Challenges in the Measurement of Long Run Risk

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Recursive utility and risk

Under a Cobb-Douglas specification ($\rho = 1$), CES recursion is

$$V_t = (C_t)^{(1-\beta)} \mathcal{R}_t(V_{t+1})^{\beta}.$$

Stochastic discount factor:

$$s_{t+1,t} \equiv \log S_{t+1,t} = -\delta - \gamma(L)w_{t+1} - \mu_c + (1-\alpha)\gamma(\beta)w_{t+1} - \frac{(1-\alpha)^2\gamma(\beta)\cdot\gamma(\beta)}{2}$$

where $\beta = \exp(-\delta)$ and

$$\gamma(z) = \sum_{j=0}^{\infty} \gamma_j z^j$$

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where $\beta = \exp(-\delta)$ and

$$\gamma(z) = \sum_{j=0}^{\infty} \gamma_j z^j$$

- $\gamma(0) = \gamma_0$ - Lucas (1978)-Breeden (1979) adjustment for risk
- $(\alpha - 1)\gamma(\beta)$ - Kreps and Porteus (1978) recursive utility adjustment for risk - featured by Bansal and Yaron (2004)
- remainder contributes to the risk free rate

Measured Consumption Dynamics

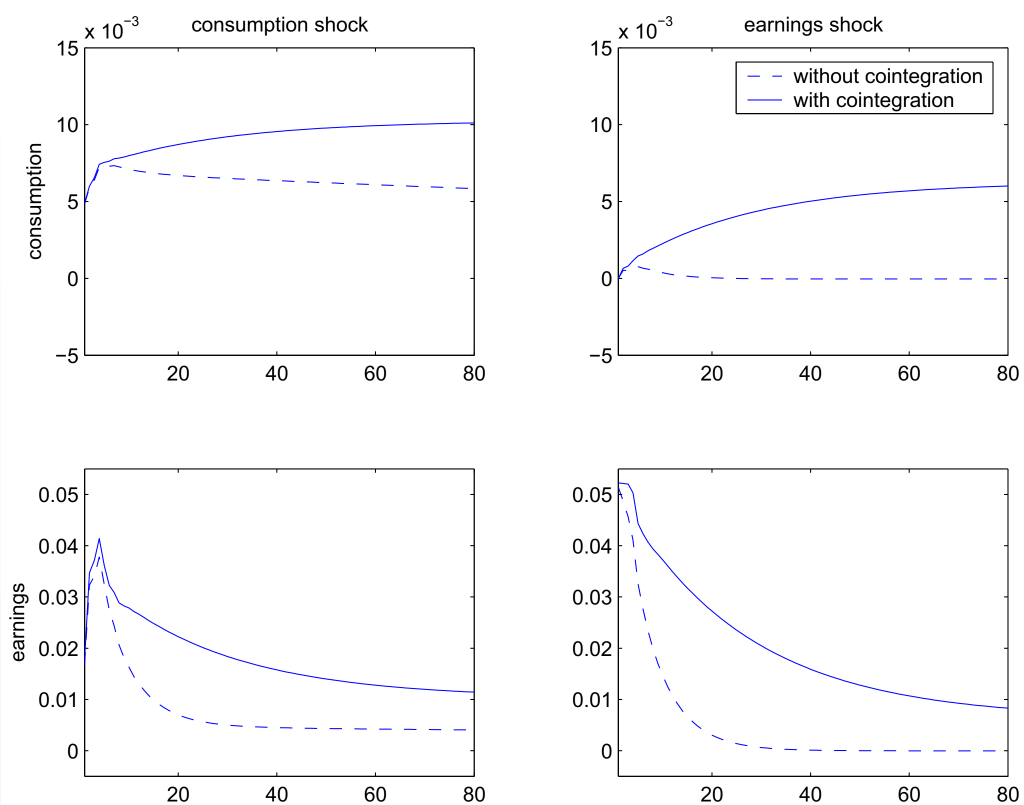
$$A_0 y_t + A_1 y_{t-1} + A_2 y_{t-2} + ... + A_\ell y_{t-\ell} + B_0 = w_t \,,$$

- Shock vector w_t has mean zero and covariance matrix I and normally distributed. A_0 lower triangular with positive entries on the diagonals.

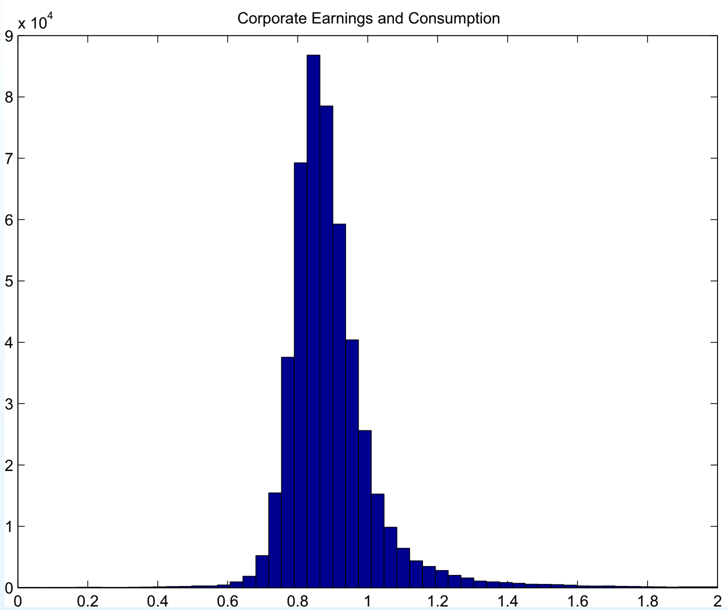
$$A(z) \doteq A_0 + A_1 z + A_2 z^2 + ... + A_\ell z^\ell.$$

  where $A(z)$ is nonsingular for $|z| < 1$.
- Log consumption is the first coordinate of y_t and corporate earnings is the second (used as an investor information variable);
- Cointegration: $A(1) = a_1 \begin{bmatrix} 1 & -\lambda \end{bmatrix}$. $\lambda = 1$ balanced growth restriction.

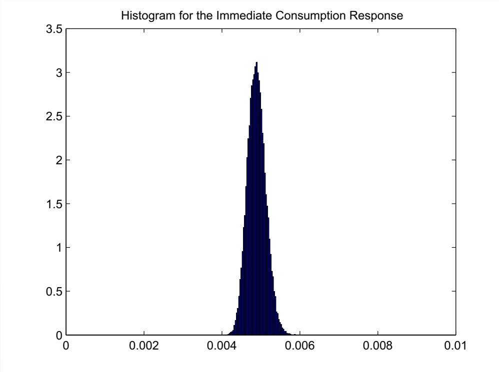
Consumption-Earnings VAR

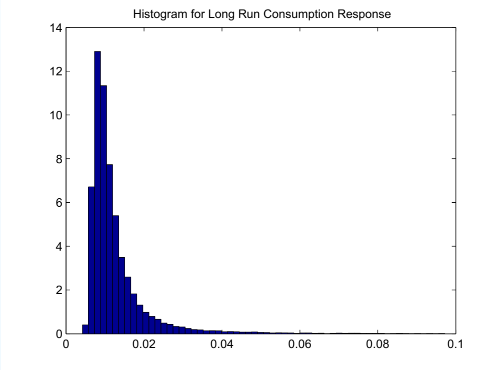


Approximate Probabilities for λ



Statistical Accuracy of Responses





Three operators

1. One-period valuation-growth operator:

$$\mathcal{P}\psi(x) = E\left[\exp\left(s_{t+1,t} + \zeta + M_{t+1} - M_t\right)\psi(x_{t+1})|x_t = x\right].$$

   $\{M_t\}$ is the martingale component for cash flow.

2. One-period growth operator (abstracts from valuation)

$$\mathcal{G}\psi(x) = E\left[\exp\left(\zeta + M_{t+1} - M_t\right)\psi(x_{t+1})|x_t = x\right].$$

3. One-period valuation operator (abstracts from growth)

$$\mathcal{P}^f\psi(x) = E\left[\exp\left(s_{t+1,t}\right)\psi(x_{t+1})|x_t = x\right].$$

Long run value accounting

Three eigenvalue problems:

1. Solve $\mathcal{P}\phi^* = \exp(-\nu)\phi^*$.
   ν asymptotic rate of decay in value.
2. $\mathcal{G}\phi^+ = \exp(\epsilon)\phi^+$.
   ϵ asymptotic growth rate of cash flow.
   $\nu + \epsilon$ is an expected rate of return.
3. Solve $\mathcal{P}^f\phi_f = \exp(-\nu_f)\phi_f$.
   $\nu + \epsilon - \nu_f$ expected excess rate of return.

Change martingales trace out long run risk-return relation.

Example Economy

Cash flow growth:

$$d_t - d_{t-1} = \mu_d + \iota(L) w_t$$

where $\iota(z) = \sum_{j=0}^{\infty} \iota_j z^j$.

- Construct martingale M_t to have increment $\iota(1) w_t$.
- Deterministic growth rate is μ_d.
- Growth eigenvalue is $\epsilon = \mu_d + \frac{1}{2}|\iota(1)|^2$

Stochastic Discount Factor

$$s_{t+1,t} = -\delta - \gamma(L)w_{t+1} - \mu_c + (1-\alpha)\gamma(\beta)w_{t+1} - \frac{(1-\alpha)^2\gamma(\beta)\cdot\gamma(\beta)}{2}$$

Dominant eigenvalue for valuation:

$$\nu = \delta - \mu_d - \frac{1}{2}|-\gamma(1) + (1-\alpha)\gamma(\beta) + \iota(1)|^2$$

Rate of return:

$$\nu + \epsilon = \delta - \frac{1}{2}|-\gamma(1) + (1-\alpha)\gamma(\beta)|^2 + [\gamma(1) + (\alpha - 1)\gamma(\beta)]\cdot\iota(1)$$

Excess return:

$$[\gamma(1) + (\alpha - 1)\gamma(\beta)]\cdot\iota(1)$$

Robustness and Intertemporal Substitution

- Robustness without hidden states $\theta = \frac{\beta}{\alpha - 1}$. Tallarini (1998)
- Intertemporal substitution - compute derivatives of dominant eigenvalue with respect to ρ.

Return Properties of Portfolios

	Portfolio					
	1	2	3	4	5	Market
Avg. Return (%)	6.48	6.88	8.90	9.32	11.02	7.23
Std. Return (%)	18.8	16.4	14.8	15.8	17.8	16.5
Avg. B/M	0.32	0.62	0.84	1.12	2.00	0.79
Sharpe Ratio	0.18	0.20	0.28	0.28	0.30	0.21
Correlation with Consumption	0.20	0.18	0.20	0.20	0.21	0.20

Implied Cash Flows

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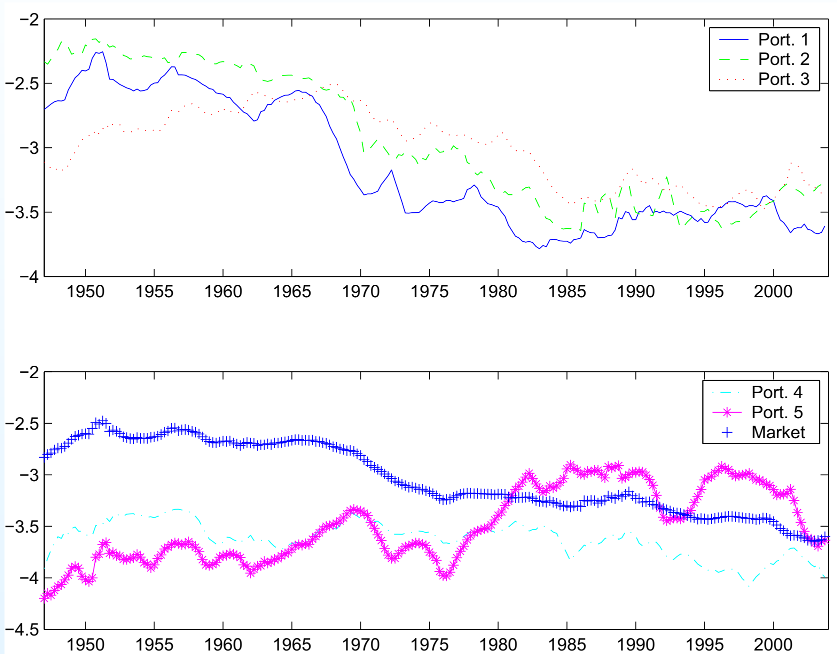


Tail Returns

$\alpha = 1$				
Portfolio	Return	Excess Return	Return Derivative	Excess Return Derivative
1	6.48	-.06	3.51	.00
5	6.78	.22	3.53	.02
market	6.60	.06	3.52	.00
long bond	6.54	.00	3.51	.00

$\alpha = 5$				
Portfolio	Return	Excess Return	Return Derivative	Excess Return Derivative
1	6.11	-.27	3.48	.03
5	7.43	1.03	3.35	-.10
market	6.69	.30	3.42	-.03
long bond	6.39	.00	3.45	.00

Tail Returns

$\alpha = 10$

Portfolio	Return	Excess Return	Return Derivative	Excess Return Derivative
1	5.65	-0.54	3.44	.07
5	8.25	2.05	3.12	-.25
market	6.79	0.60	3.29	-.08
long bond	6.19	0.00	3.37	.00

$\alpha = 20$

Portfolio	Return	Excess Return	Return Derivative	Excess Return Derivative
1	4.74	-1.72	3.34	.14
5	9.90	4.09	2.70	-.50
market	7.00	1.19	3.06	-.15
long bond	5.81	0.00	3.21	.00

VAR Growth Models

We append a dividend equation

$$A_0^* y_t^* + A_1^* y_{t-1} + A_2^* y_{t-2} + \ldots + A_\ell^* y_{t-\ell} + B_0^* + B_1^* t = w_t^* ,$$

to VAR. The vector of inputs is

$$y_t^* \doteq \begin{bmatrix} y_t \\ d_t \end{bmatrix} = \begin{bmatrix} c_t \\ e_t \\ d_t \end{bmatrix}$$

and the shock w_t^* is scalar with mean zero and unit variance.

Three Alternative Models

1. Cointegration: $B_1^* = 0$ suppose the vector $A^*(1)$ is orthogonal to $\begin{bmatrix} 1 & 1 & \lambda^* \end{bmatrix}$ where

$$A^*(z) \doteq \sum_{j=0}^{\ell} A_j^* z^j.$$

   For this model, the vector π is

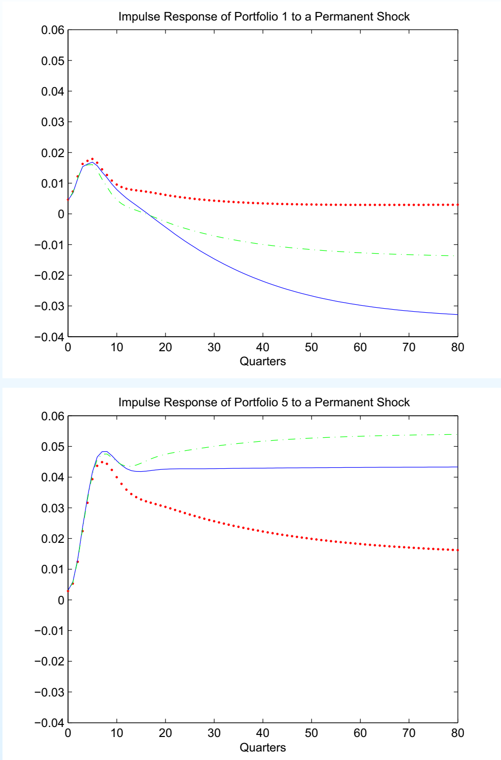
$$\pi = \iota(1) = \lambda^* \gamma(1)$$

   and $\zeta = \mu_d = \lambda^* \mu_c$.

2. Time trend: $B_1^* \neq 0$ but same rank restriction. Leaves μ_d unrestricted.

3. Unit root: $B_1^* = 0$ and $A^*(1) = \begin{bmatrix} a_1^* & -a_1^* & 0 \end{bmatrix}$. New dividend shock is permanent.

Estimated cash flow responses to macro shocks



Long Run Responses

	No Time Trends				
			Quantile		
Portfolio	.05	.25	.5	.75	.95
1	-1.35	0.08	0.35	0.53	0.83
2	0.18	0.47	0.66	0.98	2.46
3	0.54	0.69	0.78	0.88	1.09
4	0.55	0.65	0.70	0.77	0.89
5	0.65	1.21	1.44	1.71	3.00
market	0.36	0.44	0.49	0.53	0.61

Long Run Responses

Time Trend Included

Portfolio	Quantile .05	.25	.5	.75	.95
1	-6.24	-4.26	-3.42	-2.72	-1.77
2	-16.47	-7.28	-4.70	-3.15	-1.52
3	-0.64	1.21	2.09	2.84	3.87
4	1.10	1.70	2.07	2.44	3.05
5	-1.66	1.66	3.87	7.32	21.18
market	-1.60	-0.43	0.10	0.53	1.12

Long Run Responses

Dividend Growth Rate					
	Quantile				
Portfolio	.05	.25	.5	.75	.95
1	-4.84	-2.61	-1.35	-0.20	1.60
2	-2.13	-0.68	0.18	1.03	2.43
3	0.34	1.82	2.97	4.32	7.05
4	1.82	3.05	3.97	5.01	6.94
5	1.42	3.64	5.18	6.87	9.93
market	-0.20	0.84	1.49	2.20	3.49

Heteroskedasticity

Consider an environment with stationary linear dynamics:

$$x_{t+1} = Gx_t + Hw_{t+1}$$

where w_{t+1} is a multivariate normal with mean zero and covariance matrix I. Introduce a stochastic discount factor of the form $\exp(s_{t+1})$ where

$$s_{t+1,t} = \frac{1}{2} w_{t+1}{}' \Theta_0 w_{t+1} + x_t{}' \Theta_1 w_{t+1} + \frac{1}{2} x_t{}' \Theta_2 x_t + \Lambda_0 w_{t+1} + \Lambda_1 x_t + \lambda.$$

Find a dominant eigenfunction of the form:

$$\phi(x) = \exp\left(-\frac{1}{2} x' P^* x - Q^* x\right)$$

Lettau-Wachter consider heteroskedastic cash flows and constant interest rates. Eigenfunction is the exponential of a linear function of the state. Selects volatility contribution to risk premia as a source of long run risk

Why Heteroskedasticity?

Let

$$c_{t+1} - c_t = \mu_c + U_c x_t + x_t' V_c w_{t+1} + \gamma_0 w_{t+1}.$$

Consider again the recursive utility model:

$$v_t = \frac{\beta}{1-\alpha} \log E\left(\exp\left[(1-\alpha)(v_{t+1} + c_{t+1} - c_t)\right] | \mathcal{F}_t\right)$$

- Value function

$$v^* = -\frac{1}{2} x^{*\prime} \Omega^* x^* + \Delta^* x^* + \omega^*$$

- Worst case model:

$$m_{t+1} = \frac{\exp\left(-\frac{v_{t+1}+c_{t+1}-c_t}{\theta}\right)}{E\left[\exp\left(-\frac{v_{t+1}+c_{t+1}-c_t}{\theta}\right) |x_t\right]}$$

Alternative Formulations of Consumption Volatility

- Tauchen adopts an alternative continuous state specification for consumption volatility.
- Lettau, Ludvigson and Wachter adopt a regime shift formulation.
- Robustness gives a different way to assign probabilities for asset pricing.

Hidden State Formulation

Let log consumption evolve as:

$$c_{t+1} - c_t = Gx_t + Hw_{t+1}$$

where

$$x_{t+1} = Ax_t + Cw_{t+1}$$

and

$$x_t = \begin{bmatrix} y_t \\ z_t \end{bmatrix}.$$

Examples:

1. hidden predictor variables for log consumption
2. long run growth rate is estimated (constant hidden state)

Basic Idea

Two robustness recursions:

1. Allow for misspecified dynamics as before using m_{t+1} conditioned on x_t.
2. Allow for robustness in the probabilities used for averaging over the hidden states as in prior sensitivity analysis in Bayesian statistics and econometrics - perturb normal distributions for hidden states

Observations

1. hidden states can be time invariant - consumption growth rate
2. exploit tools for solving hidden state Markov chain models - Kalman filter
3. minimizing agent has an informational advantage - observe z_t
4. both ”games” easy to solve

Multiple Models

1. Model 1

$$y_{t+1} = A_1 y_t + C_1 w_{t+1}$$

2. Model 2

$$y_{t+1} = A_2 y_t + C_2 w_{t+1}$$

Observations

- Hidden state is a model indicator.
- Make robust adjustments to posterior probabilities used to mix models in addition to a robustness adjustment for each model

Technology Processes with Hidden Growth States

- Technology shock evolves as:

$$y_{t+1} - y_t = \kappa z_t + \sigma_y w_{t+1}$$

  where $\{z_t\}$ is a hidden growth regime process that evolves as a Markov chain.
- Given adjustment costs in investment and lack of perfect knowledge of the technology growth state, consumption response will be sluggish
- Make robust adjustments to the shock process conditioned on z_t and robust adjustments to the probabilities assigned to hidden states.
- Additional sources of ”risk premia”.