Subsidy Design and Asymmetric Information: Wealth versus Benefits

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Abstract

A government or public organization would like to subsidize an indivisible good. Consumers' valuations of the good vary according to their wealth and benefits from the good. Education, medical care, and housing are common examples. A subsidy scheme may be based on consumers' wealth or benefit information. We present a method to translate a wealth-based policy to a benefit-based policy, and vice versa. We give a necessary and sufficient condition for the wealth-based policy and translated benefit-based policy to implement the same assignment: consumers choose to purchase the good under the wealth-based policy if and only if they choose to do so under the translated benefit-based policy. General taxation allows equivalent wealth-based and benefit-based policies to generate the same revenue from consumers.

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1 Introduction

Governments and public organizations often subsidize goods and services such as education, health care and housing. Many common subsidies are based on partial information. In Canada and many European countries, health care subsidies under national health insurance are based on illness severity or benefit, but not on wealth. In the United States, the Medicaid program provides health insurance for low-income individuals or families, but Medicaid subsidies are usually not rationed according to illness severity or benefits. In European public universities subsidized fees are mostly based on family income and wealth, while in American private universities scholarships are mostly based on test scores and benefits from education. One may expect that wealth-based subsidies work rather differently from benefit-based subsidies. Usually, income-based subsidies help the poor while benefit-based subsidies help the more deserved.

In this paper, we investigate the relationship between subsidy policies that are based on different kinds of information in the context of an indivisible good. The key concept we develop for connecting between wealth-based and benefit-based policies is *the assignment set*. A wealth-based subsidy policy induces a set of consumers to purchase the good. Likewise, a benefit-based subsidy policy induces another set of consumers to do the same. We define an assignment set as the subset of all consumers who choose to purchase the good under a subsidy scheme.

When will a pair of wealth-based and benefit-based policies implement the same assignment set? What are the properties of a subset of consumers which can be implemented as an assignment set by a wealth-based policy, a benefit-based policy, or both? If an assignment set is implemented simultaneously by a wealth-based and a benefit-based policy, are the required budgets different? In this paper, we answer all these questions.

Our surprising result is that in terms of assignment sets, wealth-based and benefit-based subsidy schemes can be equivalent. Whoever consume the good in a wealth-based subsidy regime will consume the good in a benefit-based subsidy regime if and only if these subsidy schemes are equivalent. We provide necessary and sufficient conditions for wealth-based and benefit-based subsidy schemes to be assignment-equivalent.

What is the interest in assignment sets then? The utility of a consumer consists of the benefit from the subsidized good and the payment he makes. An assignment set with respect to a subsidy policy is a description of consumers who purchase the good. An assignment set together with the policy describe consumers' payments as a function of observable information. Therefore, an assignment set alone does not describe consumers' utilities.

An assignment set represents one important dimension of the effect of a policy. If a wealth-based policy and a benefit-based policy implement the same assignment set, each of these two policies induces the same decision from each consumer, even though each consumer receives a different subsidy depending on the policy. It does appear to us, however, that ensuring a set of consumers to obtain the good is an important and practical objective of many policies.

Many public policies are framed in terms of assignment sets when in fact the actual implementation of the policy requires the consideration of more dimensions. An example is illustrated by President Kennedy's speech to Congress on May 25, 1961. He said that "This nation should commit itself to achieving the goal, before this decade is out, of landing a man on the moon and returning him safely to the earth." In the same speech President Kennedy did ask Congress to allocate more resources for the space program, but the actual implementation of the Apollo program was complicated and very costly.[1] It did appear that financing issues were to be resolved only later, while the goal of a successful, manned lunar landing was regarded as more important.

As another example, in 2002 President Bush signed the "No Child Left Behind" Act for primary education reform. The main goals of the policy as broadly stated in the "No Child Left Behind" executive summary include increased accountability for schools; more choice for parents and students; more flexibility in the use of Federal education budget; and a stronger emphasis on reading, especially for youngest children.[2] Many final implementation issues are left to be decided by states, cities, and local authorities and schools. The reform is aimed to improve the quality of the schools, with a special concern for underprivileged children. In President Bush's words,"too many of our neediest children are being left behind." The point is that the policy can be broadly understood to aim at providing subsidies to underprivileged children to achieve certain

[1] See "Special Message to the Congress on Urgent National Needs, May 25, 1961" at http://www.jfklibrary.org/Historical+Resources/Archives/Reference+Desk/Speeches/

[2] http://www.ed.gov/nclb/overview/intro/execsumm.html

education standards. For example, the goal of children achieving literacy by Grade Three appears to be more important than the exact subsidies that children are to receive.

As these examples illustrate, there is often a hierarchical or decentralized approach to policies. Politicians and policy makers may determine the main goal, and let regulators be in charge of the implementation. We may interpret the main goal as setting up an assignment set. Financing issues such as required budgets, tax burdens, and actual subsidies to consumers are important, but may be second order, and are often left to be fully resolved in the implementation phase. We model this approach by our focus on assignment sets.

We present a way to translate a wealth-based subsidy policy to an equivalent benefit-based policy, and vice versa. Under a monotonicity condition, we show that our translation gives rise to equivalent wealth-based and benefit-based policies implementing the same assignment set. We then show that the monotonicity condition is also necessary.

The translation method can be described as follows. Suppose that a consumer's wealth is observed by the regulator, and suppose that it is $10,000. Suppose that the wealth-based subsidized price for the good at this wealth level is $200. The regulator does not have the benefit information but can compute the smallest benefit level (measured in utility units) at which this consumer with wealth $10,000 is willing to buy the good at $200. Suppose that this benefit level is 300. The benefit-based subsidized price for the good at benefit level 300 will be set at $200, the price at which a consumer with wealth $10,000 and benefit 300 is indifferent between purchasing and not. Now we can change the wealth level from $10,000, and repeat the procedure for other values.[3] The translation from a wealth-based policy to the equivalent benefit-based policy works through this *indifferent boundary* at which a consumer with a combination of wealth and benefit is the marginal consumer. The monotonicity condition refers to the indifference boundary inducing a negative relationship between wealth and benefit.

Our results yield new insights and interpretations. We show that for wealth-based and benefit-based schemes that are equivalent, the subsidized price is increasing in wealth if and only if it is decreasing in benefit. That is, a benefit-based policy that encourages consumption for the more deserved can only be

[3] Say, at wealth level $15,000, the price of the good is $300, and the consumer with benefit 350 is indifferent between between purchasing and not. Then we set the benefit-based price at $300 for benefit level 350.

assignment-equivalent to a wealth-based policy that charges a higher price for the rich.

Generally, with limited tax and subsidy instruments, equivalent wealth-based and benefit-based schemes result in different revenues for the regulator. The budgets required under each scheme for the implementation of an assignment set are different. With more tax and subsidy instruments, such as lump sum or general taxes, equivalent schemes that require the same budget can be constructed.

Our approach is different from the conventional cost-benefit analysis for comparing between wealth-based and benefit-based schemes. Given a social welfare function, one can compute the optimal wealth-based policy subject to any relevant incentive or participation constraints. This yields the optimized value of social welfare from wealth information. One can repeat the same exercise with respect to the benefit-based policy. One then picks the scheme that yields a higher welfare.

There are two problems with the standard, cost-benefit approach. First, a social welfare function is a convenient theoretical construct. In practice, a regulator may not be clearly instructed to make the tradeoff between benefits and costs. Furthermore, a given policy in practice may not be construed as an optimal choice for the maximization of a social welfare function. Second, the relevance of optimal schemes is limited by the kind of missing information that a particular model admits. Robustness is an issue because most models concentrate on one or two incentive problems while assuming away others.

We depart from the conventional methodology. We neither assume a social welfare function nor focus on a fixed set of incentive issues. We take the policies or assignment sets as given. Our method, however, complements the cost-benefit approach. Given an assignment, one can calculate consumers' cost burden and utilities under alternative equivalent subsidy schemes. This information may help a regulator to refine the choice among subsidy schemes.

Our theory yields insight about information collection and processing. For any assignment set satisfying the monotonicity condition, there is no need for a regulator to obtain both wealth and benefit information. Each of the equivalent wealth-based and benefit-based schemes will implement the given assignment set. For example, if wealth information is unreliable due to tax evasion, the regulator can rely on benefit information alone for the implementation of an assignment.

The literature on the public provision of private goods is extensive. The main focus has been on the reasons for such provisions and the optimal design of tax and transfer schemes when consumers are heterogenous and possess private information. The missing information gives rise to the relevant incentive constraints. The equivalence between subsidy schemes that are based on different information has not been studied before.

Arrow (1971) gives a benchmark for the optimal public provision of private goods under perfect information. He lets individuals be different in abilities. For a utilitarian social welfare function he derives the optimal expenditure policy. The subsequent literature focuses on asymmetric information and incentive problems. Blackorby and Donaldson (1988) show that when the social planner cannot observe individuals' abilities or illness, in-kind transfers may be preferred over monetary transfers. Assuming that income information cannot be used by the social planner, Besley and Coate (1991) justify the public provision of private goods as a way to redistribute income from the rich to the poor.

Following Mirrlees (1971), Boadway and Marchand (1995) model the public provision of a private good in the context of optimal income taxation, where individuals have private information about their labor productivities. More recently, De Fraja (2002) investigates the design of optimal education policies, when children in households have different abilities, and when household incomes differ. In De Fraja's model, income is observable, but ability is private information.

Some of the literature on the optimal tax and subsidy design deals with inequality under asymmetric information. It is recognized that inequality depends on wealth, as well as characteristics such as age, health status, gender, etc. As a consequence, transfers should take into account these characteristics. Atkinson (1992) concludes that "the issue of policy design is not therefore a confrontation between fully universal benefits and pure income testing; rather the question is that of the appropriate balance of categorical and income tests." Blackorby and Donaldson (1994) study the optimal transfers between groups ("people with serious illness, the disabled, racial and ethnic groups," etc., p.440) when the planner does not know the distribution of income within groups.

All of the above literature is concerned with the optimal design. Each paper focuses on a specific issue.

The case of missing information about abilities and incomes has been a major focus while other issues such as distribution and inequality have also been studied. In line with the literature, we assume that the regulator must design a subsidy under partial information; either income or benefit information is available. Our investigation about the relationship between subsidy policies that are based on different information appears to be new.

We introduce the model in the next section. In Section 3, we first define an assignment set for a subsidy scheme. We then present the translation between wealth-based schemes and benefit-based schemes. A necessary and sufficient condition is presented for an assignment set to be implemented simultaneously by wealth-based and benefit-based policies. In Section 4, we first show that equivalent schemes that implement the same assignment set generally collect different revenues and therefore require different budgets. Then we show that if the regulator can use more general tax or subsidy schemes, equivalent subsidy schemes can be constructed to collect the same revenue. In Section 5, we compare our approach with the conventional cost-benefit, constrained optimization approach. We show that our necessary and sufficient condition for equivalent implementation of assignment sets is not inconsistent with socially optimal schemes. The last section draws some conclusions.

2 The Model

We consider a regulator allocating a private good to a set of consumers. The good is indivisible, but each unit of the good may give different benefits to different consumers. In both the education and health markets, there are many such examples. A course of study confers different benefits depending on students' abilities; a course of treatment or surgery may heal an illness, but consumers may experience different utility recoveries. Nevertheless, the cost of a study or treatment program may not vary according to consumer characteristics.[4]

We normalize the total mass of consumers to 1. Each consumer may get at most one unit of the good. Each unit of the good costs $c > 0$. The regulator has available a budget B to pay for these goods. We assume that $0 \leq B < c$; that is, the regulator's budget is insufficient to supply the good to all consumers at zero

[4] We do not consider cost selection issues here. For some services, the provision cost may well depend on consumer characteristics. We leave this for our future research.

price. It is unimportant for the analysis whether the government actually produces the good or contracts with a firm to do so.

A consumer has wealth w and obtains some benefit ℓ when he consumes the good. We let w and ℓ be random variables. Respectively, the variables w and ℓ have supports on the positive intervals $[\underline{w}, \overline{w}]$ and $[\underline{\ell}, \overline{\ell}]$. Let F and f be the distribution and density functions of w; let G and g be the distribution and density functions of ℓ. We will assume that these distributions are independent.[5] We say that a consumer is type (w, ℓ) if he has wealth w and obtains benefit ℓ from the good.

If a type (w, ℓ) consumer pays p to obtain the good, his utility is $U(w-p)+\ell$, where U is a strictly increasing and strictly concave function. If a consumer does not obtain the good (and pays nothing), his utility is $U(w)$. In education for example, the variable ℓ measures his (expected) benefit from a course of study. The model works in a slightly different but isomorphic way in the health care setting. Here ℓ represents the (expected) loss of illness. If a sick consumer goes without a course of treatment, his utility is $U(w)-\ell$; if he pays p to obtain treatment his utility becomes $U(w-p)$.

The benefit ℓ is assumed to be separable from the utility of wealth. This assumption is made for convenience and has no conceptual consequence for the analysis. The utility from benefit ℓ is measured linearly, but this is without any loss of generality.[6]

The variables w and ℓ determine how much a consumer is willing to pay for the good. A type (w, ℓ) consumer is willing to pay for the good at a price p if

$$U(w-p)+\ell \geq U(w). \tag{1}$$

The consumer's willingness to pay exhibits monotonicity with respect to both wealth and benefit. If a type (w, ℓ) consumer is willing to pay for the good at price p, so are those who have higher wealth and those who derive higher benefits. For a type (w', ℓ') consumer, where $w' > w$ and $\ell' > \ell$, the following inequalities

[5] We do not use the properties of F and G. Our theory is not about trying to infer the (conditional) distribution of w given ℓ, or vice versa.

[6] The utility from benefit ℓ can be written generally as $V(\ell\ )$, where V is strictly increasing. We define a new benefit variable $\tilde{\ell} \equiv V(\ell)$ and adjust the distribution and density functions G and g accordingly.

follow from (1):

$$U(w'-p)+\ell \geq U(w') \qquad\qquad U(w-p)+\ell' \geq U(w). \tag{2}$$

Our basic hypothesis is that a consumer's willingness to pay is his private information, because either ℓ or w is assumed to be private information. We consider each of these two possibilities, and call these cases "unknown benefit" and "unknown wealth."

The regulator offers subsidy schemes to consumers. If w is public information while ℓ unknown, a wealth-based policy is a function $t : [\underline{w}, \overline{w}] \to \mathbb{R}^+$; a consumer pays $t(w)$ if he purchases the good. If ℓ is public information while w unknown, a policy is a function $s : \left[\underline{\ell}, \overline{\ell}\right] \to \mathbb{R}^+$; a consumer pays $s(\ell)$ if he purchases the good. If the regulator intends to give subsidies, $s(\ell)$ and $t(w)$ will be less than c, the cost of the good, which is borne by the regulator or government.[7]

For most of the analysis, we let consumers pay nothing if they do not purchase the good from the regulator. Nevertheless, in subsection 4.1, we also allow general subsidies. There if w is public information while ℓ unknown, a policy is a pair of functions $t_1 : [\underline{w}, \overline{w}] \to \mathbb{R}$ and $t_2 : [\underline{w}, \overline{w}] \to \mathbb{R}$, where $t_1(w)$ is the payment when the consumer does not buy the good and $t_2(w)$ is the payment when the consumer does. Similarly, if ℓ is public information while w unknown, a policy is a pair $s_1 : \left[\underline{\ell}, \overline{\ell}\right] \to \mathbb{R}$ and $s_2 : \left[\underline{\ell}, \overline{\ell}\right] \to \mathbb{R}$, where $s_1(\ell)$ is the payment when the consumer does not get the good and $s_2(\ell)$ is the payment when the consumer does. These payments are allowed to be positive or negative. The payments that are imposed on consumers when they do not purchase the good may be regarded as general taxation or subsidy, or they may even be lump sum or poll taxes (setting t_1 and s_1 to be constant functions).

3 Equivalent Assignments

We begin with the case where consumers' wealth w is public information, while their benefits ℓ are their private information, unknown to the regulator. Consider a wealth-based policy $t : [\underline{w}, \overline{w}] \to \mathbb{R}^+$. We define

[7] If consumers have access to a private market, the regulator's policies will be constrained by the market price. For example, if a consumer can purchase the good in the private market at d (which may be higher than c), then $s(\ell)$ and $t(w)$ must not be higher than d. The results are unaffected by this restriction.

the assignment set $\alpha(t)$ due to the payment scheme t by

$$\alpha(t) \equiv \{(w, \ell) : U(w - t(w)) + \ell \geq U(w)\}. \tag{3}$$

The inequality in (3) says that a type (w, ℓ) consumer prefers to buy the good at price $t(w)$. We will only consider those wealth-based policies that implement nonempty, proper subsets of consumers.

Next, suppose that consumers' benefit ℓ is public information, but w is unknown to the regulator. A benefit-based policy is a function: $s : [\underline{\ell}, \overline{\ell}] \to \mathbb{R}^+$. We define the assignment set $\beta(s)$ due to the payment scheme s by

$$\beta(s) \equiv \{(w, \ell) : U(w - s(\ell)) + \ell \geq U(w)\}. \tag{4}$$

The inequality in (4) says that a type (w, ℓ) consumer prefers to buy the good at price $s(\ell)$. Again, we only consider those benefit-based policies that implement nonempty, proper subsets of consumers.

When the wealth-based policy t implements the assignment $\alpha(t)$, we can compute the required subsidy. The regulator is responsible for the balance $c - t(w)$. Hence the total subsidy under policy t is

$$\int_{\alpha(t)} [c - t(w)] \, \mathrm{d}F(w)\mathrm{d}G(\ell). \tag{5}$$

Similarly, when the benefit-based policy s implements the assignment $\beta(s)$, the total subsidy is

$$\int_{\beta(s)} [c - s(\ell)] \, \mathrm{d}F(w)\mathrm{d}G(\ell). \tag{6}$$

Our first set of results concerns the relationship between the policy schemes and their assignment sets.

3.1 Unknown Benefit

First for a given t, and for any $w \in [\underline{w}, \overline{w}]$, define $\hat{\ell}$ by the equation:

$$U(w - t(w)) + \hat{\ell} = U(w). \tag{7}$$

For each w, a type $(w, \hat{\ell})$ consumer is indifferent between purchasing the good at $t(w)$ and the status quo. Equation (7) defines a functional relationship between $\hat{\ell}$ and w; we denote this function by θ:

$$\hat{\ell} = \theta(w; t) \equiv U(w) - U(w - t(w)). \tag{8}$$

We suppress the policy t in the argument of θ. From (2), at each w, consumers with benefits $\ell > \hat{\ell} = \theta(w)$ strictly prefer to purchase the good. We call θ the "indifference boundary" with respect to t.

The shape of the indifference boundary depends on the policy t. If t is differentiable, the slope of the indifference boundary is

$$\frac{\mathrm{d}\hat{\ell}}{\mathrm{d}w} = U'(w) - U'(w - t(w))[1 - t'(w)] \tag{9}$$

$$= [U'(w) - U'(w - t(w))] + U'(w - t(w))t'(w) \tag{10}$$

from the differentiation of (8). Because U is strictly concave, the term inside the square brackets in (10) must be negative. If $t(w)$ is decreasing $t'(w) \leq 0$, the slope of the indifference boundary is negative. If t is (strictly) increasing but the value of $t'(w)$ is sufficiently small, the indifference boundary remains negatively sloped.

Figures 1, 2, and 3 show indifference boundaries for various policies. In Figure 1, the boundary is increasing. Under this policy if a consumer with wealth w and benefit ℓ is indifferent between paying $t(w)$ to obtain the good and not, a consumer with wealth $w' > w$ actually declines to pay $t(w')$ to get the same benefit. The wealth-based policy is progressive and increases so rapidly that the consumer must receive more benefit than ℓ to be willing to pay $t(w')$. In Figure 2, the boundary is decreasing, and the comparison between decisions made by consumers with wealth levels w and w' goes exactly the opposite way. In Figure 3, the boundary is generated by a discontinuous policy: $t(w) = \frac{1}{4}c$ for $c < \widetilde{w}$ and $t(w) = \frac{3}{4}c$ otherwise.

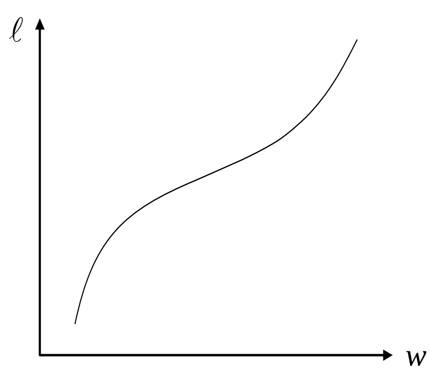

Figure 1: Increasing Indifference Boundary

For a given utility function U and a policy t, the indifference boundary θ defined above is a function

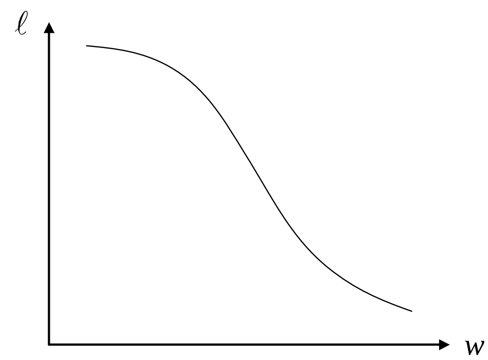

Figure 2: Decreasing Indifference Boundary

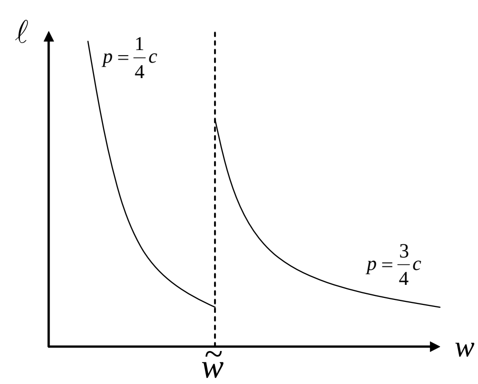

Figure 3: Discontinuous Indifference Boundary

$\theta : [\underline{w}, \overline{w}] \to \mathbb{R}$. The following refers to conditions of the policy t and its associated indifference boundary.

Condition 1 (Decreasing Indifference Boundary) *The wealth-based policy $t : [\underline{w}, \overline{w}] \to \mathbb{R}^+$ is continuous (equivalently the function θ is continuous). The indifference boundary θ is strictly decreasing.*

The indifference boundary in Figure 2 satisfies Condition 1, while the one in Figure 1 is continuous but not decreasing. The indifference boundary in Figure 3 is neither continuous nor monotone. Now we show that under Condition 1, we can translate a wealth-based policy t to a benefit-based policy s in such a way that the assignment sets under the two policies are identical.

Condition 1 implies that the inverse of θ exists for the set of benefits $[\underline{\ell}', \overline{\ell}'] \equiv \theta([\underline{w}, \overline{w}])$, the range of the function θ. Let this inverse be $\phi : [\underline{\ell}', \overline{\ell}'] \to [\underline{w}, \overline{w}]$. That is, for any benefit level in ℓ in $[\underline{\ell}', \overline{\ell}']$, the function ϕ gives the wealth level at which the consumer will be just willing to pay $t(w)$ to purchase the good. Note that under Condition 1, $\theta(\underline{w}) = \overline{\ell}'$ and $\theta(\overline{w}) = \underline{\ell}'$.

The range of θ, $[\underline{\ell}', \overline{\ell}']$, need not be exactly $\left[\underline{\ell}, \overline{\ell}\right]$, but because the assignment set $\alpha(t)$ is nonempty and a proper subset of all consumers, it must intersect $\left[\underline{\ell}, \overline{\ell}\right]$. The next two diagrams illustrate two possibilities. In Figure 4, the range of θ contains $\left[\underline{\ell}, \overline{\ell}\right]$, while in Figure 5, the range of θ is a proper subset of $\left[\underline{\ell}, \overline{\ell}\right]$.

Now we construct a benefit-based policy, $s : \left[\underline{\ell}, \overline{\ell}\right] \to \mathbb{R}^+$, which yields the same indifference boundary as t. For each $\ell \in \left[\underline{\ell}, \overline{\ell}\right] \cap [\underline{\ell}', \overline{\ell}']$, we define a payment $s(\ell)$ by

$$U(\phi(\ell) - s) + \ell = U(\phi(\ell)). \tag{11}$$

We replace the wealth variable in the definition of the indifference boundary (7) by $\phi(\ell)$. The equation in (11) yields an implicit function $s(\ell)$, a benefit-based policy. The construction of such an $s(\ell)$ yields an identical boundary: $U(w - s(\ell)) + \ell = U(w)$, but now the policy defined by (11) is written in terms of benefits instead of wealth. Consumer type $(w, \theta(w))$ is just willing to pay $t(w)$ to get the good for the benefit $\theta(w)$. Equivalently, consumer type $(\phi(\ell), \ell)$ is just willing to pay $s(\ell) = t(w)$ to get the good for the benefit ℓ.

There remain possible values of benefits which are not in the range $[\underline{\ell}', \overline{\ell}']$. These cases, when they exist,

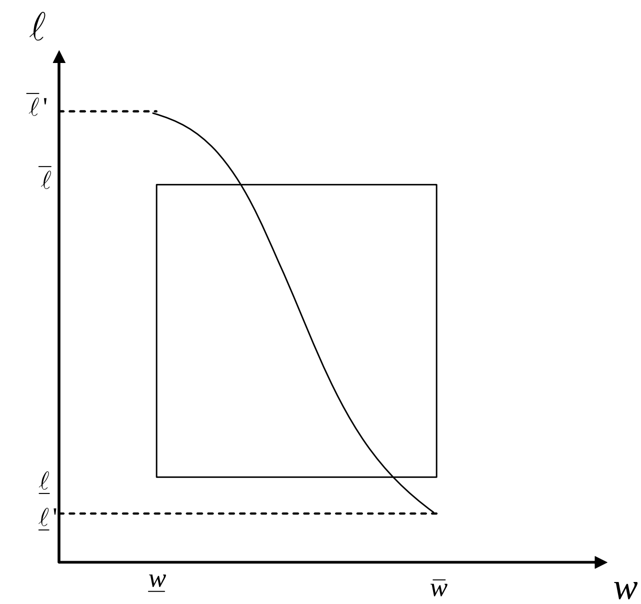

Figure 4: $[\underline{\ell}', \overline{\ell}']$ contains $[\underline{\ell}, \overline{\ell}]$

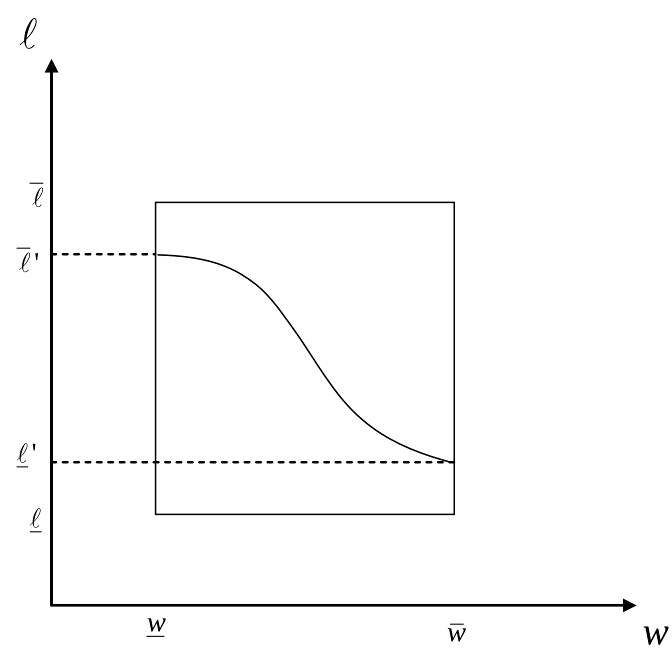

Figure 5: $[\underline{\ell}, \overline{\ell}]$ contains $[\underline{\ell}', \overline{\ell}']$

correspond to $\underline{\ell} < \underline{\ell}'$, $\overline{\ell}' < \overline{\ell}$, or both (see Figures 4 and 5). We complete the definition of s by the following.

$$\text{For } \ell \in \left[\underline{\ell}, \underline{\ell}'\right], \text{ let } s(\ell) = s(\underline{\ell}'). \qquad \text{For } \ell \in [\overline{\ell}', \overline{\ell}], \text{ let } s(\ell) = s(\overline{\ell}'). \tag{12}$$

The two sets, $\left[\underline{\ell}, \underline{\ell}'\right]$ and $[\overline{\ell}', \overline{\ell}]$, contain consumers with very low or very high benefits. Under $t(w)$, those consumers with very low benefits will not purchase the good at $s(\underline{\ell}')$ no matter how high their wealth w; those with very high benefits will always purchase at $s(\overline{\ell}')$. This completes the translation of a wealth-based policy $t(w)$ to a benefit-based policy $s(\ell)$.

Proposition 1 *Suppose that a wealth-based policy $t : [\underline{w}, \overline{w}] \to \mathbb{R}^+$ satisfies Condition 1 (Decreasing Indifference Boundary). The benefit-based policy $s : \left[\underline{\ell}, \overline{\ell}\right] \to \mathbb{R}^+$ defined in (11) and (12) implements the same assignment as the wealth-based policy t. That is, assignment sets $\alpha(t)$ and $\beta(s)$ are identical.*

Proposition 1 (whose proof is in the Appendix) makes use of the decreasing monotonicity of the indifference boundary. Given a wealth-based policy, to each wealth level, we associate a benefit threshold at which the consumer is indifferent between purchasing and not. The strict monotonicity of the boundary allows us to invert this relationship. So now for each benefit level, we are able to associate a wealth threshold.

Monotonicity alone does not guarantee that the assignments are identical when a wealth-based policy is translated to a benefit-based policy according to the method just described. An indifference boundary divides the space of all consumers into two half spaces. Consumers' preferences determine in which one of these half spaces the assignment set resides. Given a boundary θ on w-ℓ space, at a point (w, ℓ) on the boundary, by (2), those points above (w, ℓ) are consumers with higher benefits, and these belong to the set $\alpha(t)$. Conversely, given the equivalent boundary ϕ (the inverse of θ) on the same w-ℓ space, then at a point (w, ℓ) on the boundary, those points to the right of (w, ℓ) are consumers with higher wealth, and they belong to the set $\beta(s)$. Figure 6 illustrates this. When an indifference boundary is strictly decreasing, the assignment sets of the wealth-based and translated benefit-based policies coincide.

What are the implications of Proposition 1? Consider a wealth-based policy t. Suppose that wealth information may be unreliable due to audit difficulties and tax evasion, or perhaps the cost of wealth information processing and verification is high. According to Proposition 1, a regulator can use the equivalent

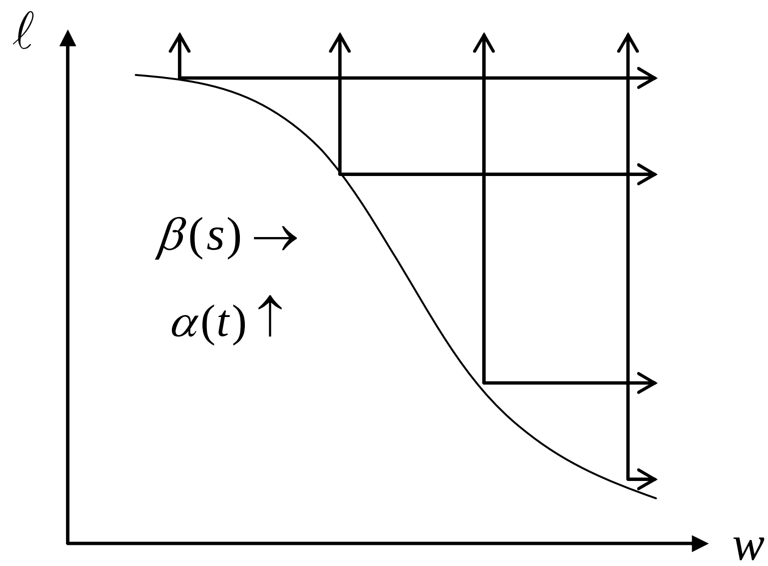

Figure 6: Downward Sloping Boundary: Direction of Preferences

benefit-based scheme $s(\ell)$ for the implementation of the same assignment set if the indifference boundary for $t(w)$ is decreasing.

A further implication of Proposition 1 is described in Corollary 1.

Corollary 1 *Suppose that a wealth-based policy t satisfies Condition 1 (Decreasing Indifference Boundary) and is increasing. The assignment-equivalent benefit-based policy s is decreasing in ℓ.*

Proof of Corollary 1: Consider two types of consumers (w_1, ℓ_2) and (w_2, ℓ_1) who are on the indifference boundary under t and the equivalent s. Without loss of generality, let $w_1 < w_2$. Because the indifference boundary is decreasing, $\ell_1 < \ell_2$; see Figure 7. By assumption $t(w_1) < t(w_2)$. Under the equivalent benefit-based policy s, type (w_1,ℓ_2) pays $s(\ell_2) = t(w_1)$, and type (w_2,ℓ_1) pays $s(\ell_1) = t(w_2)$. It follows that $s(\ell_2) < s(\ell_1)$.

Corollary 1 illustrates an interesting implication of our method. Consider a regulator whose policy makes richer consumers pay higher prices for the good. If this policy t generates a decreasing indifference boundary, it simultaneously provides a stronger incentive to consumers who derive higher benefits for buying the good. This is because the equivalent benefit-based policy s makes consumers who obtain more benefits pay less.

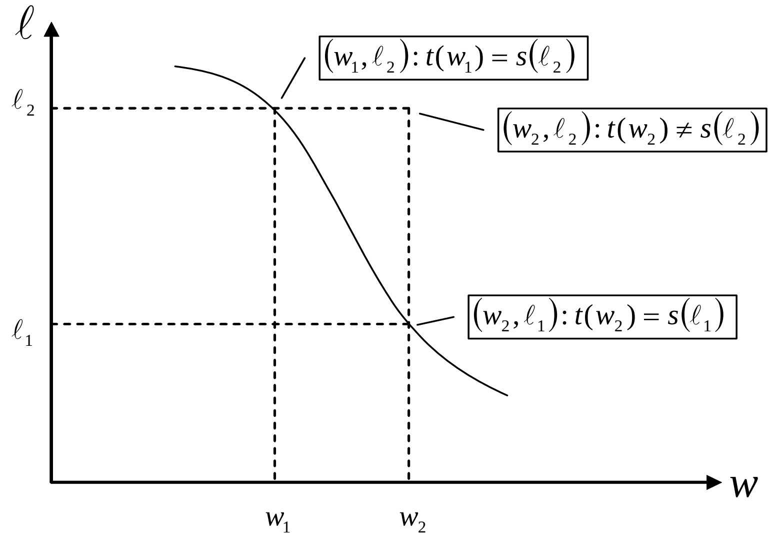

Figure 7: Types of Consumers on a Decreasing Indifference Boundary

Consumers will generally not be indifferent between whether their subsidies are based on wealth or benefit. Suppose that a type (w, ℓ) consumer is charged $t(w)$ to obtain benefit ℓ. Now if he is faced with the equivalent benefit-based scheme, the payment $s(\ell)$ may be much smaller. He still prefers to pay this to buy the good, and his surplus is increased. On the other hand, some consumers stand to lose when the scheme $t(w)$ is replaced by the equivalent $s(\ell)$. A regulator can be sure that consumers will not change their purchase behaviors under the equivalent wealth-based and benefit-based schemes. A regulator may have to consider the change in consumers' inframarginal surplus. Without assessing the change in the profile of consumer utilities, we only proceed in an orthogonal fashion. We can, however, study the revenues collected under equivalent policies, as we will do in the next section.

Our next condition says that the indifference boundary is strictly increasing. Then the union of the assignment sets $\alpha(t)$ and $\beta(s)$ of equivalent boundaries is the set of all consumers and the intersection contains only the indifference boundary. Figure 8 illustrates the direction of the preferences when the boundary is upward sloping.

Condition 2 (Increasing Indifference Boundary) *The wealth-based policy t is continuous (equivalently the function θ is continuous). The indifference boundary θ is strictly increasing.*

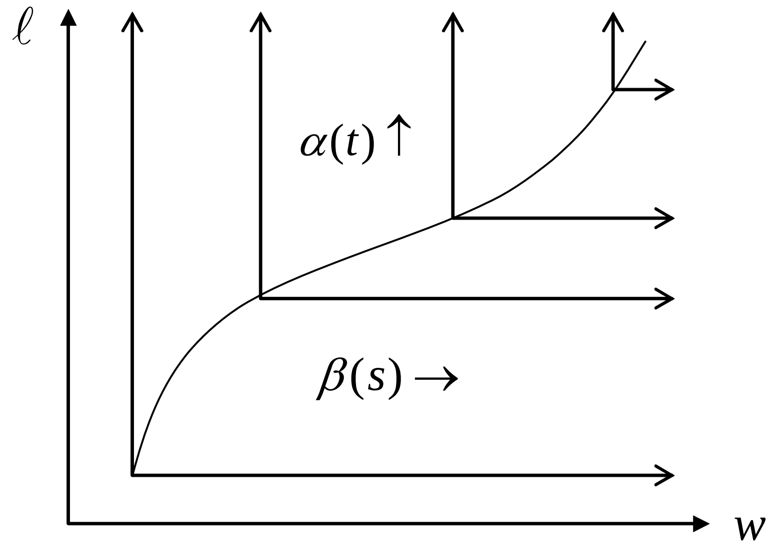

Figure 8: Upward Sloping Boundary: Direction of Preferences

Corollary 2 *Suppose that a wealth-based policy t satisfies Condition 2 (Increasing Indifference Boundary). The benefit-based policy s defined in (11) and (12) implements an assignment $\beta(s)$ whose intersection with the assignment $\alpha(t)$ implemented by the wealth-based policy $t(w)$ is the set of indifferent consumers $\{(w,\ell) : U(w - t(w)) + \ell = U(w)\}$, and whose union with $\alpha(t)$ is the set of all consumers $[\underline{w}, \overline{w}] \times \left[\underline{\ell}, \overline{\ell}\right]$.*

We omit the proof; it is symmetric to the proof of Proposition 1. We present an example of the translation of a wealth-based policy to a benefit-based policy.

Example 1 *Let U be the logarithmic function. Let a wealth-based policy be quasi-linear, $t(w) = a + bw$. The indifference boundary is given by $\ln(w - a - bw) + \ell = \ln w$, or $\ell = \theta(w) \equiv \ln \dfrac{w}{(w - a - bw)}$. We assume that the denominator is strictly positive; that is, $w - a - bw > 0$. The derivative of θ is*

$$\frac{d\theta}{dw} = -\frac{a}{w(w - a - bw)}.$$

The boundary is strictly decreasing if and only if $a > 0$ (conditional on $w - a - bw > 0$). The inverse of θ is $w = \phi(\ell) \equiv \dfrac{ae^{\ell}}{(1-b)e^{\ell} - 1}$. Using in (11) to solve for s, we obtain the benefit-based policy

$$s(\ell) = \frac{a\left(e^{\ell} - 1\right)}{(1-b)e^{\ell} - 1}.$$

3.2 Unknown Wealth

In this subsection, we assume that benefit information is available, but wealth is not. We consider the translation of a benefit-based policy to a wealth-based policy. The analysis is similar to the previous subsection. A benefit-based policy is a function $s : \left[\underline{\ell}, \overline{\ell}\right] \to \mathbb{R}^+$.

For a given s, and for any $\ell \in \left[\underline{\ell}, \overline{\ell}\right]$, define $\hat{w}$ by the equation:

$$U(\hat{w} - s(\ell)) + \ell = U(\hat{w}). \tag{13}$$

Equation (13) defines a functional relationship between $\hat{w}$ and ℓ; we denote this function by φ. From (13), at each ℓ, consumers with wealth $w > \hat{w}$ strictly prefer to purchase the good. At each ℓ, consumers with wealth more than $\varphi(\ell)$ prefer to purchase at $s(\ell)$. We call $\varphi : \left[\underline{\ell}, \overline{\ell}\right] \to \mathbb{R}$ the "indifference boundary" with respect to s.

If s is differentiable, then total differentiation of (13) yields

$$\frac{\mathrm{d}\hat{w}}{\mathrm{d}\ell} = \frac{s'(\ell)U'(\hat{w} - s(\ell)) - 1}{U'(\hat{w} - s(\ell)) - U'(\hat{w})}. \tag{14}$$

The denominator of (14) is positive due to the concavity of U. If s is decreasing $s'(\ell) \leq 0$, the slope of the indifference boundary is negative. Furthermore, even when $s(\ell)$ is increasing, the indifference boundary remains negatively sloped if the value of $s(\ell)$ is small.

Condition 3 (Decreasing Indifference Boundary) *The benefit-based policy s is continuous (equivalently the function φ is continuous). The indifference boundary φ is strictly decreasing.*

Condition 3 implies that the inverse of φ exists for the set of benefits $[\underline{w}', \overline{w}'] \equiv \varphi(\left[\underline{\ell}, \overline{\ell}\right])$, the range of the function φ. Let this inverse be $\vartheta : [\underline{w}', \overline{w}'] \to \left[\underline{\ell}, \overline{\ell}\right]$. That is, for any wealth level in $[\underline{w}', \overline{w}']$, the function ϑ gives the benefit level at which the consumer will be just willing to pay $s(\ell)$ to purchase the good.[8] Under Condition 3, $\varphi(\underline{\ell}) = \overline{w}'$ and $\varphi(\overline{\ell}) = \underline{w}'$. As in the case of unknown benefit, the range $[\underline{w}', \overline{w}']$ need not be identical to $[\underline{w}, \overline{w}]$, but because the assignment $\beta(s)$ is nonempty and a proper subset of all consumers, the intersection between $[\underline{w}', \overline{w}']$ and $[\underline{w}, \overline{w}]$ is nonempty.

[8] We assume that the domain of the function U can be extended beyond $[\underline{w}, \overline{w}]$, say to $[\underline{w}', \overline{w}']$, and that on the extended domain, U remains strictly increasing and strictly concave.

Now we construct a wealth-based policy $t : [\underline{w}, \overline{w}] \to \mathbb{R}^+$ that implements the same indifference boundary as s. For each $w \in [\underline{w}, \overline{w}] \cap [\underline{w}', \overline{w}']$, we construct a payment $t(w)$ that satisfies

$$U(w - t) + \vartheta(w) = U(w). \tag{15}$$

That is, we replace the variable ℓ in equation (13) by $\vartheta(w)$ The above equation (15) defines a functional relationship between t and w. For $w \notin [\underline{w}, \overline{w}] \cap [\underline{w}', \overline{w}']$, if it exists, we define the following payments:

$$\text{For } w \in [\underline{w}, \underline{w}'] \text{, let } t(w) = t(\underline{w}'). \qquad \text{For } w \in [\overline{w}', \overline{w}] \text{, let } t(w) = t(\overline{w}'). \tag{16}$$

Note that $t(\underline{w}') = s(\overline{\ell})$ and $t(\overline{w}') = s(\underline{\ell})$. We now state the following proposition (whose proof is omitted since it is identical to the one for Proposition 1).

Proposition 2 *Suppose that a benefit-based policy $s : \left[\underline{\ell}, \overline{\ell}\right] \to \mathbb{R}^+$ satisfies Condition 3 (Decreasing Indifference Boundary). The wealth-based policy $t : [\underline{w}, \overline{w}] \to \mathbb{R}^+$ defined in (15) and (16) implements the same assignment as the benefit-based policy s. That is, the two sets $\alpha(t)$ and $\beta(s)$ are identical.*

Corollary 3 *Suppose that a benefit-based policy s satisfies Condition 3 (Decreasing Indifference Boundary) and is decreasing. The equivalent wealth-based policy t is increasing in w.*

The proof of Corollary 3 is omitted since it is identical to Corollary 1.

The following Condition 4 and Corollary 4 describe the relationship between the sets $\alpha(t)$ and $\beta(\ell)$ when the indifference boundary is strictly increasing.

Condition 4 (Increasing Indifference Boundary) *The benefit-based policy s is continuous (equivalently the function φ is continuous). The indifference boundary φ is strictly increasing.*

Corollary 4 *Suppose that a benefit-based policy s satisfies Condition 4 (Increasing Indifference Boundary). The wealth-based policy t defined in (15) and (16) implements an assignment $\alpha(t)$ whose intersection with the assignment $\beta(s)$ implemented by the benefit-based policy s is the set of indifferent consumers $\{(w, \ell) : U(w - s(\ell)) + \ell = U(w)\}$, and whose union with $\beta(s)$ is the set of all consumers $[\underline{w}, \overline{w}] \times \left[\underline{\ell}, \overline{\ell}\right]$.*

We present an example to illustrate the translation from s to t.

Example 2 *Again let U be the logarithmic function. Let a benefit-based policy be quasi-linear in $e^{-\ell}$, $s(\ell) = a + be^{-\ell}$. The indifference boundary is given by $\ln\left(w - a - be^{-\ell}\right) + \ell = \ln w$, or $w = \varphi(\ell) \equiv e^{\ell}\dfrac{a + be^{-\ell}}{e^{\ell} - 1}$. We assume that $w - a - be^{-\ell} > 0$. The derivative of φ is*

$$\frac{d\varphi}{d\ell} = -\frac{(a+b)e^{-\ell}}{(e^{-\ell} - 1)^2}.$$

The boundary is strictly decreasing if and only if $(a + b) > 0$. The inverse of φ is $\ell = \vartheta(w) \equiv \ln \dfrac{w + b}{(w - a)}$. By substituting ℓ in $U(w - t) + \ell = U(w)$ by $\vartheta(w)$ and solving for t, we obtain the wealth-based policy

$$t(w) = \frac{w}{w + b}\,(a + b)\,.$$

3.3 Implementable Assignments

In the previous subsections, the analysis builds upon a given wealth-based or benefit-based policy. In this subsection, we provide a characterization of assignment sets that are implementable by wealth-based policies, benefit-based policies, and both simultaneously. We will see that Conditions 1 and 3 (Decreasing Indifference Boundary) are also necessary for assignment sets that are simultaneously implementable by wealth-based and benefit-based policies.

Let Ω be a nonempty and proper subset of $[\underline{w}, \overline{w}] \times \left[\underline{\ell}, \overline{\ell}\right]$. The set Ω is said to be implementable by a wealth-based policy if there exists $t : [\underline{w}, \overline{w}] \to \mathbb{R}^+$ such that $\Omega = \{(w, \ell) : U(w - t(w)) + \ell \geq U(w)\}$. Similarly, the set Ω is said to be implementable by a benefit-based policy if there exists $s : \left[\underline{\ell}, \overline{\ell}\right] \to \mathbb{R}^+$ such that $\Omega = \{(w, \ell) : U(w - s(\ell)) + \ell \geq U(w)\}$. Finally, the set Ω is said to be simultaneously implementable by a wealth-based policy and a benefit-based policy if there exist $t : [\underline{w}, \overline{w}] \to \mathbb{R}^+$ and $s : \left[\underline{\ell}, \overline{\ell}\right] \to \mathbb{R}^+$ such that $\Omega = \{(w, \ell) : U(w - t(w)) + \ell \geq U(w)\} = \{(w, \ell) : U(w - s(\ell)) + \ell \geq U(w)\}$.

We use the monotonicity properties in (2) to obtain some characterization of implementable sets. Suppose that Ω is implementable by a wealth-based policy. So there is $t(w)$ such that $U(w - t(w)) + \ell \geq U(w)$ for $(w, \ell) \in \Omega$. For any $\ell' > \ell$, we have $U(w - t(w)) + \ell' > U(w)$; if $(w, \ell) \in \Omega$, $(w, \ell') \in \Omega$. Similarly, suppose that Ω is implementable by a benefit-based policy, and $(w, \ell) \in \Omega$, then $(w', \ell) \in \Omega$ whenever $w' > w$

because $U(w - s(\ell)) + \ell \geq U(w)$ implies $U(w' - s(\ell)) + \ell \geq U(w')$. Figure 3 above illustrates a set that is implementable by a wealth-based policy but not by a benefit-based policy.

The following two lemmas characterize assignment sets that are simultaneously implementable by wealth-based and benefit-based policies. Their proofs are in the Appendix.

Lemma 1 *Let Ω be simultaneously implementable by a wealth-based policy and a benefit-based policy. Then Ω is a closed set, and the wealth-based and benefit-based policies that implement Ω are continuous.*

Intuitively, if Ω is implementable by a wealth-based policy, we have $\Omega = \{(w, \ell) : U(w - t(w)) + \ell \geq U(w)\}$, for some t. Now the utility $U(w - t(w)) + \ell$ is continuous in ℓ. So if we consider a sequence $(w, \ell_i) \in \Omega$, and if the limit of ℓ_i is ℓ, then (w, ℓ) must also belong to Ω by continuity. We can repeat the same argument for a similar sequence (w_i, ℓ) when Ω is implementable by a benefit-based policy. It follows that if Ω is simultaneously implementable, any converging sequence (w_i, ℓ_i) must have a limit in Ω if all elements of the sequence belong to Ω.

Lemma 2 *Let Ω be simultaneously implementable by a wealth-based policy and a benefit-based policy. The indifference boundary, ℓ as a function of w or vice versa, defined implicitly by either $U(w - t(w)) + \ell = U(w)$ or $U(w - s(\ell)) + \ell = U(w)$, must be strictly decreasing.*

Intuitively, the utility $U(w - t(w)) + \ell$ is strictly increasing in ℓ when Ω is implementable by a wealth-based policy, and the utility $U(w - s(\ell)) + \ell$ is strictly increasing in w when Ω is implementable by a benefit-based policy. So when Ω is simultaneously implementable, the boundary cannot remain constant over an interval of w or ℓ. So it must be increasing or decreasing over any interval. We already have seen from Figure 8 that wealth-based and subsidy-based schemes that give rise to a strictly increasing boundary never induce the same assignment set. Hence, the indifference boundary must be strictly decreasing if the assignment set is simultaneously implementable.

Lemmas 1 and 2 together establish the following:

Proposition 3 *If Ω is implementable simultaneously by wealth-based policy $t : [\underline{w}, \overline{w}] \to \mathbb{R}^+$ and benefit-*

based policy $s : [\underline{\ell}, \overline{\ell}] \to \mathbb{R}^+$, each of these policies must be continuous, and each must induce a decreasing indifference boundary.

Proposition 3 says that Conditions 1 and 3 are necessary and sufficient for an assignment set to be implemented by both wealth-based and benefit-based policies. Observing either wealth or benefit information is sufficient for the implementation of any assignment set with a decreasing indifference boundary. Conversely, when the indifferent boundary of an assignment set is not decreasing, the implementation of the assignment set requires specific information. For example, if the indifference boundary is U-shaped in w-ℓ space, or like the one in Figure 3, then wealth information is required, but no subsidy scheme based on benefit can implement the assignment.

4 Equivalent Policies and Revenues

Propositions 1 and 2 relate the wealth-based and benefit-based policies that implement the same assignment. Consumers pay different costs according to whether payment is based on wealth or benefit, and are generally not indifferent across these regimes. The required budgets (see (5) and (6)) for each of the two equivalent sets are different because each of the equivalent policies generates a different level of revenue.

Consider three consumer types: (w_1, ℓ_2), (w_2, ℓ_1), and (w_2, ℓ_2), with $w_1 < w_2$ and $\ell_1 < \ell_2$. Suppose that there is a wealth-based policy, $t : [\underline{w}, \overline{w}] \to \mathbb{R}^+$, and it generates a strictly decreasing indifference boundary. Let types (w_1, ℓ_2), (w_2, ℓ_1) be on the indifference boundary, and, therefore, type (w_2, ℓ_2) is in the interior of the assignment set. See Figure 7. Let $s : [\underline{\ell}, \overline{\ell}] \to \mathbb{R}^+$ be the benefit-based policy that implements the same assignment set. By construction, we have $t(w_1) = s(\ell_2)$, and $t(w_2) = s(\ell_1)$. Under the wealth-based policy, consumer type (w_2, ℓ_2) pays $t(w_2)$; under the equivalent benefit-based policy, the same consumer (type (w_2, ℓ_2)) pays $s(\ell_2)$. Unless t and s are constant functions, $t(w_2) \neq t(w_1) = s(\ell_2)$. So any consumer (w_2, ℓ_2) in the interior of the assignment set pays a different amount for the good. The regulator collects different revenues from any type of consumer who is in the interior of an assignment set under equivalent wealth-based and benefit-based policies.

Proposition 4 (Revenue Nonequivalence) *If a wealth-based policy $t : [\underline{w}, \overline{w}] \to \mathbb{R}^+$ and a benefit-based* *policy* $s : [\underline{\ell}, \overline{\ell}] \to \mathbb{R}^+$ *implement the same assignment, they generate different revenues (and therefore require different budgets) for generic distributions of wealth* ($F(w)$) *and benefits* ($G(\ell)$) *except when* $t(w) = s(\ell) = k$, *a constant.*

In general, it is not possible to rank revenues according to only properties of F and G. The reason is that the assignment set is defined with respect to the policies without any reference to these distributions.

An example below illustrates that the difference in collected revenues can be significant.

Example 3 (Revenue Nonequivalence) *Let* w *and* ℓ *be uniformly and independently distributed over the interval* $[0.5, 1.5]$. *Consider the wealth-based policy* $t(w) = 0.1 + 0.3w$. *We obtain the equivalent benefit-based policy* $s(\ell) = \dfrac{0.1\left(e^{\ell} - 1\right)}{0.7e^{\ell} - 1}$. *>From these, we compute the revenue that will be collected under each of these schemes:*

$$R(t) \equiv \int_{\alpha(t)} t(w)\,dF(w)\,dG(\ell) = .393 \quad \text{and} \quad R(s) \equiv \int_{\beta(t)} s(\ell)\,dF(w)\,dG(\ell) = .246.$$

In percentage terms we have:

$$\frac{R(t) - R(s)}{R(s)} = 60\% \quad \text{and} \quad \frac{R(t) - R(s)}{R(t)} = 37\%.$$

Example 3 shows a large difference (in percentage) of revenues collected by the regulator under wealth-based and benefit-based policies. In turn these policies imply a large difference of required budgets to implement the same assignment set.

Proposition 4 has a straightforward policy implication. Suppose that the regulator has to decide which information, wealth or benefits, needs to be collected. If there is any (fixed) cost in information collection in order to implement a policy, the total cost of a policy under an information regime can be computed. This cost is the sum of the required budget and the information collection cost. Cost savings for the implementation of a given assignment set can be achieved by selecting the policy that requires a smaller total cost.

4.1 Equivalent Revenue and General Subsidy

We have assumed up to now that a consumer makes a payment to the regulator only when he purchases the good. Now, we expand the regulator's policies, and allow the regulator to impose a tax or subsidy even when the consumer decides not to purchase. This can be regarded as a general taxation scheme. We have seen that equivalent benefit-based and wealth-based policies (those that implement the same assignment set) may generate different revenues. We show that with general taxation, equivalent benefit-based and wealth-based policies may be so chosen that they generate the same revenue. General taxation is assumed to be feasible and consumers cannot opt out of the system altogether.

We will only consider the case where wealth is known while benefit remains consumers' private information. The case of unknown wealth is similar. A wealth-based policy is now a pair of payment functions $[t_1 : [\underline{w}, \overline{w}] \to \mathbb{R},\ t_2 : [\underline{w}, \overline{w}] \to \mathbb{R}]$; a consumer with wealth w pays $t_1(w)$ when he does not purchase the good, and $t_2(w)$ when he does. We allow the values $t_1(w)$ and $t_2(w)$ to be positive or negative. A type (w, ℓ) consumer decides to purchase the good if

$$U(w - t_2(w)) + \ell \geq U(w - t_1(w)). \tag{17}$$

For a given policy, we define the assignment set analogously:

$$\alpha(t_1, t_2) \equiv \{(w, \ell) : U(w - t_2(w)) + \ell \geq U(w - t_1(w))\}. \tag{18}$$

Again, consider the equation $U(w - t_2(w)) + \ell = U(w - t_1(w))$. This defines a relationship between benefit ℓ and wealth w, and generates the indifference boundary $\ell = \theta(w; t_1, t_2)$.

When $[t_1, t_2]$ satisfies Condition 1 ($[t_1, t_2]$ continuous and $\theta(w; t_1, t_2)$ strictly decreasing in w), Proposition 1 applies. There is a benefit-based policy $[s_1 : [\underline{\ell}, \overline{\ell}] \to \mathbb{R},\ s_2 : [\underline{\ell}, \overline{\ell}] \to \mathbb{R}]$ implementing the same assignment set:

$$\beta(s_1, s_2) \equiv \{(w, \ell) : U(w - s_2(\ell)) + \ell \geq U(w - s_1(\ell))\} = \alpha(t_1, t_2) \tag{19}$$

where $s_1(\ell)$ is the payment by a consumer with benefit ℓ when he does not purchase, and $s_2(\ell)$ is the payment when he does. The construction of the benefit-based policy uses the same procedure: we replace w in $U(w - t_2(w)) + \ell = U(w - t_1(w))$ by the inverse of the (strictly decreasing) indifference boundary θ, say ϕ. So for each ℓ we choose $s_1(\ell)$ and $s_2(\ell)$ to satisfy

$$U(\phi(\ell) - s_2(\ell)) + \ell = U(\phi(\ell) - s_1(\ell)). \tag{20}$$

Clearly many benefit-based policies satisfy (20).

The revenue collected under $[t_1, t_2]$ is

$$R(t_1, t_2) \equiv \int_{\alpha(t_1,t_2)^c} t_1(w)\mathrm{d}F(w)\mathrm{d}G(\ell) + \int_{\alpha(t_1,t_2)} t_2(w)\mathrm{d}F(w)\mathrm{d}G(\ell). \tag{21}$$

The revenue collected under $[s_1, s_2]$ is

$$R(s_1, s_2) \equiv \int_{\beta(s_1,s_2)^c} s_1(\ell)\mathrm{d}F(w)\mathrm{d}G(\ell) + \int_{\beta(s_1,s_2)} s_2(\ell)\mathrm{d}F(w)\mathrm{d}G(\ell). \tag{22}$$

(Here, the superscript c over the sets $\alpha(t_1, t_2)$ and $\beta(s_1, s_2)$ denotes their complements.)

Proposition 5 (Revenue Equivalence) *Suppose that a wealth-based policy* $[t_1 : [\underline{w}, \overline{w}] \to \mathbb{R},\ t_2 : [\underline{w}, \overline{w}] \to \mathbb{R}]$ *satisfies Condition 1 (Decreasing Indifference Boundary). There exists a benefit-based policy* $[s_1 : [\underline{\ell}, \overline{\ell}] \to \mathbb{R},\ s_2 : [\underline{\ell}, \overline{\ell}] \to \mathbb{R}]$ *that implements the same assignment set and generates the same revenue:* $\alpha(t_1, t_2) = \beta(s_1, s_2)$ *and* $R(t_1, t_2) = R(s_1, s_2)$.

The intuition for Proposition 5 (whose proof is in the Appendix) is best illustrated by a two-part tariff policy. Consider a wealth-based policy $[t_1, t_2]$ and the assignment that it implements. Let $[s_1,s_2]$ be the policy that implements the same assignment. There are many such policies. Let $s_1(\ell) = M$, a constant, and $s_2(\ell) = M + s(\ell)$, where M can be regarded as a lump sum tax or subsidy for each consumer, and $s(\ell)$ is the incremental payment for purchasing the good. The revenue that is collected by $[s_1(\ell),s_2(\ell)] = [M, M + s(\ell)]$ is now

$$M + \int_{\beta(M,M+s)} s(\ell)\mathrm{d}F(w)\mathrm{d}G(\ell). \tag{23}$$

The level of M uses up one degree of freedom in the choice of the benefit-based policy $[s_1(\ell),s_2(\ell)]$. For each given value of M, we can set $s(\ell)$ to satisfy

$$U(\phi(\ell) - M - s(\ell)) + \ell = U(\phi(\ell) - M), \tag{24}$$

maintaining the same assignment set. So now we can adjust the level of M to achieve $R(M, M+s) = R(t_1, t_2)$, and the same revenue will be collected.[9]

Proposition 5 still does not say that consumers derive the same utility whether they are subsidized or taxed according to their wealth or benefit. Nevertheless, suppose that the social objective is focused on an

[9] The result in Proposition 5 assumes implicitly that the lump sum payment M is feasible.

assignment set to ensure that a subset of all consumers will be provided with the good. If the assignment is characterized by strictly decreasing boundary, then it is implementable simultaneously by wealth-based and benefit-based policies. Proposition 5 then says that the total subsidy required will be the same for these policies. The deciding factor becomes the relative cost of soliciting the wealth and benefit information.

5 Optimal Subsidy and Indifference Boundary

In this section, we characterize an optimal subsidy policy. Here we adopt a conventional approach, employing a utilitarian social welfare function for the analysis. We consider only the case of unknown benefit, and assume that consumers' wealth information is available to a regulator. A total budget B is to be allocated to consumers. A subsidy policy is a function $t : [\underline{w}, \overline{w}] \to \mathbb{R}^+$. A consumer with wealth w pays $t(w)$ to the regulator for the purchase of the good.

Given a subsidy policy, those consumers with wealth w prefer to purchase the good if and only if their benefits are above a threshold $\hat{\ell}(w)$ given by:

$$U(w - t(w)) + \hat{\ell} = U(w). \tag{25}$$

This threshold $\hat{\ell}(w)$ is the indifference boundary we have discussed in the previous sections. Consumers with benefits above $\hat{\ell}(w)$ purchase the good. Therefore, the mass of consumers with wealth w who purchase is $[1 - G(\hat{\ell})]$. Each of these consumers pays $t(w)$ while the regulator pays the balance $c - t(w)$. Out of the budget B consumers with wealth w use $[1 - G(\hat{\ell})][c - t(w)]$. Therefore, the budget constraint is

$$\int_{\underline{w}}^{\overline{w}} [1 - G(\hat{\ell})][c - t(w)] \, \mathrm{d}F(w) = B. \tag{26}$$

We let the regulator maximize a utilitarian social welfare function. An optimal wealth-based policy is $t(w)$ and the associated $\hat{\ell}(w)$ that maximize

$$\int_{\underline{w}}^{\overline{w}} \left\{ \int_{\underline{\ell}}^{\widehat{\ell}} U(w) \, \mathrm{d}G(\ell) + \int_{\widehat{\ell}}^{\overline{\ell}} [U(w - t(w)) + \ell] \, \mathrm{d}G(\ell) \right\} \mathrm{d}F(w) \tag{27}$$

subject to (25) and (26). In the objective function above, for each wealth level w, consumers with benefit below $\hat{\ell}(w)$ do not buy, and their utility is given by the integral of $U(w)$ from $\underline{\ell}$ to $\widehat{\ell}$, while those consumers with benefit above $\hat{\ell}(w)$ buy and their utility is given by the integral of $U(w - t(w)) + \ell$ from $\widehat{\ell}$ to $\overline{\ell}$.

From the constraint (25), we obtain $U(w-t(w)) = U(w) - \hat{\ell}$, which can then be substituted into the objective function. Furthermore, from constraint (25) $t(w) = w - h(U(w) - \hat{\ell})$, where $h \equiv U^{-1}$, the inverse of U. We substitute for $t(w)$ in (26) to obtain an equivalent constrained optimization program: choose $\widehat{\ell}(w)$ to maximize

$$\int_{\underline{w}}^{\overline{w}} \left\{ U(w) + \int_{\widehat{\ell}}^{\overline{\ell}} [\ell - \widehat{\ell}]\, \mathrm{d}G(\ell) \right\} \mathrm{d}F(w) \tag{28}$$

subject to

$$\int_{\underline{w}}^{\overline{w}} [1 - G(\hat{\ell})][c - w + h(U(w) - \hat{\ell})]\, \mathrm{d}F(w) = B. \tag{29}$$

We use pointwise optimization to characterize the optimal threshold, but will not present the first-order conditions here.

The optimal threshold and subsidy follow the following equation:

$$1 = \lambda \left[\frac{g(\widehat{\ell})}{1 - G(\widehat{\ell})}[c - t(w)] + \frac{1}{U'(w - t(w))} \right] \tag{30}$$

where $\lambda > 0$ is the multiplier for the constraint (29). Because λ is a constant, the total derivative of the term in the square brackets in (30) must be zero. So we can obtain an expression for the derivative of $\widehat{\ell}$ with respect to w, which is the slope of the indifference boundary.[10] This derivative may be positive or negative depending on the value of w. It is quite possible that the optimal wealth-based policy induces an indifference boundary that is nonmonotone. We provide some interpretation of these possibilities now.

Instead of using a policy $t(w)$ as a choice instrument, we can let the regulator decide on a budget allocation rule as a function of wealth. Let this be $B(w)$, the resource allocated to those consumers with wealth w. If the resource $B(w)$ is to be exhausted by these consumers with wealth w, the regulator must consider a threshold and a payment, $\widehat{\ell}$ and t, such that

$$U(w - t) + \hat{\ell} = U(w) \tag{31}$$

$$[1 - G(\hat{\ell})][c - t] = B(w) \tag{32}$$

which are similar to (25) and (26) above. A payment policy is equivalent to a budget allocation rule. We have seen that an optimal policy may give rise to a nonmonotone indifference boundary. This can also be

[10] This result is available from the authors.

understood in terms of the properties of the optimal budget allocation rule.

Assume that $B(w)$ is differentiable. From the implicit function theorem, $\widehat{\ell}$ and t defined in (31) and (32) can be regarded as functions of w. Again, the function $\widehat{\ell}(w)$ is the indifference boundary. After total differentiation, we obtain

$$\frac{\mathrm{d}\widehat{\ell}}{\mathrm{d}w} = -\frac{U'(w-t)B'(w) + [1-G(\widehat{\ell})][U'(w-t) - U'(w)]}{1 - G(\widehat{\ell}) + g(\widehat{\ell})U'(w-t)(c-t)}.$$

The indifference boundary is negatively sloped if $B(w)$ is (weakly) increasing. That is, if the optimal allocation rule sets a budget (weakly) increasing in wealth, then the indifference boundary is strictly decreasing.

Figure 9 illustrates this property. The upward sloping lines are equation (31) for $w = w_1$ and w_2, while the downward sloping lines are equation (32) for $w = w_1$ and w_2. An intersection point represents a pair of $\hat{\ell}$ and t satisfying both (31) and (32). Figure 9 shows two intersection points, $\hat{\ell}_1$ and $\hat{\ell}_2$. An increase in w shifts equation (31) upward because U is concave. If $B(w)$ is increasing in w, then an increase in w shifts equation (32) downward. The intersection $\hat{\ell}_2$ must then moves to the left of $\hat{\ell}_1$. That is, $\hat{\ell}$ is decreasing in w, or the indifference boundary is decreasing. If, however, $B(w)$ is decreasing, the downward sloping equation (32) may shift upward when w increases. In this case, the intersection point may well move to the right of $\hat{\ell}_1$ as w increases, and the indifference boundary may not be decreasing.

Proposition 6 *Suppose that a regulator sets a wealth-based policy $t(w)$, or equivalently, a budget allocation policy $B(w)$ for consumers with wealth w, to maximize a social welfare function. Suppose that the optimal budget allocation policy is increasing in w. Then the optimal wealth-based policy must give rise to a strictly decreasing indifference boundary.*

For a utilitarian social welfare function, an optimal policy $t(w)$ may well be consistent with a decreasing indifference boundary. For some other social welfare functions, the optimal wealth-based policy may also induce a decreasing indifference boundary. Our study on assignment sets and the policies that implement them complements the conventional welfare analysis.

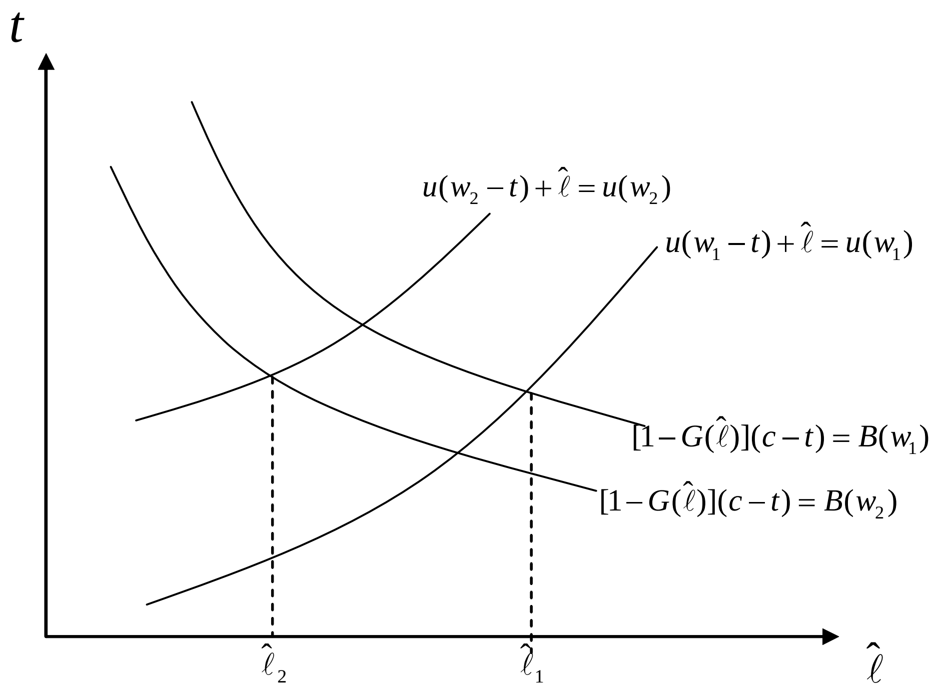

Figure 9: Budget and benefit threshold for consumers with wealth w

6 Conclusion

In this paper, we offer a new perspective on subsidy policies for an indivisible good. Our approach does not impose optimality properties on subsidy schemes, as they may have originated from historical, political, or sociological considerations. Instead, we relate subsidy policies that are based on wealth information to those that are based on benefit, and vice versa. Under a monotonicity condition, we construct equivalent wealth-based and benefit-based schemes that implement the same assignment of the good to consumers. This monotonicity condition is also necessary for any assignment set to be implemented simultaneously by wealth-based and benefit-based schemes. In any setting where consumers' willingness to pay for an indivisible good depends on two dimensions, the translation method can be applied.

We show that equivalent subsidy schemes require different budgets to support the same assignment set unless general taxes or subsidies (such as lump sum or poll taxes) are available. However, when general taxes or subsidies are infeasible, we provide a method to compare the implementation cost of equivalent subsidy schemes. One can also include for comparison the cost of collecting information on wealth and on benefits.

By adding the information collection cost to the implementation cost one obtains the total costs of subsidy schemes for a given assignment set.

Our approach, however, necessarily yields a sort of partial ordering. Since we do not postulate a social welfare objective, we are unable to compare across different assignment sets. Such a comparison must involve a full trade-off analysis through a social welfare function. Moreover, consumers may receive more or less subsidies under wealth-based subsidies compared to benefit-based subsidies.

Focusing on assignment sets rather than the allocation is natural for an indivisible good. Assignment sets appear to be important in practice. Many policies put more emphasis on one particular dimension of the environment, and relegate other dimensions. We implicitly argue that whether a consumer should consume a good is of primary importance, while the tax burden or actual subsidy amount for consumers is of secondary concern to policy makers.

We have taken as exogenous the possible information structures. We have ignored the possibility that consumers can manipulate information before it is made available to the regulator. A benefit-based policy may provide incentives for consumers to invest in effort to change their distributions of benefits. For example, if college scholarships are based on test scores, students may work harder to make their test results more favorable. For the short run, such incentives probably are unimportant, but they may be the dominant factor in the longer run.

The methodology presented here may be applied to different problems. In on-going research, we are working on a model where patients differ in wealth and treatment costs needed to obtain a given benefit level. A wealth-based policy generates an assignment set whose elements are consumer types defined by wealth and cost dimensions. Given a subsidy scheme, one can think about how a private sector supplying the same good at a given price may affect the assignment set. In other words, assignment sets can result from the interaction between public and private sectors. Selection and cream skimming issues can be studied.

Appendix

Proof of Proposition 1: Consider those $\ell \in [\underline{\ell}, \overline{\ell}] \cap [\underline{\ell}', \overline{\ell}']$. By construction, the policy $s(\ell)$ satisfies $U(w - s(\ell)) + \ell = U(w)$. Any type (w, ℓ) consumer is indifferent between purchasing the good at price $t(w)$ and not if and only if he is indifferent at price $s(\ell)$. We show that (w, ℓ) belongs to $\alpha(t)$ if and only if it belongs to $\beta(s)$. Suppose $(w, \ell) \in \alpha(t)$ where $\alpha(t) \equiv \{(w, \ell) : U(w - t(w)) + \ell \geq U(w)\}$. We have

$$U(w - t(w)) + \ell \geq U(w - t(w)) + \theta(w) = U(w), \tag{33}$$

with a strict inequality if $\ell > \theta(w)$. Let $\hat{\ell} = \theta(w) \leq \ell$. By the definition of s, we have $U(w - s(\hat{\ell})) + \hat{\ell} = U(w)$. Because θ is strictly decreasing, for $\ell \geq \hat{\ell}$, there exists $\hat{w} \leq w$ such that

$$U(\hat{w} - s(\ell)) + \ell = U(\hat{w}). \tag{34}$$

But $w \geq \hat{w}$. Hence equation (34) implies

$$U(w - s(\ell)) + \ell \geq U(w),$$

which says that $(w, \ell) \in \beta(s)$. We can use a symmetric argument to show that if (w, ℓ) does not belong to $\alpha(t)$, then it does not belong to $\beta(s)$.

Next consider $\ell \in [\underline{\ell}, \underline{\ell}']$ (if it exists). According to $\alpha(t)$, a type (w, ℓ) consumer does not purchase the good when $\ell < \underline{\ell}'$, for all $w \in [\underline{w}, \overline{w}]$. By construction, $U(\overline{w} - s(\underline{\ell}')) + \underline{\ell}' = U(\overline{w})$. Hence, $U(\overline{w} - s(\underline{\ell}')) + \ell < U(\overline{w})$, and therefore, $U(w - s(\underline{\ell}')) + \ell < U(w)$, all w. A type (w, ℓ) consumer does not purchase the good under $s(\ell)$ either.

Finally, consider $\ell \in [\overline{\ell}', \overline{\ell}]$ (if it exists). According to $\alpha(t)$, a type (w, ℓ) consumer purchases the good when $\ell > \overline{\ell}'$, for all w. By construction, $U(\underline{w} - s(\overline{\ell}')) + \overline{\ell}' = U(\underline{w})$. Hence $U(\underline{w} - s(\overline{\ell}')) + \ell > U(\underline{w})$, and therefore, $U(w - s(\overline{\ell}')) + \ell > U(w)$, all w. A type (w, ℓ) consumer always purchases the good under $s(\ell)$. This concludes the proof that $\alpha(t) = \beta(s)$.

Proof of Lemma 1: Consider a sequence (w_i, ℓ_i) that converges to (w, ℓ), and let $(w_i, \ell_i) \in \Omega$, each i. By assumption $U(w_i - t(w_i)) + \ell_i \geq U(w_i)$. Because ℓ_i converges to ℓ, we have $U(w_i - t(w_i)) + \ell \geq U(w_i)$ and $(w_i, \ell) \in \Omega$, each i. Because Ω is also implementable by a benefit-based policy, we have $U(w_i - s(\ell)) + \ell \geq$

$U(w_i)$. Now by assumption w_i converges to w. So the continuity of U implies that $(w,\ell) \in \Omega$. Since the graph of the boundary of a closed set is continuous, s and t must also be continuous.

Proof of Lemma 2: Let Ω be implementable simultaneously by $t(w)$ and $s(\ell)$. Let (w,ℓ) be a point on the indifference boundary so that $U(w-t(w))+\ell = U(w)$ and $U(w-s(\ell))+\ell = U(w)$. Consider a point (w',ℓ'), where $w' < w$ and $\ell' < \ell$. Any (w,ℓ'), $\ell' < \ell$, does not belong to Ω because $U(w-t(w))+\ell' < U(w)$. Because Ω is implementable simultaneously by t and s, we have $U(w-s(\ell'))+\ell' < U(w)$, which implies $U(w'-s(\ell'))+\ell' < U(w')$ for $w' < w$. Hence $(w',\ell') \notin \Omega$, and does not belong to the indifference boundary. A similar argument establishes that (w',ℓ') does not belong to the indifference boundary when $w' > w$ and $\ell' > \ell$.

We conclude that if (w,ℓ) and (w',ℓ') belong to the indifference boundary, either $w > w'$ and $\ell < \ell'$, or $w < w'$ and $\ell > \ell'$.

Proof of Proposition 5: Let $s_1(\ell)$ and $s_2(\ell)$ satisfy equation (20). For each ℓ choose ϵ and δ such that

$$U(\phi(\ell) - s_2(\ell) + \epsilon) + \ell = U(\phi(\ell) - s_1(\ell) + \delta). \tag{35}$$

By the continuity of U, such ϵ and δ exist. From (35), for each ℓ the values of ϵ and δ must follow:

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}\delta} = \frac{U'(\phi(\ell) - s_2(\ell) + \epsilon(\ell))}{U'(\phi(\ell) - s_1(\ell) + \delta(\ell))} > 0. \tag{36}$$

As ℓ varies over its range, we obtain the functions $\epsilon(\ell)$ and $\delta(\ell)$. Now let the policy be $[s_1(\ell)+\epsilon(\ell), s_2(\ell)+\delta(\ell)]$, which also satisfies (20). The collected revenue under $[s_1(\ell)+\epsilon(\ell), s_2(\ell)+\delta(\ell)]$ is

$$\begin{aligned} R(s_1(\ell)+\epsilon(\ell), s_2(\ell)+\delta(\ell)) &\equiv \int_{\beta(s_1,s_2)^c} [s_1(\ell)+\epsilon(\ell)]\mathrm{d}F(w)\mathrm{d}G(\ell) + \int_{\beta(s_1,s_2)} [s_2(\ell)+\delta(\ell)]\mathrm{d}F(w)\mathrm{d}G(\ell) \\ &= R(s_1,s_2) + \int_{\beta(s_1,s_2)^c} \epsilon(\ell)\mathrm{d}F(w)\mathrm{d}G(\ell) + \int_{\beta(s_1,s_2)} \delta(\ell)\mathrm{d}F(w)\mathrm{d}G(\ell). \end{aligned}$$

For a given policy $[s_1(\ell), s_2(\ell)]$, $R(s_1(\ell)+\epsilon(\ell), s_2(\ell)+\delta(\ell))$ is monotone in ϵ due to (36). So there exist $\epsilon(\ell)$ and $\delta(\ell)$ such that $R(s_1(\ell)+\epsilon(\ell), s_2(\ell)+\delta(\ell)) = R(t_1,t_2)$.

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