Les 2èmes
« TOULOUSE LECTURES IN ECONOMICS »
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Toulouse Lectures

Lecture II

A Theory of Coalition Formation

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Model

n players $N = \{1, \ldots, n\}$

players divide up into coalitions, forming *partition* of N

call partition a *coalition structure* $\mathscr{C}$

To capture externalities, what a coalition can get can depend on $\mathscr{C}$, i.e., for $S \in \mathscr{C}$,

$v(S;\mathscr{C})$ = what coalition S can get if coalition structure $\mathscr{C}$ has formed

- Coalition's *composition* public knowledge
- Coalition's agreement (contract) is secret
  (but predictable by other coalitions)
- Hence, partition function reduced form of game of (simultaneous) contract selection

- Continue to assume superadditivity: if $S_1 \cap S_2 = \varnothing$

  $$v(S_1;\mathscr{C}) + v(S_2;\mathscr{C}) \le v(S_1 \cup S_2;\mathscr{C}_{12}),$$

  $\mathscr{C}_{12}$ is $\mathscr{C}$ with S_1 and S_2 replaced by $S_1 \cup S_2$

- No longer so innocuous as in characteristic function case
  - If S_1 and S_2 merge, remaining coalitions in $\mathscr{C}_{12}$ may behave differently
  - Classic example: Cournot oligopoly

    if $n = 3$ and 2 firms merge, their total profit is reduced (because remaining firm expands output)
  - Must assume that if S_1 and S_2 merge, can credibly commit to act independently (say, by remaining as separate divisions)

Suppose S_1 and S_2 merge (coalition structure changes from $\mathscr{C}$ to $\mathscr{C}_{12}$). Consider effect on S

Three cases:

(1) $v(S;\mathscr{C}) = v(S;\mathscr{C}_{12})$ no externality (characteristic function good representation)

(2) $v(S;\mathscr{C}) < v(S;\mathscr{C}_{12})$ positive externality

(3) $v(S;\mathscr{C}) > v(S;\mathscr{C}_{12})$ negative externality

- Solution concept
  - prediction of payoffs
  - prediction of coalitions

- Need a pair of functions φ, ψ

$$\varphi_i(v) = \text{ player } i\text{'s predicted payoff}$$

$$\psi(v) = \text{ predicted coalition structure}$$

- Which players get to move determined *randomly*
  - some realized order of players $R = (k_1, \ldots, k_n)$

- φ and ψ will be indexed by R

$$\varphi^R \qquad \psi^R$$

- *Overall* solution given by randomization over R

- $\left(\varphi^R, \psi^R\right)$ doesn't fully specify solution
- Given that some coalitions already partially formed, want to predict *ultimate* payoffs and coalition structure
- Suppose $\mathscr{P}$ is a *partial partition*, i.e., partition of $\{1, \ldots, i-1\}$

$\varphi_j^R\left(v \mid \mathscr{P}\right)$ predicts ultimate payoff $j \notin \{1, \ldots, i-1\}$

$\psi^R\left(v \mid \mathscr{P}\right)$ predicts ultimate coalition structure

- Solution is pair $\left(\varphi^R\left(\cdot \mid \cdot\right), \psi^R\left(\cdot \mid \cdot\right)\right)$
- $\varphi^R\left(v\right) = \varphi^R\left(v \middle| \varnothing\right) \qquad \psi^R\left(v\right) = \psi^R\left(v \middle| \varnothing\right)$
- Henceforth, assume $R = 1, 2, \ldots, n$ and drop superscript

Axioms

- Want to capture idea that player can *commit* not to negotiate with certain other players
- Consider $\{1,\ldots,i-1\}$ and partial partition $\mathscr{P}$
- Suppose $j \in S \in \mathscr{P}$ and $k \in \{1,\ldots,i-1\}$ but $k \notin S$
  - fact that j and k are separate, implies communication line cut between them
  - so at later date, can't find them in same coalition together

Formally

(1) *Non-negotiation Commitment*:

Given $j, k \in \{1, \ldots, i-1\}$ and partial partition $\mathscr{P}$, suppose that $j \in S \in \mathscr{P}$ but $k \notin S$.

Then if $j \in S^* \in \psi(v \mid \mathscr{P})$, we have $k \notin S^*$.

If negotiation *does* occur and results in coalition, then coalition is *binding*

(3) *Binding Coalitions*: Given $\{1,\ldots i-1\}$ and partial partition $\mathscr{P}$, if $S \in \mathscr{P}$, then there exists $S' \in \psi\left(v \middle| \mathscr{P}\right)$ such that $S \subseteq S'$.

- Too strong: coalition may break apart if all members gain from doing so

- Major qualitative results not much changed if rule out renegotiation of coalitions
- Analysis simpler without renegotiation, so will invoke Binding Coalitions
- Still, more satisfactory axiom would allow coalitions to break up if members collectively gain

In 3-player model (like yesterday's), suppose

- {1} and {2} already exist
- 3 could be allocated to {1}, {2} or could remain by self
- competition implies 3 will be allocated to maximize welfare of those involved: 1, 2, and 3.

Given {1, …, i-1} and partial partition $\mathscr{P}$, i could join any coalition $\hat{S} \in \mathscr{P} \cup \{\varnothing\}$

("i joins $\varnothing$" means starts own coalition)

- Welfare of coalition if $S \in \mathscr{P} \cup \{\varnothing\}$ if i joins $\hat{S} \in \mathscr{P} \cup \{\varnothing\}$
  - if $\hat{S} = S$

$$\Phi\left(S \middle| \mathscr{P}\left(S \cup \{i\}\right)\right) = v\left(S^*; \psi\left(v \middle| \mathscr{P}\left(S \cup \{i\}\right)\right)\right) - \sum_{j>i} t_j\left(S \middle| \mathscr{P}\left(S \cup \{i\}\right)\right)$$

where

- $S \cup \{i\} \subseteq S^* \in \psi\left(v \middle| \mathscr{P}\left(S \cup \{i\}\right)\right)$
- $\mathscr{P}\left(S \cup \{i\}\right)$ is $\mathscr{P}$ with S replaced by $S \cup \{i\}$,
- $t_j\left(S \middle| \mathscr{P}\left(S \cup \{i\}\right)\right)$ is payment that coalition S later makes to player j if i joins S

- if $S \in \mathscr{P},\ \hat{S} \in \mathscr{P} \cup \{\varnothing\} \quad S \neq \hat{S}$

  $$\Phi\left(S \middle| \mathscr{P}\left(\hat{S} \cup \{i\}\right)\right) = v\left(S^{*}; \psi\left(v \middle| \mathscr{P}\left(\hat{S} \cup \{i\}\right)\right)\right) - \sum_{j>i} t_j\left(S \middle| \mathscr{P}\left(\hat{S} \cup \{i\}\right)\right)$$

  where

  - $S \subseteq S^{*} \in \psi\left(v \middle| \mathscr{P}\left(\hat{S} \cup \{i\}\right)\right)$
  - $\mathscr{P}\left(\hat{S} \cup \{i\}\right)$ is $\mathscr{P}$ with $\hat{S}$ replaced by $\hat{S} \cup \{i\}$

    (if $\hat{S} = \{\varnothing\}$, then just add $\{i\}$ to $\mathscr{P}$)
  - $t_j\left(S \middle| \mathscr{P}\left(\hat{S} \cup \{i\}\right)\right)$ is payment S (or successor coalition) makes to j if i joins $\hat{S}$

- if $S = \varnothing, \quad S \neq \hat{S}$

  $$\Phi\left(S \middle| \mathscr{P}\left(\hat{S} \cup \{i\}\right)\right) = 0.$$

- Recall public good game

  {1, 2, 3} can produce 24

  {1, 2} can produce 12, and 3 obtains 9

  {1, 3} can produce 13, and 2 obtains 9

  {2, 3} can produce 14, and 1 obtains 9

  if no coalitions, everyone gets 0

In public good example, if $\mathscr{P} = \{\{1\},\{2\}\}$

$$\Phi\left(\{1\} \middle| \{1,3\},\{2\}\right) = 13$$

$$\Phi\left(\{1\} \middle| \{1\},\{2,3\}\right) = 9$$

$$\Phi\left(\{2\} \middle| \{1\},\{2,3\}\right) = 14$$

$$\Phi\left(\{2\} \middle| \{1,3\},\{2\}\right) = 9$$

$$\Phi\left(\{1\} \middle| \{1\},\{2\},\{3\}\right) = \Phi\left(\{2\} \middle| \{1\},\{2\},\{3\}\right) = \Phi\left(\{\varnothing\} \middle| \{1\},\{2\},\{3\}\right) = 0$$

- Player i should be allocated to coalition $\hat{S}$ (including possibly $\varnothing$) for which total welfare of coalitions (including $\varnothing$) maximized:

(4) *Competitive Allocation*: Given $\{1, \ldots, i-1\}$ and partial partition $\mathscr{P}$

if

$$S^{\circ} = \underset{\hat{S} \in \mathscr{P} \cup \{\varnothing\}}{\arg\max} \sum_{S \in \mathscr{P} \cup \{\varnothing\}} \Phi\left(S \middle| \mathscr{P}\left(\hat{S} \cup \{i\}\right)\right),$$

then there exists $S^{\circ *} \in \psi\left(v \middle| \mathscr{P}\right)$ such that

$$i \in S^{\circ} \subseteq S^{\circ *}.$$

In the public good example, if $\mathscr{P} = \{\{1\},\{2\}\}$

$$23 = \underbrace{\Phi\left(\{1\}\middle|\{1\},\{2,3\}\right)}_{9} + \underbrace{\Phi\left(\{2\}\middle|\{1\},\{2,3\}\right)}_{14}$$

$$> \underbrace{\Phi\left(\{1\}\middle|\{1,3\},\{2\}\right)}_{13} + \underbrace{\Phi\left(\{2\}\middle|\{1,3\},\{2\}\right)}_{9} = 22$$

$$> \underbrace{\Phi\left(\{1\}\middle|\{1\},\{2\},\{3\}\right)}_{0} + \underbrace{\Phi\left(\{2\}\middle|\{1\},\{2\},\{3\}\right)}_{0} + \underbrace{\Phi\left(\{\varnothing\}\middle|\{1\},\{2\},\{3\}\right)}_{0}$$

- Thus, according to Competitive Allocation, 3 gets allocated to $\{2\}$.

Consider example where player 3 can stand alone

{1, 2, 3} can get 28

{1, 3} can get 18 and 2 obtains 8

{2, 3} can get 16 and 1 obtains 9

{1, 2} can obtain 8 and 3 obtains 13

$$v\left(1;\{1\},\{2\},\{3\}\right)=v\left(2;\{1\},\{2\},\{3\}\right)=0 \qquad v\left(3;\{1\},\{2\},\{3\}\right)=12$$

$$26=\Phi\left(\{1\}\middle|\underbrace{\{1,3\}}_{18},\{2\}\right)+\Phi\left(\{2\}\middle|\{1,3\},\underbrace{\{2\}}_{8}\right)$$

$$>\Phi\left(\{1\}\middle|\underbrace{\{1\}}_{9},\{2,3\}\right)+\Phi\left(\{2\}\middle|\{1\},\underbrace{\{2,3\}}_{6}\right)=25$$

$$>\Phi\left(\{1\}\middle|\underbrace{\{1\}}_{0},\{2\},\{3\}\right)+\Phi\left(\{2\}\middle|\{1\},\underbrace{\{2\}}_{0},\{3\}\right)+\Phi\left(\{\varnothing\}\middle|\{1\},\{2\},\underbrace{\{3\}}_{12}\right)$$

$$=12$$

- Thus, 3 gets allocated to {1}.

- Just as players are allocated to coalitions competitively should be *paid* competitively

- In public good game when $\mathscr{P} = \{\{1\},\{2\}\}$
  - 1 willing to pay 4 for player 3
  - 2 willing to pay 5 for player 3
  - so $\varphi_3\left(v \middle| \{1\},\{2\}\right) = 4$ (opportunity wage)

- In public good game when $\mathscr{P} = \{1,2\}$
  - player 3 can get 9 by remaining on own
  - so $\varphi_3\left(v \middle| \{1,2\}\right) = 9$ (opportunity wage)

To generalize:

- Given $\{1,\ldots,i-1\}$ and $\mathscr{P}$, suppose competitive allocation of i is to $S^{\circ}\in\mathscr{P}$
- Coalition S willing to pay
  $$\max\left\{\left[\Phi\left(S\middle|\mathscr{P}\left(S'\cup\{i\}\right)\right)-\Phi\left(S\middle|\mathscr{P}\left(S^{\circ}\cup\{i\}\right)\right)\right],0\right\}$$
  to get i allocated to S' instead
- Total amount that players will pay to get i allocated to S' rather than S°

$$\sum_{S\in\mathscr{P}\cup\{\varnothing\}}\max\left\{\left[\Phi\left(S\middle|\mathscr{P}\left(S'\cup\{i\}\right)\right)-\Phi\left(S\middle|\mathscr{P}\left(S^{\circ}\cup\{i\}\right)\right)\right],0\right\}$$

(4) *Competitive (Opportunity) Wages*: Given $\{1,\ldots,i-1\}$ and $\mathscr{P}$,

suppose i is competitively allocated to $S^\circ \in \mathscr{P} \cup \{\varnothing\}$

- Then,

$$\varphi_i\left(v|\mathscr{P}\right) = \max_{S' \in \mathscr{P} \cup \{\varnothing\}} \sum_{S \in \mathscr{P} \cup \{\varnothing\}} \max\left\{\left[\Phi\left(S|\mathscr{P}\left(S' \cup \{i\}\right)\right) - \Phi\left(S|\mathscr{P}\left(S^\circ \cup \{i\}\right)\right)\right], 0\right\}$$

= total amount that players would pay to get i allocated to "favorite" alternative S' rather than S°.

In public good example, with $\mathscr{P} = \{\{1\},\{2\}\}$

- player 3 should be allocated to $S^{\circ} = \{2\}$
- next best alternative $S' = \{1\}$

$$\varphi_3 = \max\left\{\left[\Phi\left(\{1\}\middle|\{1,3\},\{2\}\right) - \Phi\left(\{1\}\middle|\{1\},\{2,3\}\right)\right],0\right\}$$

$$+\max\left\{\left[\Phi\left(\{2\}\middle|\{1,3\},\{2\}\right) - \Phi\left(\{2\}\middle|\{1\},\{2,3\}\right)\right],0\right\}$$

$$= \max\{13-9,0\} + \max\{9-14,0\}$$

$$= 4+0$$

Example where player 3 can stand alone, with $\mathscr{P} = \{\{1\},\{2\}\}$

- player 3 should be allocated to $S^\circ = \{1\}$
- next best alternative is $S' = \{\varnothing\}$, *not* $S' = \{2\}$
  - player 2 willing to pay only 16 – 8 = 8 to change allocation from $\{1\}$ to $\{2\}$
  - but $\varnothing$ is willing to pay 12 to change allocation from $\{1\}$ to $\{3\}$.

- Hence, $\varphi_3\left(v \middle| \{1\},\{2\}\right) = 12$

allocations

coalitions	$\{1,3\}$	$\{2,3\}$	$\{3\}$
$\{1\}$	18	9	0
$\{2\}$	8	16	0
$\{\varnothing\}$	0	0	12

First row, second column is $\Phi\left(\{1\} \middle| \{1\},\{2,3\}\right)$

allocations

coalitions	$\{1,3\}$	$\{2,3\}$	$\{3\}$
1	~~18~~ 8	~~9~~ 0	0
2	8	16	0
$\emptyset$	0	0	12

How will coalitions 1 and 2 divide up $\varphi_3 = 12$ in this example?

- $\{1\}$ won't pay for it all, because $\{2\}$ gains from 3 being allocated to $\{1\}$ rather than $\{\varnothing\}$
- If coalition $\{1\}$ not present, then optimal to allocate 3 to $\{2\}$
- $\{1\}$'s marginal impact is to switch allocation from $\{2,3\}$ to $\{1,3\}$
  - marginal effect on others, $\{2\}$ and $\{\varnothing\}$, is $16 - 8 = 8$
  - so this is what $\{1\}$ must pay to get $\{1,3\}$

- replace $\{1\}$'s payoffs by his payment 8
- if coalition $\{2\}$ not present, then optimal to allocate 3 to $\{\varnothing\}$
- $\{2\}$'s marginal impact is to switch allocation from $\{3\}$ to $\{1,3\}$
  - marginal effect on others, $\{1\}$ and $\{\varnothing\}$, is $12-8=4$
  - so this is what $\{2\}$ will pay

- Thus $t_3\left(\{1\} \middle| \{1,3\},\{2\}\right)=8$ and $t_3\left(\{2\} \middle| \{1,3\},\{2\}\right)=4$

Given $S, T \in \mathscr{P} \cup \{\varnothing\}$, $S < T$ if $\min\{i | i \in S\} < \min\{i | i \in T\}$
or $T \in \{\varnothing\}$

(5) *Vickrey Payments*: Given $\{1, \ldots, i-1\}$ and $\mathscr{P}$,
suppose i is competitively allocated to $S^\circ \in \mathscr{P} \cup \{\varnothing\}$

for all $S \in \mathscr{P}$, $\hat{S} \in \mathscr{P} \cup \{\varnothing\}$

- $t_i\left(S | \mathscr{P}\left(S^\circ \cup \{i\}\right)\right)$

$$= \max\left\{0, \max_{S' \in \mathscr{P} \cup \{\varnothing\}} \left[\sum_{S'' > S} \left(\Phi\left(S'' | \mathscr{P}\left(S' \cup \{i\}\right)\right) - \Phi\left(S'' | \mathscr{P}\left(S^\circ \cup \{i\}\right)\right)\right) + \sum_{S'' < S} \left(t_i\left(S'' | \mathscr{P}\left(S' \cup \{i\}\right)\right) - t_i\left(S'' | \mathscr{P}\left(S^\circ \cup \{i\}\right)\right)\right)\right]\right\}$$

- if $t_i\left(S \middle| \mathscr{P}\left(S^{\circ} \cup \{i\}\right)\right) > 0$

  $$t_i\left(S \middle| \mathscr{P}\left(S' \cup \{i\}\right)\right) = 0, \text{ for all } S' \neq S^{\circ}$$

- if $t_i\left(S \middle| \mathscr{P}\left(S^{\circ} \cup \{i\}\right)\right) = 0$

$$t_i\left(S \middle| \mathscr{P}\left(S' \cup \{i\}\right)\right) = \max\left\{0, \left[\Phi\left(S \middle| \mathscr{P}\left(S' \cup \{i\}\right)\right) - \Phi\left(S \middle| \mathscr{P}\left(S^{\circ} \cup \{i\}\right)\right) + \sum_{S'' < S}\left(t_i\left(S'' \middle| \mathscr{P}\left(S' \cup \{i\}\right)\right) - t_i\left(S'' \middle| \mathscr{P}\left(S^{\circ} \cup \{i\}\right)\right)\right)\right]\right\}$$

- $t_i\left(\{\varnothing\}\middle|\mathscr{P}\left(S'\cup\{i\}\right)\right)=0,\ \text{for all } S'\in\mathscr{P}$

- $t_i\left(\{\varnothing\}\middle|\mathscr{P}\left(\{\varnothing\}\cup\{i\}\right)\right)=\Phi\left(\{\varnothing\}\middle|\mathscr{P}\left(\{\varnothing\}\cup\{i\}\right)\right)$

- *Lemma*: Given $\{1,\ldots, i-1\}$ and $\mathscr{P}$

if i competitively allocated to $S^\circ \in \mathscr{P} \cup \{\varnothing\}$

$$\sum_{S \in \mathscr{P} \cup \{\varnothing\}} t_i \left(S \middle| \mathscr{P} \left(S^\circ \cup \{i\}\right)\right) = \varphi_i \left(v \middle| \mathscr{P}\right).$$

- Consider public good game and suppose $\mathscr{P} = \{\{1\}\}$

$$\Phi\left(\{1\}\middle|\{1,2\}\right) = v\left(\{1,2,3\};\{1,2,3\}\right) - t_3\left(\{1,2\}\middle|\{1,2\}\right)$$

$$= v\left(\{1,2,3\};\{1,2,3\}\right) - \varphi_3\left(v\middle|\{1,2\}\right) \text{ (from Lemma)}$$

$$= \; 24 - 9 = 15$$

$$\Phi\left(\{1\}\middle|\{1\},\{2\}\right) = v\left(\{1\};\{1\},\{2,3\}\right) - t_3\left(\{1\}\middle|\{1\},\{2\}\right)$$

$$= \; 9 - 0 = 9$$

$$\Phi\left(\{\varnothing\}\middle|\{1,2\}\right) = 0, \text{ by definition}$$

$$\Phi\left(\{\varnothing\}\middle|\{1\},\{2\}\right) = v\left(\{2,3\};\{1\},\{2,3\}\right) - t_3\left(\{2\}\middle|\{1\},\{2\}\right)$$

$$= v\left(\{2,3\};\{1\},\{2,3\}\right) - \varphi_3\left(v\middle|\{1\},\{2\}\right) \text{ (from Lemma)}$$

$$= \; 14 - 4 = 10$$

allocations

		$\{1,2\}$	$\{2\}$
coalitions	$\{1\}$	15	9
	$\{\varnothing\}$	0	10

- Competitive allocation is {2}
- I.e., partial partition is {1}, {2}
- Hence, final coalition structure: {1}, {2,3}

Suppose i competitively allocated to $S^\circ \in \mathscr{P} \cup \{\varnothing\}$

- predictions about payoffs, coalitions, and payments should be same before and after allocation

(6) *Consistency*: Given $\{1, \ldots, i-1\}$ and $\mathscr{P}$, if i is competitively allocated to S°, then

$$\varphi_j\left(v \middle| \mathscr{P}\right) = \varphi_j\left(v \middle| \mathscr{P}\left(S^\circ \cup \{i\}\right)\right)$$

$$\psi\left(v \middle| \mathscr{P}\right) = \psi\left(v \middle| \mathscr{P}\left(S^\circ \cup \{i\}\right)\right)$$

$$t_j\left(S \middle| \mathscr{P}\right) = \begin{cases} t_j\left(S \middle| \mathscr{P}\left(S^\circ \cup \{i\}\right)\right), & \text{for } S \in \mathscr{P}, S \neq S^\circ \\ t_j\left(S^\circ \cup \{i\} \middle| \mathscr{P}\left(S^\circ \cup \{i\}\right)\right), & \text{if } S = S^\circ \end{cases}$$

for all $j > i$.

Theorem 1: For generic v, there exist unique $\varphi, \psi,$ and t satisfying

- Nonnegotiation Commitment (NC)
- Binding Coalitions (BC)
- Competition Allocation (CA)
- Competitive Wages (CW)
- Vickrey Payments (VP)
- Consistency (C)

Proof: Let $n = 3$

- Consider $\mathscr{P} = \{\{1\},\{2\}\}$
  - From NC and BA,

$$\psi\left(v \middle| \{1\},\{2\}\right) = \{1,3\},\{2\}$$

or

$$\{1\},\{2\},\{3\}$$

or

$$\{1\},\{2,3\}$$

  - CA determines which of three holds
  - CW and VP determine φ_3 and t_3
- Can do same thing for $\mathscr{P} = \{\{1,2\}\}$
- Can use C, ψ, and t_3 to calculate

$$\Phi\left(S \middle| \{1,2\}\right) \text{and } \Phi\left(S \middle| \{1\},\{2\}\right)$$

- Hence, CA determines whether 2 allocated to $\{1\}$ or $\{\varnothing\}$
- continue iteratively