Heavy-tailedness and diversification disasters: Implications for models in economics, finance and insurance

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Objectives and key results

• *(Sub-)*Optimality of diversification under heavy tails & dependence

• *(Non-)*robustness of models in economics & finance to heavy tails, heterogeneity & dependence

• Implications for financial & (re-)insurance markets: Diversification traps & disasters


Stylized Facts of Real-World Returns

Daily % changes in the Dow Jones Industrial Average, Jan. 1980 - Sept. 2007
Dependence vs. margins in economic and financial problems

• Problems in finance, economics & risk management:
  Solution is affected by both

  • Marginal distributions (Heavy-Tailedness, Skewness)
  • Dependence (Positive or Negative, Asymmetry)

• Portfolio choice & value at risk (VaR)
  • Marginal effects under independence: Heavy-Tailedness
    Moderately HT vs. extremely HT \(\implies\) Opposite solutions
  • Different solutions: Positive vs. negative dependence

• Similar conclusions on (non-)robustness to heavy-tailedness:
  other models in economics, finance & econometrics:
    • Optimal bundling, firm growth theory, efficiency of statistical & econometric estimators, time series models
Normal vs. Heavy-tailed Power Laws
Heavy-tailed margins

• Many economic & financial time series: power law tails:
  \[ P(|X| > x) \approx \frac{c}{x^\alpha}, \alpha > 0 \] : tail index

• Moments of order \( p \geq \alpha \) : infinite; \( E|X|^p < \infty \) iff \( p < \alpha \)
  
  • \( \alpha \leq 4 \implies \text{Infinite fourth moments: } E|X|^4 = \infty \)
  • \( \alpha \leq 2 \implies \text{Infinite variances: } E|X|^2 = \infty \)
  • \( \alpha \leq 1 \implies \text{Infinite first moments: } E|X| = \infty \)

• Returns on many stocks & stock indices: \( \alpha \in (2, 4) \)
  \( \Rightarrow \text{finite variance, infinite fourth moment} \)
A tale of two tails

Light vs. heavy tails

Figure: Tails of Cauchy distributions are heavier than those of normal distributions. Tails of Lévy distributions are heavier than those of Cauchy or normal distributions.
A tale of two tails

Simulated data from Normal, Cauchy and Levy distributions, n=25

Figure: Heavy-tailed distributions: more extreme observations
Heavy-tailed margins

\[ P(|X| > x) \approx \frac{c}{x^\alpha} \]

- **Income**: \( \alpha \in [1.5, 3] \Rightarrow \text{infinite } EX^4 \), possibly infinite variances
- **Wealth**: \( \alpha \approx 1.5 \Rightarrow \text{infinite variances!} \)
- **Returns** from technological innovations, **Operational risks**: \( \alpha < 1 \Rightarrow \text{infinite means } E|X| = \infty! \)
- **Firm sizes, sizes of largest mutual funds, city sizes**: \( \alpha \approx 1 \)
- **Economic losses** from earthquakes: \( \alpha \in [0.6, 1.5] \Rightarrow \text{infinite variances, possibly infinite means} \)
- **Economic losses** from hurricanes: \( \alpha \approx 1.56; \alpha \approx 2.49 \)
Stable distributions

- $X \sim S_\alpha(\sigma)$: symmetric stable distribution, $\alpha \in (0, 2]
  \quad \text{CF: } E(e^{ixX}) = \exp\{-\sigma^\alpha |x|^\alpha\}$

- Normal $\mathcal{N}(0, \sigma)$: $\alpha = 2$

- Cauchy: $\alpha = 1$, $f(x) = \frac{\sigma}{\pi(\sigma^2 + x^2)}$

- Lévy: $\alpha = 1/2$, support $[0, \infty)$, $f(x) = \frac{\sigma}{\sqrt{2\pi}}x^{-3/2}\exp\left(-\frac{1}{2x}\right)$

- Power laws: $P(|X| > x) \approx \frac{C}{x^\alpha}$, $\alpha \in (0, 2)$

- Moments $E|X|^p$: finite iff $p < \alpha$
  - Infinite variances for $\alpha < 2$

- Portfolio formation: $\sum_{i=1}^n w_i X_i =_d (\sum_{i=1}^n w_i^\alpha)^{1/\alpha} X_1$
  - $\alpha = 2$ (normal): $\frac{1}{\sqrt{n}}(X_1 + \ldots + X_n) =_d X_1$
Value at risk (VaR)

• VaR
  • Risk $X$; positive values = losses
  • Loss probability $q$
  • $\text{VaR}_q(X) = z : P(X > z) = q$

• Risks $X_1, \ldots, X_n$

• $Z_w = \sum_{i=1}^{n} w_i X_i$: return on portfolio with weights $w = (w_1, \ldots, w_n)$

• Problem of interest:

  \[ \text{Minimize} \, \text{VaR}_q(Z_w) \]

  s.t. $w_i \geq 0$, $\sum_{i=1}^{n} w_i = 1$

• When diversification $\Rightarrow$ decrease in portfolio riskiness (VaR)?
Diversification & risk

- **Most diversified:** $w = (1/n, 1/n, \ldots, 1/n) \Rightarrow Z_w = \frac{1}{n} \sum_{i=1}^{n} X_i$

- **Least diversified:** $\bar{w} = (1, 0, \ldots, 0) \Rightarrow Z_{\bar{w}} = X_1$

- $X_1, \ldots, X_n \sim \mathcal{N}(0, \sigma)$ ($\alpha = 2$)

- $Z_w = \frac{1}{n} \sum_{i=1}^{n} X_i = d \frac{1}{\sqrt{n}} X_1 = \frac{1}{\sqrt{n}} Z_{\bar{w}}$

- $\text{VaR}_q(Z_w) = \frac{1}{\sqrt{n}} \text{VaR}_q(Z_{\bar{w}}) < \text{VaR}_q(Z_{\bar{w}})$

- $\text{VaR}_q(Z_w) \downarrow$ as $n \uparrow$ (Diversification $\nearrow$)
Diversification & risk

- \( X_1, \ldots, X_n \sim S_{1/2}(\sigma), \alpha = 1/2, \) Lévy distribution
  - \( Z_w = \frac{1}{n} \sum_{i=1}^{n} X_i = d \left[ \sum_{i=1}^{n} \left( \frac{1}{n} \right)^{1/2} \right]^2 X_1 = nX_1 = nZ_w \)
  - \( \text{VaR}_q(Z_w) = n\text{VaR}_q(Z_w) > \text{VaR}_q(Z_w) \)
  - \( \text{VaR}_q(Z_w) \uparrow \) as \( n \uparrow \) (Diversification \( \uparrow \))

- Heavy tails (margins) matter:
  - diversification \( \implies \) opposite effects on portfolio riskiness

- Skewness: typically priced
Heavy-tailedness & diversification

- **Moderate** heavy tails $\alpha > 1$: finite first moments

  $\text{VaR}_q(Z_w) < \text{VaR}_q(Z_{\bar{w}}) \quad \forall q > 0$

  Optimal to **diversify for all** loss probabilities $q$

- **Extremely** heavy tails $\alpha < 1$: infinite first moments

  $\text{VaR}_q(Z_w) < \text{VaR}_q(Z_{\bar{w}}) \quad \forall q > 0$

  Diversification: **suboptimal for all** loss probabilities $q$

- **Similar** conclusions: **Many other models in economics & finance**
  
    - Firm growth theory, optimal bundling, monotone consistency of sample mean, efficiency of linear estimators
  
    - Robust to moderate heavy tails
  
    - Properties: reversed under extremely heavy tails
What happens for intermediate heavy-tails?

- $X_1, \ldots, X_n$ i.i.d. stable with $\alpha = 1$: Cauchy distribution
  - Density $f(x) = \frac{\sigma}{\pi(\sigma^2+x^2)}$
  - Heavy power law tails: $P(|X| > x) \approx \frac{C}{x}$
  - Infinite first moment

- $Z_w = \sum_{i=1}^{n} w_i X_i =_d X_1 \forall w = (w_1, \ldots, w_n) : w_i \geq 0,$

- Diversification: no effect at all!
Summary so far: Diversification for heavy-tailed and bounded distributions

- A. Light-tailed i.i.d. $Z_i$ with $\alpha > 1$.
  Example: Traditional situation with normal $Z_i$

- B. Extremely heavy-tailed i.i.d. $Z_i$ with $\alpha < 1$.
  Example: Levy distribution with $\alpha = 1/2$

- C. Specific boundary case: i.i.d. Cauchy $Z_i$ with $\alpha = 1$

- D. Bounded $Z_i$

Figure: $N = 10$ risks/insurer; $M = 7$ insurers

- D: Individual/non-diversification corners vs insurer and reinsurer equilibrium
1st example: full risk pooling with normally distributed risks

Assume:
1 ≤ s ≤ M (= 5) insurers
N (= 20) risks/insurer
1 ≤ j ≤ Ns total risks
   i.i.d. normal X_i
CARA utility, Unlimited liability

Results:
If M − 1 insurers are pooling, so will Mth
If no insurers pool, each still has N risks

\[ z_{j,s} = \left( \sum_{i=1}^{j} X_i \right) / s \]
2nd example: Bernoulli-Lévy distribution with limited liability

Assume:
Limited liability:
maximum loss \( k = 80 \)
\( M = 5 \) insurers
\( N = 20 \) max risks/insurer
\[ u(x) = (x + k)^{3/4} \]
\[ z_{j,s} = \left( \sum_{i=1}^{j} X_i \right) / s \]

Results:
If insurers can coordinate, they can reach
\( MN = 100 \) reinsurance equilibrium
But if not, each insurer reverts to the \( N = 0 \) corner
Implications for markets for catastrophic risks

- **Equilibria in re-insurance markets for catastrophe risks** (Ibragimov, Jaffee and Walden, RFS)
  - A *diversification equilibrium* with *full risk pooling* for normally distributed (*light-tailed*) risks
  - No risk pooling & no insurance or reinsurance activity (*market collapse*) for extremely heavy-tailed cat risks
  - Intermediate cases (*heavy tails*): both
    - *Diversification equilibria*, in which insurers offer catastrophe coverage and reinsurance their risks
    - *Non-diversification equilibria* with no insurance or re-insurance
  - A *coordination problem* must be solved to shift from the bad to the good equilibrium

Government regulations or well functioning capital markets
Implications for markets for catastrophic risks

- **Catastrophic risks** have many **features favorable** to the provision of insurance
  - Generally **independent** over **risk types** and **geography**
  - **Few issues** of **asymmetric information** at the risk level
  - So a **complete failure** of these markets is puzzling

- We have shown that **market failures** (non-diversification traps) may arise when risks are **fat-tailed** and there is **limited liability**
  - **Diversification** may not be **beneficial** for the **single insurer**, although a **full reinsurance equilibrium** may exist.
  - **Government** programs (or diversified equity owners) may allow the **system to reach** the **full diversification** outcome
Diversification & dependence

- Minimize $VaR_q(w_1X_1 + w_2X_2)$ s.t. $w_1, w_2 \geq 0, w_1 + w_2 = 1$

- Independence:
  - Optimal portfolio: $(\tilde{w}_1, \tilde{w}_2) = (\frac{1}{2}, \frac{1}{2})$ (diversified) if $\alpha > 1$ (not extremely heavy-tailed, finite means)
  - $(\tilde{w}_1, \tilde{w}_2) = (1, 0)$ (not diversified, one risk) if $\alpha < 1$ (extremely heavy-tailed, infinite means)
Diversification & dependence

- Extreme **positive dependence**: $X_1 = X_2$ (a.s.) comonotonic risks
  
  - $\text{VaR}_q(w_1 X_1 + w_2 X_2) = \text{VaR}_q(X_1)$ \(\forall w\)
  
  - Diversification: no effect at all (similar to Cauchy) regardless of heavy-tailedness

- Extreme **negative dependence** $X_1 = -X_2$ (a.s.) countermonotonic risks
  
  - $\text{VaR}_q(w_1 X_1 + w_2 X_2) = (w_1 - w_2) \text{VaR}_q(X_1)$
  
  - Optimal portfolio: $\overline{w} = (1/2, 1/2)$ (most diversified regardless of heavy-tailedness

- Optimal **portfolio choice**: affected by both dependence & properties of margins
Copulas and dependence

- **Main idea**: separate effects of *dependence* from effects of *margins*
  
  - What *matters* more in *portfolio choice*: heavy-tailedness & skewness or (positive or negative) *dependence*?

- **Copulas**: functions that *join together marginal* cdf’s to form *multidimensional* cdf
Copulas and dependence

- Sklar’s theorem

- Risks $X$, $Y$:
  
  - Joint cdf $H_{XY}(x, y) = P(X \leq x, Y \leq y)$: affected by dependence and by marginal cdf’s $F_X(x) = P(X \leq x)$ and $G_Y(y) = P(Y \leq y)$

  - $C_{XY}(u, v)$: copula of $X$, $Y$:
    
    $$H_{XY}(x, y) = C_{XY} \left( F_X(x), G_Y(y) \right)$$

    dependence marginals

- $C_{XY}$: captures all dependence between risks $X$ and $Y$
Copulas and dependence

Advantages:

- **Exists for any risks** (correlation: finiteness of second moments)

- Characterizes **all dependence** properties

- **Flexibility in dependence modeling**
  
  - Asymmetric dependence: **Crashes vs. booms**
  
  - **Positive vs. negative** dependence

- **Independence**: Nested as a particular case: **Product** copula, particular values of parameter(s)

- **Extreme dependence**: $X = Y$ or $X = -Y$ $\iff$ extreme copulas; dependence in $C_{XY}$ varies in between
Copula structures

- **Archimedean** copulas

\[ C(u, v) = \phi^{-1}(\phi(u) + \phi(v)) \]

- **Contagion**: Non-zero tail dependence coeff.

\[
\begin{align*}
\lambda_L &= \lim_{u \to 0^+} P[Y \leq F^{-1}(u) | X \leq F_X^{-1}(u)] = \lim_{u \to 0^+} \frac{C(u, u)}{u} \\
\lambda_U &= \lim_{u \to 1^-} P[Y > F^{-1}(u) | X > F_X^{-1}(u)] = \lim_{u \to 1^-} \frac{1 - 2u + C(u, u)}{1 - u}
\end{align*}
\]

- **Clayton & Gumbel** copulas
Copula structures

- **Eyraud-Farlie-Gumbel-Morgenstern (EFGM):**

  \[ C(u, v) = uv[1 + \gamma (1 - u)(1 - v)] \]

  \( \gamma \in [-1, 1] \): dependence parameter Tail independent: no contagion

- **Heavy-tailed** Pareto marginals:

  \[ P(X > x) = \frac{1}{x^\alpha}, \quad x \geq 1 \]

- **Power laws**, tail index \( \alpha \)
Diversification: Copulas & heavy tails

Embrechts, Nešlehová & Wüthrich (2009): **Archimedean** copulas

- **Moderate** heavy tails $\alpha > 1$: finite first moment

\[
\text{VaR}_q\left(\frac{X + Y}{2}\right) < \text{VaR}_q(X) \quad \text{for sufficiently small } q
\]

Optimal to **diversify** for sufficiently small loss probabilities $q$

- **Extremely** heavy tails $\alpha < 1$: infinite first moments

\[
\text{VaR}_q\left(\frac{X + Y}{2}\right) > \text{VaR}_q(X) \quad \text{for sufficiently small } q
\]

Diversification: suboptimal for suff. small loss prob. $q$

Ibragimov & Prokhorov (2013): Similar conclusions for **EFGM**

- Tail **independent** EFGM & tail **dependent** Archimedean
  (Clayton, Gumbel): *same* boundary $\alpha = 1$ as in the case of independence
When dependence helps: Student-$t$ copulas

- Conclusions similar to independence: Models with common shocks

  \[ X_1 = ZY_1, X_2 = ZY_2, \ldots, X_n = ZY_n \]

- Common shock $Z > 0$ affecting all risks $X_1, \ldots, X_n$

- $Y_1, \ldots, Y_n : \text{i.i.d. normal or heavy-tailed with tail index } \alpha$

  \[ Z : \text{heavy-tailed with tail index } \beta \]

  Then $X_i : \text{heavy-tailed with tail index } \gamma = \min(\alpha, \beta)$

- Important particular case: (Dependent) Multivariate Student-$t$

  $X_1, X_2, \ldots, X_n$ with $\alpha$ d.f. (tail index) $\Rightarrow$ Optimal to diversify for all loss probabilities $q$ regardless of tail index $\alpha$

  - Tail dependent Student-$t$ copula and heavy-tailed margins with arbitrary tail index $\alpha$ : diversification pays off

- Contrast: Independent Student-$t$ $X_1, X_2, \ldots, X_n$ with $\alpha$ d.f. (tail index): diversification optimal for $\alpha > 1$; suboptimal for $\alpha < 1$
Diversification: Heavy-tailedness & dependence matter

- **Independence, Tail dependent** models with **common shocks** (e.g., Student-\(t\) distr. = Student-\(t\) copula with Student-\(t\) marginals):
  - Diversification always **pays off** for all loss probabilities \(q\)

- **Tail independent** EFGM, possibly **tail dependent** Archimedean copulas (e.g., Clayton & Gumbel):
  - **Dividing boundary** \(\alpha = 1\) for sufficiently small loss probability \(q\)

- **Numerical** results on interplay of **heavy-tailedness & dependence** (copula) assumptions and **loss probability** \(q\) in **diversification** decisions:
  - **Deviations** from threshold \(\alpha = 1\) for different **copulas** and **loss probabilities** \(q\)

- **Theoretical** results for **general** copulas = ?

- **(Non-)robustness** of other models in economics & finance
Key results

• **(Sub-)Optimality** of diversification under **heavy tails & dependence**

• **(Non-)robustness** of models in economics & finance to heavy tails, heterogeneity & dependence

• Implications for **financial & (re-)insurance markets**:
  Diversification traps & disasters


Key results

- (Sub-)Optimality of diversification under heavy tails & dependence

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- Implications for financial & (re-)insurance markets: Diversification traps & disasters
Characterizations of copulas & dependence

- $V_1, \ldots, V_n$: i.i.d. $\mathcal{U}([0, 1])$

- $C$: $n-$copula iff $\exists \tilde{g}_{i_1, \ldots, i_c}$ s.t.
  
  **A1** (integrability):
  
  \[
  \int_0^1 \cdots \int_0^1 |\tilde{g}_{i_1, \ldots, i_c}(t_{i_1}, \ldots, t_{i_c})| dt_{i_1} \cdots dt_{i_c} < \infty
  \]

  **A2** (degeneracy):
  
  \[
  E_{V_{i_k}} \left[ \tilde{g}_{i_1, \ldots, i_c} (V_{i_1}, \ldots, V_{i_{k-1}}, V_{i_k}, V_{i_{k+1}}, \ldots, V_{i_c}) \right] = 0
  \]

  **A3** (positive definiteness):
  
  \[
  \tilde{U}_n(V_1, \ldots, V_n) \equiv \sum_{c=2}^{n} \sum_{1 \leq i_1 < \ldots < i_c \leq n} \tilde{g}_{i_1, \ldots, i_c} (V_{i_1}, \ldots, V_{i_c}) \geq -1
  \]
• Representation for $C$:

$$C(u_1, ..., u_n) = \int_0^{u_1} ... \int_0^{u_n} (1 + \tilde{U}_n(t_1, ..., t_n)) \prod_{i=1}^n dt_i$$

• $\tilde{U}_n$: sum of **degenerate** $U-$statistics
Device for **constructing* $n$–copulas and cdf’s**

- **Bivariate Eyraud-Farlie-Gumbel-Morgenstern copulas & cdf’s:**
  \[
  C_\theta(u, v) = uv (1 + \theta(1 - u)(1 - v)) \\
  H_\theta(x, y) = F(x)G(y)\left(1 + \theta(1 - F(x))(1 - G(y))\right)
  \]

  $n = 2; \; \tilde{g}_{1,2}(t_1, t_2) = \theta(1 - 2t_1)(1 - 2t_2), \; \theta \in [-1, 1]$

- **Multivariate EFGM copulas & cdf’s:**
  \[
  C_\theta(u_1, u_2, ..., u_n) = \prod_{i=1}^{n} u_i \left(1 + \theta \prod_{i=1}^{n} (1 - u_i)\right)
  \]
  \[
  \tilde{g}_{i_1, ..., i_c}(t_{i_1}, ..., t_{i_c}) = \theta_{i_1, ..., i_c} (1 - 2t_{i_1})(1 - 2t_{i_2})...(1 - 2t_{i_c})
  \]
• **Generalized multivariate EFGM copulas** (Johnson and Kotz, 1975, Cambanis, 1977)

\[
C(u_1, \ldots, u_n) = \prod_{k=1}^{n} u_k \left( 1 + \sum_{c=2}^{n} \sum_{1 \leq i_1 < \cdots < i_c \leq n} \theta_{i_1, \ldots, i_c} (1 - u_{i_k}) \right)
\]

\[
\tilde{g}_{i_1, \ldots, i_c}(t_{i_1}, \ldots, t_{i_c}) = 0, \; c < n - 1
\]

\[
\tilde{g}_{1,2,\ldots,n}(t_1, t_2, \ldots, t_n) = \theta(1 - 2t_1)(1 - 2t_2)\cdots(1 - 2t_n)
\]

• **Generalized EFGM copulas**: complete **characterization** of joint **cdf’s** of **two-valued r.v.’s** (Sharakhmetov & Ibragimov, 2002)
From dependence to independence through 
\( U \)-statistics

\( G_n: \) sums of \( U \)-statistics

\[
U_n(\xi_1, \ldots, \xi_n) = \sum_{c=2}^{n} \sum_{1 \leq i_1 < \ldots < i_c \leq n} g_{i_1, \ldots, i_c}(\xi_{i_1}, \ldots, \xi_{i_c})
\]

\( g_{i_1, \ldots, i_c} \): satisfy A1-A3

- Arbitrarily dependent r.v.'s:
  sum of \( U \)-statistics in independent r.v.'s with canonical kernels

- Reduction of problems for dependence to well-studied objects

- Transfer of results for \( U \)-statistics under independence
From dependence to independence through $U$–statistics

- $X_1, ..., X_n$: 1-cdf’s $F_k(x_k)$
- $\xi_1, ..., \xi_n$: independent copies (1-cdf’s $F_k(x_k)$)

$\exists U_n \in G_n$ s.t. $\forall f : \mathbb{R}^n \to \mathbb{R}$

$$Ef(X_1, ..., X_n) = Ef(\xi_1, ..., \xi_n) \left(1 + U_n(\xi_1, ..., \xi_n) \right)$$

- Representation for c.f.’s:

$$E\exp \left( i \sum_{k=1}^{n} t_k X_k \right) = E\exp \left( i \sum_{k=1}^{n} t_k \xi_k \right) + E\exp \left( i \sum_{k=1}^{n} t_k \xi_k \right) U_n(\xi_1, ..., \xi_n)$$

‡ CLT for bivariate r.v.’s
Characterizations of dependence

- **Canonical** $g'$s: complete **characterizations** of dependence properties

- $X_1, ..., X_n$: $r$-**independent** if $\forall$ $r$ jointly independent $\iff$ $g_{i_1, ..., i_c}(V_{i_1}, ..., V_{i_c}) = 0$ (a.s.) $1 \leq i_1 < ... < i_c \leq n$, $c = 2, ..., r$

\[
g_{i_1, ..., i_{r+1}}(u_{i_1}, ..., u_{i_{r+1}}) = \frac{\alpha_1 \cdots \alpha_n}{\alpha_{i_1} \cdots \alpha_{i_{r+1}}} \left( (k + 1)u_{i_1}^k - (k + 2)u_{i_1}^{k+1} \right) \times \cdots \times \left( (k + 1)u_{i_c}^k - (k + 2)u_{i_c}^{k+1} \right)
\]

\[
C(u_1, ..., u_n) = \prod_{i=1}^{n} u_i \left( 1 + \sum_{1 \leq i_1 < ... < i_{r+1} \leq n} \frac{\alpha_1 \cdots \alpha_n}{\alpha_{i_1} \cdots \alpha_{i_{r+1}}} \times \left( u_{i_1}^k - u_{i_1}^{k+1} \right) \times \cdots \times \left( u_{i_{r+1}}^k - u_{i_{r+1}}^{k+1} \right) \right)
\]

Extensions of Wang (1990) $(k = 0)$
Copulas and Markov processes

- Darsow, Nguyen and Olsen, 1992: copulas and first-order Markovness

- $A, B : [0, 1]^2 \rightarrow [0, 1] :$

$$ (A \ast B)(x, y) = \int_0^1 \frac{\partial A(x, t)}{\partial t} \cdot \frac{\partial B(t, y)}{\partial t} dt $$

- $A : [0, 1]^m \rightarrow [0, 1], B : [0, 1]^n \rightarrow [0, 1] : \ast - product$

$$ A \ast B(x_1, \ldots, x_{m+n-1}) = $$

$$ \int_0^{x_m} \frac{\partial A(x_1, \ldots, x_{m-1}, \xi)}{\partial \xi} \cdot \frac{\partial B(\xi, x_{m+1}, \ldots, x_{m+n-1})}{\partial \xi} d\xi $$
Copulas and Markov processes

- Transition probabilities

$$P(s, x, t, A) = P(X_t \in A | X_s = x)$$ satisfy CKE's

iff $$C_{st} = C_{su} \ast C_{ut} \ \forall s < u < t$$

- $$X_t$$: first-order Markov iff

$$C_{t_1, \ldots, t_n} = C_{t_1 t_2} \ast C_{t_2 t_3} \ast \ldots \ast C_{t_{n-1} t_n}$$
New results: Higher-order Markovness and copulas

• $\{X_t\}_{t \in T}$: $k$-order Markov $\iff$

$$P(X_t < x_t | X_{t_1}, \ldots, X_{t_{n-k}}, X_{t_{n-k+1}}, \ldots, X_{t_n}) =$$

$$P(X_t < x_t | X_{t_{n-k+1}}, \ldots, X_{t_n})$$

• Complete characterization in terms of $(k+1)$-copulas

• $C_{t_1, \ldots, t_k}$: copulas of $X_{t_1}, \ldots, X_{t_k}$

• $\{X_t\}_{t \in T}$: $k$-order Markov iff $\forall t_1 < \ldots < t_n, \ n \geq k + 1$

$$C_{t_1, \ldots, t_n} = C_{t_1, \ldots, t_{k+1}} \ast^k C_{t_2, \ldots, t_{k+2}} \ast^k \ldots \ast^k C_{t_{n-k}, \ldots, t_n}$$
Stationary case

- \( X_t \): stationary \( k \)-order Markov iff

\[
C_{1,...,n}(u_1, ..., u_n) = C \star^k C \star^k ... \star^k C(u_1, ..., u_n)
= C^{n-k+1}(u_1, ..., u_n) \quad \forall n \geq k + 1
\]

\( C \): \( (k+1) \)-copula s.t.

\[
C_{i_1+h, ..., i_l+h} = C_{i_1, ..., i_l}, \quad 1 \leq j_1 < ... < j_l \leq k + 1
\]

- \( C^s \): \( s \)-fold product \( \star^k \) of \( C \)
Advantages of copula-based approach

- **Modeling higher order Markov** processes by using an alternative to transition matrices.

  - Instead of initial distribution & transition probabilities:
    - Prescribe marginals & $(k + 1)$–copulas

- Generate **copulas of higher order** & finite-dimensional cdf’s

  - **Advantage**: separation of properties of marginals (fat-tailedness) & dependence properties (conditional symmetry, $m$–dependence, $r$–independence, mixing)
Advantages of copula-based approach

- Inversion method:

**New** $k-$Markov with dependence similar to a given Markov process

Different marginals

† $X_t$: stationary $k-$Markov

$(k + 1)-\text{cdf } \tilde{F}(x_1, \ldots, x_{k+1}), 1-\text{cdf } F$

$\Rightarrow (k + 1)-\text{copula:}$

$$C(u_1, \ldots, u_{k+1}) = \tilde{F}\left( F^{-1}(u_1), \ldots, F^{-1}(u_{k+1}) \right)$$
Another 1–cdf $G$:

**Stationary $k$–Markov, same** dependence as $\{X_t\}$, **different** 1-marginal $G$:

$(k + 1)$–copula:

$$ C(u_1, ..., u_{k+1}) = \tilde{F}(G^{-1}(u_1), ..., G^{-1}(u_{k+1})) $$

Representation $\Rightarrow$ **Higher-order copulas & cdf’s**

$\{X_t\}$: stationary $C$–based $k$–Markov chain
Advantages of copula-based approach

- $C$: all dependence properties of the time series
  - $k$-independence, $m$-dependence, martingaleness, symmetry
  - On-going project with Johan Walden: characterizations of time-irreversibility; focus on $C_{t_1,...,t_k} = C_{t_k,...,t_1}$
  - Applications: forward-looking vs. backward-looking market participants ("fundamentalists" vs. noise traders or "chartists")
  - "Compass rose" for $P_{t-1}$ and $P_t$: symmetry in copulas
Combining higher-order Markovness with other dependence properties

- A number of studies in dependence modeling: Higher-order Markovness + \( m \)-dependence & \( r \)-independence

Lévy (1949): 2nd order Markovness + pairwise independence

Rosenblatt & Slepian (1962): \( N \)-order \( N \)-independent stationary Markov

- Impossibility/reduction:
  \( N \)-order Markov + \( N \)-independence + two-valued \( \Leftrightarrow \) joint independence

‡ Testing sensitivity to WD in DGP Rosenblatt & Slepian (1962)
Combining Markovness with other dependencies

‡ Examples:

Not 1-order Markovian

But 1-st order transition probabilities

\[ P(s, x, t, A) = P(X_t \in A|X_s = x) \]

satisfy C-K SE

\[ P(s, x, t, A) = \int_{-\infty}^{\infty} P(u, \xi, t, A)P(s, x, u, d\xi) \]

(other examples: Feller, 1959, Rosenblatt, 1960)
Combining Markovness with other dependencies

† 1-dependent Markov: Aaronson, Gilat and Keane (1992)
Burton, Goulet and Meester (1993), Matúš (1996)

† Matúš (1998): $m$–dependent discrete-space Markov

† Impossibility/Reduction:

‡ stationary $m$–dependent Markov if

\[ \text{card}(\Omega) < m + 2 \]
Markovness of higher-order and $k-$independence

- Characterization of stationary $k-$\textbf{independent} $k-$\textbf{Markov} processes

- $\{X_t\}$: $C-$based $k-$\textbf{independent} stationary $k-$\textbf{Markov} iff

$$\frac{\partial^{k+1} C(u_1, ..., u_{k+1})}{\partial u_1 ... \partial u_{k+1}} = 1 + g(u_1, ..., u_{k+1})$$

$g : [0, 1]^{k+1} \rightarrow [0, 1]:$ \textbf{canonical} $g-$\textbf{function}

(Integrability + more degeneracy + positive definiteness)
Markovness of higher-order and $k$–independence

\[ \int_0^1 \cdots \int_0^1 |g(u_1, \ldots, u_{k+1})| \, du_1 \cdots du_{k+1} < \infty \]

\[ \int_0^1 \cdots \int_0^1 g(u_1, \ldots, u_{k+1}) g(u_2, \ldots, u_{k+2}) \cdots g(u_s, \ldots, u_{k+s}) \, du_1 \cdots du_s = 0 \]

\forall s \leq u_{i_1} < \ldots < u_{i_s} \leq k + 1, \ s = 1, 2, \ldots, \left\lceil \frac{k+1}{2} \right\rceil

\[ g(u_1, \ldots, u_{k+1}) \geq -1 \]

- Integration: w.r. to all $s$ among $u_s, u_{s+1}, \ldots, u_{k+1}$ common to all $g$–functions

$g(u_1, \ldots, u_{k+1}), g(u_2, \ldots, u_{k+2}), \ldots, g(u_s, \ldots, u_{k+s})$

$k$–marginals: product copulas, independence

$k$–independence: satisfied
Markovness of higher-order and $m$–independence

- $\{X_t\}$: $C$–based $m$–dependent 1-Markov iff

$$\frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2} = 1 + g(u_1, u_2)$$

$g : [0, 1]^2 \rightarrow [0, 1]$: canonical $g$–function:

$$\int_0^1 \int_0^1 |g(u_1, u_2)| du_1 du_2 < \infty$$

$$\int_0^1 g(u_1, u_2) du_i = 0, \ g(u_1, u_2) \geq -1$$

$$\int_0^1 g(u_1, u_2) g(u_2, u_3) ... g(u_m, u_{m+1}) du_2 du_3 ... du_m = 0$$

‡ Integration: w.r. to $u_2, u_3, ..., u_m$ more than once among $g(u_1, u_2), g(u_2, u_3), ... , g(u_m, u_{m+1})$

$X_1, X_{m+1}$: independent; Process: $m$–dependent
New examples via existing constructions

- Higher-order Markovness + martingaleness
- Inversion method + existing examples ⇒

$k$–independent, $m$–dependent Markov processes

different marginals
Reduction & impossibility for $k$–order Markov processes

- $\{X_t\}$: $C$–based $k$–independent stationary $k$–Markov

\[ \frac{\partial^{k+1} c(u_1, \ldots, u_{k+1})}{\partial u_1 \cdots \partial u_{k+1}} = 1 + g(u_1, \ldots, u_{k+1}) \]

$g$ : product form (EFGM-type):

\[ g(u_1, u_2, \ldots, u_{k+1}) = \alpha f(u_1)f(u_2)\cdots f(u_{k+1}) \]

$\Leftrightarrow \{X_t\}$: jointly independent
Examples: EFGM and power copulas

• \((k + 1)-\text{EFGM}\) copulas:

\[
C(u_1, u_2, \ldots, u_{k+1}) = \prod_{i=1}^{k+1} u_i \left(1 + \alpha (1 - u_1)(1 - u_2)\ldots(1 - u_{k+1})\right)
\]

\[
g(u_1, u_2, \ldots, u_{k+1}) = \alpha (1 - 2u_1)(1 - 2u_2)\ldots(1 - 2u_{k+1})
\]

• \((k + 1)-\text{power}\) copulas

\[
C(u_1, u_2, \ldots, u_{k+1}) = \prod_{i=1}^{k+1} u_i \left(1 + \alpha (u_1^l - u_1^{l+1})(u_2^l - u_2^{l+1})\ldots(u_{k+1}^l - u_{k+1}^{l+1})\right)
\]

\(l \geq 0\) (EFGM: \(l = 0\))
Impossibility/reduction for \(m\)-dependence

- \(\{X_t\}\): \(C\)-based \(m\)-dependent Markov

\[ \frac{\partial^2 C(u_1,u_2)}{\partial u_1 \partial u_2} = 1 + \alpha f(u_1)f(u_2) \]

(separable product form)

\[ \Leftrightarrow X_t: \text{jointly independent} \]

- Representations \(\Rightarrow\)

\[ \int_0^1 \ldots \int_0^1 \alpha^m f(u_1)f^2(u_2)\ldots f^2(u_m)f(u_{m+1}) du_2\ldots du_m = 0; \]

\[ \alpha^m f(u_1)f(u_{m+1}) \left[ \int_0^1 f^2(u_2) du_2 \right]^{m-1} = 0 \]

\[ \Rightarrow f = 0 \Leftrightarrow \text{Independence} \]
Examples, new and old

† EFGM copulas, $k = 1$:

\[
C(u_1, u_2) = u_1 u_2 \left( 1 + \alpha (1 - u_1)(1 - u_2) \right)
\]

\[
g(u_1, u_2) = \alpha (1 - 2u_1)(1 - 2u_2)
\]

● Limitations of EFGM copulas,

separable copulas:

Complement & generalize existing results
Examples, new and old

† Cambanis (1991): **common dependencies**

**cannot be exhibited** by multivariate EFGM

\[
C_{j_1,...,j_n}(u_{j_1},...,u_{j_n}) = \\
\prod_{s=1}^{n} u_{j_k} \left( 1 + \sum_{1 \leq l < m \leq n} \alpha_{l m} (1 - u_{j_l})(1 - u_{j_m}) \right)
\]

† Rosenblatt & Slepian (1962): **non-existence** of bivariate \( N \)-independent \( N \)-Markov

Sharakhmetov & Ibragimov (2002):

**EFGM copulas for two-valued r.v.’s**

† **Technical difficulties in modeling**
Solution: New flexible copula classes

- **Copula-based TS with flexible dependencies**

† Copulas based on **Fourier polynomials**

- **$k$–independent $k$–Markov: Conditions satisfied for**

$$g(u_1, ..., u_{k+1}) = \sum_{j=1}^{N} \left[ \alpha_j \sin(2\pi \sum_{i=1}^{k+1} \beta_{ij} u_i) + \gamma_j \cos(2\pi \sum_{i=1}^{k+1} \beta_{ij} u_i) \right]$$

† $\alpha_j, \gamma_j \in \mathbb{R}, \beta_{ij} \in \mathbb{Z}, i = 1, ..., k + 1, j = 1, ..., N$:

† $\beta_{11}^{ij} + \sum_{l=2}^{s} \epsilon_{l-1} \beta_{l}^{ij} \neq 0$

$\epsilon_1, ..., \epsilon_{s-1} \in \{-1, 1\}, s = 2, ..., k + 1$

† $1 + \sum_{j=1}^{N} [\alpha_j \epsilon_j + \gamma_j \epsilon_{j+N}] \geq 0, \epsilon_1, ..., \epsilon_{2N} \in \{-1, 1\}$
Fourier copulas

\[ C(u_1, \ldots, u_{k+1}) = \int_{0}^{u_1} \cdots \int_{0}^{u_{k+1}} (1 + g(u_1, \ldots, u_{k+1})) \, du_1 \cdots du_{k+1} \]

\((k + 1)-\text{Fourier copulas}\)
Fourier copulas

- \(1\)-dependent \(1\)-Markov:

**Conditions satisfied** for Fourier copulas

\[
C(u_1, u_2) = \int_0^{u_1} \int_0^{u_2} (1 + g(u_1, u_2))du_1du_2
\]

\[
g(u_1, u_2) = \sum_{j=1}^{N} \left[ \alpha_j \sin(2\pi(\beta_1^j u_1 + \beta_2^j u_2)) + \gamma_j \cos(2\pi(\beta_1^j u_1 + \beta_2^j u_2)) \right]
\]

\(\downarrow\) \(\alpha_j, \gamma_j \in \mathbb{R}, \beta_1^j, \beta_2^j \in \mathbb{Z} :\)

\[
\beta_1^j + \beta_2^j \neq 0
\]

\[
\beta_1^j - \beta_2^j \neq 0
\]

\[
1 + \sum_{j=1}^{N} \left[ \alpha_j \epsilon_j + \gamma_j \epsilon_{j+N} \right] \geq 0
\]

\(\forall \epsilon_1, ..., \epsilon_{2N} \in \{-1, 1\}\)
Concluding remarks

- (Sub-)Optimality of diversification under heavy tails & dependence
- (Non-)robustness of models in economics & finance to heavy tails, heterogeneity & dependence
- General representations for joint cdf’s and copulas of arbitrary r.v.’s
  - Joint cdf’s and copulas of dependent r.v.’s = sums of $U$—statistics in independent r.v.’s
  - Similar results: expectations of arbitrary statistics in dependent r.v.’s
  - New representations for multivariate dependence measures
  - Complete characterizations of classes of dependent r.v.’s
  - Methods for constructing new copulas
  - Modeling different dependence structures
Concluding remarks

- **Copula-based** modeling for time series
- **Characterizations** of dependence in terms of copulas
  - Markovness of arbitrary order
  - Combining Markovness with other dependencies:
    - \( m \)-dependence, \( r \)-independence, martingaleness, conditional symmetry
  - Non-Markovian processes satisfying **Kolmogorov-Chapman SE**
Concluding remarks

- New flexible copulas to combine dependencies
- Expansions by linear functions (Eyraud-Fairlie-Gumbel-Morgensten copulas)
  - power functions (power copulas); Fourier polynomials (Fourier copulas)
- Impossibility/reduction: Copula-based dependence + specific copulas
  ⇔ Independence
Copula memory

- **Long-memory** via copulas: various definitions
- Dependence measures & copulas
- Gaussian & EFGM $\Rightarrow$ short-memory Markov
- Fast exponential decay of dependence between $X_t$ & $X_{t+h}$
- Numerical results $\Rightarrow$ Clayton copula-based Markov $\{X_t\}$: can behave as long memory (copulas) in finite samples
  - High persistence important for finance & economics
- Long memory-like: $X_t$ & $X_{t+h}$: slow decay of dependence for commonly used lages $h$
- **Volatility** modeling & **Nonlinear dependence** in finance
- Non-linear CH & long memory-like volatility
- Generalizations of GARCH
Copula memory


Beare (2008): $\alpha$, $\beta$ & $\phi$–mixing

- $\kappa(h) \leq \alpha(h) \leq \beta(h) \leq 0.5\phi(h)$
- **Numerical** results $\Rightarrow$ **Clayton**: exponential decay in $\beta(h) \Rightarrow$ short $\kappa$–memory in copulas

**Theoretical results** in Chen, Wu & Yi (2008):

- **Clayton**: weakly dependent & short memory in terms of mixing properties!
- Our numerical results $+$ Chen, Wu & Yi (2008): **Non-robustness** of procedures for detecting long memory in copulas
Objectives and key results

• (Sub-)Optimality of diversification under heavy tails & dependence

• (Non-)robustness of models in economics & finance to heavy tails, heterogeneity & dependence
  
  

• General representations for joint cdf’s and copulas of arbitrary r.v.’s

• Copula-based modeling for time series

• Characterizations of time series dependence in terms of copulas

• New flexible copulas to combine dependencies

• Long-memory via copulas: various definitions

• Non-robustness of procedures for detecting long memory in copulas