t} x^2 \nu_s^{\mathbb{P}}(dx) ds$$

$$LJV_{[t,t+\tau]}^{\mathbb{Q}} = \int_t^{t+\tau} \int_{x < -k_t} x^2 \nu_s^{\mathbb{Q}}(dx) ds \quad RJV_{[t,t+\tau]}^{\mathbb{Q}} = \int_t^{t+\tau} \int_{x > k_t} x^2 \nu_s^{\mathbb{Q}}(dx) ds$$

- Parallels the definition of $VRP_{t,\tau}$
- $VRP_{t,\tau} - (LJP_{t,\tau} + RJP_{t,\tau})$ portion of the variance risk premium due to “normal” sized price fluctuations
- “Tails” and “large” jumps defined in a relative sense k_t

Jump Tail Risk Premia Decompositions

- Mimicking the previous decomposition for VRP:

$$LJP_{t,\tau} = [\mathbb{E}_t^{\mathbb{P}}(LJV_{[t,t+\tau]}^{\mathbb{P}}) - \mathbb{E}_t^{\mathbb{Q}}(LJV_{[t,t+\tau]}^{\mathbb{P}})] + [\mathbb{E}_t^{\mathbb{Q}}(LJV_{[t,t+\tau]}^{\mathbb{P}}) - \mathbb{E}_t^{\mathbb{Q}}(LJV_{[t,t+\tau]}^{\mathbb{Q}})]$$

$$RJP_{t,\tau} = [\mathbb{E}_t^{\mathbb{P}}(RJV_{[t,t+\tau]}^{\mathbb{P}}) - \mathbb{E}_t^{\mathbb{Q}}(RJV_{[t,t+\tau]}^{\mathbb{P}})] + [\mathbb{E}_t^{\mathbb{Q}}(RJV_{[t,t+\tau]}^{\mathbb{P}}) - \mathbb{E}_t^{\mathbb{Q}}(RJV_{[t,t+\tau]}^{\mathbb{Q}})]$$

- First term naturally associated with investors willingness to hedge against changes in the investment opportunity set
- Second term reflects the “special” treatment of jump tail risk
- The \mathbb{P} jump intensity process appears approximately symmetric for “large” (absolute) sized jumps [Bollerslev and Todorov (2011, Econometrica)]
- The first term drops out in the difference $LJP_{t,\tau} - RJP_{t,\tau}$
- This difference may be interpreted as a proxy for investor **fears** [Bollerslev and Todorov (2011, Journal of Finance)]

Jump Tail Risk Premia Decompositions

- Jump tail “fear” measure:

$$LJP_{t,\tau} - RJP_{t,\tau} \approx \mathbb{E}_t^{\mathbb{Q}}(RJV_{[t,t+\tau]}^{\mathbb{Q}}) - \mathbb{E}_t^{\mathbb{Q}}(LJV_{[t,t+\tau]}^{\mathbb{Q}})$$

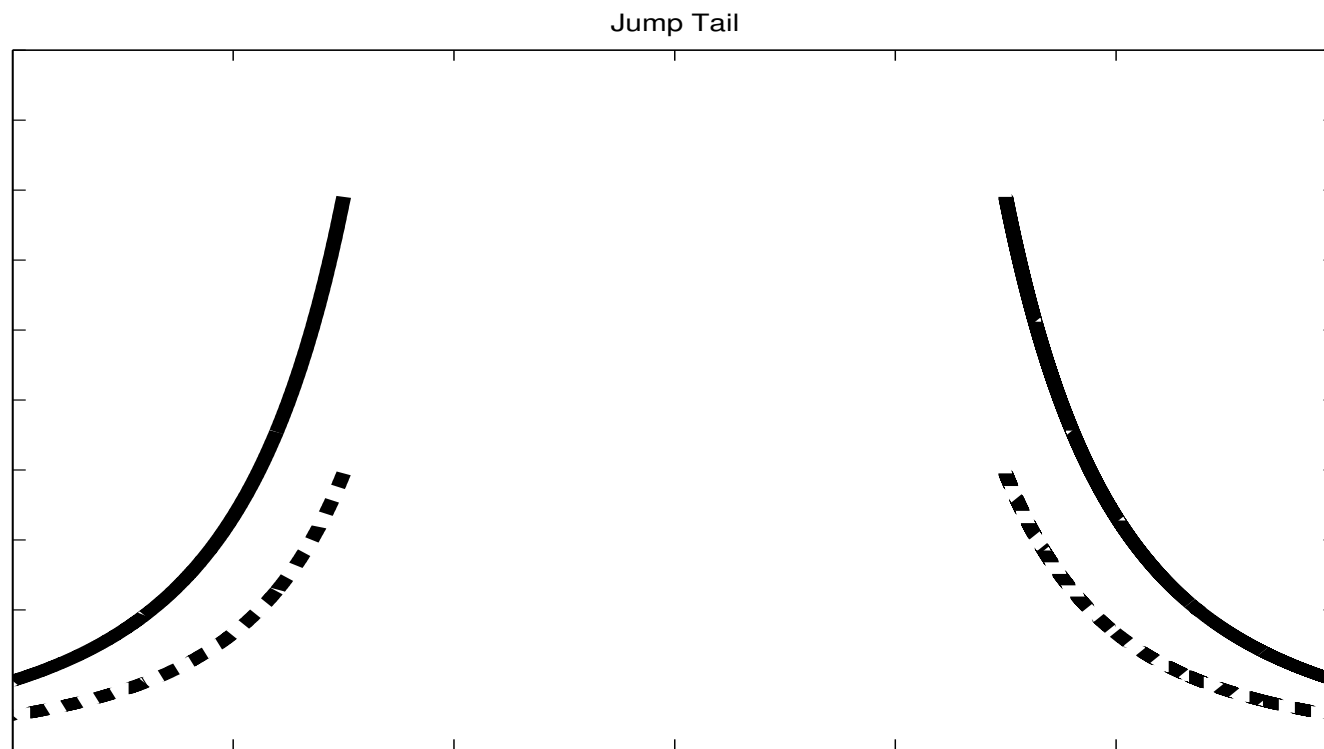
- Conveniently avoids Peso type problems and any “tail” estimation under \mathbb{P}
- Our estimation of $\mathbb{E}_t^{\mathbb{Q}}(RJV_{[t,t+\tau]}^{\mathbb{Q}})$ and $\mathbb{E}_t^{\mathbb{Q}}(LJV_{[t,t+\tau]}^{\mathbb{Q}})$ is based on:
 - A very general specification for the jump tail intensity process $\nu_t^{\mathbb{Q}}(dx)$
 - Close-to-maturity out-of-the-money options
 - Close-to-maturity out-of-the-money options are essentially bets on rare tail events
 - Empirically $LJP_{t,\tau} - RJP_{t,\tau} \approx -\mathbb{E}_t^{\mathbb{Q}}(LJV_{[t,t+\tau]}^{\mathbb{Q}})$

Jump Tail Estimation

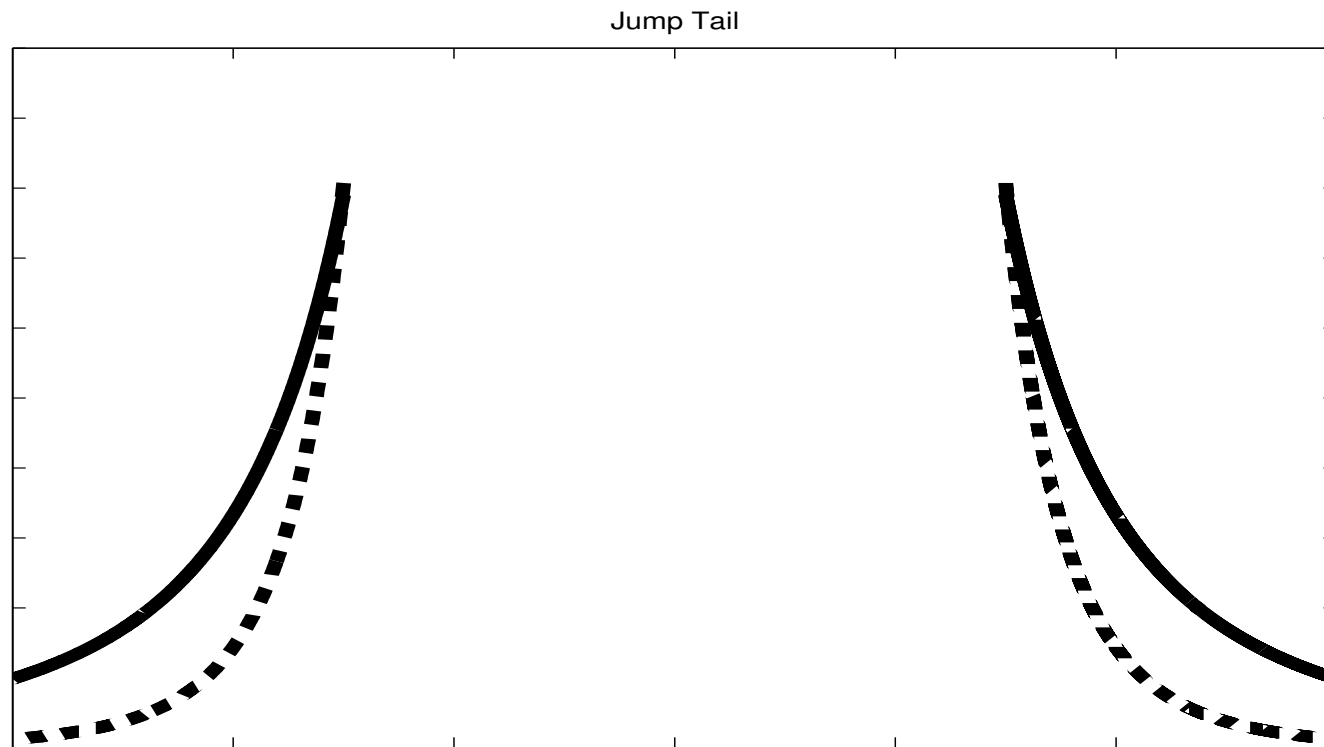
- Jump Tail Intensity process:

$$\nu_t^{\mathbb{Q}}(dx) = \left(\phi_t^+ \times e^{-\alpha_t^+ x} 1_{\{x>0\}} + \phi_t^- \times e^{-\alpha_t^- |x|} 1_{\{x<0\}} \right) dx, \quad |x| > k_t$$

- Explicitly allows the left (-) and right (+) jump tails to differ
- Explicitly allows both the “shape” (α_t^{\pm}) and “level shift” (ϕ_t^{\pm}) parameters to change over time
- Puts no restriction on the behavior of the “small” to “medium” sized jumps
- Existing models that do allow for temporal variation fix $\alpha_t^+ = \alpha_t^- = \alpha$ and further restrict $\phi_t^+ = \phi_t^-$ to be an affine function of σ_t^2
- Nests almost all models hitherto used in the literature, including the double jump model of Duffie, Pan and Singleton (2000, Econometrica) and the time-changed tempered stable models of Carr, Geman, Madan and Yor (2003, Mathematical Finance)



Different level shift parameters ϕ^\pm



Different shape parameters α^{\pm}

Jump Tail Estimation

Formally estimation is based on the following approximation for short maturity out-of-the-money option with moneyness $k = \log(K/F_t)$

$$\frac{e^{rt, \tau} O_{t, \tau}(k)}{F_{t-, \tau}} \approx \begin{cases} \frac{\tau \phi_t^+ e^{k(1-\alpha_t^+)}}{\alpha_t^+ (\alpha_t^+ - 1)}, & \text{if } k > 0, \\ \frac{\tau \phi_t^- e^{k(1+\alpha_t^-)}}{\alpha_t^- (\alpha_t^- + 1)}, & \text{if } k < 0, \end{cases}$$

for $\tau \downarrow 0$.

\implies the tail “shape” parameters may be estimated from the way in which option prices decay as a function of their strikes:

$$\hat{\alpha}_t^\pm = \operatorname{argmin}_{\alpha_t^\pm} \frac{1}{N_t^\pm} \sum_{i=2}^{N_t^\pm} \left| \frac{\log \left(\frac{O_{t, \tau_t}(k_{t,i})}{O_{t, \tau_t}(k_{t,i-1})} \right)}{k_{t,i} - k_{t,i-1}} - \left(1 \pm (-\alpha_t^\pm) \right) \right|$$

Jump Tail Estimation

For given α_t^\pm -s the “level shift” parameters may be estimated by:

$$\begin{aligned} \hat{\phi}_t^\pm = \operatorname{argmin}_{\phi^\pm} \frac{1}{N_t^\pm} \sum_{i=1}^{N_t^\pm} & \left| \log \left(\frac{e^{r_{t,\tau}} O_{t,\tau_t}(k_{t,i})}{\tau F_{t-, \tau}} \right) - \left(1 - \hat{\alpha}_t^\pm \right) k_{t,i} \right. \\ & \left. + \log \left(\hat{\alpha}_t^\pm - 1 \right) + \log \left(\hat{\alpha}_t^\pm \right) - \log(\phi^\pm) \right| \end{aligned}$$

\implies we “pool” together out-of-the-money options once we know by how much they should decay as they get deeper out of the money.

Jump Tail Estimation

- All of the previously discussed **jump variation measures** may be expressed as functions of the α_t^\pm and ϕ_t^\pm **tail parameters**
 - Left and right jump variation:

$$LJV_t = \tau \phi_t^- e^{-\alpha_t^- k_t} (\alpha_t^- k_t (\alpha_t^- k_t + 2) + 2) / (\alpha_t^-)^3,$$

$$RJV_t = \tau \phi_t^+ e^{-\alpha_t^+ k_t} (\alpha_t^+ k_t (\alpha_t^+ k_t + 2) + 2) / (\alpha_t^+)^3.$$

- Left and right jump intensity:

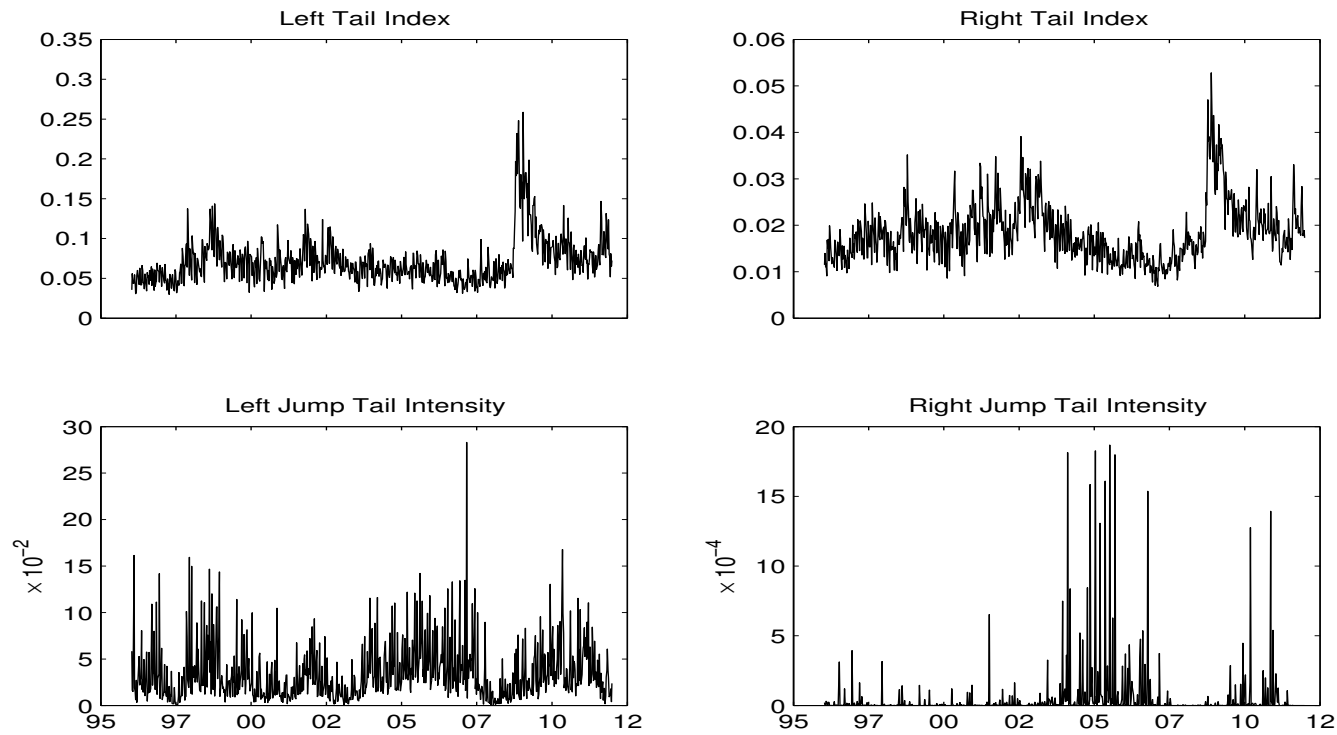
$$LJI_t = \hat{\phi}_t^- e^{-k_t / \hat{\alpha}_t^-} / \hat{\alpha}_t^-, \quad RJI_t = \hat{\phi}_t^+ e^{-k_t / \hat{\alpha}_t^+} / \hat{\alpha}_t^+.$$

- We implement all of these estimators on a **weekly** basis using S&P 500 index options

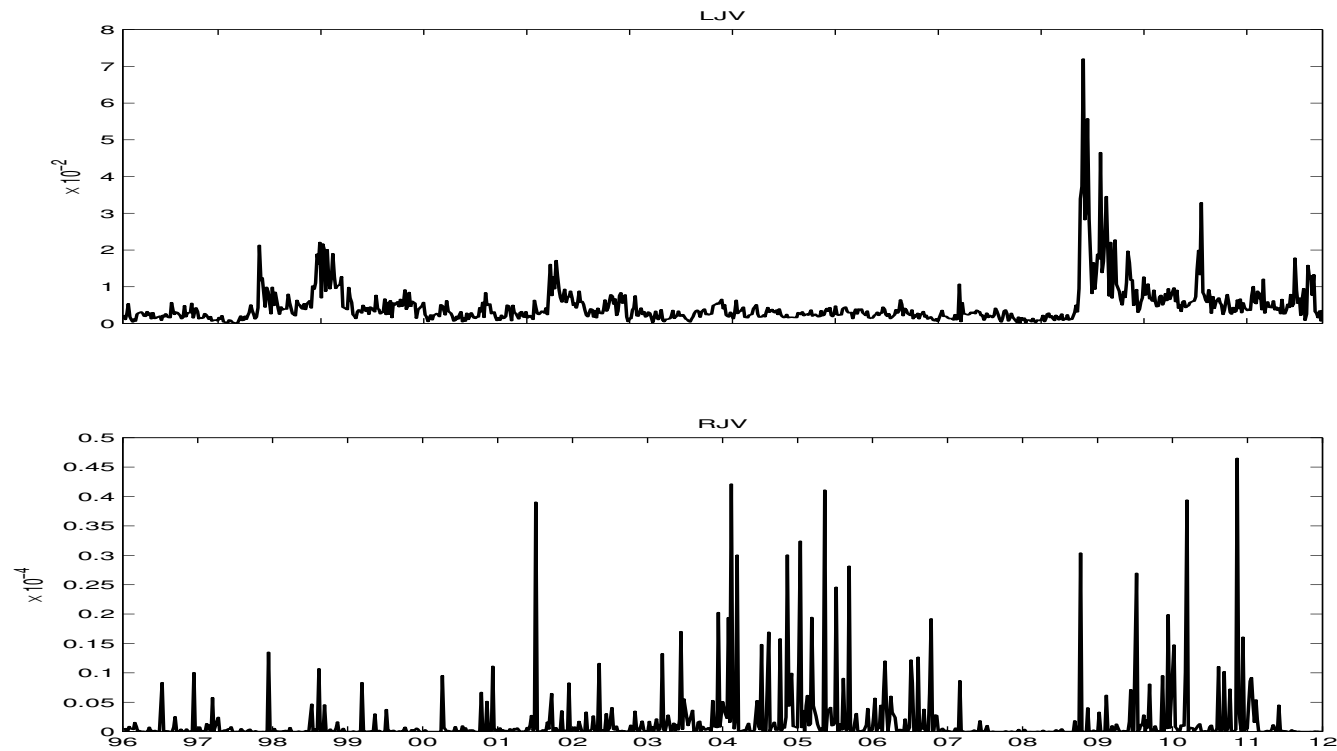
Data

- Sample period: January 1996 to December 2011
- S&P 500 options data from OptionMetrics
 - Standard “cleaning” procedures
 - Maturities 8-45 days
 - All puts with moneyness less than $-2.5 \times \text{BS volatility}$ (≈ 18.20 obs. per day)
 - All calls with moneyness greater than $1.0 \times \text{BS volatility}$ (≈ 7.61 obs. per day)
- S&P 500 high-frequency data from Tick Data Inc.
 - Standard “cleaning” procedures
 - Five-minute returns (81 obs. per day)
- Portfolio returns from Ken French’s website
 - Market portfolio of all publicly traded U.S. equities
 - Various portfolio sorts

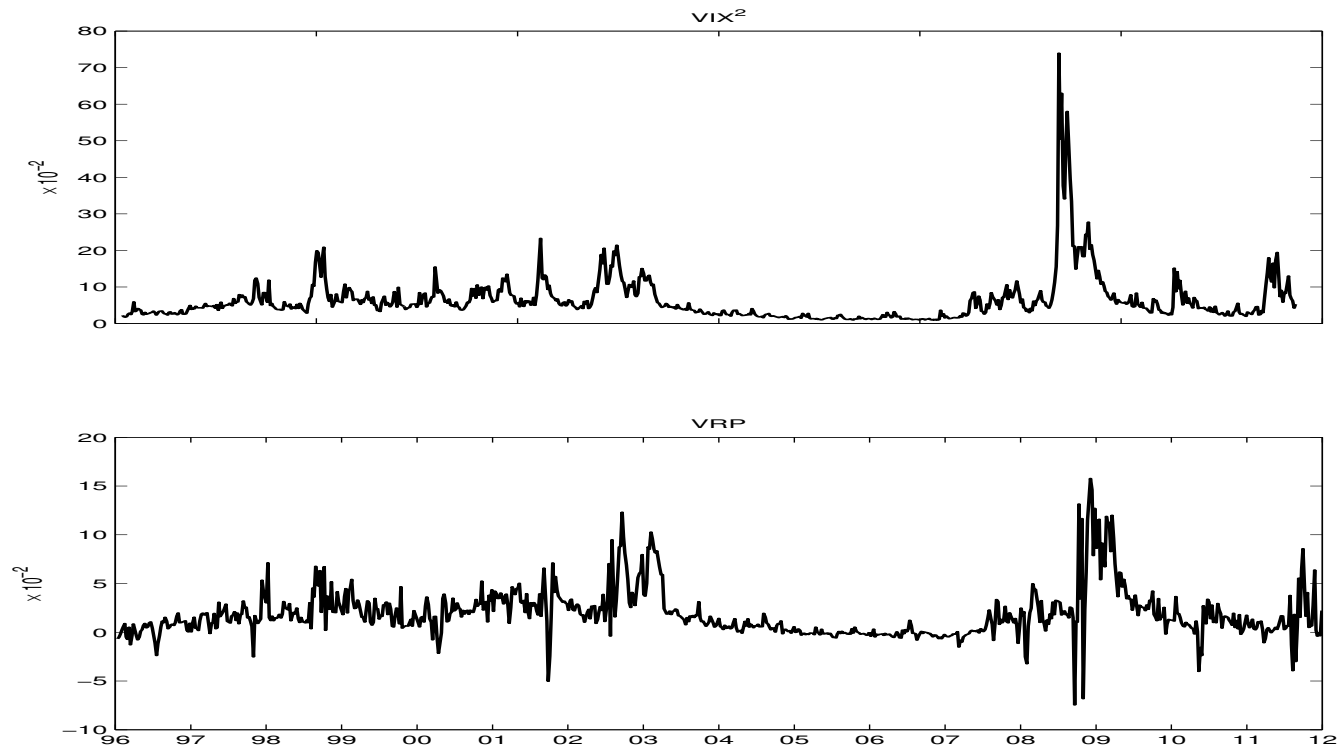
Tail Estimates



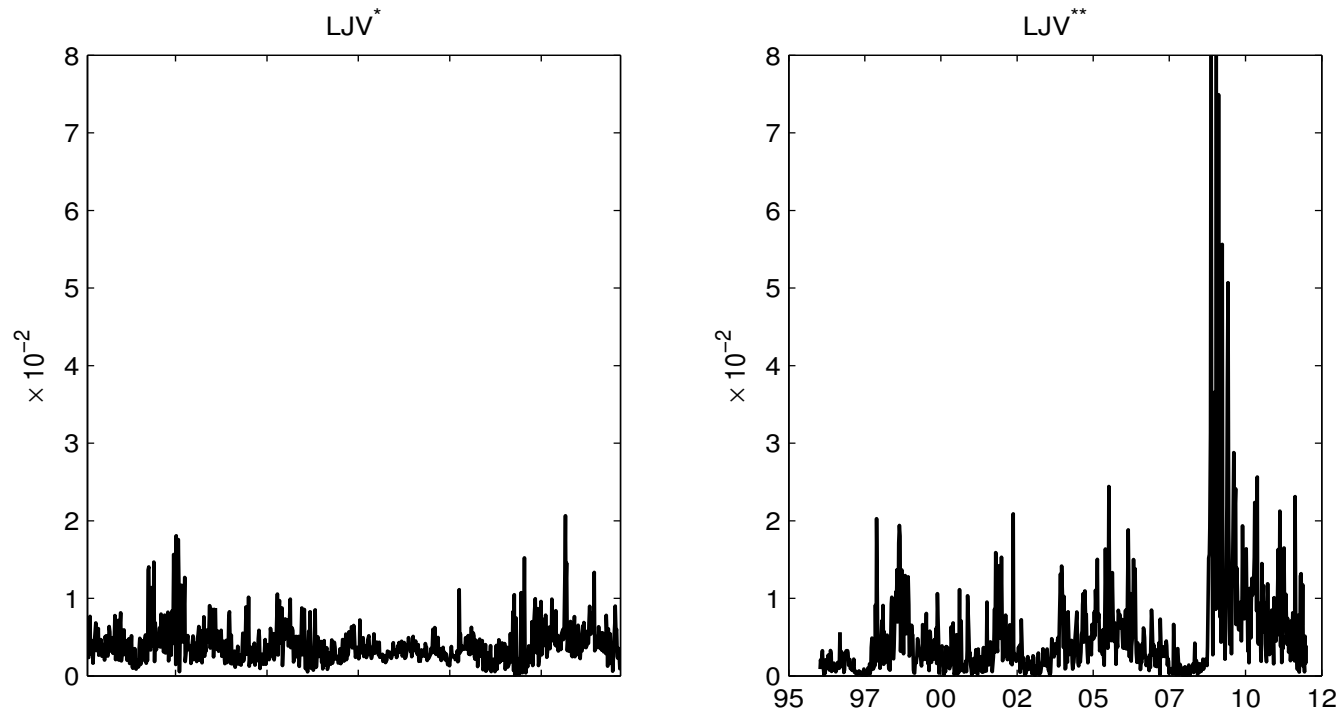
Jump Variation Measures



Traditional Variation Measures



Restricted Jump Variation Measures



LJV_t^* ($\alpha_t^- = \alpha$) and LJV_t^{**} ($\phi_t^- = \phi$): allowing for temporal variation in both α_t^- and ϕ_t^- importantly affects the jump variation estimates

Summary Statistics

	LJV	RJV	LJV*	LJV**	VIX2	VRP	VRP-LJV
Mean	0.47	0.02	0.40	0.52	6.30	1.86	1.39
St.Dev.	0.57	0.05	0.25	0.77	6.50	2.51	2.35
Skewness	5.12	5.28	2.11	6.10	4.65	1.73	1.38
Kurtosis	43.36	35.63	10.17	59.96	36.31	8.49	8.38
Max.	7.20	0.46	2.07	10.23	73.94	15.78	14.96
Min.	0.01	0.00	0.00	0.00	0.93	-7.45	-9.63
AR(1)	0.70	0.05	0.26	0.50	0.89	0.72	0.72

- LJV ($k_t = 6.868 \times \text{BS vol.}$) accounts for roughly one-fourth of VRP
- LJV completely dominates RJV

	LJV	RJV	LJV*	LJV**	VIX2	VRP	VRP-LJV
LJV	1.00	0.06	0.47	0.73	0.70	0.39	0.17
RJV		1.00	0.18	0.08	-0.02	-0.04	-0.06
LJV*			1.00	0.24	0.10	0.05	-0.06
LJV**				1.00	0.32	0.20	0.04
VIX2					1.00	0.66	0.54
VRP						1.00	0.97
VRP-LJV							1.00

- LJV and VRP only weakly correlated

Summary Statistics

- Contemporaneous weekly market return (MRK) correlations:

	LJV	RJV	LJV*	LJV**	VIX2	VRP	VRP-LJV
MRK	-0.24	-0.06	-0.28	-0.13	-0.18	-0.20	-0.16

- Parallels traditional “leverage” effect
- One-week-ahead market return correlations:

	LJV	RJV	LJV*	LJV**	VIX2	VRP	VRP-LJV
MRK	0.05	-0.01	0.04	-0.00	-0.05	0.12	0.11

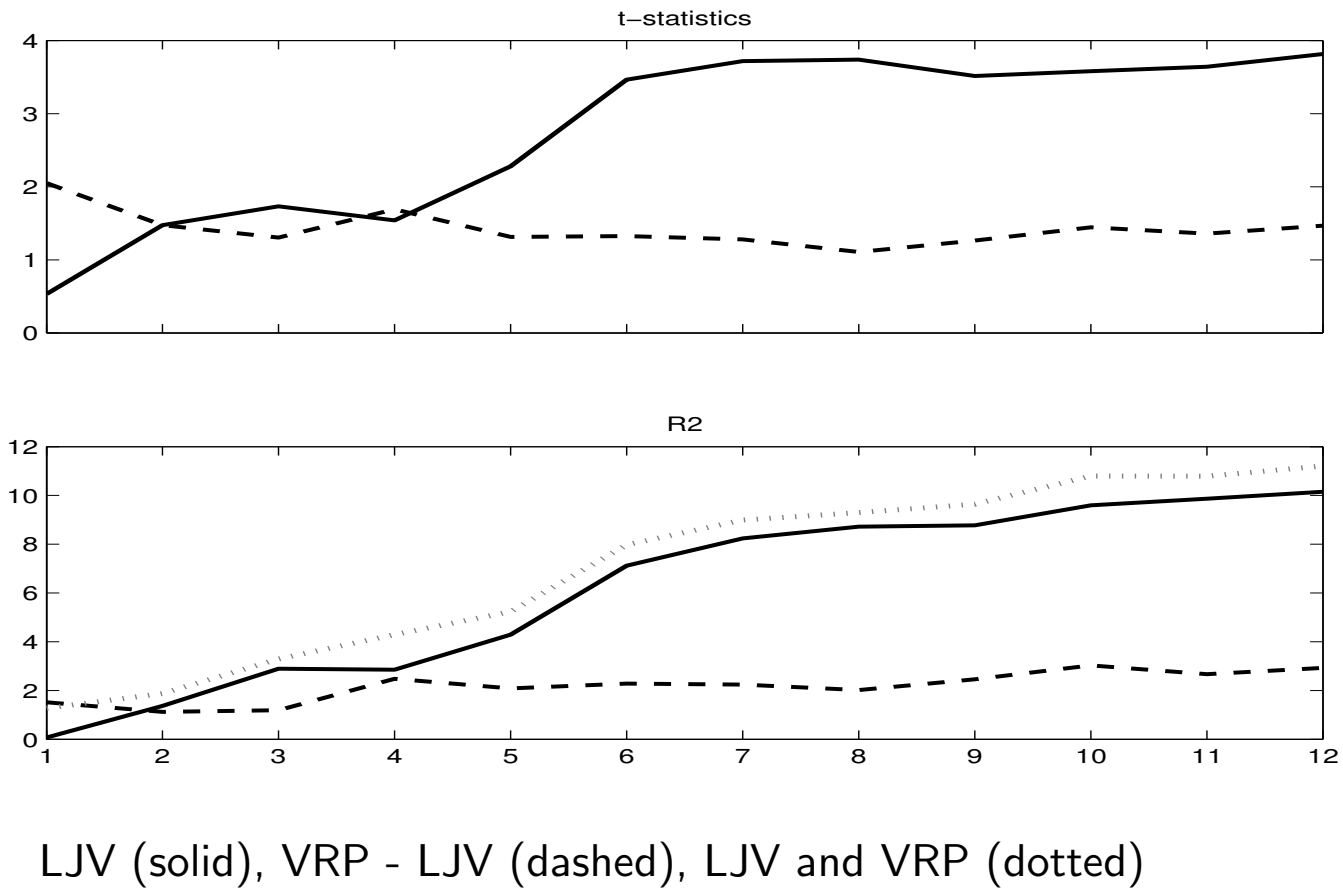
- Suggestive of “volatility feedback” type effect

Six-months market return predictability regressions

Constant	-0.168 (2.405)	2.407 (2.254)	0.174 (3.013)	0.656 (2.473)	0.366 (2.112)	0.643 (2.204)	1.407 (2.051)	-0.242 (2.223)	-0.789 (2.049)
LJV	5.933 (1.713)							5.906 (1.996)	5.507 (2.028)
RJV		9.139 (9.712)						5.038 (8.318)	
LJV*			6.018 (3.842)						
LJV**				3.656 (1.120)					
VIX2					0.356 (0.150)				
VRP						1.033 (0.565)			
VRP-LJV							0.827 (0.625)		0.583 (0.550)
R^2	7.119	0.134	1.401	4.919	3.314	4.089	2.283	7.160	8.217

- The LJV **fear** proxy works better than, and essentially drives out, VRP
- No predictability in RJV
- Allowing both the “shape” and the “level” of the jump tails to change over time **significantly** increases the predictability
- What about other return horizons?

Market return predictability regressions



Market return predictability regressions

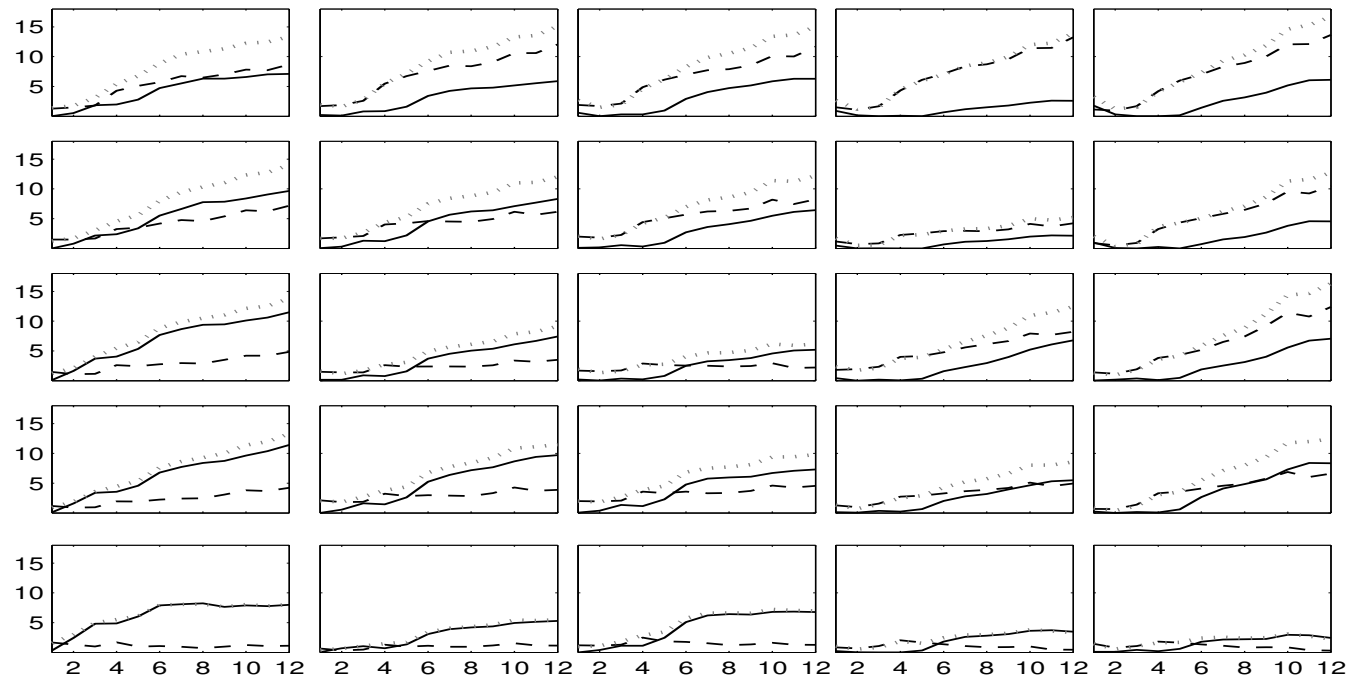
- VRP - LJV provides the most predictability over shorter 1-2 months horizons
- LJV provides the most predictability over longer 4-12 months horizons
 - Possibly related to the rare disasters literature: Barro (2006, QJE), Gabaix (2012, QJE), Reitz (1988, JME)
- But, where is the predictability coming from?
 - Are LJV and VRP-LJV priced risk factors?
 - Do they affect only time-varying risk aversion?
 - Lets look at some popular **portfolio sorts**
 - May help sort out where the predictability is coming from

Six-months portfolio returns predictability regressions

	Small	Big	SMB	High	Low	HML	Winners	Losers	WML
Constant	0.346 (2.413)	1.120 (2.114)	-0.774 (1.575)	2.092 (2.359)	0.534 (2.047)	1.558 (1.777)	7.828 (4.505)	-19.372 (7.376)	27.200 (5.429)
LJV	3.640 (1.982)	4.841 (2.069)	-1.201 (1.603)	2.802 (1.774)	6.229 (2.115)	-3.427 (1.936)	5.496 (5.628)	33.159 (6.504)	-27.663 (7.900)
VRP-LJV	1.826 (0.677)	0.319 (0.512)	1.507 (0.373)	0.987 (0.657)	0.400 (0.531)	0.587 (0.454)	0.064 (0.933)	6.169 (2.726)	-6.104 (2.189)
R^2	9.547	5.837	9.383	4.340	8.859	4.528	1.298	26.535	33.114

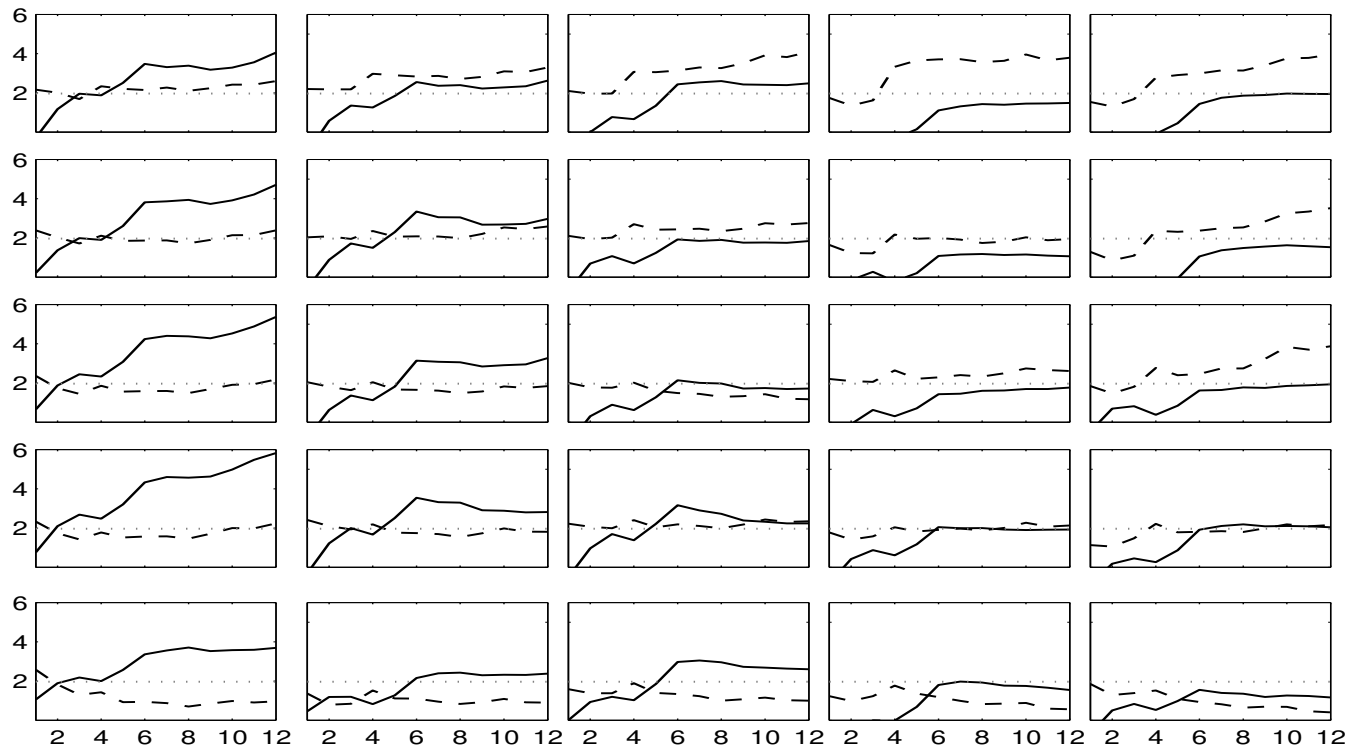
- Very **high** R^2 -s for certain portfolios
- VRP - LJV works best for **small**-stock portfolios
- LJV works best for portfolios of past **losers**
- Lets look at some double sorted portfolios

Size and book-to-market double-sort portfolio regression R^2 -s



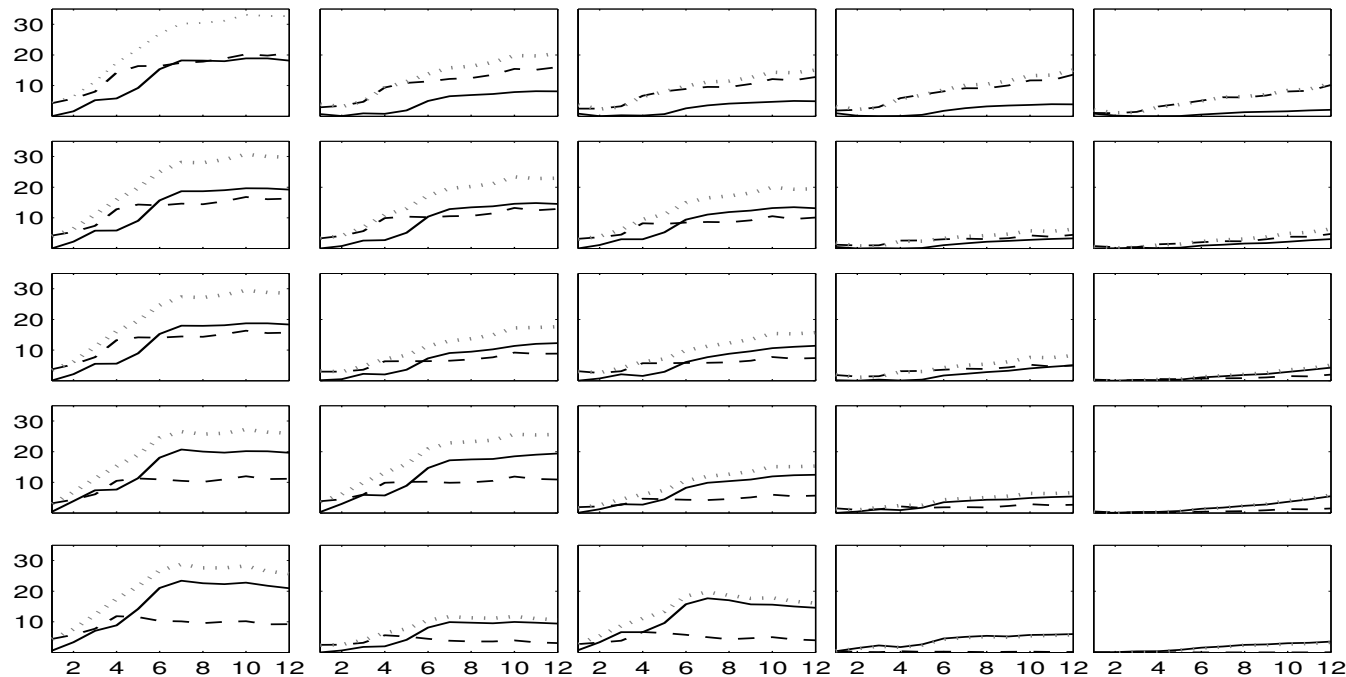
- LJV (solid), VRP - LJV (dashed)
- Small (top), large (bottom), growth (left), value (right)

Size and book-to-market double-sort portfolio regression t-statistics



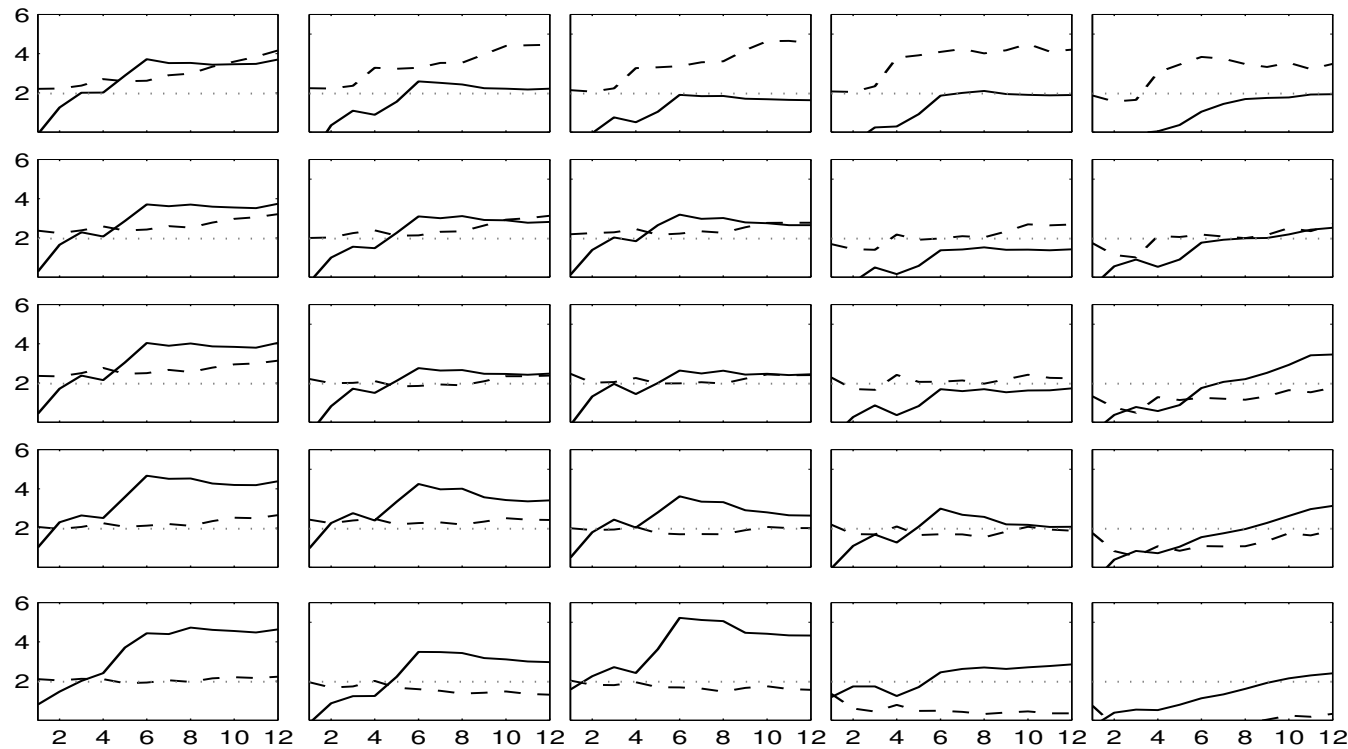
- LJV (solid), VRP - LJV (dashed)
- Small (top), large (bottom), growth (left), value (right)

Size and momentum double-sort portfolio regression R^2 -s



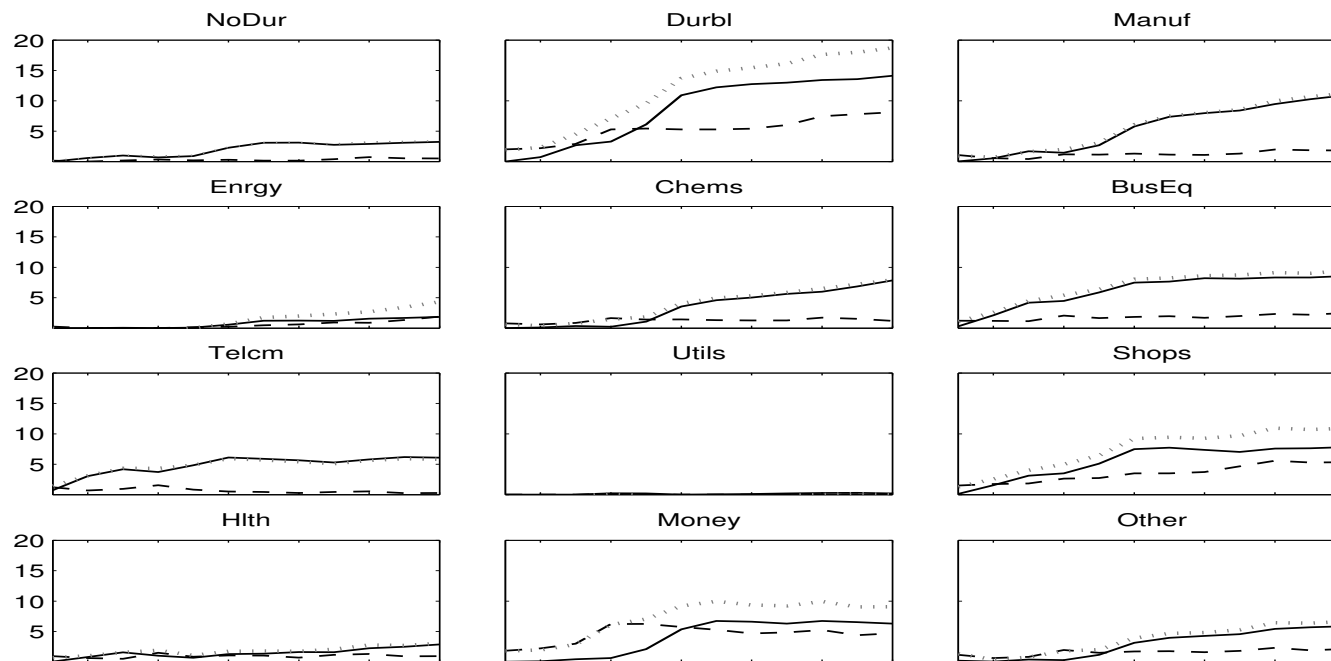
- LJV (solid), VRP - LJV (dashed)
- Small (top), large (bottom), losers (left), winners (right)

Size and momentum double-sort portfolio regression t-statistics



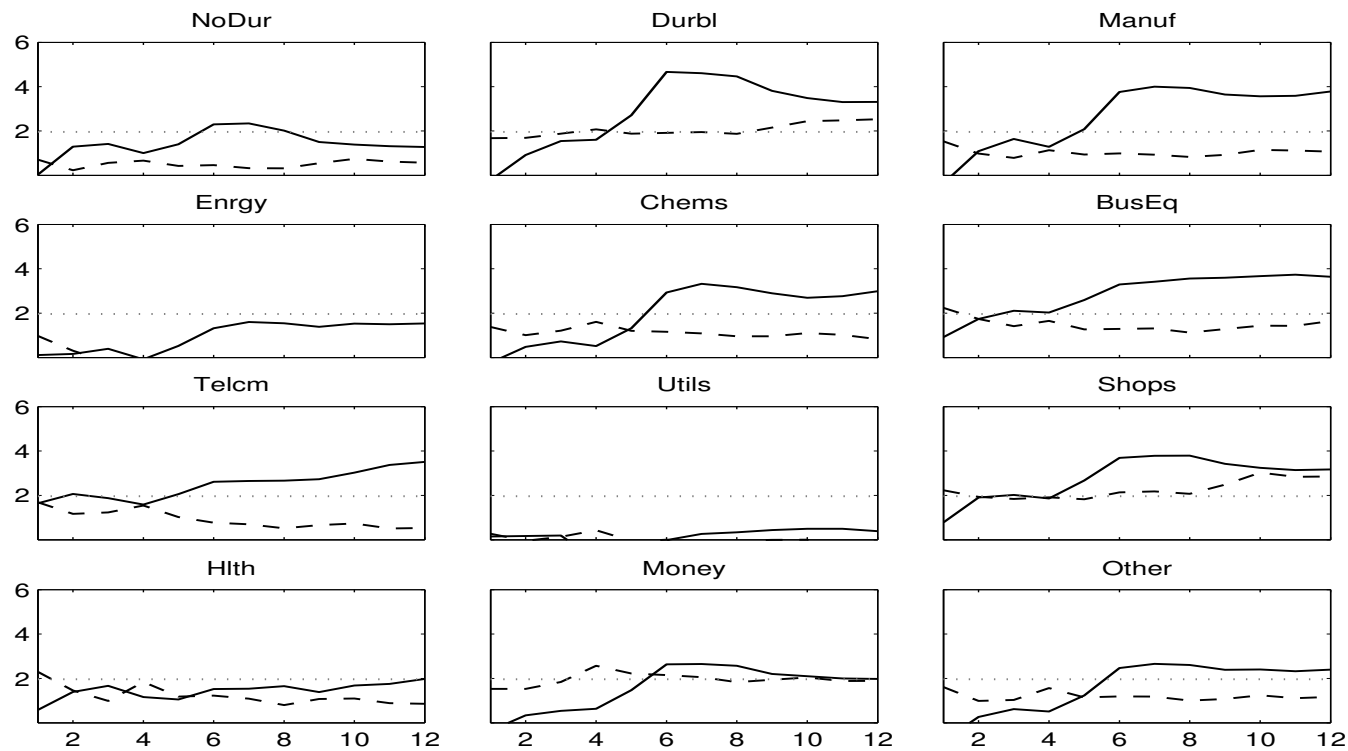
- LJV (solid), VRP - LJV (dashed)
- Small (top), large (bottom), losers (left), winners (right)

Industry portfolio regression R^2 -s



- LJV (solid), VRP - LJV (dashed)

Industry portfolio regression t-statistics



- LJV (solid), VRP - LJV (dashed)

Portfolio return predictability regressions

- Very high R^2 -s for certain portfolios
 - No predictability for large-value and large-winners
 - No predictability for non-durables, utilities, and healthcare
- VRP - LJV works best for small and value
 - At the industry level works best for financials
- LJV works best for growth and losers
 - At the industry level works best for durables, manufacturing, and business equipment
- So, what to make of all this?

Portfolio return predictability regressions

- VRP - LJV works best for small and value
 - Small firms more strongly affected by credit market conditions [Perez-Quiros and Timmermann (2000, JF)]
 - More distressed companies among value stocks [Fama and French (1992, JF), Gomes and Schmid (2010, JF)]
 - Small and value firms more susceptible to general economic conditions
 - Consistent with the idea that VRP - LJV captures economic uncertainty
- LJV works best for growth and losers
 - Growth and momentum returns both related to funding liquidity risk [Asness, Moskowitz and Pedersen (2013, JF) Nagel (2012, RFS), Korajczyk and Sadka (2004, JF)]
 - Liquidity conditions depend on market sentiment [Garleanu, Pedersen and Poteshman (2009, RFS)]
 - Consistent with the idea that LJV captures attitudes to risk, or investor fears

Cross-sectional relations

- More formal **cross-sectional pricing** relations:

$$\hat{\lambda}_{\text{VRP-LJV}} = 0.521_{(3.566)} \quad \hat{\lambda}_{\text{LJV}} = 0.009_{(0.347)}$$

- Connections with other **macro-finance variables**:

$$|\text{Corr}(\text{VRP-LJV}, \Delta \text{Ind. pro.})| = 0.186_{(3.349)} > 0.042_{(0.981)} = |\text{Corr}(\text{LJV}, \Delta \text{Ind. pro.})|$$

$$|\text{Corr}(\text{VRP-LJV}, \Delta \text{Sentiment})| = 0.122_{(1.596)} > 0.309_{(5.081)} = |\text{Corr}(\text{LJV}, \Delta \text{Sentiment})|$$

- Consistent with the idea that LJV captures investor fears

Concluding remarks

- New flexible estimation procedures based on out-of-the-money options for characterizing time-varying jump tails
- Much of the return predictability for the aggregate market portfolio previously attributed to the variance risk premium “sits” in the tails and the part of the jump tail variation naturally associated with investor fears
- Even stronger return predictability for certain portfolio sorts and industry portfolios
- Empirical results consistent with the idea that VRP-LJV captures time-varying risk, or economic uncertainty, while LJV is more closely associated with changes in risk aversion, or investor fears