Regulating an agent with different beliefs[1]

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Abstract

There is some evidence that people have biased perceptions of risks, like health or environmental risks. Hence their behavior is based on beliefs which may differ from the 'objective' beliefs used by a risk regulator. We set up a general framework to study whether and how this difference in beliefs affects the regulator's policy. It turns out that, in many situations, the regulatory change only depends on the absolute 'distance' between beliefs, and not on whether agents over-estimate or under-estimate risks. We then characterize the cases where the difference in beliefs justify more, or less, public intervention. We also discuss why, in the context of risk regulation, uniform norms may dominate uniform taxes. Finally, we relate our results to various settings, including the question of scientific uncertainty.

1 Introduction

There is a lot of evidence that people misperceive the risks they face.[1] This may be rooted in people's psychology, or due to bounded rationality.[2] This often implies that individuals' beliefs on hazard risks differ from quantitative estimates and scientific evidences (Slovic, 1986, Viscusi, 1998). This paper is interested in the implications of this observation for risk regulatory policies.[3] Specifically: do public risks misperception call for more regulatory intervention? And how does this affect the choice of regulatory instruments?

From a policy viewpoint, the inability of citizens to accurately evaluate the consequences of their own decisions is often a justification for extensive regulatory programs. This justification is actually based on a merit good argument (Musgrave, 1959). For example, education policies or compulsory insurance programs should be developed by the government, on the basis of faulty choices by consumers. The main prediction of merit good theory for public intervention is clearcut. The consumption of merit goods should be subsidized, while demerit goods should be taxed (Besley, 1988). The key theoretical instrument here is taxation. The positive effect of the tax policy

[1] A famous example is the bias in mortality risks perception. Individuals systematically overestimate the rare causes of death such as cataclysmic storms or plane crashes and underestimate more common causes of death like cancers or automobile accidents (Lichtenstein et al., 1978).

[2] Empirical evidence of misperceptions is very well documented since the seminal paper by Tversky and Kahneman (1974). Individuals have difficulties to evaluate small probabilities. Individuals also use heuristics or rules of thumbs that are useful but misleading. For instance, they are subject to 'availability heuristic'. People assess the risks of heart attack by recalling such occurences among one's acquaintances. They 'anchor' their estimates to easily retrievable events in memory such as sensational stories in the medias. People can also manipulate their own beliefs in order to confirm their desired beliefs (see, e.g., Akerlof and Dickens, 1982). This process of beliefs misperceptions can be exacerbated at a collective level by a chain reaction that gives the perception increasing plausibility through its rising availability in public discourse (Kuran and Sustein, 1999).

[3] Our approach does not propose another model to understand how individuals' probability misperceptions affect their own choices. See the developments of Non-Expected Utility models based on systematic distorsions of probabilities, e.g., Starmer, 2000. Our paper is about the impact of the difference in belief on the regulatory policy within an Expected Utility framework based on subjective beliefs. Also, our paper does not examine methods to aggregate heterogeneous beliefs in complete contingent markets. See Varian(1985), Gollier(2003), or Calvet, Grandmont and Lemaire(2003) for an analysis of this question.

is immediate: people consume less demerit goods and consume more merit goods.

Merit good arguments for public intervention are however quite controversial since they rely on a paternalistic view of consumers' preferences. At the root of these criticisms is the fact that the regulator must recognize that the preferences used to determine private choices corresponds to a misrepresentation of well-being. Yet, this criticism has less weight in the present paper, as the only difference in well-being comes from the difference in beliefs.

This paper builds on empirical evidence on risks misperceptions to assume that the regulator knows better what is the risk faced by the public, and has thus some legitimacy to correct faulty individuals' choices.[4] An even less demanding case is when agents do not misperceive risk 'on average', though each agent's beliefs is a noisy version of the true beliefs. Notice that in the context of risk misperceptions the taxation policy remains straightforward. The government must tax the consumption of those goods whose risky consequences are underestimated by consumers, and subsidize the consumption of goods whose risky consequences are overestimated. Such a monotonic pattern between the optimal public taxation as a function of the difference in beliefs has been derived in Sandmo (1983).

Nevertheless public intervention is not limited to the use of taxes or subsidies. Many regulations take the form of direct risk-reduction programs such as safety standards or prevention expenditures; examples abound in areas such as health or food safety. In this paper these instruments are gathered under the term of norms. Norm analysis is more complex than taxation analysis because the consumer response to the new, safer situation may offset the direct beneficial effect.[5] Introducing risk misperceptions then modifies the trade-off between the direct effect and the spillover effect; and, as we shall

[4] A simple interpretation of our model is that the regulator has superior information about the distribution of the risk but, for some reasons, cannot signal this information to the public; Spence (1977) offers an early reference on this problem. For a principal-agent approach, see Wirl (1999). For an economic analysis of the signalling issue in consumption regulation, see the recent paper of Barigozzi and Villeneuve (2002).

[5] Consider for example the consequences of making automobile seatbelt compulsory. The direct effect is that seatbelts reduce the gravity of injury in case of accident; the associated spillover effect is to make drivers driving faster, maybe offsetting the benefits of the safety regulatory policy (Peltzman, 1975). Similarly, Viscusi (1998) found a correlation between child-resistant packages and an increase in accidental poisonings. He postulated that consumers might have become less safety conscious due to the existence of safety caps.

see, it makes more complex the analysis of optimal regulation and raises the question of the superiority of price instruments over direct risk-reduction instruments.

This paper proposes the first general analysis of the impact of public misperceptions on the optimal prevention policy. This analysis is developed within a general framework that can accommodate almost any type of models of paternalistic risk regulation. We obtain comparative statics results about the effect of the difference in beliefs on the norms level. This allows us to answer the question of whether misperceptions support the setting of higher (more stringent) or lower norms by the regulator.

We derive three main messages from our analysis. First, the trade-off between the direct effect and the spillover effect is modified by risk misperceptions; this may call for setting higher or lower norms, depending on a simple condition on consumers' preferences. For example, in the next section we discuss an example in which agents misperceive the risk of drinking tap water. With a demand characterized by a constant elasticity, we show that more stringent norms are called for if water demand is inelastic, an empirically plausible feature. Conversely norms should be made less stringent if demand is elastic; a case-by-case study is thus needed if one applies the same model to other goods. Hence, the differences in risk beliefs certainly do not justify any systematic increase in the stringency of norms, compared to the case when the regulator and the agents share the same beliefs.

Second, and more strikingly, the optimal norm level is monotonic with the *absolute distance* between beliefs, but does not depend on whether the agent is pessimistic or optimistic compared to the regulator. We call this property 'radial monotonicity'. It is in sharp contrast with the monotonic pattern for the optimal taxation policy as a function of the difference in beliefs. Radial monotonicity is a quite general property; we shall characterize all models in which it always calls for more, or for less, stringent norms, whatever the beliefs.

Third, and relatedly, we offer a simple explanation for the prevalence of norms in the domain of risk policies. A tax (or a subsidy) can correct an over-exposure (or under-exposure) to risk if it set at the appropriate level. In the presence of multiple agents with heterogeneous beliefs, the regulator then has to set individualized taxes, a difficult task in practice. It is therefore likely that a uniform tax shall be set at an average level. In particular, if agents do not misperceive the risk *on average*, then the optimal uniform tax will be shown to be close to zero, and thus quite useless. Conversely a norm may be useful, since it can be chosen irrespective of whether agents underestimate or over-estimate the risk. Consequently, our analysis suggests that uniform taxes (or uniform subsidies) may be poor instruments to regulate misperceived risks compared to direct risk prevention programs (uniform 'norms'), when the heterogeneity in beliefs is high enough.

The paper proceeds as follows. The next section builds on a simple example in order to illustrate the above-mentioned policy questions, and to isolate the contribution of the present paper. Section 3 introduces the general framework, with a single agent and norms. Section 4 derives the radial monotonicity property. Section 5 offers necessary and sufficient conditions for the difference in beliefs to increase (or reduce) the optimal norm level. Section 6 introduces the most general and policy relevant problem, that is the analysis of optimal regulation with heterogeneous agents and several regulatory instruments (tax or norms). Section 7 shows that our analysis is conceptually equivalent to the analysis of optimal sequential decision when uncertainty resolves over time, thus solving an important problem formulated in Epstein(1980).

2 Regulation in Happyville

The following problem was introduced by Portney (1992, p. 131). It is called 'Trouble in Happyville':

You are Director of Environmental Protection in Happyville (...). The drinking water supply in Happyville is contaminated by a naturally occurring substance that each and every resident believes may be responsible for the above-average cancer rate observed there (...).

You have asked the top ten risk assessors in the world to test the contaminant for carcinogenicity (...). These risk assessors tell you that while one could never prove that the substance is harmless, they would each stake their professional reputations on it being so.

You have repeatedly and skillfully communicated this to the Happyville citizenry, but because of the deep-seated skepticism of all government officials, they remain completely unconvinced and truly frightened.

The mirror image of Happyville is Blissville (Viscusi, 2000). In Happyville, the risk is low but perceived as large. In Blissville, the risk is large but perceived as low. The question becomes: You are the Director of Environmental Protection both in Happyville and Blissville, where do you allocate your cleanup efforts? To Viscusi, the choice is clear-cut. Efforts have to be spent in Blissville. If efforts are spent in Happyville, this is a 'statistical murder' since lives are sacrificed to focus instead on illusory fears. Viscusi (2000)'s view is probably shared by most economists.[6] This is the point of view we adopt through the paper.

Let us now present more formally our approach. Let the utility of a representative Happyville citizen be

$$U(x, a, b) = u(b) - (1 - a)\delta bx - c(a),$$

where

$u(.)$	is the agent's utility from drinking water,
b	is water consumption,
a	is cleanup effort, $0 \leq a \leq 1$,
δ	is the desutility from getting a cancer,
x	is the unknown dose-response risk of carcinogenicity, $0 \leq x \leq 1$,
$c(a)$	is the cleanup cost function.

Assume simple functional forms

$$\begin{aligned} u(b) &= -(1-b)^2/2 \\ \delta &= 1 \\ c(a) &= a^2/2. \end{aligned}$$

Under expected utility, only the expected value of x matters. The objective expected value used by the regulator is denoted by r. Yet, the agent does not share the same beliefs as the regulator. The agent thinks that this expected value is $s \neq r$. Happyville (resp. Blissville) is then simply characterized by a society where $s > r$ (resp. $s < r$).

For a given cleanup effort a, the agent simply chooses b to maximize

$$E_sU(x, a, b) = -(1 - b)^2/2 - (1 - a)sb - c(a),$$

[6] Yet, the reader may find related theoretical discussions and counter-arguments in Hammond (1981), Sandmo (1983) and Marshall (1988). Clearly the choice is not so clear-cut (Pollack, 1995, 1998). If efforts are spent in Happyville, people who were worried feel protected, and so feel better. Such a view is called the 'populist approach' to risk regulation (Breyer, 1993, Hird 1994).

so that we get

$$b(a, s) = 1 - (1 - a)s.$$

According to the intuition, optimal water consumption b is decreasing in the perceived probability of getting a cancer s and increasing in the level of cleanup efforts a. We arrive now to the question of the optimal cleanup level. Our assumption will be that the objective of the government is to maximize the *ex-post* expected utility of the individual $E_r U$, i.e. the expected utility based on the objective risk r. Nevertheless the regulator must take into account the faulty consumer's response, so that a is chosen to maximize

$$E_r U(x, a, b(a, s)) = -((1 - a)s)^2/2 - (1 - a)r(1 - (1 - a)s) - a^2/2, \quad (1)$$

the solution being

$$a(r, s) = \frac{r - 2rs + s^2}{1 - 2rs + s^2} \in [0, 1]. \quad (2)$$

This framework thus captures a complex channel for why the individual's perception s affects the regulatory choices. This channel is related to the anticipation of the irrational response of individuals, i.e. the response based on s, not on r.

Furthermore, our approach allows us to consider three polar cases. The regulator may select:

$$\begin{array}{rl} a(r, s): & \text{the 'second-best' policy,} \\ a(r, r): & \text{the 'rationalist' policy,} \\ a(s, s): & \text{the 'populist' policy.} \end{array}$$

The 'rationalist' approach is clearly inefficient because it does not anticipate correctly the agent's reactions. The 'populist' approach is intuitively inefficient because it does not make use of the regulator's information. The 'second-best' approach is adopted in this paper, but we briefly discuss the other approaches for completeness. Figure 1 represents cleanup efforts as a function of individual's misperceptions s.

Let us first examine the 'rationalist' regulator decision $a(r, r)$. This decision does not internalize individual's misperceptions, so it is a straight line on the figure. The 'rationalist' decision is insensitive to public beliefs.

Then, turn to the opposite case, the 'populist' decision which is equal to

$$a(s, s) = \frac{s}{1 + s}.$$

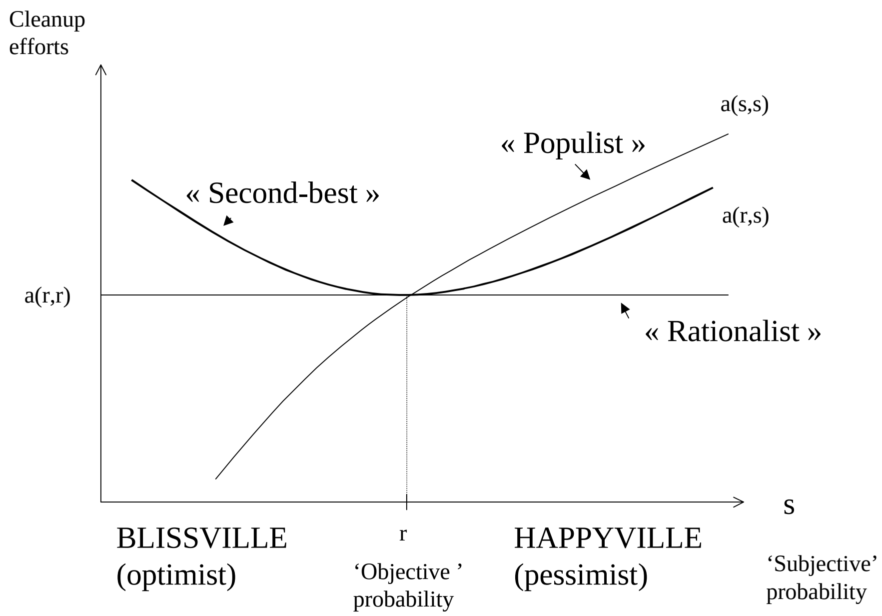

Figure 1:

This decision ignores the objective risk r. In Happyville, i.e. in the city where the perceived risk is large, $s > r$, cleanup efforts are high. In Blissville, where the perceived risk is low, $s < r$, cleanup efforts are low.

Finally, let us turn to the optimal decision, that is the 'second best' decision. From equation (2), decision $a(r, s)$ is decreasing in s then increasing in s. Importantly, this function takes a minimum at $s = r$. Thus we have $a(r, s) \geq a(r, r)$. This shows that the optimal decision is always larger than the 'rationalist' decision. Why is it so ?

In Happyville, individuals are pessimistic and do not consume enough water. Cleanup efforts gives an incentive for the population to consume more water, which is a source of welfare in Happyville where people overestimated the risk. In Blissville, the reason for why cleanup efforts increase is different. People are optimistic and consume too much water. Risk-exposure to cancer is thus too large in Blissville. Hence, cleaning water simply reduces risk-exposure.[7]

[7] This interpretation suggests that this result is model-dependent. We will precisely examine this question in the paper.

Finally, note that the difference between the 'second-best' policy $a(r,s)$ and the 'rationalist' policy $a(r,r)$ increases as the absolute value $|r-s|$ increases. Moreover, they increase exactly at the same rate. Indeed replace s by $r+u$ in (2) to get

$$a(r,r+u) = \frac{r - r^2 + u^2}{1 - r^2 + u^2}$$

so that the value of a is independent of the sign of u. This means that cleanup efforts are the same in Blissville and Happyville.[8] Hence, an important lesson from that example is that the *difference* between the public and the regulator beliefs is more important than the direction of the misperception. In other words, it is not so important for the regulator to know whether he is in Blissville or Happyville. What is important is to know 'how large' is the misperception.

To summarize: because the population's response is 'irrational', regulation may depart strongly from a myopic Cost-Benefit Analysis. Yet, this example has shown that this departure displays a strong 'regularity' property. The regulatory policy depends on the absolute 'distance' between beliefs, not on whether agents over-estimate or under-estimate risks. This raises the question of the effect of different beliefs on regulatory policies in general. This is the question we study in the next two sections.

3 The General Framework with Norms

This Section focuses on the case of a single agent, when regulation consists in setting a norm (these limitations are relaxed in Section 6). Consider the following game. A regulator chooses a norm level a. An agent reacts to a by choosing a decision b. These choices are performed under uncertainty on the true state of nature $x \in X$. The Von Neumann-Morgenstern preferences of the agent are given by the utility function $U(x,a,b)$. Because the regulator is benevolent, he shares the same preferences under certainty.

We assume that x takes a finite number of values. Decision a is a real number, constrained to belong to a closed interval. Decision b is a real vector of finite dimension. Also, U is three-times continuously differentiable with

[8]The decisions are the same in the sense that $a(r,s)$ is symmetric around r. This symmetry is due to the selection of the parameters. For a different set of parameters, the symmetry is lost. The message remains though.

respect to (a, b), and U is strictly concave with respect to b.

Let us endow the agent with beliefs q, defined in the usual manner[9]:

$$\forall\ x \qquad q(x) > 0 \qquad \sum_{x \in X} q(x) = 1.$$

Once a is chosen, the agent chooses b to maximize

$$\sum_{x \in X} q(x) U(x, a, b) \tag{3}$$

whose unique solution is denoted by $b(a, q)$.

Now suppose that the beliefs p used by the regulator differs from the beliefs q used by the agent. Acting as a Stackelberg leader, the regulator should adjust his first-period decision consequently, by maximizing over a

$$\sum_{x \in X} p(x) U(x, a, b(a, q)). \tag{4}$$

Our objective in the following is to study how the solution(s) to this program vary with the agent's beliefs.[10] To do so, we need to introduce a measure for the difference in beliefs. Let us introduce two scalars $r, s \in [0, 1]$. Suppose that the regulator uses the beliefs $(1-r)p+rq$, while the agent uses the beliefs $(1-s)p+sq$. Also, an increase in s makes the latter more distant from the former if $s > r$, and closer otherwise. Hence the absolute value $|s-r|$ is a measure for the absolute value of the beliefs difference. Define the regulator's expected payoff as

$$K(a, r, s) = \sum_{x \in X} [(1-r)p(x) + rq(x)] U(x, a, b(a, (1-s)p + sq)). \tag{5}$$

Notice that these linear forms for the weights appear quite naturally in many cases. For example, suppose that initially agents share the same beliefs p, but an experiment is performed, giving additional information on the

[9] In what follows we could suppose that weights belong to an open convex subset of all possible weights. The assumption that weights are strictly positive plays a role.

[10] Existence of a solution obtains thanks to our regularity assumptions.

true state of nature. Nevertheless, there is an exogenous probability that the experiment has failed, and in that case its results are uninformative. Moreover there is no way to tell whether the experiment has failed or not. If the regulator and the agent do not agree on the probability of failure, their revised beliefs take these linear forms.

In what follows, we shall make s vary in order to capture the impact of the difference in beliefs. We will say that beliefs are more *distant* if $|s - r|$ increases, r being given.

4 The Radial Monotonicity Property

The Happyville example has derived a remarkable property for the optimal norm level : it is monotonic with more distant beliefs; in other words, what matters is the distance, not the direction. We call this property **radial monotonicity**, so as to distinguish it from the usual monotonicity. Radial monotonicity means that the regulator's decision always increase, or always decrease, when s moves away from r no matter the direction. This section shows that this property is in a sense the only regularity property one can obtain; we also give equivalent conditions in terms of the properties of the objective function K. These results are quite general since to derive them we only use basic properties of K. Notice also that in this Section we focus on what happens on a given axis (p, q); only r and s are allowed to vary.

First let us underline that radial monotonicity is at the heart of our framework. This can be illustrated when one studies the expected payoff of the regulator. By definition of K and $b(a, q)$, we have

$$K(a, r, r) = \max_b \sum_x [(1 - r)p(x) + rq(x)]U(x, a, b)$$

so that

$$\forall\ r, s \qquad K(a, r, s) \leq K(a, r, r). \tag{6}$$

This result states that the regulator's expected utility reaches its maximum when the agent and the regulator have the same beliefs, $s = r$. Using only these inequalities and the linearity of K with respect to r, we get[11]

[11] Proofs are given in appendix.

Proposition 1 *The regulator's expected utility decreases with more distant beliefs, i.e.* $K(a, r, s)$ *weakly increases with* s *for* $s < r$*, and weakly decreases with* s *for* $s > r$*.*

Hence the regulator would anyway prefer that the agent's beliefs be closer to his own beliefs. Let us now turn to the effect of the difference in beliefs on the regulator's decision. Recall that the regulator maximizes the value function $K(a, r, s)$ as defined in (5). Hence the properties of the derivative K_a are essential here. In particular, we would like to know how this derivative varies with s. Notice that by linearity of K we have

$$K(a, r, s) = K(a, s, s) + (r - s)K_r(a, r, s).$$

Moreover from the Envelope theorem we know that $K_s(a, s, s) = 0$. Differentiating with respect to s and a we thus get

$$K_{as}(a, r, s) = (r - s)K_{ars}(a, r, s). \tag{7}$$

In words, if for example $K_{ars}(a, r, s) > 0$ an increase in s reduces K_a if $r < s$ and increases K_a if $r > s$; this indicates that a should be reduced by more distant beliefs, and once more this property is the radial monotonicity property.

Finally suppose for example that we have

$$\forall\ a, r, s,\ \ K_a(a, r, s) \leq K_a(a, r, r). \tag{8}$$

Then it is clear that the optimal norm choice is reduced when different beliefs are introduced. Reciprocally, if this condition does not hold, then it is possible to build a counter-example in which a is made higher with different beliefs. Therefore (8) is equivalent to the fact that a difference in beliefs reduces the optimal a. Since (8) is similar to (6), as in Proposition 1 we obtain that a is reduced by more distant beliefs. A bit of algebra then yields the following result :

Proposition 2 *Given* p *and* q*, define the regulator's beliefs as* $(1 - r)p + rq$ *and the agent's beliefs as* $(1 - s)p + sq$*. The three following statements are equivalent:*

i) For any r*, the optimal norm level decreases with more distant beliefs;*
ii) For any a*,* s*,* $K_{as}(a, 0, s) \leq 0$*;*
iii) For any a*,* r*,* s*,* $K_{ars}(a, r, s) \geq 0$*.*

The last two statements are clearly equivalent since from (7) these terms have opposite signs.[12] We included statement ii) because it yields a nice interpretation. Consider a regulator with beliefs p and an agent with beliefs $(1-s)p+sq$. Assume monotonicity of a in one direction: that is, when s increases the optimal norm level is reduced. This is the most simple regularity property we may think of. But since it is equivalent to ii), the Proposition shows that i) must hold. In words, monotonicity in one direction q from a given reference point p implies that radial monotonicity holds on the whole axis defined by p and q. This striking result vindicates our focus on radial monotonicity as the only regularity property one can obtain.

Let us finally give an example of the usefulness of that Proposition.

Example 1 *Water Demand in Happyville*

In the example presented in Section 2, we introduced several parametric assumptions which we now relax. Consider the following model

$$U(x,a,b)=u(b)-g(a)bx-c(a)$$

where u is concave and g is decreasing. Now $b(a,s)$ is defined by

$$u'(b(a,s))=g(a)s,$$

where s stands for the perceived probability of getting cancer. We have

$$K(a,r,s)=u(b(a,s))-g(a)b(a,s)r-c(a),$$

where r is objective risk-probability level. We then compute

$$K_{rs}(a,r,s)=-g(a)\frac{\partial b}{\partial s}(a,s)=-\frac{g(a)^2}{u''(b(a,s))}=-\frac{1}{s^2}\frac{u'^2(b(a,s))}{u''(b(a,s))}.$$

Because water consumption b is increasing with a, K_{ars} will be negative if and only if

$$\frac{u'^2(b)}{u''(b)}$$

[12] Notice also that in iii) K_{ars} does not depend on r, by linearity of K. Also the equivalence still holds if one reverses all the comparisons in the statements; for axample, $K_{as}\geq 0$ is the necessary and sufficient condition for the difference in beliefs to increase the stringency of norms.

is increasing with b. Therefore the impact of more distant beliefs only depends on water on u, and not on g or c. Note that the property holds when u is quadratic, as in Section 2. To go further one may define a demand function $D(\rho)$ by the familiar identity $u'(D(\rho)) = \rho$; ρ is interpreted as the implicit unit price for water. From Proposition 2 this shows that the norm level will be higher due to the difference in beliefs if and only if

$$\rho^2 D'(\rho)$$

is decreasing with ρ. This holds for linear demand functions, as well as for constant elasticity demand functions with an elasticity less than one. Empirical studies strongly support this feature (see e.g. Nauges and Thomas (2000)). Therefore we obtain the general result that in Happyville the optimal norm level should be increasing with more distant beliefs. A crucial point is that this is true in Happyville, but this is true in Blissville as well. Hence, public misperceptions always calls for an higher public intervention here. Yet, when applied to another consumption good, this conclusion may be reversed if the elasticity of demand is more than one.

Interestingly, observe that one would also obtain the radial monotonicity in an even larger class of models. Indeed, through a simple change of variable, any model such as

$$U(x, a, b) = u(b) - g(a, b)x - c(a)$$

may be transformed into

$$V(x, a, B) = v(a, B) - Bx - c(a)$$

where B is the level of exposure to risk chosen by the agent. We directly build the function

$$K(a, r, s) = v(a, B(a, s)) - B(a, s)r - c(a)$$

so that K_{ars} is negative if and only if $\frac{\partial B}{\partial a}$ is increasing with s. In words, this means that the exposure to risk of more pessimistic agents is more sensitive to a change in the safety level. This makes sense, and thus supports the idea that norms should be increased due to misperceptions, from Proposition 2.[13]

[13]On the other hand, the opposite property may hold, in particular if the exposure to risk of pessimistic agents is already low. Once more, a case-by-case study is needed.

The general intuition may be presented as follows. Optimistic agents are over-exposed to risks, and their level of exposure is not very sensitive to the safety level. Therefore increasing the water safety yields a beneficial direct effect, while the spillover effect is weak. Conversely the spillover effect is strong for pessimistic agents. This justifies an increase in safety, even though the direct beneficial effect is negative.

5 Characterization Results

The previous section has identified the necessary and sufficient condition so that more distant beliefs leads to decrease the regulator's decision. This condition reduced to examining the property of the value function $K(a, r, s)$. Notice that K is the value of function of a Stackelberg game, i.e. the value function of the regulator's problem when the agent has different beliefs, s. Analyzing the properties of K is thus quite a technical problem. We solved one simple example which displayed some linearity properties - U linear in x -. This raises a more general question: Is it possible to solve the comparative statics analysis for any problem (that is, a function U and a set X)? In other words, under which condition on the primitives of the model one would obtain the radial monotonicity property?

5.1 Quasi-Independent Problems

Let us first focus on a particular class of problems :

Definition 1 *A problem (U, X) is quasi-independent if and only if there exists a change of variable $B = f(b, a)$ such that, for any q, $f(b(a, q), a)$ is independent from a.*

Here a change of variable is defined as a one-to-one, twice continuously differentiable function.[14] Intuitively a problem is quasi-independent if when a varies the agent's choice of b preserves an invariant quantity $f(b, a)$. Another interpretation obtains when using the reciprocal function f^{-1} of f in b : we get

$$b(a, q) = f^{-1}(B(q), a)$$

[14] It may be local or global, without any change in our results.

so that b depends on q only through a statistic $B(q)$. The constraint is that since the change of variable must be one-to-one, this statistic must have the same dimension as b. This is the case in particular when U is linear in x, as in the Happyville example : indeed consider

$$U(x, a, b) = u(a, b) + v(a, b).x$$

where the dot denotes a scalar product. The agent's choice $b(a, q)$ is characterized by

$$u_b(a, b) + v_b(a, b)E_q x = 0$$

so that $b(a, q)$ depends on q only through the statistic $E_q x$. The problem is then quasi-independent if the change of variable

$$f(a, b) = -[v_b(a, b)]^{-1}u_b(a, b)$$

is well-defined.

Therefore when U is linear the invariant is the marginal rate of substitution between the risk-free part u and the exposure to risk v; for given beliefs the agent always make the same trade-off between these two elements. As we shall see quasi-independent problems are not necessarily linear. The following result characterizes such problems, and offers a sufficient condition for radial monotonicity :

Proposition 3 *The problem (U,X) is past-independent if and only if for any a and b, there exists a vector $d(a, b)$ and a matrix $M(a, b)$ such that*

$$\forall\ x, a, b \qquad U_{ab} + U_{bb}d = MU_b. \tag{9}$$

Moreover, if this property holds, then

$$\forall\ a, b, p \qquad b = b(a, p) \Rightarrow d(a, b) = \frac{\partial b}{\partial a}(a, p).$$

Finally, if $M + d_b$ is positive (resp. negative) semi-definite, then whatever the regulator's beliefs the optimal norm level is reduced (resp. increased) by more distant beliefs.

This characterization can be applied to a variety of problems with a multi-dimensional second-period decision. The second statement in the Proposition makes it easier to find d. The generality of the result is illustrated in the following example:

Example 2 *A Rotten-Kid in a Risky Environment*

Consider a father having to decide the size a of a transfer he shall leave to his son. The son will use the money to invest in some risky projects (such as education, travelling, addictive drugs, ...). Suppose there are N such projects, with net stochastic returns $x = (x_1, .., x_N)$. The problem is that the father and the son do not share the same beliefs over the probability distributions of the risky projects. Moreover the father is altruistic and takes into account the well-being of his child. How does the difference in beliefs affect the transfer a given to the son by the father?[15]

In our framework, the story can be modelled by setting the Von-Neumann Morgenstern father's utility as

$$U(x, a, b) = u(W_0 - a) + kv(a + b.x)$$

where $k > 0$ is the altruistic weight the father puts on the son's utility[16]

$$v(x, a, b).$$

Here b is chosen by the son, and can be interpreted as the level of risk-exposure. Assume that the son is risk-averse, so that his portfolio problem is well-defined. Now we will apply Proposition 3.

Let us first check that the condition (9) is verified if and only if v displays a linear tolerance to risk[17]. Indeed compute

$$U_b = kv'x \quad U_{ab} = kv''x \quad U_{bb} = kv"(xx')$$

(note that (xx') is a matrix) so that (9) requires to find d and M such that

$$v"x + v"(xx')d = v'Mx$$

or equivalently

[15] Compared to the literature on Rotten Kid Theorems introduced by Becker(1974) (see e.g. Bergstrom(1989) for an overview), the new ingredient that will make the Rotten-Kid Theorem to fail in general is of course the risky aspect of the problem.

[16] The child may be altruistic toward his father as well, without any change in the results.

[17] That is, $-v'/v''$ is linear. These functions are called HARA (for Harmonic Absolute Risk Aversion), and include all the usual exponential, power, logarithm and quadratic functions.

$$x + (xx')d + T(a + b.x)Mx = 0$$

where $T(c)$ is the tolerance to risk of v at c. Notice that the following identity holds :

$$(xx')d = (d.x)x$$

so that our equation becomes

$$\forall\ x \qquad [I(1 + d.x) + T(a + b.x)M]x = 0$$

where I is the identity matrix. This implies as announced that T must be linear.

To go further, set $T(c) = y + zc$. $T(c)$ is positive because by assumption the son is risk-averse. Now our equation is verified if one sets

$$d = \frac{z}{y + za}b \qquad M = -\frac{1}{y + za}I.$$

There remains to compute

$$M + d_b = \frac{z - 1}{y + za}I$$

so that we only need to compare z to 1. Therefore we have obtained that the difference in beliefs increases the size of the bequest if and only if $z = T'(c)$ is below one.[18] This comparison is well-known and even almost ubiquitous in risk theory. For an agent with a constant relative risk-aversion σ, it amounts to $\sigma > 1$. The empirical evidence for households tends to indicate a value of σ between 1 and 4. Nevertheless one may argue (in a loosely manner) that sons are on average younger than the typical household and may be less risk-averse. Overall we obtain the following result : if the father thinks the son has a constant relative risk-aversion less than one, he shall reduce the bequest's size due to misperceptions; while the opposite is true if the son has a constant relative risk-aversion above one. (The son should therefore try to convince his father the he is risk-averse).

[18] Hence the logarithmic case $v(c) = \log(c+k)$ is a knife-edged case in which the difference in beliefs has no effect at all on savings.

5.2 When does Radial Monotonicity Hold?

Let us now present our last characterization result. This result justifies our early focus on quasi-independent problems :

Proposition 4 *For any problem (U, X), if whatever the regulator's beliefs radial monotonicity holds, then the problem must be quasi-independent.*

This Proposition allows us to conclude on a generic ambiguity. Indeed quasi-independent problems are non-generic, even locally, as shown in Proposition 3. This means that if one chooses a problem 'at random' it is never quasi-independent, even locally.[19] Our result thus shows that ambiguity is the rule : for given regulator's beliefs p, one may find two different beliefs q_1 and q_2 arbitrarily close to p such that the optimal a is reduced when the agent adopts q_1 and is increased when the agent adopts q_2, compared to the case when the agent shares the regulator's beliefs.[20]

Nevertheless the examples in this paper show that quasi-independent problems include some economically sound cases. For quasi-independent problems, there is no ambiguity locally (as soon as $M + d_b$ is continuous and different from zero). Nevertheless, there may be some subsets of beliefs for which the decision is increased, and some other for which it is reduced; to avoid this one needs to show that $M + d_b$ does not change sign.

6 Regulation in Happyville with Heterogeneous Agents and Taxation

Until now we have focussed on the case of a single agent. Moreover, we have assumed that the regulator can only choose a norm level, represented by the choice of the scalar a. Two important questions thus remain: i) Does the previous analysis extend to a population of agents with heterogeneous beliefs? ii) What would be the optimal regulatory policy if the regulator could set a tax on individual's consumption?

First, observe that the answer to question i) is direct. If the optimal norm level chosen in the single-agent case decreases with more distant beliefs,

[19] The qualifier locally means that beliefs are constrained to lie in a neighbourhood of some reference point.

[20] p, q_1 and q_2 are not necessarily on the same axis; that is, radial monotonicity on each given axis may well hold.

whatever the agent considered, then this must be *a fortiori* true when one considers a population of agents with heterogeneous beliefs. The implicit assumption is that these agents share similar preferences, while their beliefs may be arbitrary; for example, in the Happyville example what is needed is that all agents have a demand elasticity which is less than one. The next Section offers more precise results concerning what happens when beliefs become more heterogeneous.

Question ii) requires to introduce taxation into the model. Observe that it would make no sense to investigate these questions for any problem $U(x, a, b)$, as b does not necessarily represent a consumption level. For simplicity reasons, let us again analyze this issue within the Happyville society.

Consider a population of N agents where agent $i = 1, .., N$ has beliefs s_i and utility u_i from water consumption. Agent i's payoff is

$$U_i(x, a, b_i) = u_i(b) - (1 - a)b_i x - t_i b$$

where t_i is the tax that the regulator set on agent's i water consumption. Agent i's optimal consumption b is then characterized by

$$u_i'(b_i) = (1 - a)s_i + t_i$$

or equivalently $b = D_i((1 - a)s_i + t_i)$, where $D_i(\rho)$ is the demand function of agent i for water sold at an implicit price ρ. Because u is concave, D_i is decreasing with ρ, so that water demand is increasing with a and decreasing with the tax and the pessimism index s.

In an otherwise first-best world, the regulator's program is simply given by

$$\max_{a,(t_i)} \sum_{i=1}^{N} \lambda_i [u_i(D_i((1 - a)s_i + t_i)) - (1 - a)D_i((1 - a)s_i + t_i)r] - c(a), \quad (10)$$

where λ_i represents the proportions of individuals i in the economy. If the regulator uses personalized taxes, then he will clearly choose to correct the impact of erroneous beliefs by setting

$$t_i^* = (1 - a)(r - s_i).$$

Hence the optimal tax is monotonic with the difference in beliefs (but not radially monotonic, a key difference with norms). This monotonicity property of the tax is emphasized in the literature (see, e.g., Belsey, 1988), and corresponds to the idea that a well-chosen tax corrects erroneous beliefs, from the viewpoint of the regulator. As a consequence each agent now behaves as if he shared the same beliefs r as the regulator. This implies that the optimal norm level is not distorted by the heterogeneity in beliefs. We then get the first-best cleanup effort $a(r, r)$: the introduction of personalized taxation has allowed the regulator to restaur efficiency in the Happyville society.

Such a personalized taxation is however difficult to implement in practice. As is well-known, it requires information on both demand functions and beliefs, and it assumes that agents cannot arbitrate by reselling water to other agents. It is therefore necessary to turn to a second-best analysis of the economy, by constraining the regulator to set a uniform tax on water consumption. Taking into account the constraints $t_i = t$, the first-order condition in program (10) now becomes

$$\sum_i^N \lambda_i \frac{\partial D_i}{\partial \rho}(u_i'(D_i) - (1-a)r) = 0$$

and since $u_i'(D_i) = (1-a)s_i + t$ we get

$$t^{**} = (1-a)\frac{\sum_i \lambda_i \frac{\partial D_i}{\partial \rho}(r - s_i)}{\sum_i \lambda_i \frac{\partial D_i}{\partial \rho}}$$

or equivalently

$$t^{**} = \frac{\sum_i \lambda_i \frac{\partial D_i}{\partial \rho} t_i^*}{\sum_i \lambda_i \frac{\partial D_i}{\partial \rho}}.$$

Therefore the optimal second-best tax is an average of the first-best taxes, computed with weights which indicate the sensitivity of individual water demand to price. Now assume that agents do not misperceive the risk on average; that is, some of them are optimistic while others are pessimistic. If moreover there is no correlation between the sensitivity to the price of individual water demand with the degree of optimism or pessimism, the optimal second-best tax must then be set to zero. This shows that uniform norms may dominate uniform taxes, in the domain of risk policies. The effect of the difference in beliefs on the uniform norm has been analyzed in the previous sections.[21]

[21] Previous results show that a similar argument may apply against the use uniform

7 Different Beliefs and Scientific Uncertainty

Increasingly, the Society faces the difficult problem of managing risks whose consequences are imperfectly known. Examples abound: climate change, 'mad cow' disease, electromagnetic fields, hazardous wastes, cellular phones, GMO and the list could go on. These risks pose two main questions for current policy-makers. One question is the question of the optimal regulatory effort to develop today in face of important current scientific uncertainties.[22] The second question is the question of the acceptability of the risk policy given the different belief perception that public forms over the risk he faces. This paper has been concerned with the second question. In this section, we will show that this second question is conceptually similar to the first one. This allows us, in turn, to complete the research program introduced by Epstein (1980).

Let us first posit the following problem. Suppose that a decision-maker with *a priori* beliefs p and Von Neumann-Morgenstern preferences $U(x, a, b)$ sequentially chooses a and b. For some prior beliefs p on x, its objective function when choosing a is

$$j(a, p) \equiv \max_b \sum_x p(x)U(x, a, b). \tag{11}$$

Being a maximum of linear functions, j is convex in p. It is then easy to formalize the arrival of information, after decision a is taken but before decision b. Consider a random variable $\tilde{y}$ whose distribution conditional to x is known. Given the prior beliefs p, a realization y of $\tilde{y}$ makes the decision-maker update his prior p into posterior beliefs q_y. Bayesian updating only requires that

norms. If some agents exhibit an elasticity of demand above one while others have an elasticity of demand below one, then the optimal norm is likely to be an average of the optimal individual norms, and therefore to be close to $a(r, r)$. As above this requires an assumption on the correlation between elasticity of demand and the absolute value of the distance; this also requires that 'on average' the elasticity of demand is equal to one. Such a feature is clearly particular; assuming that agents do not misperceive the risk on average may make more sense as an economic benchmark.

[22] This question classically amounts to analyze how the prospect of increasing information over time affect the optimal regulatory effort today. The most general treatment of this question has been derived in Epstein (1980), and was extended in several papers since (see, e.g., Jones and Ostroy, 1984, Ulph and Ulph, 1997, and Gollier-Jullien-Treich, 2000).

$$p = E_y q_y \tag{12}$$

and the objective function of the decision-maker changes from (11) to

$$E_y \max_b \sum_{x \in X} q_y(x) U(x, a, b) = E_y j(a, q_y). \tag{13}$$

Now, from (12) and the convexity of j, it immediately follows that (13) is above $j(a, p)$: the prospect of information is always beneficial.

Similarly, consider the more general problem in which it is the informativeness of future information which is learnt to be increased. A more precise information is defined as a random variable $\tilde{y}'$, such that any decision-maker prefers $\tilde{y}'$ to $\tilde{y}$. As is well-known, this is equivalent to the requirement that $\tilde{y}$ can be obtained from $\tilde{y}'$ by using a 'garbling machine', which adds a noise uncorrelated with the true state of nature.

Interestingly one manner to obtain a more precise information is to consider that the decision-maker faces a given experiment $\tilde{y}$, and to reduce his confidence in his prior beliefs. Then the decision-maker will revise his beliefs differently, by giving more weight to the new information and less to his prior beliefs. It can be shown that this leads to a new structure of posteriors corresponding to a more precise information (see Jones and Ostroy, 1984)). Hence switching to a more informative experiment may be interpreted as introducing more uncertainty in the decision-maker's prior beliefs.

Another useful result about the comparison of information structures is the following: for any prior p, the distribution of posteriors $q_{y'}$ forms a mean-preserving spread of the distribution of posteriors q_y, in the (multi-dimensional) space of posteriors. In other words, one can consider that the joint distribution of $(x,\ y,\ y')$ is such that

$$\forall\ y \qquad q_y = \sum_{y'} prob(y'|y) q_{y'}.$$

We then get

$$E_y j(a, q_y) = E_y \sum_x q_y(x) U(x, a, b(a, q_y))$$

$$= \sum_{y,y'} Prob(y) prob(y'|y) \left[\sum_x q_{y'}(x) U(x, a, b(a, q_y)) \right]. \tag{14}$$

We also have

$$E_{y'}j(a, q_{y'}) = \sum_{y'} Prob(y') \sum_{x} q_{y'}(x) U(x, a, b(a, q_{y'}))$$

$$= \sum_{y,y'} Prob(y) Prob(y'|y) \left[\sum_{x} q_{y'}(x) U(x, a, b(a, q_{y'})) \right].$$

Comparing this last expression to (14), we see that the change from y' to a less informative y can be decomposed as a weighted sum of changes in the bracketed terms. Each of these changes is similar to a change in the second-period agent's beliefs we have studied until now. Therefore our results apply; in fact, it turns out that there is an equivalence between both classes of problems:

Proposition 5 *The following properties are equivalent:*
i) The optimal norm level decreases with more distant beliefs.
ii) The prospect of more information makes the first-period decision increase.

As explained above, the second statement can be interpreted as saying that the decision-maker should exert more effort today if there is more scientific uncertainty. Together with Proposition 3 this result characterizes the models in which the manner with which scientific uncertainty resolves over time exerts a systematic effect on today's policy. It can be directly applied to the global warming models in Ulph and Ulph (1997) and Gollier-Jullien-Treich (2000) to show that the answer is typically ambiguous : more uncertainty does not necessarily implies that one should make more preventive efforts today, in contradiction with Precautionary Principle. The answer depends on a precise property of preferences. Other applications include for example the impact of scientific uncertainty in the Happyville society. More scientific uncertainty would then (under the conditions derived above) *reduce* the optimal norm level.

The idea here is to trade flexibility and risk in the future. The condition found at the end of Example 1 expresses that the optimal exposure to risk $B(a, s)$ chosen in period 2 when beliefs have been revised into s is such that $\partial B/\partial s$ is increasing with a. In such a case, an increase in a makes the decision-maker more flexible since he can better adapt to new informations. Nevertheless the risk on his water consumption is made higher, and this second effect is stronger here.

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Appendix

Proof of proposition 1: for any $(r,\, s)$, one has from (6)

$$K(a,r,s) \leq K(a,r,r)$$

$$K(a,s,r) \leq K(a,s,s)$$

and by substracting we get

$$K(a,s,s) - K(a,r,s) \geq K(a,s,r) - K(a,r,r).$$

Denote $K_r(a,.,s)$ the slope of K in r (by linearity it is independent from r). We get

$$(s-r)(K_r(a,.,s) - K_r(a,.,r)) \geq 0$$

so that $K_r(a,.,s)$ is weakly increasing with s.

Then choose any $s_1 < s_2$. We have $K_r(a,.,s_2) \geq K_r(a,.,s_1)$, so that $K(a,r,s_2) - K(a,r,s_1)$ is weakly increasing with r. Apply at $r < s_1 < s_2$ to get

$$K(a,r,s_2) - K(a,r,s_1) \leq K(a,s_1,s_2) - K(a,s_1,s_1)$$

and we know that the right-hand side is non-positive from (6). Therefore $K(a,r,s_2) \leq K(a,r,s_1)$, for $r < s_1 < s_2$. This shows the second part of the result. The case $r > s_2 > s_1$ is treated similarly.■

Proof of Proposition 2 : equation (7) shows that $K_{as}(a,0,s)$ and $K_{ars}(a,r,s)$ have opposite signs. Therefore ii) is equivalent to iii). Also we have shown in the text the equivalence of i) to the system of inequalities (8). Notice that (7) implies

$$K_a(a,r,r) - K_a(a,r,s) = \int_s^r K_{as}(a,r,\sigma)d\sigma = \int_s^r (r-\sigma)K_{ars}(a,\sigma,\sigma)d\sigma$$

so that (8) is equivalent to iii). This concludes the proof.■

Proof of Proposition 3 : if the problem is quasi-independent, then there must exist $f^{-1}(B,a)$ and $B(q)$ such that $b(a,q) = f^{-1}(B(q),a)$, for any a and q. Because the agent's program is well-behaved, this is equivalent to

$$\sum q(x)U_b(x,a,f^{-1}(B(q),a)) = 0 \qquad \forall\; a, q.$$

Differentiating with respect to a we get

$$\sum q(x)[U_{ab}(x,a,f^{-1}(B(q),a)) + U_{bb}(x,a,f^{-1}(B(q),a))\frac{\partial f^{-1}}{\partial a}(B(q),a)] = 0$$

or equivalently, for any a, B and q :

$$\sum q(x)U_b(x,a,f^{-1}(B,a)) = 0$$

$$\Rightarrow \sum q(x)[U_{ab}(x,a,f^{-1}(B,a)) + U_{bb}(x,a,f^{-1}(B,a))\frac{\partial f^{-1}}{\partial a}(B,a)] = 0.$$

Define

$$d(a,b) = \frac{\partial f^{-1}}{\partial a}(f(a,b),a).$$

Because f^{-1} is a one-to-one change of variable, we get for any a, b and q

$$\sum q(x)U_b(x,a,b) = 0$$

$$\Rightarrow \sum q(x)[U_{ab}(x,a,b) + U_{bb}(x,a,b)d(a,b)] = 0.$$

This implication must hold for any q. By a well-known property of matrixes, this implies that there exists a matrix $M(a,b)$ such that (9) holds.

Reciprocally, if (9) holds, then summing over x at $b = b(a,p)$ yields

$$\sum_x p(x)(U_{ab} + U_{bb}d) = \sum_x p(x)MU_bd = M\sum_x p(x)U_b = 0.$$

Because by assumption the hessian matrix

$$H(a,p) = \sum_x p(x)U_{bb}(x,a,b(a,p))$$

is negative definite, this implies the second statement in the Proposition. Now choose f such that

$$f_a(a,b) + f_b(a,b)d(a,b) = 0.$$

Then $f(a,b(a,p))$ is indeed independent from p. This shows that the problem is past-independent.

Now we can compute

$$K_{ar}(a,r,s) = \sum[q(x)-p(x)][U_a(x,a,b(a,(1-s)p+sq))$$
$$+U_b(x,a,b(a,(1-s)p+sq))d(a,b(a,(1-s)p+sq))]$$

which depends on s only through $b(a,(1-s)p+sq)$. Using (9) we get

$$K_{ars}(a,r,s) = \sum(q(x)-p(x))[U_{ab}+U_{bb}d+d_bU_b]\frac{\partial}{\partial s}b(a,(1-s)p+sq)$$
$$=[\sum(q(x)-p(x))U_b]'[M+d_b]'\frac{\partial}{\partial s}b(a,(1-s)p+sq).$$

where the prime stands for transposition. And from the first-order condition characterizing $b(a,(1-s)p+sq)$

$$\sum_x[(1-s)p(x)+sq(x)]U_b(x,a,b(a,(1-s)p+sq)) = 0 \qquad (15)$$

we have

$$\frac{\partial}{\partial s}b(a,(1-s)p+sq) = -H^{-1}(\sum(q(x)-p(x))U_b). \qquad (16)$$

Replacing yields

$$K_{ars}(a,r,s) = -[\sum(q(x)-p(x))U_b]'[M+d_b]'H^{-1}(\sum(q(x)-p(x))U_b).$$

This shows the result, from Proposition 2.■

Proof of Proposition 4: compute

$$K_{rs}(a,r,s) = [\sum(q(x)-p(x))U_b(x,a,b(a,(1-s)p+sq))].\frac{\partial}{\partial s}b(a,(1-s)p+sq)$$

which from (16) is equal to

$$-[\sum(q(x)-p(x))U_b(x,a,b(a,(1-s)p+sq))]'H^{-1}(a,(1-s)p+sq)$$
$$\times[\sum(q(x)-p(x))U_b(x,a,b(a,(1-s)p+sq))].$$

Now from the first-order condition (15) we have

$$\sum(q(x)-p(x))U_b(x,a,b(a,(1-s)p+sq)) = -\frac{1}{s}\sum p(x)U_b(x,a,b(a,(1-s)p+sq))$$

so that the expression above is equal to

$$-\frac{1}{s^2}[\sum p(x)U_b(x,a,b(a,(1-s)p+sq))]'$$

$$H^{-1}(a,(1-s)p+sq)[\sum p(x)U_b(x,a,b(a,(1-s)p+sq))].$$

From Proposition 2 radial monotonicity is equivalent to the fact that this expression is monotonic with a, for any p, s, q. This is equivalent to saying that

$$f(a,p,q) \equiv [\sum p(x)U_b(x,a,b(a,q))]'[H(a,q)]^{-1}[\sum p(x)U_b(x,a,b(a,q))]$$

is monotonic with a, for any p and q.

Now suppose that $b(a,p) = b(a,q)$ at some $(a,\ p,\ q)$. Then not only $f(a,p,q) = 0$, but also $f_a(a,p,q) = 0$ because all terms in the derivative vanish. Since by assumption f is monotonic in a, f_a cannot change sign. Then it must be that $f_{aa} = 0$. Computing this second derivative, all terms vanish but

$$[\frac{\partial}{\partial a}\sum p(x)U_b(x,a,b(a,q))]'[H(a,q)]^{-1}[\frac{\partial}{\partial a}\sum p(x)U_b(x,a,b(a,q))].$$

Since H^{-1} is negative definite, this temr can be zero only if

$$\frac{\partial}{\partial a}\sum p(x)U_b(x,a,b(a,q)) = 0. \tag{17}$$

So we have proven that $b(a,p) = b(a,q)$ implies

$$\frac{\partial b}{\partial a}(a,q) = \frac{\partial b}{\partial a}(a,p).$$

We can then define a vector $d(a,b)$ such that

$$b(a,p) = b \Rightarrow \frac{\partial b}{\partial a}(a,p) = d(a,b).$$

This can be rewritten

$$\sum p(x)U_b = 0 \Rightarrow \sum_x p(x)[U_{ab} + U_{bb}d] = 0.$$

Because this must hold for all p, we get the existence of a matrix M such that (9) in Proposition 3 holds. This concludes the proof. ■

Proof of Proposition 5 : Epstein(1980) shows that ii) is equivalent to the derivative $j_a(a,p)$ being convex in p. For p and q given, define

$$J(a,r) \equiv j(a,(1-r)p + rq).$$

Notice that Epstein's condition is equivalent to J_a being convex in r. Notice also that

$$J(a,r) = K(a,r,r)$$

so that

$$J_r(a,r) = K_r(a,r,r) + K_s(a,r,r) = K_r(a,r,r)$$

since $K_s(a,r,r) = 0$ from the Envelope theorem. Since K is linear with r, we get

$$J_{rr}(a,r) = K_{rs}(a,r,r)$$

so that J_{arr} and K_{ars} share the same sign. This concludes the proof, from Proposition 2.■