NORMEX
- a new method for evaluating the VaR of aggregated heavy tailed risks -

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This study has been done to a great extent at the
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Motivation

In banks and insurances, one always considers portfolio of risks $\Rightarrow$ aggregation of risks (modeled with rv's) = basis of the internal model.

In practice, when assuming aggregation of iid observations in the portfolio model, distribution of the yearly log returns of financial assets : often approximated by a normal distribution (CLT) .

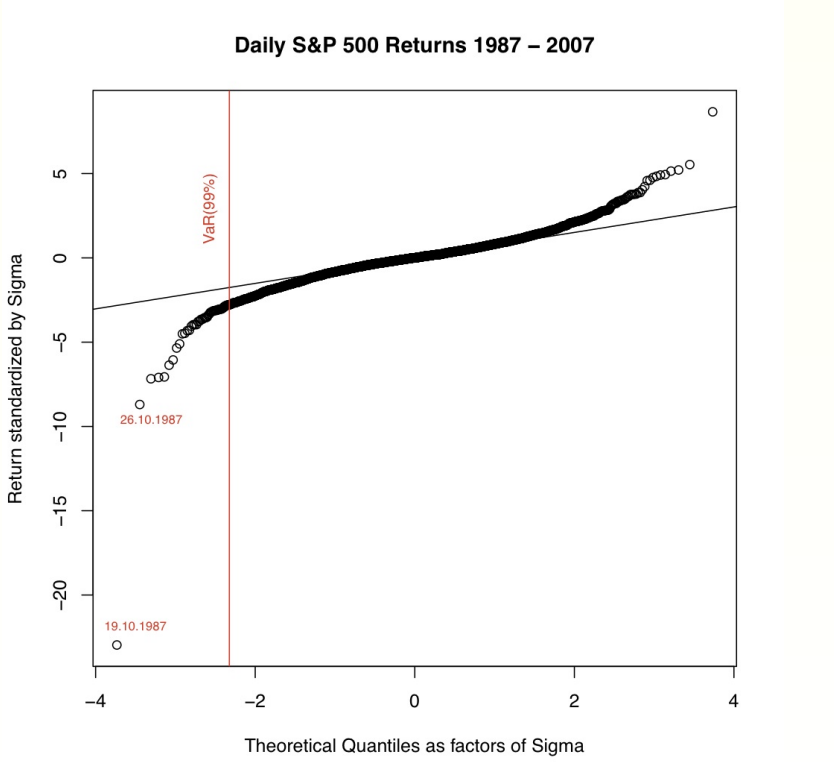
Two main drawbacks when using the CLT for moderate heavy tail distributions (e.g. Pareto with a shape parameter larger than 2).

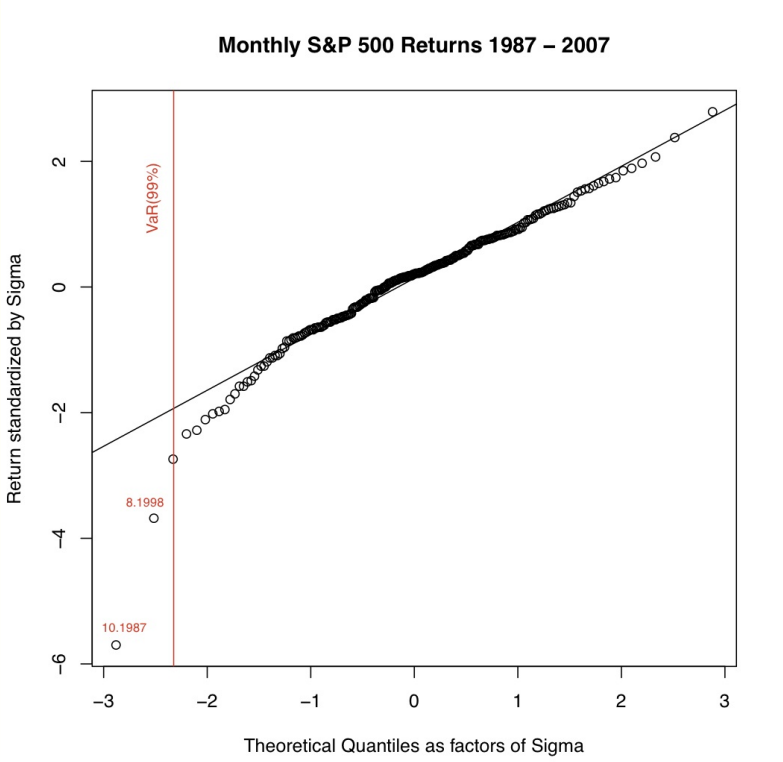
$\hookrightarrow$ if the CLT may apply to the sample mean because of a finite variance, it also provides a normal approximation with a very slow rate of convergence ; may be improved when removing extremes from the sample (see e.g. Hall).
Even if we are interested only in the sample mean, samples of small or moderate sizes will lead to a bad approximation. To improve the approximation, existence of moments of order larger than 2 may appear as necessary.

↪ With aggregated data, a heavy tail may appear :

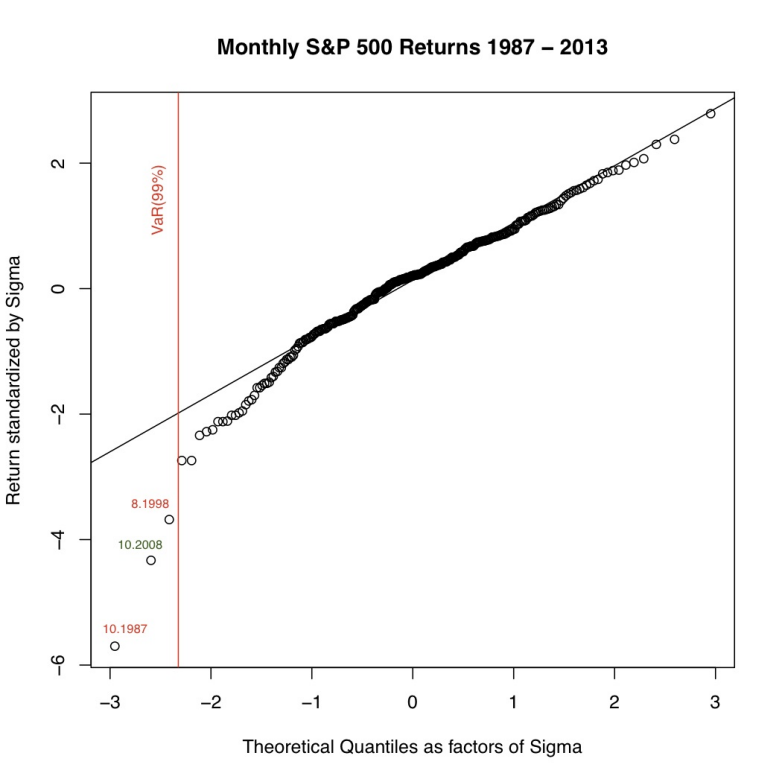
- clearly on high frequency data (e.g. daily ones)
- not visible anymore when aggregating them in e.g. yearly data (i.e. short samples),

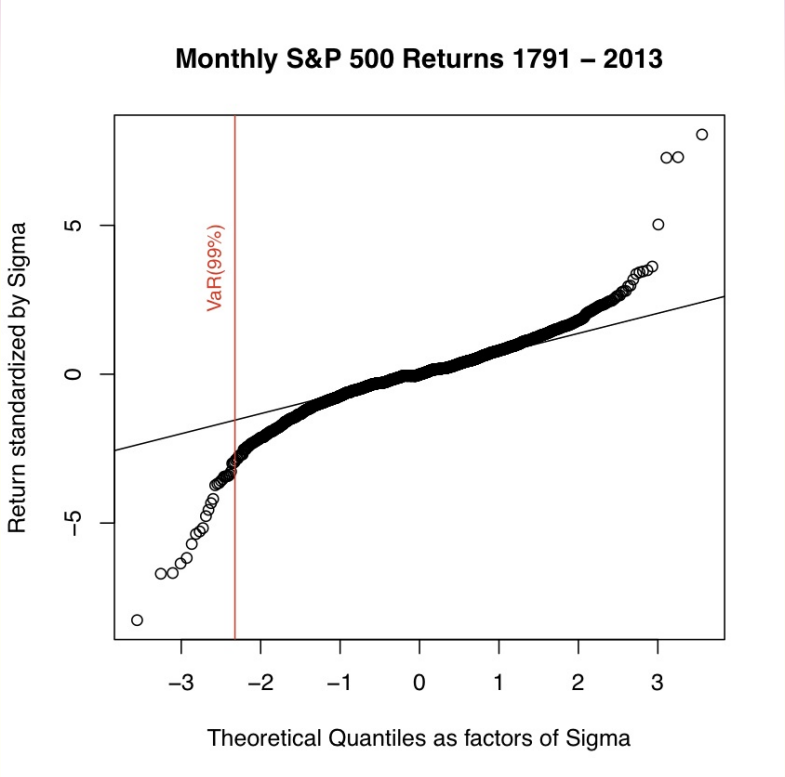
although known that the tail index of the underlying distribution remains constant under aggregation.





(data from https ://www.globalfinancialdata.com)





Main objective : to obtain the most accurate evaluations of risk measures when working on financial data under the presence of fat tail. We explore various approaches to handle this problem, theoretically and numerically.

With financial/actuarial applications in mind, we use power law models, such as Pareto, for the marginal distributions of the risks.

Some questions :

- Aggregation $\Rightarrow$ the number of observations decreases so, is it reasonable to use a limit distribution as an approximation of the true distribution ?

- Which type of approximation can be used whenever we are under the presence of heavy tails ? (issue of heavy vs moderately heavy) ?

- Why considering Pareto distribution ?
  $\hookrightarrow$ justified by the EVT :

  - Recall Pickands theorem : for sufficiently high threshold u, the GPD $G_{\xi,\sigma(u)}$ (with shape parameter ξ and scale parameter $\sigma(u)$) is a very good approximation to the excess cdf defined by $F_u(x) = \mathbb{P}[X - u \leq x | X > u]$:

$$F_u(y) \underset{u\to\infty}{\approx} G_{\xi,\sigma(u)}(y)$$

  - Recall also that, for $\xi > 0$,

$$\overline{G}_{\xi,\sigma(u)}(y) \underset{y\to\infty}{\sim} cy^{-1/\xi} \quad (c > 0 \text{ some constant})$$

When considering heavy-tailed risks, it implies that the extreme risks follow a GPD with a positive shape parameter $\xi > 0$, so it is natural and quite general to consider a Pareto distribution for heavy-tailed risks.

- About the iid condition
  - ▷ In our practical example of log returns (the motivation of this work), the independence condition is satisfied, hence is not a restriction in this case of time aggregation.
  - ▷ In the general case :
    - ↪ For different tail indices : EVT argument $\Rightarrow$ the tail index of the aggregated distribution corresponds to the one of the marginal with the heaviest tail, hence depends only weakly on considering the dependence
    - ↪ For the other cases, the influence of dependence on the VaR is not known, although it will have an impact, in particular increasing the VaR for positive dependence. We plan to further study this effect.

Outline

- Introduction - existing methods
- Normex : a mixed normal and extremes limit
- Application to risk measures - Comparison
- Conclusion : further development

Introduction

- Notation

X : (type I) Pareto r.v., with shape parameter α , df f, cdf F
$(\overline{F}(x) := 1 - F(x) = x^{-\alpha}, \quad \alpha > 0,\ x \geq 1)$.

Inverse function of F : $F^{\leftarrow}(z) = (1-z)^{-\frac{1}{\alpha}}, \quad$ for $0 < z < 1$.
Recall that

$$\mathbb{E}(X) < \infty \text{ for } \alpha > 1 \quad (\mathbb{E}(X) = \frac{\alpha}{\alpha - 1})$$

$$var(X) < \infty \text{ for } \alpha > 2 \quad (var(X) = \frac{\alpha}{(\alpha-1)^2(\alpha-2)})$$

Portfolio of heavy-tailed risks : modeled by a Pareto sum
$S_n := \sum_{i=1}^n X_i$, with $(X_i, i = 1, \ldots, n)$ an n-sample with parent r.v.X

$X_{(1)} \leqslant \cdots \leqslant X_{(n)}$ denote the order statistics of $(X_i)_{1 \leqslant i \leqslant n}$.

Φ, φ denote, respectively, the cdf and df of $\mathcal{N}(0,1)$.

Risk measures we consider :

- the Value-at-Risk VaR of order q of X, $q \in (0,1)$:

  $VaR_q(X) = \inf\{y \in \mathbb{R} : P[X > y] \leq 1 - q\} = F_X^{\leftarrow}(q)$ (quantile of F_X, order q)

- if $\mathbb{E}|X| < \infty$, the Expected Shorfall ES (or Tail VaR) at confidence level $q \in (0,1)$:
  $ES_q(X) = \frac{1}{1-q}\int_q^1 VaR_\beta(X)\, d\beta$ or $ES_q(X) = \mathbb{E}[X \mid X \geq VaR_q]$

▷ Existing methods to approximate the distribution of the Pareto sum S_n

- *A GCLT approach* (see e.g. Samorodnitsky et al. 1994, Petrov 1995, Zaliapin et al. 2005, Furrer 2012)
  The distribution of S_n can be approximated by
  - a stable distribution whenever $0 < \alpha < 2$ (via the GCLT)
  - a standard normal distribution for $\alpha \geq 2$ (via the CLT for $\alpha > 2$; for $\alpha = 2$, comes back to a normal limit with a variance different from $var(X) = \infty$) :

$$\text{If } 0 < \alpha < 2, \quad \frac{S_n - b_n}{n^{1/\alpha} C_\alpha} \xrightarrow{d} G_\alpha \quad \text{normalized } \alpha\text{-stable distribution}$$

$$\text{If } \alpha \geq 2, \quad \frac{1}{d_n}\left(S_n - \frac{n\alpha}{\alpha - 1}\right) \xrightarrow{d} \Phi$$

with

$$b_n = \begin{cases} 0 & \text{if} \quad 0 < \alpha < 1 \\ \frac{\pi n^2}{2} \int_1^\infty \sin\left(\frac{\pi x}{2n}\right) dF(x) \simeq n\left(\log n + 1 - C - \log(2/\pi)\right) & \text{if} \quad \alpha = 1 \\ n\,\mathbb{E}(X) = n\alpha/(\alpha - 1) & \text{if} \quad 1 < \alpha < 2 \end{cases}$$

($C =$ Euler constant 0.5772)

$$C_\alpha = \begin{cases} (\Gamma(1-\alpha)\cos(\pi\alpha/2))^{1/\alpha} & \text{if } \alpha \neq 1 \\ \pi/2 & \text{if } \alpha = 1 \end{cases} \quad ; \quad d_n = \begin{cases} \sqrt{n\, var(X)} = \sqrt{\frac{n\alpha}{(\alpha-1)^2(\alpha-2)}} & \text{if } \alpha > 2 \\ \inf\left\{x : \frac{2n \log x}{x^2} \leq 1\right\} & \text{if } \alpha = 2 \end{cases}$$

- *An EVT approach*

  Under the assumption of regular variation of the tail distribution (with non negative tail index), the tail of the cdf of the sum of iid rv's is mainly determined by the tail of the cdf of the maximum of these rv's :

$$\mathbb{P}[S_n > x] \simeq \mathbb{P}[\max_{1\leq i\leq n} X_i > x] \quad \text{as } x \to \infty$$

- *A mixed approach by Zaliapin et al., in the case* $2/3 < \alpha < 2$ $(var(X) = \infty)$.

  - Idea of the method : to rewrite the sum of the X_i's as the sum of the order statistics $X_{(i)}$ and to separate it into two terms, one with order statistics having finite variance and the other as the complement

$$S_n = \sum_{i=1}^{n} X_i = \sum_{i=1}^{n-2} X_{(i)} + \left(X_{(n-1)} + X_{(n)}\right)$$

- *Results* :

  Compared with the GCLT method, this 'approximative' approach provides

  $\hookrightarrow$ a better approximation for the Pareto sum, for any n, with a higher degree of accuracy ;
  $\hookrightarrow$ a better result for the evaluation of the VaR

- *Main drawbacks :*

  $\hookrightarrow$ assuming a condition of independence between the two dependent subsums
  $\hookrightarrow$ approximating the quantile of the Pareto sum as the sum of the quantiles of each subsum
  $\hookrightarrow$ when considering the case $\alpha > 2$, we still remain with a poor normal approximation for the tail distribution

▷ A general mixed approach

- *Main idea*, inspired by the Zaliapin et al.'s method : to separate mean behavior and extreme behavior, writing S_n as

$$S_n = \sum_{i=1}^{n} X_{(i)}$$

- *Main goal* : to improve approximations of the distribution of S_n and of the risk measures, when
  - taking into account the dependence of the order statistics
  - for any shape parameter α, in particular for the case $2 < \alpha < 4$ (for financial application, e.g. market risk data known to have α in this range)

- *Choice of the threshold* k for the trimmed sum by removing the k largest order statistics from the sample

  k selected in order to use the CLT, but also to improve its fit since we want to approximate the behavior of T_k by a normal one.

  - The finitude of the 2nd moment of X may lead to a poor normal approximation, if higher moments do not exist, as occurs for instance with financial market data.
  - The existence of the third moment provides a better rate of convergence to the normal distribution in the CLT (Berry Esséen inequality)
  - Another information useful to improve the approximation of the distribution of S_n with its limit distribution, is the Fisher index (kurtosis), defined by the ratio $\gamma = \frac{\mathbb{E}[(X-\mathbb{E}(X))^4]}{(var(X))^2}$

Therefore, fixing $p = 4$, we select $k = k(\alpha)$ such that

$$\mathbb{E}[X_{(j)}^p] \begin{cases} < \infty & \forall\, j \leq n-k \\ = \infty & \forall\, j > n-k \end{cases}$$

In our case of α-Pareto rv's : $k > \dfrac{p}{\alpha} - 1$

Note that the choice of k is independent of the sample size n

Value of the threshold $k = k(\alpha)$ for which the 4th moment is finite, according to the set of definition of α :

$\alpha \in I(k)$ with $I(k) =$	$]\frac{1}{2};\frac{4}{7}]$	$]\frac{4}{7};\frac{2}{3}]$	$]\frac{2}{3};\frac{4}{5}]$	$]\frac{4}{5};1$	$]1;\frac{4}{3}]$	$]\frac{4}{3};2[$	[2,4]
$k = k(\alpha)$ =	7	6	5	4	3	2	1

Normex - A mixed normal and extremes limit

- Main steps

  - *A conditional decomposition*

    Because of the dependence between the two subsums $T_k := \sum_{j=1}^{n-k} X_{(j)}$ and $U_{n-k} := \sum_{j=0}^{k-1} X_{(n-j)}$, we decompose the Pareto sum S_n in a slightly different way as

    $$S_n = T_k + X_{(n-k+1)} + U_{n-k+1}$$

    to use the property of conditional independence between $T_k/X_{(n-k+1)}$ and $U_{n-k+1}/X_{(n-k+1)}$.

  - *A normal approximation for the conditional trimmed sum*

    Now, since $T_k/X_{(n-k+1)} \underset{n\to\infty}{\overset{d}{\sim}} \sum_{j=1}^{n-k} Y_j$ with (Y_j) an $(n-k)$-sample with parent cdf defined by $F_Y(.) = \mathbb{P}\Big(X_i \leq . \ / \ X_i < X_{(n-k+1)}\Big)$, the CLT applies ; we have to compute the conditional first two moments.

Proposition

$$\mathcal{L}\Big(T_k/(X_{(n-k+1)}=y)\Big) \underset{n\to\infty}{\overset{d}{\sim}} \mathcal{N}\Big(m_1(\alpha,n,k,y),\sigma^2(\alpha,n,k,y)\Big)$$

where $$m_1(\alpha,n,k,y) = \frac{n-k(\alpha)}{1-y^{-\alpha}} \times \begin{cases} \frac{1-y^{1-\alpha}}{1-1/\alpha} & \text{if } \alpha \neq 1 \\ \ln(y) & \text{if } \alpha = 1 \end{cases}$$

$$\begin{aligned}
\sigma^2(\alpha,n,k,y) &:= \big(m_2(\alpha,n,k,y) - m_1^2(\alpha,n,k,y)\big) \qquad (y>1) \\
&= (n-k(1))y\left(1-\frac{y\ln^2(y)}{(y-1)^2}\right)\mathbb{1}_{(\alpha=1)} \\
&\quad + 2(n-k(2))\frac{y^2}{y^2-1}\left(\ln(y) - 2\,\frac{y-1}{y+1}\right)\mathbb{1}_{(\alpha=2)} \\
&\quad + \frac{n-k(\alpha)}{1-y^{-\alpha}}\left(\frac{1-y^{2-\alpha}}{1-2/\alpha} - \frac{1}{(1-1/\alpha)^2}\times\frac{(1-y^{1-\alpha})^2}{1-y^{-\alpha}}\right)\mathbb{1}_{(\alpha\neq 1,2)}
\end{aligned}$$

- *A Pareto distribution for the conditional sum of the largest order statistics*

$U_{n-k+1}/\left(X_{(n-k+1)} = y\right)$ can be written as
$U_{n-k+1}/\left(X_{(n-k+1)} = y\right) = \sum_{j=1}^{k-1} Z_j$
with (Z_j) iid rv's with parent cdf defined by
$F_Z(.) = \mathbb{P}\Big[X \leq . \ / \ (X > X_{(n-k+1)} = y)\Big] =$ Pareto cdf with parameters α and $y(> 1)$.

Hence the density function of $U_{n-k+1}/\left(X_{(n-k+1)} = y\right)$ is the convolution product of order $k-1$ of the df of Z :

$$f_{U_{n-k+1}/(X_{(n-k+1)}=y)} = h_y^{\star(k-1)}, \quad \text{with } h_y(x) = \frac{\alpha\, y^{\alpha}}{x^{\alpha+1}}\, \mathbb{1}_{(x \geq y)}$$

- Main result - an approximation of the distribution of the Pareto sum via Normex

  Theorem. The cdf of S_n can be approximated, for any n, by $G_{n,\alpha,k}$ defined for any $x \geq 1$ by

$$G_{n,\alpha,k}(x) = \begin{cases} \int_1^x f_{(n-k+1)}(y) \int_0^{x-y} \varphi_{m_1(y),\sigma(y)} \star h_y^{\star(k-1)}(v) dv\, dy & \text{if } k \geq 2 \\ \int_1^x \frac{f_{(n)}(y)}{\sigma(y)} \int_0^{x-y} \varphi\left(\frac{v-m_1(y)}{\sigma(y)}\right) dv\, dy & \text{if } k = 1 \end{cases}$$

For $k = 1$, the cdf of S_n is given by

$$G_{n,\alpha,1}(x) = n\alpha \int_1^x \frac{1}{\sigma(y)} y^{-(1+\alpha)} (1 - y^{-\alpha})^{n-1} \int_0^{x-y} \varphi\left(\frac{v - m_1(y)}{\sigma(y)}\right) dv\, dy$$

For $k \geq 2$ (but small), we have

$$G_{n,\alpha,k}(x) = \int_1^x \frac{f_{(n-k+1)}(y)}{\sigma(y)} \int_0^{x-y} \left(\int_0^v \varphi\left(\frac{v - u - m_1(y)}{\sigma(y)}\right) h_y^{\star(k-1)}(u) du \right) dv\, dy$$

where the convolution product $h_y^{\star(k-1)}$ can be numerically evaluated using the recursive convolution equation applied to h, or, explicitly for $\alpha = 1, 2$

- On the quality of the approximation of the distribution of the Pareto sum S_n

Consider for instance the case $2 < \alpha \leq 3$

- When applying the CLT directly to S_n, for any x,

$$F_n(x) - \Phi(x) = \frac{1}{\sqrt{n}} Q_1(x) + \frac{1}{n} Q_2(x) + o(1/n)$$

with
$$Q_1(x) = -\varphi(x) \frac{H_2(x)}{6} \frac{\mathbb{E}[(X - \mathbb{E}(X))^3]}{(var(X))^{3/2}} = \infty$$

$$Q_2(x) = -\varphi(x) \left\{ \frac{H_5(x)}{72} \frac{\left(\mathbb{E}[(X - \mathbb{E}(X))^3]\right)^2}{(var(X))^3} + \frac{H_3(x)}{24} (\gamma - 3) \right\} = \infty$$

- With Normex, using simply Berry-Esséen on the trimmed sum, we can write the error as

$$|\mathbb{P}(S_n \leq x) - G_{n,\alpha;1}(x)| \leq K(x) = \frac{c}{\sqrt{n-1}} \int_1^x \frac{C(y)}{\left(1 + \left|\frac{x - y - (n-1)\mu_y}{\sqrt{n-1}\ \gamma_y}\right|\right)^3} f_{(n)}(y)\, dy$$

with $c = 0.4693$, $C(y)$, μ_y and γ_y computed explicitly.
Moreover, for any $n \geq 52$ and $\alpha \in (2; 3]$, $0 \leq \max_{x>1} K(x) < 5\%$ and K decreases very fast to 0 after having reached its maximum ; the larger n, the faster to 0.

Application to VaR and Comparison

Possible approximations of VaR

Approximations $z_q^{(i)}$ of the VaR of order q , deduced from the various limit theorems (case $0 < \alpha \leq 4$) :

▷ via the GCLT , for $\alpha \leq 2$:

- for $\alpha < 2$:

$z_q^{(1)} \;=\; n^{1/\alpha} C_\alpha \, G_\alpha^{\leftarrow}(q) + b_n \quad (G_\alpha \; (\alpha, 1, 1, 0)$-stable distribution$)$

for $1/2 < \alpha < 2,$ and for $q > 0.95,$

$z_q^{(1bis)} \;=\; n^{1/\alpha} q^{-1/\alpha} + b_n$

- for $\alpha = 2$:

$z_q^{(1)} \;=\; d_n \, \Phi^{\leftarrow}(q) + 2n$

▷ via the CLT, for $\alpha > 2$:

$$z_q^{(2)} \;=\; \frac{\sqrt{n\alpha}}{(\alpha-1)\sqrt{\alpha-2}} \, \Phi^{\leftarrow}(q) + \frac{n\alpha}{\alpha-1}$$

▷ via the Max (EVT) approach, for high order q, for any α

$$z_q^{(3)} \;=\; n^{1/\alpha}\Big(\log(1/q)\Big)^{-1/\alpha} + b_n$$

▷ via Normex, for any α

$$z_q^{(5)} \;=\; G^{\leftarrow}_{n,\alpha,k}(q) \quad \text{with}$$

$$G_{n,\alpha,k}(x) = \int_1^x \frac{f_{(n-k+1)}(y)}{\sigma(y)} \int_0^{x-y}\Big(\int_0^v \varphi\Big(\frac{v-u-m_1(y)}{\sigma(y)}\Big) h_y^{\star(k-1)}(u)du\Big)\; dv\, dy$$

Numerical comparison - examples

Simulation of samples $(X_i, i = 1, \ldots, n)$ with parent r.v. X for different shape parameters, namely $\alpha = 3/2; 2; 5/2; 3; 4$, respectively.

For each n and each α, we aggregate the x_i's $(i = 1, \ldots, n)$. We repeat the operation $N = 10^7$ times, thus obtaining 10^7 realizations of the Pareto sum S_n, from which we can deduce its quantiles.

Three possible order q : 95%, 99% (Basel II) and 99.5% (Solvency 2)

Approximative relative empirical error :

$$\delta^{(i)} = \delta^{(i)}(q) = \frac{z_q^{(i)}}{z_q} - 1$$

- Case $\alpha = 5/2$

$n = 52$ q	Simul z_q	CLT $z_q^{(2)}$ $\delta^{(1)}$ (%)	Max $z_q^{(3)}$ $\delta^{(3)}(\%)$	Normex $z_q^{(5)}$ $\delta^{(5)}(\%)$
95%	103.23	104.35 1.08	102.60 -0.61	103.17 **-0.06**
99%	119.08	111.67 -6.22	117.25 -1.54	119.11 **0.03**
99.5%	128.66	114.35 -11.12	127.07 -1.24	131.5 2.21

$n = 100$ q	Simul z_q	CLT $z_q^{(2)}$ $\delta^{(1)}$ (%)	Max $z_q^{(3)}$ $\delta^{(3)}(\%)$	Normex $z_q^{(5)}$ $\delta^{(5)}(\%)$
95%	189.98	191.19 0.63	187.37 -1.38	189.84 **-0.07**
99%	210.54	201.35 -4.36	206.40 -1.96	209.98 **-0.27**
99.5%	222.73	205.06 -7.93	219.14 -1.61	223.77 **0.47**

$n = 250$ q	Simul z_q	CLT $z_q^{(2)}$ $\delta^{(1)}$ (%)	Max $z_q^{(3)}$ $\delta^{(3)}$ (%)	Normex $z_q^{(5)}$ $\delta^{(5)}$ (%)
95%	454.76	455.44 0.17	446.53 -1.81	453.92 **-0.18**
99%	484.48	471.5 -2.68	473.99 -2.17	483.27 **-0.25**
99.5%	501.02	477.38 -4.72	492.38 -1.73	501.31 **0.06**

$n = 500$ q	Simul z_q	CLT $z_q^{(2)}$ $\delta^{(1)}$ (%)	Max $z_q^{(3)}$ $\delta^{(3)}$ (%)	Normex $z_q^{(5)}$ $\delta^{(5)}$ (%)
95%	888.00	888.16 0.02	872.74 -1.72	886.07 -0.22
99%	928.80	910.88 -1.93	908.97 -2.14	925.19 **-0.39**
99.5%	950.90	919.19 -3.33	933.23 -1.86	948.31 **-0.27**

Comments

- Normex always gives sharp results (error less than 0.5% and often extremely close) ; it appears more or less independent of n. Among the various methods we tested, it is the one giving the most accurate evaluations, for the entire distribution of the sum and for the extreme quantiles, for any n and any $\alpha \in (0, 4]$.
- The max-method overestimates for $\alpha < 2$ and underestimates for $\alpha \geq 2$; it improves a bit when n increases.
- The GCLT method ($\alpha < 2$) overestimates the quantiles but improves with higher quantiles and when n increases.
- The CLT method underestimates the quantiles and the higher the quantile, the higher the underestimation ; it improves slightly when n increases.

Normex in practice

We dispose of a sample $(X_1, \cdots, X_n)$, with unknown heavy tailed cdf and positive tail index α. We consider the aggregated risks

$$S_n := \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} X_{(i)}, \quad \text{with} \quad X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$$

1. Preliminary step : estimation $\hat{\alpha}$ of α, with standard EVT methods (e.g. Hill estimator, QQ-estimator, ...)
2. Define $k = [p/\hat{\alpha} - 1] + 1$ with $p = 4$, such that $\mathbb{E}[X_{(i)}^4] = \infty$ iff $i > n - k$
3. The $n - k$ first order statistics and the $k - 1$ last ones being, conditionally on $X_{(n-k+1)}$, independent,
   - apply the CLT to the sum of the $n - k$ first order statistics conditionally on $X_{(n-k+1)}$
   - and compute the distribution of the sum of the last $k - 1$ ones conditionally on $X_{(n-k+1)}$ assuming a Pareto distribution for the rv's.
4. Then approximate the cdf of S_n by $G_{n,\alpha,k}$ defined in our Theorem, which provides a sharp approximation, easily computable whatever the size of the sample is.
5. Deduce any quantile z_q of order q of S_n as $z_q = G^{\leftarrow}_{n,\alpha,k}(q)$; allows in part. an accurate evaluation of risk measures of aggregated heavy tailed risks.

Conclusion

- Advantage of Normex : a quite general method
  - Fitting a normal distribution for the mean behavior can apply, not only for the Pareto distribution, but for any underlying distribution, without having to know about it, hence the method is quite general
  - For the extreme behavior, we have already seen that a Pareto type is standard in this context
  - Trimming the total sum by taking away extremes having infinite moments (of order $p \geq 3$) is always possible and allows to better approximate the distribution of the trimmed sum with a normal one (via the CLT), whatever underlying distributions

- This mixed distribution could be used to find out a range for the tail index α when fitting it to the empirical distribution (type of inverse problem).

- Next steps (ongoing work)

  ↪ Application to real data

  ↪ Extension to the dependent case, via
  - GCLT method : using the theorem on stable limits for sums of dependent infinite variance r.v. (Bartkiewicz et al., 2010) / LDP (Mikosch et al.)
  - CLT under weak dependence theorem
  - Max method (no need of independence)

  ↪ Study of the scaling behavior of VaR under aggregation

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