Toulouse Lectures

# Lecture III

## General Results

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- Have characterized (generically) unique solution concept using axioms:
  - Nonnegotiation Commitment
  - Binding Coalitions
  - Competitive Allocation
  - Competitive Wages
  - Vickrey Payments
  - Consistency
- Today - - some general results and, noncooperative implementation

*Theorem 2*: If no externalities, solution reduces to Shapley value (always true if $n \leq 3$; requires additional conditions if $n \geq 4$ )

*Proof*: Assume $n = 3$



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Shapley value for player $1 \leftrightarrow$ 1's expected marginal contribution

| | | |
|---|---|---|
| • 1, 2, 3 | $v(1)$ | $\frac{1}{6}$ |
| • 1, 3, 2 | $v(1)$ | $\frac{1}{6}$ |
| • 2, 1, 3 | $v(1,2) - v(2)$ | $\frac{1}{6}$ |
| • 3, 1, 2 | $v(1,3) - v(3)$ | $\frac{1}{6}$ |
| • 2, 3, 1 | $v(1,2,3) - v(2,3)$ | $\frac{1}{6}$ |
| • 3, 2, 1 | $v(1,2,3) - v(2,3)$ | $\frac{1}{6}$ |



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Assume $(*)\quad v(1,3)-v(1)>v(2,3)-v(2)$

$(**)\quad v(1,2)-v(1)>v(2,3)-v(3)$

- 1, 2, 3 — if 1 and 2 compete for 3, 1 wins (from (*))
  2 left with $v$ (2)
  1 gets $v$ (1, 2, 3) – $v$ (2) – $v$ (3)
- 1, 3, 2 — if 1 and 3 compete for 2, 1 wins (from (**))
  1 gets $v$ (1, 2, 3) – $v$ (2) – $v$ (3)
- 2, 1, 3 — $v$ (1, 3) – $v$ (2, 3) + $v$ (2)
- 3, 1, 2 — $v$ (1, 2) – $v$ (2, 3) + $v$ (3)
- 2, 3, 1 — $v$ (1)
- 3, 2, 1 — $v$ (1)



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Idea:

- 1 not paid marginal product by 2 and 3 under competitive bargaining
  - paid a bit less (opportunity wage)
- But can pay 2 and 3 a bit less than *their* marginal products
- So *on average* gets marginal product

*Theorem 3*: If all externalities nonpositive, then grand coalition forms, i.e.,

$$\psi(v) = \{N\}.$$

- if no externalities, result follows from Theorem 2



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$$v(1;\{1\},\{2\},\{3\}) = v(2;\{1\},\{2\},\{3\}) = v(3;\{1\},\{2\},\{3\}) = 10$$

$$v(\{1,2\};\{1,2\},\{3\}) = v(\{1,3\};\{1,3\},\{2\}) = v(\{2,3\};\{2,3\},\{3\}) = 28$$

$$v(3;\{1,2\},\{3\}) = 8,\ \ v(2;\{1,3\},\{2\}) = 6,\ \ v(1;\{2,3\},\{1\}) = 4$$

$$v(\{1,2,3\};\{1,2,3\}) = 40$$

1, 2, 3

- if 1 and 2 compete for 3,
  - 1 wins
  - Needs to pay only 6 to sign up player 2
- {1, 2} need pay only 8 to player 3
- 1 gets 40 – 6 – 8 = 26

With negative externalities

- player fares badly if left out of coalition
- so can be “bought up” cheaply
- grand coalition cheap to form

## Connection with core

- With externalities, multiple definitions of core possible
  - whether coalition can block may depend on $\mathscr{C}$

Let

$$\text{core }(v) = \left\{ (x_1, \ldots, x_n) \text{ feasible} \middle| \forall S \ \sum_{i \in S} x_i \geq v\left(S; \{S, N-S\}\right) \right\}$$

Theorem 4: Suppose $v$ symmetric. If core $(v) \neq \emptyset$, then $\psi(v) = \{N\}$

- saw in first lecture that just because core empty, cannot conclude grand coalition won't form
- however, version of converse *does* hold:
  - If core of symmetric game nonempty, grand coalition will form

Proof: $n = 3$

- If game symmetric, only numbers matter

  $v$ [3] = payoff to grand coalition

  $v$ [2] = payoff to a coalition of two

  $v$ [1] = payoff to a single player (if other two form coalition)

- if grand coalition does *not* form.

$$v\,[3] - v\,[1] - v\,[1] < v\,[1]$$

- Hence,

$$(*) \qquad v\,[3] < 3v\,[1]$$

- But if $(x_1, x_2, x_3) \in \text{core}(v)$,

$$x_i \geq v[1] \text{ for all } i$$

- $v[3] = x_1 + x_2 + x_3 \geq 3v[1]$,

contradicting (*)

## Noncooperative Implementation

- players arrive in sequence 1, 2, …, $n$
- suppose player 1, …, $i-1$ already arrived
  - formed coalitions $S_1, \ldots, S_m$
- in turn, each coalition $S_j$ makes bids

$$\left(b_{1j}, \ldots, b_{mj}, b_{m+1\,j}\right),$$

  where

  $b_{kj} =$ how much $S_j$ will pay player $i$ if $i$ joins coalition $S_k$ $\left(S_{m+1} = \varnothing\right)$

- player $i$ chooses one of coalitions (possibly $\varnothing$)
  - if joins $S_k$, gets $\sum_{j=1}^{m} b_{kj}$ from $S_1, \ldots, S_m$
- menu auction à la Bernheim - Whinston

*Theorem 5*: Generically unique trembling-hand-perfect equilibrium implements our solution $(\varphi, \psi, t)$

Recall example in which player 3 can stand alone

{1, 2, 3} can get 28

{1, 3} can get 18 and 2 obtains 8

{2, 3} can get 16 and 1 obtains 9

{1, 2} can get 8 and 3 obtains 13

$v(1;\{1\},\{2\},\{3\}) = v(2;\{1\},\{2\},\{3\}) = 0 \qquad v(3;\{1\},\{2\},\{3\}) = 12$

Suppose $\mathscr{P} = \{\{1\},\{2\}\}$

| | allocations $\{1,3\}$ | $\{2,3\}$ | $\{3\}$ |
|---|---|---|---|
| coalitions $\{1\}$ | 18 | 9 | 0 |
| $\{2\}$ | 8 | 16 | 0 |
| $\{\varnothing\}$ | 0 | 0 | 12 |

- first, $\{1\}$ makes bids $(b_{11}, b_{12}, b_{31})$ for 3
- then, $\{2\}$ makes bids $(b_{12}, b_{22}, b_{32})$ for 3

- clear that won't have {3}(3 allocated to{∅})
  - both {1} and {2} willing to bid at least 12 to ensure {1,3} or {2, 3}
- {2} inclined to try to get {2, 3}
  - gets 16 rather than 8

- {1} can make {2} indifferent between {1, 3} and {2, 3} by bidding
  $$b_{11} = b_{21} + 8$$
  - cheapest to take $b_{11} = 8 \quad b_{21} = 0 \quad b_{31} = 0,$
  - 1 will do this because
  $$18 - 8 > 9$$
- {2} will then bid $b_{12} = 4 \quad b_{21} = 0 \quad b_{31} = 0,$
  to get {1, 3}
- But this is exactly what solution concept predicts
  - get {1, 3} because 18 + 8 > 9 + 6
  - 1 makes Vickrey payment of 8 to shift allocation from{2, 3} to {1, 3}
  - 2 makes Vickrey payment of 4 to shift allocation from {3} to {1, 3}

*Extensions*

- multiple memberships
- networks
- nontransferable utility
- incomplete information