Equilibrium Valuation with Growth Rate Uncertainty

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Portfolio dividends relative to consumption

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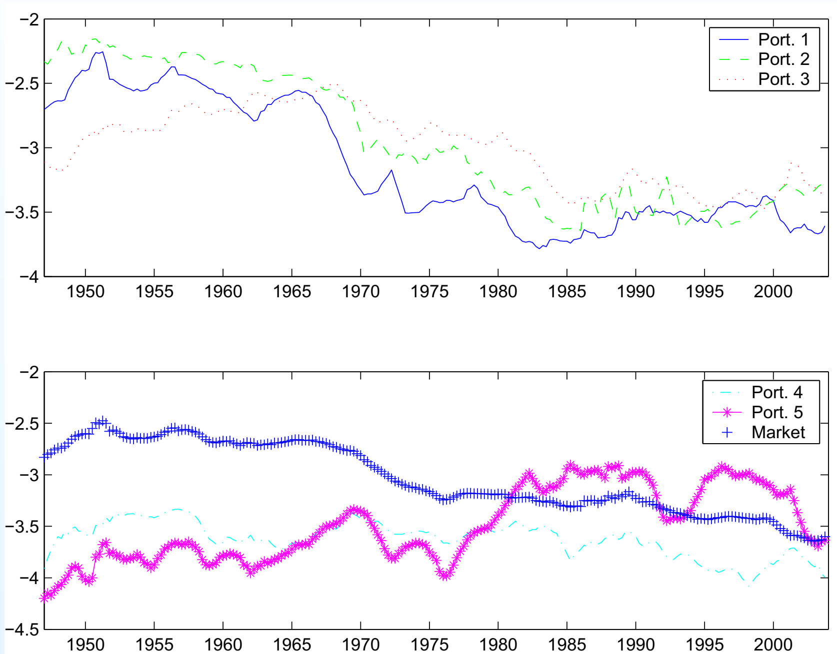


Dominant Eigenvalues: Matrix digression

Growth state of the economy at time t is d $x_t \equiv C_t/C_{t-1}$, where $x_t \in \{\mathsf{x}_n : n = 1, 2, \ldots, N\}$. The probabilities of transiting from one state to another are given by:

$$a_{m,n} = \text{Prob}(x_{t+1} = \mathsf{x}_n | x_t = \mathsf{x}_m)$$

One-period stochastic discount factor is assumed to be

$$S_{t+1,t} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\alpha} = \beta (x_{t+1})^{-\alpha} \ .$$

Consider a cash flow growth process:

$$D_t^* = (C_t)^\lambda$$

Construct the matrix P where the (m, n) entry of this matrix is given by:

$$p_{m,n} = \beta a_{m,n} (\mathsf{x}_n)^{\lambda - \alpha} \ .$$

Cash flow valuation

Let D_t equal:

$$D_t = (C_t)^\lambda \phi(x_t)$$

and write $\phi(x_t)$ as an N dimensional vector Φ.

- $P\Phi$ is the vector of date t prices of a payoff D_{t+1} multiplied by $D_t^* = (C_t)^\lambda$.
- The date t value of an infinite cash flow is:

$$\sum_{j=0}^{\infty} P^j \Phi$$

multiplied by D_t^*.

Long run value dictated by the behavior of P^j.

Dominant eigenvalue

Suppose that the matrix P has distinct eigenvalues, and write the eigenvalue decomposition as:

$$P = T\Delta T^{-1}$$

where Δ is a diagonal matrix of eigenvalues. Then

$$(P)^j = T\Delta^j T^{-1}.$$

Typically one eigenvalue will be positive with an associated positive eigenvector. Largest (in absolute value) eigenvalue. Write as $\exp(-\nu)$. Then

$$\lim_{j\to\infty} \exp(j\nu)(P^j)\Phi.$$

proportional to the dominant eigenvector, but not on Φ.

Value Decomposition

Suppose that Φ is strictly positive.

$$-\frac{1}{j}\log(P^j)\Phi$$

is the yield on a j period security with cash flow D_{t+j} once we adjust initial payout for the initial payout $\frac{1}{j}D_t$.

Observations

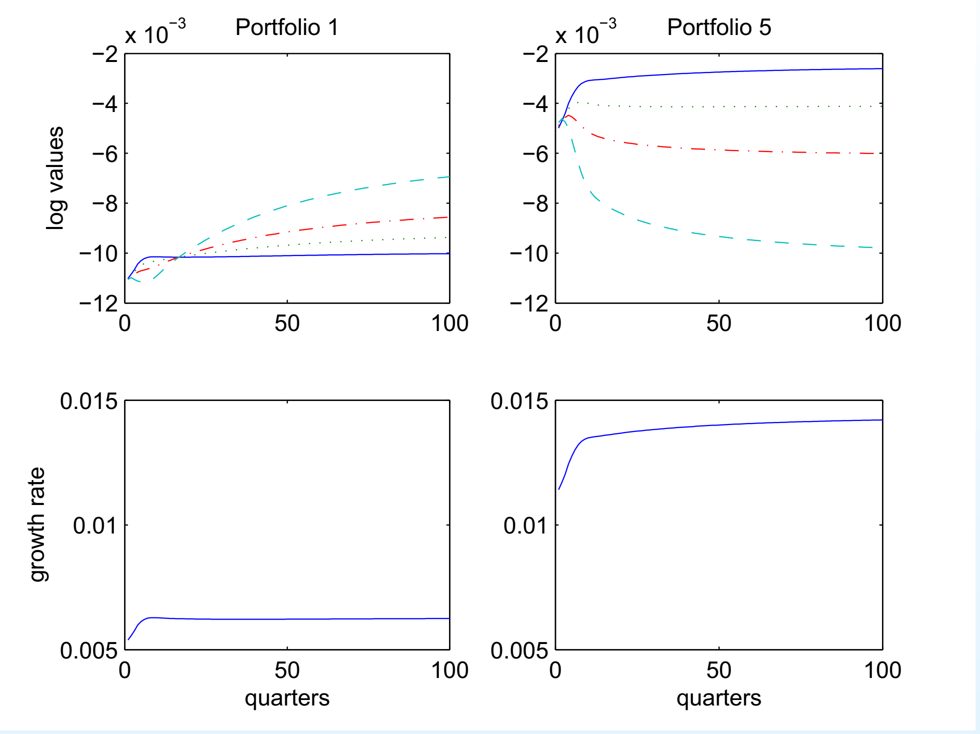
1. Decomposition of value

$$\sum_{j=0}^{\infty} P^j \Phi$$

by horizon.

2. $\lim_{j\to\infty} -\frac{1}{j}\log(P^j)\Phi = \nu 1_N$

Illustration



Avoid Markov chain approximation

Introduce valuation operators that map functions into functions.
Steps:

1. Construct a reference growth process - statistical decomposition
2. Build a family of valuation operators - economic model
3. Build a family of growth operators
4. Compute dominant eigenvalue and function
5. Construct tail returns, expected returns and expected excess returns after adjusting for expected growth

Abstract operator formulation

Ingredients:

1. $\{x_t\}$ be a stationary Markov process.
2. $s_{t+1,t}$ an economic model of a stochastic discount factor. Price of a payoff $\psi(x_{t+1}$ is:

$$E\left[\exp(s_{t+1,t})\psi(x_{t+1})|x_t\right]$$

   and depends only on x_t. Markov pricing.
3. Martingale M_t with increments that depend on x_t and shocks that influence the evolution of x_t used to build a stochastic growth process:

$$\exp(M_t + \zeta t)$$

Three operators

1. One-period valuation-growth operator:

$$\mathcal{P}\psi(x) = E\left[\exp\left(s_{t+1,t} + \zeta + M_{t+1} - M_t\right)\psi(x_{t+1})|x_t = x\right].$$

2. One-period growth operator (abstracts from valuation)

$$\mathcal{G}\psi(x) = E\left[\exp\left(\zeta + M_{t+1} - M_t\right)\psi(x_{t+1})|x_t = x\right].$$

3. One-period valuation operator (abstracts from growth)

$$\mathcal{P}^f\psi(x) = E\left[\exp\left(s_{t+1,t}\right)\psi(x_{t+1})|x_t = x\right].$$

Long run value accounting

Three eigenvalue problems:

1. Solve $\mathcal{P}\phi^* = \exp(-\nu)\phi^*$.
   ν asymptotic rate of decay in value.
2. $\mathcal{G}\phi^+ = \exp(\epsilon)\phi^+$.
   ϵ asymptotic growth rate of cash flow.
   $\nu + \epsilon$ is an expected rate of return.
3. Solve $\mathcal{P}^f\phi_f = \exp(-\nu_f)\phi_f$.
   $\nu + \epsilon - \nu_f$ expected excess rate of return.

Change martingales trace out long run risk-return relation.

Dominant Eigenfunctions

Use the dominant eigenfunction to construct a valuation process and the corresponding return. A valuation process $\{J_t : t = 1, 2, ...\}$ is one for which the date t price of the security with liquidation value J_{t+1} is J_t. Form:

$$J_{t+1} = \exp\left[(\nu + \zeta)(t+1) + M_{t+1} - M_0\right] \phi^*(x_{t+1}).$$

Since ϕ is a positive eigenfunction, the date t value of the payoff J_{t+1} is indeed J_t, verifying that J_{t+1} is indeed a valuation process.

Observations

The k-period return is:

$$R_{t+k}^{k} = \frac{J_{t+k}}{J_t} = \exp\left[(\nu + \zeta)k + M_{t+k} - M_t\right] \frac{\phi^*(x_{t+k})}{\phi^*(x_t)}$$

1. Riskiness of constructed one-period return (k=1) depends on riskiness of the stochastic component to the growth process M_{t+1} and of the logarithm of $\phi^*(x_{t+1})$.
2. Take expectations and logarithms:

$$\lim_{k\to\infty} \frac{1}{k} \log E\left(R_{t+k}^{k} | \mathcal{X}_t\right) = \nu + \epsilon$$

Constructed equity

Build a security with same returns by valuing equity with a dividend

$$\hat{D}_{t+1} = \exp\left[\zeta(t+1) + M_{t+1} - M_0\right] \phi^*(x_{t+1}).$$

- Using the eigenvalue property the price dividend ratio is given by:

$$\frac{\hat{P}_t}{\hat{D}_t} = \frac{\exp(-\nu)}{1 - \exp(-\nu)}$$

  Includes both a pure discount factor (adjusted for risk) and a dividend growth factor.
- The implied discount rate is $\nu + \epsilon$ since the asymptotic dividend growth factor for dividends is: $\exp(\epsilon)$.

Cash flow growth

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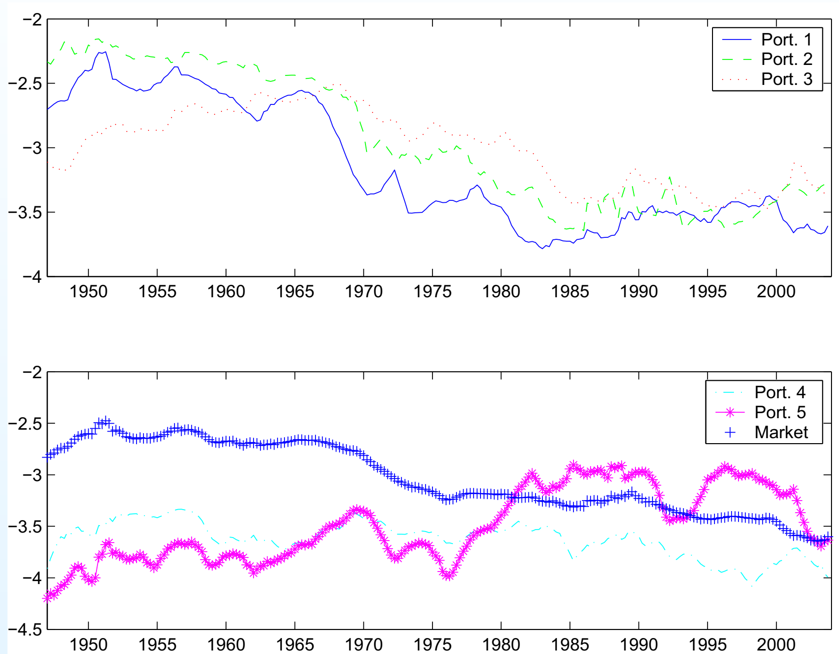


Useful time series martingale decomposition

Suppose

$$x_{t+1} = Gx_t + Hw_{t+1}$$

where G has stable eigenvalues and $\{w_{t+1}\}$ is iid vector process of standard normally distributed random variables. Stationary process. Positive cash flow process expressed in logarithms:

$$d_{t+1} - d_t = \mu_d + U_d x_t + \iota_0 w_{t+1}.$$

Stationary increments.
Alternatively, we may specify the process in moving-average form:

$$d_{t+1} - d_t = \mu_d + \iota(L) w_{t+1}.$$

where $\iota(z) = \sum_{j=0}^{\infty} \iota_j z^j$

Martingale decomposition

Commonly used in establishing central limit approximations (*e.g.* see Hall and Heyde (1980)) and it is not limited to linear processes (*e.g.* see Hansen and Scheinkman (2003) for a nonlinear Markov version.)

Write

$$d_{t+1} - d_t = \mu_d + \iota(1) w_{t+1} + U_d^* x_{t+1} - U_d^* x_t$$

Thus

1. $\{d_t\}$ has growth rate μ_d
2. $\{d_t\}$ has a martingale component has increment $\iota(1) w_{t+1}$. where

$$\iota(1) = \iota_0 + U_d G (I - G)^{-1} H$$

3. $\{D_t\}$ asymptotic growth rate for cash flow is $\mu_d + (1/2)|\iota(1)|^2$.

Model: Investor Preferences

- Relax discounted expected utility theory model, but maintain recursivity and dynamic programming. Consider a Kreps and Porteus (1978) specification with a CES recursion:

$$V_t = \left[(1-\beta)\left(C_t\right)^{1-\rho} + \beta \mathcal{R}_t(V_{t+1})^{1-\rho} \right]^{\frac{1}{1-\rho}} .$$

  where C_t is date t consumption and V_t the date t *continuation value* of the consumption profile.

- $\mathcal{R}_t$ adjusts the continuation value for risk via:

$$\mathcal{R}_t(V_{t+1}) = \left[E\left(V_{t+1}\right)^{1-\alpha} | \mathcal{X}_t \right]^{\frac{1}{1-\alpha}}$$

  where $\mathcal{X}_t$ is the current period information set.

- Special case: Cobb-Douglas specification ($\rho = 1$). The recursion becomes: $V_t = (C_t)^{(1-\beta)} \mathcal{R}_t(V_{t+1})^{\beta}$.

Investor Preferences Continued

- Recursion again

$$V_t = \left[(1-\beta)\left(C_t\right)^{1-\rho} + \beta \mathcal{R}_t(V_{t+1})^{1-\rho}\right]^{\frac{1}{1-\rho}}.$$

$$\mathcal{R}_t(V_{t+1}) = \left[E\left(V_{t+1}\right)^{1-\alpha}|\mathcal{X}_t\right]^{\frac{1}{1-\alpha}}$$

- Observations:
  1. $\frac{1}{\rho}$ is a measure of intertemporal substitution.
  2. No *reduction* of intertemporal lotteries - the intertemporal allocation of risk matters!!
  3. α is a measure of risk aversion for simple wealth gambles.

Stochastic consumption growth

An alternative recursion:

$$\frac{V_t}{C_t} = \left[(1-\beta) + \beta \mathcal{R}_t \left(\frac{V_{t+1}}{C_{t+1}} \frac{C_{t+1}}{C_t}\right)^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

Let v_t denote the logarithm of the continuation value relative to the logarithm of consumption, and let c_t denote the logarithm of consumption. Rewrite recursion as

$$v_t = \frac{1}{1-\rho} \log\left((1-\beta) + \beta \exp\left[(1-\rho)\, \mathcal{Q}_t(v_{t+1} + c_{t+1} - c_t)\right]\right),$$

where $\mathcal{Q}_t$ is the risk-sensitive recursion:

$$\mathcal{Q}_t(v_{t+1}) = \frac{1}{1-\alpha} \log E\left(\exp\left[(1-\alpha) v_{t+1}\right] | \mathcal{X}_t\right).$$

Solve when $\rho = 1$ using consumption dynamics; compute derivatives.

$\rho = 1$ limit

$$v_t = \beta \mathcal{Q}_t(v_{t+1} + c_{t+1} - c_t).$$

where

$$\mathcal{Q}_t(v_{t+1}) = \frac{1}{1-\alpha} \log E\left(\exp\left[(1-\alpha)v_{t+1}\right] | \mathcal{X}_t\right),$$

log linear consumption dynamics

$$c_t - c_{t-1} = \gamma(L)w_t + \mu_c,$$

$$\gamma(z) = \sum_{j=0}^{\infty} \gamma_j z^j.$$

$\rho = 1$ solution

Solve a linear expectational difference equation forward:

$$v_t = \sum_{j=1}^{\infty} \beta^j E\left(c_{t+j} - c_{t+j-1} - \mu_c | \mathcal{F}_t\right) + \mu_v$$

where

$$\mu_v = \frac{\beta}{1-\beta}[\mu_c + \frac{(1-\alpha)}{2}\gamma(\beta) \cdot \gamma(\beta)]$$

and $\gamma(\beta)$ is the discounted response.

- The term $\gamma(\beta)$ will be an important ingredient in our calculations.
- Easy to solve for v_t when $\rho = 1$.
- Could extend to accommodate conditional volatility as in Tauchen and Lettau-Ludvigson-Wachter.

Stochastic Discount Factor

Valuation of one-period securities:

- $$S_{t+1,t} = \frac{MV_{t+1}MC_{t+1}}{MC_t} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \left(\frac{V_{t+1}}{\mathcal{R}_t(V_{t+1})}\right)^{\rho-\alpha}$$
- The stochastic discount factor in the $\rho = 1$ case is:

$$S_{t+1,t} \equiv \beta \left(\frac{C_t}{C_{t+1}}\right) \left[\frac{(V_{t+1})^{1-\alpha}}{\mathcal{R}_t(V_{t+1})^{1-\alpha}}\right].$$

- Depends on continuation values - much of the literature finds clever ways to avoid this dependence.
- Valuation of multi period securities multiplies up stochastic discount factors.

Stochastic discount factor for $\rho = 1$.

The logarithm of the stochastic discount factor can now be depicted as:

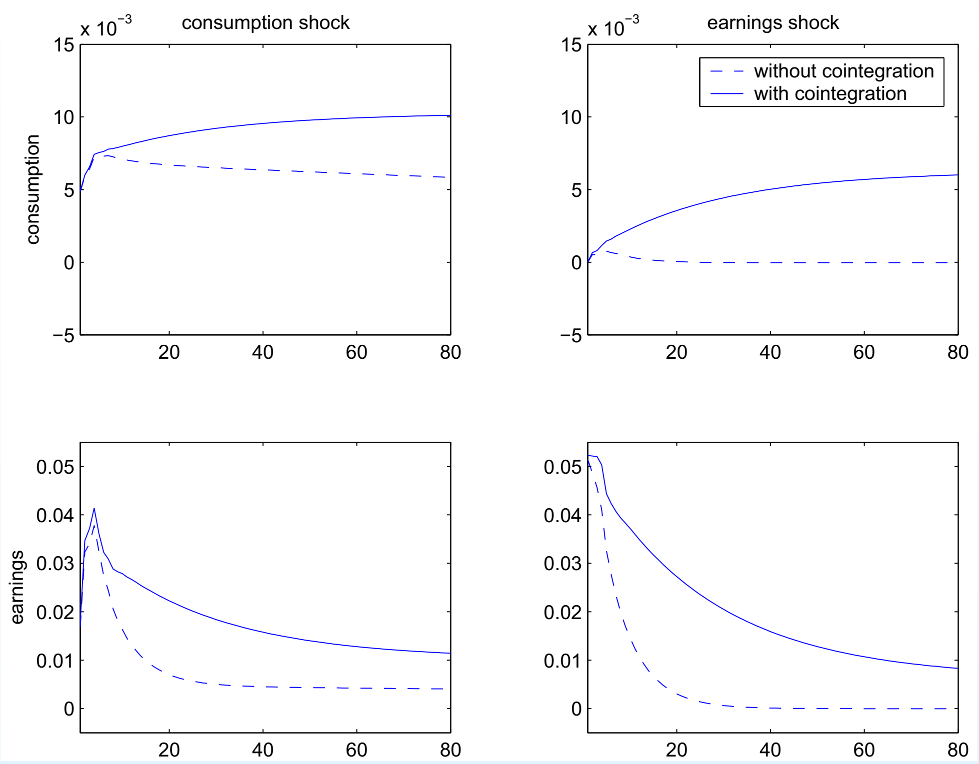
$$s_{t+1,t} \equiv \log S_{t+1,t} = -\delta - \gamma(L) w_{t+1} - \mu_c + (1-\alpha)\gamma(\beta) w_{t+1} - \frac{(1-\alpha)^2 \gamma(\beta) \cdot \gamma(\beta)}{2}$$

where $\beta = \exp(-\delta)$. The term $\gamma(\beta) w_{t+1}$ is the is the solution to

$$(1-\beta) \sum_{j=0}^{\infty} \beta^j \left[E(c_{t+j} | \mathcal{X}_{t+1}) - E(c_{t+j} | \mathcal{X}_t) \right].$$

Illustrates the role of consumption predictability as featured by Bansal and Yaron (2004).

Extraction of macro risk



Questions

- Implied long run risk return tradeoff?
- Relation to robustness θ versus $\frac{1}{\alpha-1}$?
- Learning and sensitivity to model specification?