The Provision and Pricing of Excludable Public Goods: Ramsey-Boiteux Pricing versus Bundling*

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Abstract

This paper studies the relation between Bayesian mechanism design and the Ramsey-Boiteux approach to the provision and pricing of excludable public goods. For a large economy with private information about individual preferences, the two approaches are shown to be equivalent if and only if, in addition to incentive compatibility and participation constraints, the final allocation of private-good consumption and admission tickets to public goods satisfies a condition of renegotiation proofness. Without this condition, a mechanism involving mixed bundling, i.e. combination tickets at a discount, is superior.

Key Words: Mechanism Design, Excludable Public Goods, Ramsey-Boiteux Pricing, Renegotiation Proofness, Bundling

JEL Classification: D61, H21, H41,H42

1 Introduction

This paper studies the relation between Bayesian mechanism design for the provision and pricing of excludable public goods and the Ramsey-Boiteux approach to public-sector service provision and pricing under a government budget constraint. For a large economy with private information about individual preferences, the paper shows that the Bayesian mechanism design problem with interim participation constraints is equivalent to the corresponding Ramsey-Boiteux problem if and only if, in addition to incentive compatibility

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and participation constraints, the final allocation is constrained by a condition of renegotiation proofness in the sense that participants have no scope for incentive-compatible side-trading to a Pareto superior allocation.

The analysis combines three ideas. First, the imposition of interim participation constraints (in addition to incentive compatibility and feasibility) in the Bayesian mechanism design problem with private information is equivalent to the imposition of a "government budget constraint" which requires the costs of public-good provision to be covered by the payments that people are willing to make in order to benefit from the public goods. Second, if the mechanism is unable to prevent people from side-trading, then in the large economy, in the absence of transactions costs, the final allocation of private-good consumption and admission tickets to public goods must be Walrasian, i.e. supported by a price system which does not leave any room for arbitrage. Third, if the mechanism designer is able to prevent people from side-trading, he finds it advantageous to offer admission to the different public goods on the basis of mixed bundling, i.e. a scheme involving combination tickets for multiple public goods coming at a discount relative to the sum of the individual ticket prices. Each of these ideas has been around before: The equivalence of interim participation constraints and the "government budget constraint" is discussed in Hellwig (2003 a), the constraints that frictionless side-trading imposes on mechanism design have been studied by Hammond (1979, 1987) and Guesnerie (1995), and, in the multiproduct monopoly literature, the advantages from mixed bundling have been pointed out by McAfee et al. (1989) and Manelli and Vincent (2002).

The contribution of this paper is to pull these ideas together for a precise characterization of the Ramsey-Boiteux approach from the perspective of Bayesian mechanism design.

The model studied is one where individuals have private information about their preferences for public goods, but through a large-numbers effect, the cross-section distribution of preferences is fixed. This distribution is known to the mechanism designer, so the assessment of alternative levels of public-good provision is unencumbered by information problems. However, any attempt to relate financial contributions to the benefits that people draw from the public goods is hampered by the privateness of people’s information about their preferences. If the ability of excluding people from the enjoyment of the public goods is never used, i.e. if everybody is freely admitted to all public goods, the only incentive compatible financing scheme involves lump sum payments from all individuals, whether they benefit from the public goods or not.

Under a financing scheme with lump sum payments from all individuals, people who have no taste for the public goods are negatively affected by the provision of these goods. Their participation must be therefore be based on the government’s power of coercion rather than any voluntary agreement. Using the government’s power of coercion to levy lump sum contributions from people who do not benefit from the public goods raises concerns about equity as well as the possibility of abuse of this power.1

1On equity, see Section 5 below and Hellwig (2003 b), on power abuse, Bierbrauer (2002).
Given these concerns, the paper looks at the problem of designing a mechanism for the provision and pricing of multiple excludable public goods under the additional constraint that nobody can be coerced into participating. This constraint stands in the tradition of Lindahl’s (1919) voluntary-exchange approach to public-good provision and taxation. It eliminates the possibility of raising funds through lump sum contributions so public goods have to be fully financed from payments that people are willing to make in order to enhance their prospects of benefitting from them. In the large economy, where each individual is too insignificant to affect the provision of the public good itself, such payments are entirely motivated by the desire to avoid being excluded, and public goods must be financed by admission fees. With zero marginal social costs of individual use of public goods, admission fees induce an inefficiency, but in the absence of the other sources of funds, this inefficiency is unavoidable if the public goods are to be provided at all. The argument is the same as in the Ramsey-Boiteux theory of second-best pricing for goods and services whose production involves significant fixed costs.

For the large economy with private information about individual preferences, the paper identifies a condition of renegotiation proofness under which the Bayesian mechanism design problem with interim participation constraints is actually equivalent to the Ramsey-Boiteux problem of finding an optimal vector of admission fees for the different excludable public goods. The final allocation of private goods and of admission tickets for the public goods is said to be renegotiation proof if it does not leave consumers with an incentive to engage in (incentive compatible) Pareto improving trades among each other. In the large economy, this renegotiation proofness condition is satisfied if and only if the final allocation of private goods and of admission tickets for public goods is Walrasian.

The associated price vector is precisely the vector of consumer prices that the Ramsey-Boiteux theory is concerned with. A Bayesian mechanism that satisfies renegotiation proofness as well as interim incentive compatibility and individual rationality is thus identified with a vector of admission prices. The mechanism design problem is equivalent to the corresponding Ramsey-Boiteux problem. A vector of optimal admission fees must satisfy a version of the Ramsey-Boiteux inverse-elasticities rule.

The admission fees on excludable public goods can be also used to finance some nonexcludable public goods. This cross-subsidization eliminates the problem discussed by Mailath and Postlewaite (1990) that in a large economy with private information, it may not be possible to finance the provision of nonexcludable public goods at all. The general structure of optimal admission fees for excludable public goods, in particular the inverse-elasticities rule, is unaffected.

If renegotiation proofness is not imposed, the mechanism design problem is richer than the Ramsey-Boiteux problem. In this case, a second-best mechanism

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2 See Musgrave (1959), Ch. 4, pp. 61 - 89, for an extensive discussion of the contract theory of the state and the benefit approach to public finance culminating in Lindahl’s voluntary-exchange approach.

3 The link between excludable public goods and the Ramsey-Boiteux pricing problem has been pointed out by Samuelson (1958, 1969) and Laffont (1982/1988), see also Drèze (1980).
can involve bundling of public goods and even randomized schemes under which a person’s admission to one or several public goods is the subject of a lottery. For excludable public goods, Fang and Norman (2003) have shown that under certain assumptions about the underlying data, the Ramsey-Boiteux solution may be dominated by a mechanism involving pure bundling in the sense that consumers are offered admission to all public goods at once or to none. To their finding, the present paper adds the result that, if the cross-section distributions of valuations for the different public goods are mutually independent, then the optimal renegotiation proof mechanism, i.e. the optimal Ramsey-Boiteux solution, is always dominated by a mechanism involving mixed bundling, i.e., a scheme where consumers are offered admission to public goods separately as well as in bundles, where any bundle comes at a discount relative to the sum of its individual components. The result parallels similar results of McAfee et al. (1989) and Manelli and Vincent (2002) for multiproduct monopoly profit maximization. It implies that the renegotiation proofness requirement is necessary as well as sufficient for the applicability of the Ramsey-Boiteux approach.

In the following, Section 2 lays out the basic model of the large economy with private information about individual preferences. Section 3 establishes the equivalence of the Bayesian mechanism design problem with interim participation constraints and the Ramsey-Boiteux problem under the renegotiation proofness condition. Section 4 shows that the equivalence of the two approaches breaks down and that some form of bundling dominates Ramsey-Boiteux pricing if renegotiation proofness fails. Section 5 shows that the inverse-elasticities rule of the original Ramsey-Boiteux analysis is replaced by a weighted inverse-elasticities rule if the mechanism designer is inequality averse, the weights taking account of differences in marginal social valuations attached to the consumers of the different public goods. If inequality aversion is sufficiently large, then, as in Hellwig (2003 b), the desire for redistribution may replace the interim participation constraints and induced “government budget constraint” as a rationale for admission fees. Proofs are given in the Appendix.

2 Bayesian Mechanism Design in a Model with Multiple Public Goods

I study public-good provision in a large economy with one private good and \( m \) public goods. The public goods are excludable. An allocation must determine provision levels \( Q_1,...,Q_m \) for the public goods and, for each individual \( h \) in the economy, an amount \( c^h \) of private-good consumption and a set \( J^h \) of public goods to which the individual is admitted. Given \( c^h, J^h, \) and \( Q_1,...,Q_m \), the consumer obtains the payoff

\[
c^h + \sum_{i \in J^h} \theta^h_i Q_i.
\]

(2.1)

The vector \( \theta^h = (\theta^h_1,...,\theta^h_m) \) of parameters determining the consumer’s pref-
erences for the different public goods is the realization of a random variable \( \tilde{\theta}^h \) taking values in \([0,1]^m\), which is defined on some underlying probability space \((X, \mathcal{F}, P)\). Private-good consumption and public-goods admissions will typically be made to depend on \( \theta^h \). In addition, they will also be allowed to depend on the realization \( \omega^h \) of a further random variable \( \tilde{\omega}^h \) taking values in \([0,1]\). This random variable is introduced to allow for the possibility of individual randomization in public-good admissions and private-good consumption.

The random variables \( \tilde{\theta}^h \) and \( \tilde{\omega}^h \) are assumed to be independent. Their distributions \( F \) and \( \nu \) are assumed to be the same for all agents. Moreover, the distribution \( F \) of the vector \( \tilde{\theta}^h \) of preference parameters has a strictly positive, continuously differentiable density \( f(.) \).

The set of participants is modelled as an atomless measure space \((H, \mathcal{H}, \eta)\). I assume a large-numbers effect whereby the cross-section distribution of the pair \((\tilde{\theta}^h(x), \tilde{\omega}^h(x))\) in the population is \(P\)-almost surely equal to the probability distribution \( F \times \nu \). Thus, for almost every \( x \in X \), I postulate that

\[
\frac{1}{\eta(H)} \int \varphi(\tilde{\theta}^h(x), \tilde{\omega}^h(x)) d\eta(h) = \int_{[0,1]^{m+1}} \varphi(\theta, \omega) dF(\theta) d\nu(\omega) \quad (2.2)
\]

for every \( F \times \nu \)-integrable function \( \varphi \) from \([0,1]^{m+1}\) into \( \mathbb{R}^4 \).

I restrict the analysis to allocations that satisfy an *ex-ante neutrality* or *anonymity* condition. The level \( c^h \) of an individual’s private-good consumption and the set \( J^h \) of public goods to which the individual is admitted are assumed to depend on \( h \) and on the state of the world \( x \) only through the realizations \( \tilde{\theta}^h(x) = \theta^h \) and \( \tilde{\omega}^h(x) = \omega^h \) of the random variables \( \tilde{\theta}^h \) and \( \tilde{\omega}^h \). In principle, \( c^h \) and \( J^h \) should also depend on the cross-section distribution of the other agents’ parameter realizations \( \tilde{\theta}^{h'}(x) = \theta^{h'} \) and \( \tilde{\omega}^{h'}(x) = \omega^{h'} \) in the population, but because this cross-section distribution is constant and independent of \( x \), there is no need to make this dependence explicit. This is a major advantage of working with the large-economy specification with the law of large numbers.

An *allocation* is thus defined as an array

\[
A = (Q^A, c^A(\ldots), \chi^A_1(\ldots), \ldots, \chi^A_m(\ldots)), \quad (2.3)
\]

such that \( Q^A = (Q_1^A, \ldots, Q_m^A) \) is a vector of public-good provision levels, and \( c^A(\ldots), \chi^A_1(\ldots), \ldots, \chi^A_m(\ldots) \) are functions which stipulate for each \((\theta, \omega) \in [0,1]^{m+1}\), a level \( c^A(\theta, \omega) \) of private-good consumption and indicators \( \chi^A_i(\theta, \omega) \) for admission to public goods \( i = 1, \ldots, m \), to be applied to participant \( h \) in the state \( x \) if \((\tilde{\theta}^h(x), \tilde{\omega}^h(x)) = (\theta, \omega) \). The indicator \( \chi^A_i(\theta, \omega) \) takes the value one if the consumer is admitted and the value zero, if he is not admitted to public good \( i \).

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1As discussed by Judd (1985), the law-of-large-numbers property (2.2) is consistent with, though not implied by stochastic independence of the random pairs \((\tilde{\theta}^h, \tilde{\omega}^h)\), \( h \in H \). As discussed in Alós-Ferrer (2002), in the absence of independence, one may assume that the map \((h, x) \rightarrow (\tilde{\theta}^h(x), \tilde{\omega}^h(x))\) is jointly measurable and impose (2.2) directly as an assumption; however, in this case, the question arises whether the mechanism designer shouldn’t exploit the correlations of the random pairs \((\tilde{\theta}^h, \tilde{\omega}^h)\) across agents.
The economy has an exogenous production capacity permitting the aggregate consumption $Y$ of the private good if no public goods are provided. If a vector $Q$ of public-good provision levels is to be provided, an aggregate amount $K(Q)$ of private-good consumption must be foregone. An allocation $A = (Q^A, c^A(\ldots), \chi_1^A(\ldots), \ldots, \chi_m^A(\ldots))$ is feasible if

$$\frac{1}{\eta(H)} \int_H c^A(\tilde{\theta}^h(x), \tilde{\omega}^h(x))d\eta(h) + K(Q^A) \leq Y$$

(2.4)

for almost every $x \in X$, so the sum of aggregate consumption and public-good provision costs does not exceed $Y$. By the large-numbers condition (2.2), this requirement is equivalent to the inequality

$$\int_{[0,1]^{m+1}} c^A(\theta, \omega)f(\theta)d\theta d\nu(\omega) + K(Q^A) \leq Y.$$  

(2.5)

The cost function $K(.)$ is assumed to be strictly increasing, with $K(0) = 0$, strictly convex, and twice continuously differentiable, with partial derivatives $K_i(.)$ satisfying $\lim_{k \to \infty} K_i(Q^k) = 0$ for any sequence $\{Q^k\}$ with $\lim_{k \to \infty} Q^k_i = 0$ and $\lim_{k \to \infty} K_i(Q^k) = \infty$ for any sequence $\{Q^k\}$ with $\lim_{k \to \infty} Q^k_i = \infty$.

The allocation $A = (Q^A, c^A(\ldots), \chi_1^A(\ldots), \ldots, \chi_m^A(\ldots))$, provides consumer $h$ with the ex ante expected payoff

$$\int_{[0,1]^{m+1}} \left[c^A(\theta, \omega) + \sum_{i=1}^m \chi_i^A(\theta, \omega)\theta_iQ^A_i\right]f(\theta)d\theta d\nu(\omega).$$

(2.6)

Because of the ex ante neutrality property of allocations, (2.6) is independent of $h$. All participants are therefore in agreement about the ex ante ranking of allocations. Taking this ranking as a normative standard, I refer to an allocation as being first-best if it maximizes (2.6) over the set of feasible allocations. By (2.2), the ex ante expected payoff for any one participant is equal to the aggregate per capita payoff

$$\frac{1}{\eta(H)} \int_H \left[c^A(\tilde{\theta}^h(x), \tilde{\omega}^h(x)) + \sum_{i=1}^m \chi_i^A(\tilde{\theta}^h(x), \tilde{\omega}^h(x))\tilde{\theta}_i^h(x)Q^A_i\right]d\eta(h)$$

(2.7)

for almost every $x \in X$. A first-best allocation therefore is also best if the mechanism designer is concerned with this cross-section aggregate of payoffs in the population. In taking (2.7) or (2.6) to be a suitable welfare indicator, I implicitly assume that there is no risk aversion on the side of participants and no inequality aversion on the side of the mechanism designer. This assumption will be relaxed in Section 5 below.

In a slightly more compact notation, the first-best welfare problem is to choose $A$ so as to maximize

$$\int_{[0,1]^{m+1}} \left[C^A(\theta) + \sum_{i=1}^m \pi_i^A(\theta)\theta_iQ^A_i\right]f(\theta)d\theta$$

(2.8)
under the constraint that

$$\int_{[0,1]^m} C^A(\theta) f(\theta) d\theta + K(Q) \leq Y, \quad (2.9)$$

where

$$C^A(\theta) := \int_{[0,1]} c^A(\theta, \omega) d\nu(\omega), \quad (2.10)$$

$$\pi^A_i(\theta) := \int_{[0,1]} \chi^A_i(\theta, \omega) d\nu(\omega), \quad (2.11)$$

are the conditional expectations of a consumer’s private-good consumption and admission probability for public good $i$, given the information that $\tilde{\theta}^A = \theta$.

By standard arguments, one obtains:

**Lemma 2.1** An allocation $A$ is first-best if and only if it satisfies the feasibility condition (2.5) with equality and, for $i = 1, \ldots, m$, one has $\pi^A_i(\theta) = 1$ for almost all $\theta \in [0,1]^m$ and

$$K_i(Q^A) = \int_0^1 \theta_i dF_i(\theta_i), \quad (2.12)$$

where $F_i$ is the marginal distribution of $\tilde{\theta}_i$, the $i$-th component of the random vector $\tilde{\theta}$.

In a first-best allocation, the ability to exclude people from the enjoyment of a public good is never used. Moreover, the levels of public-good provision are chosen so that for each $i$, the marginal cost $K_i(Q^A)$ of increasing the level at which public good $i$ is provided is equal to the aggregate marginal benefits that consumers in the economy draw from the increase. Given the assumption that $\lim_{k \to \infty} K_i(Q^k) = 0$ for any $i$ and any sequence $\{Q^k\}$ with $\lim_{k \to \infty} Q^k_i = 0$, it follows that in a first-best allocation provision levels of all public goods are positive, and so is $K(Q)$.

Turning to the specification of information, I assume that each consumer knows the realization $\theta$ of his own preference parameter vector, but beyond knowing the measure $\nu$, he has no further information about the random variable $\tilde{\omega}$. The information about $\theta$ is private. Apart from the distribution $\tilde{F} \times \nu$, nobody knows anything about the pair $(\tilde{\theta}, \tilde{\omega})$ pertaining to somebody else. Given this information specification, an allocation $A$ is said to be incentive compatible if and only if, for all $\theta$ and $\theta' \in [0,1]^m$,

$$v^A(\theta) \geq C^A(\theta') + \sum_{i=1}^m \pi^A_i(\theta') \theta_i Q^A_i, \quad (2.13)$$

\[\text{If there was risk aversion on the side of consumers or inequality aversion on the side of the mechanism designer, the conditions of Lemma 2.1 would have to be augmented by a condition equating the social marginal utility of private-good consumption across agents. For details, see Hellwig (2003 b, 2004).}\]
where

\[ v^A(\theta) := C^A(\theta) + \sum_{i=1}^{m} \pi_i^A(\theta) \theta_i Q_i^A. \]  

(2.14)

From Rochet (1987), one has

**Lemma 2.2** An allocation \( A \) is incentive compatible if and only if the expected-payoff function \( v^A(.) \) that is given by (2.14) is convex and has partial derivatives \( v_i^A(.) \) satisfying

\[ v_i^A(\theta) = \pi_i^A(\theta) Q_i^A \]  

(2.15)

for all \( i \) and almost all \( \theta \in [0, 1]^m \).

For a first-best allocation, with \( \pi_i^A(\theta) = 1 \) for almost all \( \theta \in [0, 1]^m \), (2.15) is equivalent to the requirement that \( C^A(\theta) \) be independent of \( \theta \). Upon combining Lemmas 2.1 and 2.2, one therefore obtains:

**Proposition 2.3** A first-best allocation \( A \) is incentive compatible if and only if

\[ C^A(\theta) = Y - K(Q^A) \]  

(2.16)

for almost all \( \theta \in [0, 1]^m \).

Proposition 2.3 indicates that in the large economy studied here a first-best allocation can be implemented if and only if public-good provision is entirely financed by a lump sum payment, which people make regardless of their preferences.\(^6\) This lump sum payment amounts to \( K(Q^A) > 0 \) per person. With such lump sum payments, public-good provision hurts people who do not care for the public goods and benefits people who care a lot for them. People who do not care for the public goods are strictly worse off than they would be if they could just have the private-good consumption \( Y \).

To articulate this concern formally, I introduce a concept of individual rationality. I assume that each participant \( h \) has the capacity to produce \( Y \) units of the private good (at no further cost to himself) and that, without any agreement on the provision of public goods, each participant simply consumes these \( Y \) units of the private good out of his own production. Given this assumption, an allocation \( A \) is said to be *individually rational* if the expected payoff (2.14) satisfies \( v^A(\theta) \geq Y \) for all \( \theta \in [0, 1]^m \). With this definition, Proposition 2.3 implies that a first-best, incentive-compatible allocation cannot be individually rational. Indeed, any incentive compatible allocation with \( K(Q) > 0 \) and \( \pi_i(\theta) = 1 \) for almost all \( \theta \in [0, 1]^m \) fails to be individually rational.\(^7\)

\(^6\)This result is moot if participants are risk averse or the mechanism designer is inequality averse. In this case, (2.16) is incompatible with the requirement that the social marginal utility of private-good consumption be equalized across agents, so a first-best allocation is never incentive compatible. For details, see Hellwig (2003 b, 2004).

\(^7\)These observations provide large-economy, multiple-public-goods extensions of the finite-economy, single-public-good results of Güth and Hellwig (1986) as well as Mailath and Postlewaite (1990).
The interpretation of the inequality $v^A(\theta) \geq Y$ as a participation constraint raises some questions. First, how robust is the analysis to a relaxation of the symmetry assumption which gives all people the same outside option? Second, why should participation constraints prevent the government from taxing incomes (the private-good production $Y$) and using the proceeds to cover the costs of public-goods provision? Third, why don’t participation constraints allow for the possibility that a person who rejects the proposed allocation gets to consume his own output $Y$ and at the same time to enjoy the public goods provided by others?

The first of these questions is easy to answer. If different people have different outside options, these differences must be reflected in the specification of the allocation, so if the outside option of consumer $h$ is given by this household’s own private-good production capacity $y^h$, his private-good consumption must take the form $c(\theta^h, \omega^h, y^h)$ and similarly, for public-goods admissions. The individual-rationality constraint then takes the form $v(\theta^h, y^h) \geq y^h$ and the impossibility result goes through unchanged. The analysis can be extended without significant change to a model with endogenous production as long as the government refrains from taxing people’s production capacities or people’s outputs.

With the assumption that public-goods provision is not financed from taxes on production capacities or production activities to finance public-goods provision, I follow tradition. As discussed in Musgrave (1959), Lindahl’s (1919) creation of the theory of public goods was designed as an interpretation of government activities in terms of voluntary exchanges, culminating the development of the benefit approach to public finance as part of a theory of the state built on contracts. In Lindahl’s analysis, the government was considered to be first providing for equity through redistribution and then providing for efficient public-goods provision through voluntary contracting on a do-ut-des basis. The voluntariness of exchange at the second stage of government activity corresponds precisely to the individual-rationality condition introduced here.\footnote{This individual-rationality condition is central to the entire literature on the difficulties caused by the relevant version of the Myerson-Satterthwaite theorem for public-goods provision; see, e.g., Güth and Hellwig (1986), Rob (1989), Mailath and Postlewaite (1990), Norman (2004).}

Given the government’s power of coercion, the voluntary-exchange approach or the benefit approach to public finance may be considered to be unrealistic, but even so, there is some interest in understanding the implications of this approach.\footnote{For a study of public-goods provision and pricing when income taxation is available as a source of finance, see Hellwig (2004).}

Finally, the possibility that a person who rejects the proposed allocation gets to consume his own output $Y$ and at the same time to enjoy the public goods provided by others is not left out of the analysis. Under the incentive compatibility condition (2.13) with $\theta' = 0$ and given the participation constraint
\( v^A(0) \geq Y \), the allocation satisfies

\[
v^A(\theta) \geq Y + \sum_{i=1}^{m} \pi^A_i(0) \theta_i Q^A_i,
\]

which is precisely the requirement in question.

Because a first-best allocation cannot be individually rational as well as incentive compatible, it is of interest to consider *second-best allocations*, which maximize the aggregate surplus \((2.8)\) over the set of all feasible, incentive-compatible and individually rational allocations. For a single excludable public good, i.e. \( m = 1 \), second-best allocations have been characterized by Schmitz (1997) and Norman (2004). Assuming that the function

\[
\theta \rightarrow g(\theta) := \theta - \frac{1 - F(\theta)}{f(\theta)}
\]

is nondecreasing, they show that for any second-best allocation \( A \) there is some \( \hat{\theta}^A \in (0, 1) \) such \( C^A(\theta) = Y, \pi^A(\theta) = 0 \) for \( \theta < \hat{\theta}^A \) and \( C^A(\theta) = Y - \hat{\theta}^A Q^A \), \( \pi^A(\theta) = 1 \) for \( \theta > \hat{\theta}^A \). There is an admission fee \( p^A \), equal to \( \hat{\theta}^A Q^A \), so that people for whom the benefit \( \theta Q^A \) from the enjoyment of the public good exceeds \( p^A \) pay the fee and are admitted, and people for whom the benefit \( \theta Q^A \) is less than \( p^A \) do not pay the fee and are not admitted. The critical \( \hat{\theta}^A \) and the fee \( p^A = \hat{\theta}^A Q^A \) are chosen so that the aggregate revenue \( p^A(1 - F(\hat{\theta}^A)) \) just covers the cost \( K(Q^A) \) of providing the public good.

For \( m > 1 \), a general characterization of second-best allocations is not as yet available. As discussed by Rochet and Choné (1998), Lemma 2.2 implies that the problem of finding such an allocation can be formulated in terms of the public-good provision levels \( Q^A_1, \ldots, Q^A_m \) and the expected-payoff function \( v^A(.) \). For any \( Q^A = (Q^A_1, \ldots, Q^A_m) \) and any convex function \( v^A(.) \) with partial derivatives satisfying \( v^A_i(\theta) \in [0, Q^A_i] \) for all \( i \) and almost all \( \theta \in [0, 1]^m \), an incentive-compatible allocation is obtained by setting

\[
\pi^A_i(\theta) = \begin{cases} 
\frac{1}{Q^A_i} \lim_{\theta_i' \rightarrow \theta_i} v^A_i(\theta_i', \theta_{-i}) & \text{if } Q^A_i > 0, \\
0 & \text{if } Q^A_i = 0,
\end{cases} \quad (2.17)
\]

and

\[
C^A(\theta) = v^A(\theta) - \sum_{i=1}^{m} \theta_i v^A_i(\theta), \quad (2.19)
\]

and specifying \( c^A(.,.) \) and \( \chi^A_1(.,.), \ldots, \chi^A_m(.,.) \) accordingly. By (2.19), (2.5) is equivalent to the inequality

\[
\int_{[0,1]^m} \left[ v^A(\theta) - \sum_{i=1}^{m} \theta_i v^A_i(\theta) \right] f(\theta) d\theta \leq Y - K(Q^A). \quad (2.20)
\]
The problem of finding a second-best allocation is therefore equivalent to the problem of choosing $Q^A$ and a convex function $v^A$ with partial derivatives satisfying $v^A_i(\theta) \in [0, Q^A_i]$ for all $i$ and almost all $\theta$ so as to maximize (2.8) subject to (2.20) and the participation constraint $v^A(\theta) \geq Y$ for all $\theta$.

As usual in problems of multi-dimensional mechanism design, the second-order conditions for incentive compatibility (convexity of the expected-payoff function $v^A(\cdot)$) and integrability conditions (equality of the cross derivatives $v^A_{ij}(\theta)$ and $v^A_{ji}(\theta)$, i.e. of the derivatives $\frac{\partial\pi^A_i(\theta)}{\partial \theta_j} Q^A_i$ and $\frac{\partial\pi^A_i(\theta)}{\partial \theta_i} Q^A_j$) are difficult to handle analytically. For the case of two excludable public goods, i.e. $m = 2$, with independent preference parameters having identical two-point distributions, a complete characterization of second-best allocations in finite economies is given by Fang and Norman (2003). The large-economy limits of these allocations involve nonseparability and genuine randomization in admission rules.

### 3 Renegotiation Proofness and the Optimality of Ramsey-Boiteux Pricing

Rather than carry the analysis of second-best allocations further, I consider third-best allocations as allocations which maximize aggregate surplus subject to feasibility, incentive compatibility, individual rationality, and an additional condition of renegotiation proofness. The latter condition reflects the idea that the agency which implements the chosen mechanism is unable to verify the identities of people who present tickets for being admitted to the enjoyment of a public good. In particular, the agency is unable to check whether the people who present tickets for admission to a public good are in fact the same people to whom the tickets have been issued. It is also unable to prevent people from trading these tickets, public good by public good, as well as the private good, among each other. If the initial allocation of tickets leaves room for a Pareto improvement through such trading, then, as discussed by Hammond (1979, 1987) and Guesnerie (1995), in the absence of transactions costs, such trading will occur, and the initial allocation will not actually be the final allocation.

Underlying the imposition of renegotiation proofness is the idea that, one may suppose that, regardless of the allocation that is initially chosen by the mechanism designer, in the absence of transactions costs, any allocation that is finally implemented must itself be renegotiation proof. If the mechanism designer is aware of the possibility of renegotiation and if he cares about the allocation that is finally implemented rather than the one that is initially chosen, his choice may be directly expressed in terms of the final, renegotiation proof allocation. Indeed if he chooses a renegotiation proof allocation from the beginning, this initial allocation will also be the final allocation. Given these considerations, I refer to an allocation as being third-best if and only if it maximizes the aggregate surplus (2.8) over the set of all feasible, incentive-
compatible, individually rational, and renegotiation proof allocations.\footnote{In contrast, Hammond (1979, 1987) and Guesnerie (1995) treat the mechanism design problem in terms of a two-stage game with a revelation game in the first stage determining an allocation which provides the starting point for side-trading in the second stage leading to a Walrasian outcome. The approach taken here collapses the two stages into one by imposing a renegotiation proofness constraint on the mechanism designer.}

To define renegotiation proofness formally, I say that a net-trade allocation for private-good consumption and public-good admission tickets is an array \((z_c(\ldots), z_1(\ldots), \ldots, z_m(\ldots))\) such that for each \((\theta, \omega)\), \(z_c(\theta, \omega)\) and \(z_1(\theta, \omega), \ldots, z_m(\theta, \omega)\) are the net additions to private-good consumption and admission ticket holdings for public goods of a consumer with preference parameter vector \(\theta\) and indicator value \(\omega\). Given \(A\), a net-trade allocation \((z_c(\ldots), z_1(\ldots), \ldots, z_m(\ldots))\) is said to be feasible if \(\chi_i^A(\theta, \omega) + z_i(\theta, \omega) \in \{0, 1\}\) for all \((\theta, \omega) \in [0, 1]^{m+1}\) and, moreover,

\[
\int_{[0,1]} z_i(\hat{\theta}^h(x), \hat{\omega}^h(x)) d\eta(h) = 0 \tag{3.1}
\]

for \(i = c, 1, \ldots, m\) and almost all \(x \in X\), which by (2.2) is equivalent to the requirement that

\[
\int_{[0,1]^{m+1}} z_i(\theta,\omega) f(\theta) d\theta d\nu(\omega) = 0 \tag{3.2}
\]

for \(i = c, 1, \ldots, m\).\footnote{To keep matters simple, I assume that \(Y\) is large enough so that nonnegativity of private-good consumption is not an issue.} Given \(A\), the net-trade allocation \((z_c(\ldots), z_1(\ldots), \ldots, z_m(\ldots))\) is said to be incentive compatible, if

\[
z_c(\theta, \omega) + \sum_{i=1}^{m} z_i(\theta, \omega) \theta_i Q_i^A \geq z_c(\theta', \omega') + \sum_{i=1}^{m} z_i(\theta', \omega') \theta_i Q_i^A \tag{3.3}
\]

for all \(\theta\) and \(\theta'\) in \([0, 1]^m\) and all \(\omega\) and \(\omega'\) in \(\Omega\) for which \(\chi_i^A(\theta, \omega) + z_i(\theta', \omega') \in \{0, 1\}\) for all \(i\). The idea is that the holdings \((c^A(\theta, \omega), \chi_1^A(\theta, \omega), \ldots, \chi_m^A(\theta, \omega))\) of private-good consumption and public-goods admission tickets of a given agent as well as the realization \((\theta, \omega)\) of his preference parameter vector \(\theta\) and randomization device \(\tilde{\omega}\) are not known by anybody else.\footnote{One might argue that the mechanism designer knows the consumer’s actual holdings as well as \(\omega\), and therefore the incentive constraints on net-trade allocations might be alleviated. Such loosening of incentive constraints would tend to enhance the scope for renegotiations and make the condition of renegotiation proofness even more restrictive. In the large economy considered here, it does not actually make a difference because the characterization of renegotiation proofness in Lemma 3.1 remains valid. In a finite economy, there would be a difference.} Therefore, if the agent claims that his preference parameter vector is \(\theta'\), he obtains the net trade \((z_c(\theta', \omega'), z_1^A(\theta', \omega'), \ldots, z_m^A(\theta', \omega'))\) that is available to an agent with parameter vector \(\theta'\) when his randomization variable takes the value \(\omega'\). Incentive compatibility of the net-trade allocation requires that such a claim must not provide the agent with an improvement over the stipulated net trade \((z_c(\theta, \omega), z_1(\theta, \omega), \ldots, z_m(\theta, \omega))\).

An allocation \(A\) is said to be renegotiation proof if, starting from \(A\), there is no feasible and incentive compatible net-trade allocation which provides a
Pareto improvement in the sense that for all \((\theta, \omega) \in [0,1]^{m+1}\), the utility gain from the net trade \((z_c(\theta, \omega), z_1(\theta, \omega), ..., z_m(\theta, \omega))\) is nonnegative, i.e.

\[
z_c(\theta, \omega) + \sum_{i=1}^{m} z_i(\theta, \omega)\theta_iQ^A_i \geq 0, \tag{3.4}
\]

and the aggregate utility gain is strictly positive, i.e.

\[
\int [z_c(\tilde{\theta}^h, \tilde{\omega}^h) + \sum_{i=1}^{m} z_i(\tilde{\theta}^h, \tilde{\omega}^h)] d\eta(h) > 0 \tag{3.5}
\]

with positive probability; by (2.2), the latter inequality is equivalent to the inequality

\[
\int_{[0,1]^{m+1}} [z_c(\theta, \omega) + \sum_{i=1}^{m} z_i(\theta, \omega)\theta_iQ^A_i] dF(\theta) d\nu(\omega) > 0, \tag{3.6}
\]

which actually implies that (3.5) holds with probability one.

As introduced here, the concept of renegotiation proofness presumes that admission tickets to all public goods can be traded separately. A weaker concept of renegotiation proofness would be obtained if the mechanism designer were unable to prevent sidetrading, but is able to prepare admission tickets to bundles of public goods in such a way that unbundling is impossible. This weaker concept is briefly discussed at the end of Section 4 and in Appendix B.

For the strong concept introduced here, the following lemma shows that renegotiation proofness holds if and only if there exists a price system which supports the allocation as a competitive equilibrium of the exchange economy in which people trade the private good as well as admission tickets for the different public goods, taking the vector \(Q^A\) of public-good provision levels as given.

**Lemma 3.1** An allocation \(A\) is renegotiation proof if and only if there exist prices \(p_1^A, ..., p_m^A\) such that for \(i = 1, ..., m\) and almost all \((\theta, \omega) \in [0,1]^{m} \times \Omega\), one has

\[
\chi_i^A(\theta, \omega) = 0 \text{ if } \theta_iQ^A_i < p_i^A \tag{3.7}
\]

and

\[
\chi_i^A(\theta, \omega) = 1 \text{ if } \theta_iQ^A_i > p_i^A. \tag{3.8}
\]

Renegotiation proofness implies, for each public good \(i\), a simple dichotomy between a set of participants with high \(\theta_i\), who get admission to public good \(i\) with probability one, and a set of participants with low \(\theta_i\), who do not get admission to public good \(i\). Renegotiation proofness leaves no room for randomized admissions.

The price characterization of renegotiation proof allocations provides for a drastic simplification of incentive compatibility. The prices \(p_1, ..., p_m\) serve as
admission fees. In an incentive compatible allocation, a consumer is admitted to public good $i$ is obtained if and only if he pays the fee $p_i$. Consumers with $\theta_i Q_i > p_i$ pay the fee and enjoy the public good for a net benefit equal to $\theta_i Q_i - p_i$, consumers with $\theta_i Q_i < p_i$ do not pay the fee and are excluded from the public good. Formally, one obtains:

**Lemma 3.2** An allocation $A$ is renegotiation proof and incentive compatible if and only if there exist prices $p_1^A, ..., p_m^A$ such that for all $\theta \in [0, 1]^m$, the admission probabilities $\pi_i^A(\theta)$, $i = 1, ..., m$, and the conditional expectation $C^A(\theta)$ of private-good consumption satisfy

$$\pi_i^A(\theta) = 0 \text{ if } \theta_i Q_i^A < p_i^A, \quad \pi_i^A(\theta) = 1 \text{ if } \theta_i Q_i^A > p_i^A, \quad (3.9)$$

and

$$C^A(\theta) = C^A(0) - \sum_{i=1}^m \pi_i^A(\theta) p_i^A. \quad (3.11)$$

The associated expected payoff is

$$v^A(\theta) = C^A(0) + \sum_{i=1}^m \max(\theta_i Q_i^A - p_i^A, 0) \quad (3.12)$$

for $\theta \in [0, 1]^m$.

For any incentive compatible and renegotiation proof allocation $A$, the aggregate surplus (2.8) and the feasibility constraint (2.5) take the form

$$C^A(0) + \sum_{i=1}^m \int_0^1 \int_{\hat{\theta}_i(p_i^A, Q_i^A)}^1 (\theta_i Q_i^A - p_i^A) dF_i(\theta_i). \quad (3.13)$$

and

$$C^A(0) - \sum_{i=1}^m p_i^A(1 - F_i(\hat{\theta}_i(p_i^A, Q_i^A))) + K(Q^A) \leq Y \quad (3.14)$$

where $p_1^A, ..., p_m^A$ are the competitive prices associated with the allocation and, for any $i$,

$$\hat{\theta}_i(p_i^A, Q_i^A) := \frac{p_i^A}{Q_i^A} \text{ if } Q_i^A > 0 \text{ and } \hat{\theta}_i(p_i^A, Q_i^A) := 1 \text{ if } Q_i^A = 0. \quad (3.15)$$

The problem of finding a third-best allocation is therefore equivalent to the problem of choosing an expected base consumption $C^A(0)$ as well as public-good provision levels $Q_1^A, ..., Q_m^A$ and prices $p_1^A, ..., p_m^A$, with associated critical preference parameter values $\hat{\theta}_1, ..., \hat{\theta}_m$ satisfying $\hat{\theta}_i Q_i^A = p_i^A$, so as to maximize
(3.13) subject to the feasibility constraint (3.14) and the participation constraint $C^A(0) \geq Y$.

In this maximization, the participation constraint is binding. Otherwise the problem would be solved by the incentive-compatible first-best allocation of Lemma 2.1 and Proposition 2.3, which is obviously renegotiation proof, but violates the participation constraint. Given that the participation constraint is binding, the base consumption $C^A(0)$ in (3.13) and (3.14) can be replaced by the constant $Y$, and one obtains:

**Proposition 3.3** The third-best allocation problem is equivalent to the problem of choosing public-good provision levels $Q_1, ..., Q_m$ and prices $p_1, ..., p_m$ so as to maximize

$$\sum_{i=1}^{m} \int_{\theta_i(p_i, Q_i)}^{1} (\theta_i Q_i - p_i) \, dF_i(\theta_i)$$

(3.16)

under the constraint that

$$\sum_{i=1}^{m} p_i (1 - F_i(\theta_i(p_i, Q_i))) \geq K(Q).$$

(3.17)

The problem of finding a third-best allocation has thus been reformulated in terms of only the public-good provision levels and prices. Given the fees $p_1, ..., p_m$, for any $i$, there are $(1 - F_i(\theta_i(p_i, Q_i)))$ participants with $\theta_i Q_i$ who are willing to pay the fee $p_i$ for admission to public good $i$. The aggregate admission fee revenue from public good $i$ is therefore $p_i (1 - F_i(\theta_i(p_i, Q_i)))$ and the aggregate admission fee revenue from all public goods is $\sum_{i=1}^{m} p_i (1 - F_i(\theta_i(p_i, Q_i)))$. The constraint (3.17) requires that this revenue cover the cost $K(Q)$.

Proposition 3.3 provides an analogue to the results of Hammond (1979, 1987) and Guesnerie (1995) in which the possibility of unrestricted side-trading reduces the general problem of mechanism design for optimal taxation to a Diamond-Mirrlees (1971) problem of finding optimal consumer prices. Here, the third-best allocation problem is equivalent to the Ramsey-Boiteux problem of choosing public-good provision levels and prices so as to maximize aggregate surplus under the constraint that admission fee revenues be sufficient to cover the costs of public-good provision. Because individual-rationality constraints preclude the imposition of lump sum taxes, the costs of public-good provision must be fully financed by payments that people make in order to gain the benefits of the public goods. Renegotiation proofness implies that these payments are characterized by admission fees $p_1, ..., p_m$, as in the Ramsey-Boiteux approach to public-sector service provision and pricing.

The equivalence stated in Proposition 3.3 indicates that the difference between the allocation problem for an excludable public good and the allocation problem for a good whose production involves significant fixed costs is purely one of semantics: The enjoyment of the public good by any one individual can be treated as a private good the production of which involves only a fixed cost and no variable costs.
The following characterization of third-best allocations in terms of first-order conditions is now straightforward.

**Proposition 3.4** Let $A$ be a third-best allocation, and let $p_A^1, \ldots, p_m^A$ be the associated admission fees. Then, for $i = 1, \ldots, m$, one has $Q^A_i > p_A^i > 0$, and, for some $\lambda > 1$,

$$\frac{1}{\lambda} \int_{\hat{\theta}^A_i}^{1} \theta_i dF_i(\theta_i) + \frac{(\lambda - 1)}{\lambda} \hat{\theta}_i^A (1 - F_i(\hat{\theta}_i^A)) = K_i(Q^A_i)$$  \hspace{1cm} (3.18)

and

$$p_A^i f_i(\hat{\theta}_i^A) \frac{1}{Q_i^A} = \frac{\lambda - 1}{\lambda} (1 - F_i(\hat{\theta}_i^A)),$$  \hspace{1cm} (3.19)

where $\hat{\theta}_i^A = \hat{\theta}_i(p_A^i, Q^A_i)$, as given by (3.15).

Condition (3.19) is the usual Ramsey-Boiteux condition for "second-best" consumer prices. The term $(1 - F_i(p_A^i Q^A_i))$ on the right-hand side indicates the level of aggregate demand for admissions to public good $i$ when the price is $p_A^i$ and the "quality", i.e. the provision level is $Q^A_i$. The term $f_i(p_A^i Q^A_i) \frac{1}{Q_i^A}$ on the left-hand side indicates the absolute value of the derivative of demand with respect to $p_A^i$. Condition (3.19) requires admission fees to be chosen in such a way that the elasticities

$$\eta_i^A := \frac{p_A^i (1 - F_i(p_A^i Q^A_i))}{f_i(p_A^i Q^A_i) \frac{1}{Q_i^A}}$$

of demands for admissions to the different public goods are locally all the same, i.e. that

$$1 = \frac{\lambda - 1}{\lambda} \eta_i^A$$

for all $i$, which is the degenerate form taken by the Ramsey-Boiteux inverse-elasticities formula when variable costs are identically equal to zero.\(^\text{13}\)

Condition (3.18) is a version of the Lindahl-Samuelson condition for public-good provision which is appropriate for the third-best allocation problem.\(^\text{14}\)

\(^{13}\)See, e.g., equation (15-23), p. 467, in Atkinson and Stiglitz (1980). If variable costs are positive, e.g., if costs take the form $K(Q, U_1, \ldots, U_m)$, where, for $i = 1, \ldots, m$, $U_i := \int \pi_i dF$ is the aggregate use of public good $i$, equation (3.19) takes the form

$$(p_i - \frac{\partial K}{\partial U_i}) f_i(\hat{\theta}_i) \frac{1}{Q_i} = \frac{\lambda - 1}{\lambda} (1 - F_i(\hat{\theta}_i)), $$

which yields the usual nondegenerate form

$$\frac{p_i - \frac{\partial K}{\partial U_i}}{p_i} = \frac{\lambda - 1}{\lambda} \eta_i$$

of the inverse-elasticities formula.

\(^{14}\)For the case $m = 1$, this condition is also obtained by Norman (2004).
Third-best public-good provision levels are determined in such a way that for each $i$, the marginal cost of providing public good $i$ is equated to a weighted average of the aggregate marginal benefits that are obtained by users and the aggregate marginal revenues that are obtained by the mechanism designer if the admission fee $p_i$ is raised in proportion to $Q_i$ so that the critical $\theta_i(p_i, Q_i)$ is unchanged. If provision costs are additively separable, i.e. if the marginal cost $K_i(Q)$ depends only on $Q_i$, then each of the third-best provision levels $Q_1, ..., Q_m$ will be lower than the corresponding first-best level given by (2.12). The reason is first, that there are fewer users of the public good than in the first-best allocation and second, that the mechanism designer is unable to fully appropriate the benefits from additional public-good provision so aggregate marginal revenues accruing to him are less than aggregate marginal benefits accruing to users.

The analysis is easily extended to a situation with nonexcludable as well as excludable public goods. Suppose, for example, that $n < m$ public goods $1, ..., n$ are nonexcludable and public goods $n+1, ..., m$ are excludable. Nonexcludability of public good $i$ is equivalent to the requirement that $\pi_i(\theta)$ be equal to one for all $\theta$ or, in terms of the Ramsey-Boiteux analysis, that the admission fee for this public good be equal to zero. The third-best allocation problem then is to maximize (3.16) subject to (3.17) and the constraint that $p_i = 0$ for $i = 1, ..., n$. Except for the fact that admission fees for nonexcludable public goods are zero, the conditions for a third-best allocation are the same as before, i.e. admission fees for excludable public goods satisfy an inverse-elasticities formula, and provision levels for all public goods satisfy an appropriate version of the Lindahl-Samuelson condition.

For the nonexcludable public goods, the Lindahl-Samuelson condition takes the form
\[
\frac{1}{\lambda} \int_0^1 \theta_i dF_i(\theta_i) = K_i(Q),
\] (3.20)
so by the same reasoning as before, a third-best allocation involves strictly positive provision levels of nonexcludable as well as excludable public goods. In the large economy, provision of the nonexcludable public goods does not generate any revenue, but nevertheless they are provided. Admission fees from the excludable public goods provide finance for the nonexcludable public goods as well. This cross-subsidization is desirable because, with $\int_0^1 \theta_i dF_i(\theta_i) > 0$ and $K_i(Q) = 0$ when $Q_i = 0$, the benefits of the first (infinitesimal) unit that is provided always exceed the costs. This finding is unaffected by the fact that the cross-subsidization of nonexcludable public goods requires higher admission fees and creates additional distortions for excludable public goods. Concern about these additional distortions will reduce but not eliminate the provision of nonexcludable public goods.\footnote{Given that $\lambda > 1$, a comparison of (3.20) and (2.12) shows that if the cost function $K$ is additively separable, the third-best provision level for public goods $i$ is strictly lower than the corresponding first-best level given by (2.12). On this point, see also Guesnerie (1995).}

This discussion of nonexcludable public goods stands in contrast to the as-
essment of Mailath and Postlewaite (1990) that in a large economy, with asymmetric information and interim participation constraints, a nonexcludable public good will not be provided at all. The Mailath-Postlewaite result presumes a single nonexcludable public good the costs of which have to be covered by revenues coming from this very public good itself. Here there is no such requirement. Interim participation constraints for individual consumers and the induced aggregate budget (feasibility) constraint allow for cross-subsidization between public goods. For nonexcludable public goods, this cross-subsidization eliminates the Mailath-Postlewaite problem.

4 The Desirability of Bundling and the Necessity of Renegotiation Proofness for the Optimality of Ramsey-Boiteux Pricing

In the preceding analysis, the requirement of renegotiation proofness has served to reduce a complex problem of multidimensional mechanism design to a simple $m$-dimensional pricing problem. The key to this simplification lies in the observation that renegotiation proofness restricts admission rules so that the expected-payoff function $v^A(.)$ takes the form (3.12), which is additively separable and convex in $\theta_1, ..., \theta_m$. The integrability condition $v^A_{ij} = v^A_{ji}$ and the second-order condition for incentive compatibility (convexity), are then automatically satisfied.

However, renegotiation proofness is restrictive. If the mechanism designer is able to control the identities of people presenting admission tickets to the different public goods or if there are some impediments to renegotiation, an optimal allocation will typically not have the simple structure that is implied by renegotiation proofness. Second-best allocations, which maximize aggregate surplus subject to feasibility, incentive compatibility and individual rationality, without renegotiation proofness, tend to involve bundling of the different public goods and, possibly, randomized admissions. These devices reduce the efficiency losses that are associated with the use of admission fees to reduce the participants’ information rents.

Thus, Fang and Norman (2003) have noted that, if the random variables $\tilde{\theta}_1, ..., \tilde{\theta}_m$ are independent and identically distributed, the weighted sum $\sum_{i=1}^m \tilde{\theta}_i Q_i$ has a lower coefficient of variation than any one of its summands, and therefore, under certain additional assumptions about the distribution of $\tilde{\theta}_i$, an allocation involving pure bundling can dominate a third-best allocation because it involves a lower incidence of participants being excluded. Pure bundling refers to a situation where participants are admitted either to all public goods at once or to none.

In the following, I show that, if $\tilde{\theta}_1, ..., \tilde{\theta}_m$ are mutually independent, a third-best allocation is always dominated by an allocation involving mixed bundling, i.e. an allocation where participants can obtain admission to each public good separately as well as admission to different public goods at the same time.
through a combination ticket which comes at a discount relative to the individual tickets. Even if tastes for the different public goods are completely unrelated, the allocation that is induced by the best pricing scheme à la Ramsey-Boiteux is not second-best. Renegotiation proofness is thus necessary as well as sufficient for Ramsey-Boiteux pricing to be equivalent to optimal mechanism design under interim incentive compatibility and individual-rationality constraints. This finding complements the results of Fang and Norman. The argument involves a straightforward adaptation of corresponding arguments in the multiproduct monopoly models of McAfee et al. (1989) or Manelli and Vincent (2002).

To fix notation and terminology, let $M = \{1, \ldots, m\}$ be the set of public goods, and let $\mathcal{P}(M)$ be the set of all subsets of $M$. As in Manelli and Vincent (2002), a function $P : \mathcal{P}(M) \rightarrow \mathbb{R}$ is called a price schedule, with the interpretation that for any set $J \subset M$, $P(J)$ is the amount of private-good consumption that a participant has to give up in order to get a combination ticket for admission to the public goods in $J$.

Given $Q$, the allocation $(Q, c_{\mathcal{P}(\theta)}(\cdot), \chi_{1\mathcal{P}(\cdot)}, \ldots, \chi_{m\mathcal{P}(\cdot)})$ is said to be induced by the price schedule $P$ if, for any $\theta \in [0, 1]^m$, there exists a vector $q_P(\theta) = (q_P(\emptyset; \theta), \ldots, q_P(M; \theta))$ of probabilities on $\mathcal{P}(M)$ such that

$$C_P(\theta) := \int_\Omega c_P(\theta, \omega) d\nu(\omega) = Y - \sum_{J \in \mathcal{P}(M)} q_P(J; \theta)P(J),$$

$$\pi_{jP}(\theta) := \int_\Omega \chi_{jP}(\theta, \omega) d\nu(\omega) = \sum_{J \in \mathcal{P}(M)} \delta_{jJ} q_P(J; \theta),$$

for $j = 1, \ldots, m$, where $\delta_{jJ} = 1$ if $j \in J$, $\delta_{jJ} = 0$ if $j \not\in J$, and, finally,

$$Y + \sum_{J \in \mathcal{P}(M)} q_P(J; \theta) \left[ \sum_{j \in J} \theta_j Q_j - P(J) \right] \geq Y + \sum_{J \in \mathcal{P}(M)} q(J) \left[ \sum_{j \in J} \theta_j Q_j - P(J) \right]$$

(4.1)

for all probability vectors $q$ on $\mathcal{P}(M)$. The idea is that, for the given $Q$ and $P$, each consumer is free to choose a set $J$ of public goods that he wants to enjoy at a price $P(J)$. He may also randomize this choice. For generic price schedules though, there is a single set $J_P(\theta)$ which he strictly prefers to all others; in this case, the incentive compatibility condition (4.1) becomes $q_P(J_P(\theta); \theta) = 1$, i.e. he simply chooses the set $J_P(\theta)$.

A price schedule $P$ is said to be arbitrage free if it satisfies the equation

$$P(J) = \sum_{j \in J} P(\{j\})$$

(4.2)

for all $J$, so each set $J \subset M$ is priced as the sum of its components. Using Lemma 3.2, one easily verifies that if an allocation is induced by an arbitrage free price

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16 In the notation used here and in (4.4) below, the dependence of consumers’ choices and payoffs on $Q$ is suppressed because it does not play a role in the analysis.
schedule $P$, then it is also renegotiation proof and incentive compatible, with
associated prices $p_j = P(J)$, $j = 1, \ldots, m$. Conversely, again by Lemma 3.2,
an allocation that renegotiation proof and incentive compatible, with associated
prices $p_1, \ldots, p_m$, is induced by the arbitrage free price schedule $P$ satisfying
\begin{equation}
P(J) = \sum_{j \in J} p_j
\end{equation}
for any $J \subset M$.

The set of renegotiation proof and incentive compatible allocations is thus a
proper subset of the set allocations that are induced by any price schedules. It
does not include, e.g., an allocation that is induced by a price schedule involving
discounts for bundles of public goods, i.e. $P(J) < \sum_{j \in J} p_j$ for some $J \subset M$.
Such an allocation leaves room for Pareto-improving renegotiation between
someone who found it barely worthwhile to buy the combination ticket for $J$
and a set of people of whom each one was on the margin to buying admission
to one of the public goods in $J$, but chose not to.

Given that the set of renegotiation proof and incentive compatible allocations
is a strict subset of the set of allocations induced by price schedules, it is of interest to compare optimal price schedules with the arbitrage free price
schedules corresponding to third-best allocations and optimal Ramsey-Boiteux
prices. For an allocation that is induced by a price schedule, the resulting payoffs
are given as
\begin{equation}
v(\theta | P) = Y + \sum_{J \in P(M)} q_p(J; \theta) \left[ \sum_{j \in J} \theta_j Q_j - P(J) \right].
\end{equation}
Given a vector $Q >> 0$ of public-good provision levels, the price schedule $P$
is said to be *optimal* if it maximizes the aggregate surplus
\begin{equation}
\int_{[0,1]^m} v(\theta | P) f(\theta) d\theta
\end{equation}
over the set of all price schedules.

The following result shows that, if the preference parameters $\theta_1, \ldots, \theta_m$ are
independent, an optimal price schedule is *never* arbitrage free, so a third-best
allocation is always dominated by another allocation that is induced by a price
schedule. In particular, a third-best allocation is always dominated by an allocation
induced by a price schedule involving some mixed bundling, i.e. an offer involving combination tickets, e.g. for the opera performance and the football
game, which come at a discount relative to the individual tickets.

**Proposition 4.1** Let $m > 1$, and assume that the density $f$ takes the form
$f(\theta) = \prod_{j=1}^m f_j(\theta_j)$. For any $Q >> 0$, an optimal price schedule is not arbitrage
free. In particular, if $A$ is a third-best allocation, with associated admission fees
$p_1^A, \ldots, p_m^A$ for the different public goods, there exists a price schedule $\hat{P}$ satisfying
$\hat{P}(M) < \sum_{j=1}^m p_j^A$ and $\hat{P}(\{j\}) > p_j^A$ for $j = 1, \ldots, m$ such that, given $Q^A$, the
allocation that is induced by \( \hat{P} \) is feasible and individually rational and generates a higher aggregate surplus than the third-best allocation \( A \).

Like its analogue in Manelli and Vincent (2002), the proof of Proposition 4.1 is based on an inspection of first-order conditions for the prices \( P^*({j}) \), \( j = 1, \ldots, m \), and \( P^*(M) \) in an optimal price schedule \( P^* \). These first-order conditions involve similar tradeoffs as the Ramsey-Boiteux first-order conditions (3.19). However, there are two important differences: First, in setting the price \( P^*({j}) \) for admission to public good \( j \) by itself, the tradeoff between price and quantity that is mirrored in the Ramsey-Boiteux condition is tempered by the consideration that at least some of the consumers who respond to an increase in \( P^*({j}) \) by not demanding the set \( J = \{ j \} \) any more will not be lost, but will in fact be demanding another set \( \hat{J} \), possibly even with \( j \in \hat{J} \).

For a given shadow price of the feasibility constraint, the individual admission prices \( P^*({1}), \ldots, P^*({m}) \) therefore tend to be higher than the corresponding Ramsey-Boiteux prices. Second, in setting the price \( P^*(M) \) for the bundle providing admission to all public goods, the usual tradeoff between price and quantity is sharpened by the fact an increase in \( P^*(M) \) induces a loss of consumers on \( m \) margins rather than just one, i.e. for each \( i = 1, \ldots, m \), there will be some consumers who respond to the increase in \( P^*(M) \) by demanding the set \( M \setminus \{ i \} \) rather than \( M \). On average therefore, the demand for the bundle \( M \) is more elastic than the demand for any one of the public goods by itself and, for a given shadow price of the feasibility constraint, the price \( P^*(M) \) for the bundle providing admission to all public goods tends to be less than the sum of the corresponding Ramsey-Boiteux prices. Both considerations together yield the conclusion that, starting from a price schedule based on Ramsey-Boiteux prices, i.e. a price schedule that is optimal among arbitrage free price schedules, one can raise aggregate surplus by simultaneously raising the individual admission prices \( P^*({1}), \ldots, P^*({m}) \) and lowering the price \( P^*(M) \) for the bundle providing admission to all public goods.

These considerations suggest that, at least under the independence assumption optimal price schedules should always involve some bundling. The following proposition, which is again inspired by Manelli and Vincent (2002), confirms this notion for the case \( m = 2 \). Unfortunately I have not been able to prove an analogous result for \( m > 2 \).

**Proposition 4.2** Let \( m = 2 \), and assume that the density \( f \) takes the form \( f(\theta_1, \theta_2) = f_1(\theta_1)f_2(\theta_2) \). Assume further that, for \( i = 1, 2 \), the function \( \theta_i \to \)
If \( f_i(\theta_i) \) is nondecreasing. Then, for any \( Q \gg 0 \), if \( P^* \) is an optimal price schedule, one has \( P^*(\{1,2\}) < P^*(\{1\}) + P^*(\{2\}) \).

Whereas the preceding analysis has contrasted third-best allocations with allocations induced by price schedules, the reader may wonder about the relation between the latter and second-best allocations. For the multiproduct monopoly problem, the examples of Thanassoulis (2001) as well as Manelli and Vincent (2002) show that price schedules can be dominated by more complicated schemes involving nontrivial randomization over admissions to the different public goods. Given that the formal structure of the second-best welfare problem is very similar to the monopoly problem, the lesson from these examples should apply in the current setting as well.

However, a focus on allocations that are induced by price schedules can be justified by the weakening of the renegotiation proofness condition that was mentioned in the preceding section. If the mechanism designer is unable to prevent side-trading, but "bundles" of admission tickets to different public goods can be prepared in such a way that "unbundling" is impossible, then an argument parallel to the one given before can be used to show that an allocation is renegotiation proof in this weaker sense as well as incentive compatible if and only if it is induced by a price schedule. An allocation that is induced by an optimal price schedule may thus be said to be 2.5\textsuperscript{th} best, i.e. optimal in the set of all feasible, incentive compatible and weakly renegotiation proof allocations.\textsuperscript{19} For details, the reader is referred to Appendix B.

5 Inequality Aversion

The Ramsey-Boiteux approach to public-sector service provision and pricing has often been criticized for paying insufficient attention to issues of equity. To conclude the paper, therefore I briefly indicate how the analysis is affected if participants are risk averse and/or the mechanism designer is inequality averse. If individual participants are risk averse, the specification (2.6) for \textit{ex ante} expected payoffs must be replaced by

\[
\int_{[0,1]^{m+1}} u(c^A(\theta, \omega) + \sum_{i=1}^{m} \chi^A_i(\theta, \omega) \theta_i Q^A_i) dF(\theta) d\nu(\omega),
\]

where \( u(\cdot) \) is an increasing, strictly concave function. Alternatively, an inequality averse mechanism designer who cares about the cross-section distribution of the payoff \( c^A(\tilde{\theta}^h(x), \tilde{\omega}^h(x)) + \sum_{i=1}^{m} \chi^A_i(\tilde{\theta}^h_i(x), \tilde{\omega}^h(x)) \tilde{\theta}^h_i(x) Q^A_i \) in the population may be interested in a welfare functional of the form

\[
\frac{1}{\eta(H)} \int_H W(c^A(\tilde{\theta}^h(x)) + \sum_{i=1}^{m} \chi^A_i(\tilde{\theta}^h_i(x)) \tilde{\theta}^h_i(x) Q^A_i) d\eta(h),
\]

\textsuperscript{19} I owe this insight to suggestions and questions from Hans Gersbach and a referee.
where $W(.)$ is an increasing, strictly concave function. In this latter case, the large-numbers assumption (2.2) implies that the mechanism designer will choose the allocation to maximize the expectation

$$
\int_{[0,1]} W(c^A(\theta, \omega) + \sum_{i=1}^{m} \chi^A_i(\theta, \omega)\theta_i Q_i^A) dF(\theta) d\nu(\omega)
$$

(5.3)

over the constraint set. Given that (5.1) and (5.3) have the same formal structure, I neglect the distinction between risk aversion and inequality and focus on (5.3).

For this welfare functional, an analogue of Proposition 3.3 shows that the third-best allocation problem is equivalent to the problem of choosing the public-good provision levels $Q_1, \ldots, Q_m$, the admission prices $p_1, \ldots, p_m$ and the base consumption $C(0)$ so as to maximize

$$
\int_{[0,1]} W \left( C(0) + \sum_{i=1}^{m} \max(\theta_i Q_i - p_i, 0) \right) dF(\theta)
$$

(5.4)

under the feasibility constraint (3.14) and the participation constraint $c(0) \geq Y$ (and the constraint $p_i = 0$ if public good $i$ is nonexcludable). The first-order conditions (3.18) and (3.19) are then replaced by the conditions

$$
\frac{1}{\lambda} \int_{\hat{\theta}_i}^{1} \int_{[0,1]} W' (\theta_i - \hat{\theta}_i) dF(\theta_i, \theta_{-i}) + \hat{\theta}_i (1 - F_i(\hat{\theta}_i)) = K_i(\xi)
$$

(5.5)

and

$$
p_i \frac{f_i(\hat{\theta}_i)}{Q_i} = \frac{1}{\lambda} \int_{\hat{\theta}_i}^{1} \int_{[0,1]} (\lambda - W') dF(\theta_i, \theta_{-i}),
$$

(5.6)

where $W'$ in each integral is evaluated at the point $C(0) + \sum_{i=1}^{m} \max(\theta_i Q_i - p_i, 0)$ and again $\hat{\theta}_i = \hat{\theta}_i(p_i, Q_i)$ is given by (3.15). Moreover, by the first-order condition for $C(0)$, the Lagrange multiplier for the feasibility constraint satisfies

$$
\lambda \geq \int_{[0,1]} W' dF(\theta),
$$

(5.7)

with equality if $C(0) > Y$.

Condition (5.6) can be rewritten as

$$
1 = \frac{\int_{\hat{\theta}_i}^{1} \int_{[0,1]} (\lambda - W') dF(\theta) \frac{1}{\eta_i}}{\lambda(1 - F_i(\hat{\theta}_i))}
$$

(5.8)

---

20 A referee has pointed out that the specification $\int W(v^A(\theta)) g(\theta) dF(\theta)$ would generalize the analysis even further. With obvious modifications, all the results of this section would go through for this more general specification, which corresponds to the approach of Diamond and Mirrlees (1971) or Ledyard and Palfrey (1999).
which provides a weighted inverse-elasticities formula, as in Diamond-Mirrlees (1971), instead of the simple one that was derived in Section 3. The weight

\[
\frac{1}{\lambda(1 - F_i(\hat{\theta}_i))} = \frac{1}{\lambda} \frac{\int_{\theta_i}^{1} \int_{0,1}^{m-1} (\lambda - W') dF(\theta_i, \theta_{-i})}{\int_{\theta_i}^{1} \int_{0,1}^{m-1} W' dF(\theta)}
\]

in (5.8) is a decreasing function of the conditional expectation of the social marginal valuation \(W'(v(\theta))\) of additional consumption for people demanding admission to public good \(i\). The admission fee \(p_i\) therefore tends to be higher for a public good with a relatively low expected value of \(W'(v(\hat{\theta}))\) conditional on the information that admission to public good \(i\) is requested.

Inequality aversion provides the mechanism designer with an additional rationale for admission fees. Because admission fees are paid by people who benefit a lot from the enjoyment of the public goods, they provide for redistribution from people who benefit a lot from the enjoyment of the public goods to people who do not benefit from the public goods. To see the role of this effect, rewrite (5.6) in the form

\[
p_i f_i(\hat{\theta}_i) \frac{1}{Q_i} = \frac{\lambda - \bar{\lambda}}{\lambda} (1 - F_i(\hat{\theta}_i)) + \frac{1}{\lambda} \int_{\theta_i}^{1} \int_{[0,1]^{m-1}} (\bar{\lambda} - W') dF(\theta_i, \theta_{-i}), \quad (5.9)
\]

where \(\bar{\lambda} := \int W' dF\) is the cross-section average social marginal valuation of additional consumption. The first term on the right-hand side of (5.9) reflects the need to meet the participation constraint \(C(0) \geq Y\). This term is independent of the mechanism designer’s inequality aversion and corresponds to the right-hand side of (3.19) in the previous analysis. The second term on the right-hand side of (5.9) reflects the fact that on average, people asking admission to public good \(i\) have a higher payoff and, for the inequality averse mechanism designer, a lower social marginal valuation of additional consumption than the population average. As discussed in Hellwig (2003 b), an admission fee \(p_i\), which is paid by people with preference parameter \(\theta_i \geq p_i/Q_i\), can be seen as a tool for lowering the private-good consumption of these people and raising the private-good consumption of people with preference parameter \(\theta_i < p_i/Q_i\). With a social marginal valuation of additional consumption that is lower for people with preference parameter \(\theta_i \geq p_i/Q_i\) than for people with preference parameter \(\theta_i < p_i/Q_i\), the inequality averse mechanism designer considers this redistribution to be beneficial. In (5.9) therefore, the marginal benefit of an increase in \(p_i\) is exhibited as the sum of benefits from the contribution of the increase to meeting the participation constraint and benefits from redistribution.

If the mechanism designer’s inequality aversion is sufficiently large, the distributive rationale for admission fees actually supersedes the budgetary one. The ”government budget constraint” \(\sum_{i=1}^{m} p_i(1 - F_i(\hat{\theta}_i)) \geq K(Q)\) ceases to be binding, and revenues from entry fees are used to raise the private-good consumption \(C(0)\) of people who do not care for the public goods at all above the level which they would have in the absence of public-good provision. In this case, one obtains \(C(0) > Y\) and \(\lambda = \bar{\lambda}\). The first term on the right-hand side of
The optimal admission fee for public good $i$ is entirely determined by the tradeoff between the efficiency loss from excluding people, exhibited on the left-hand side of (5.9), and the equity concern exhibited in the second term on the right-hand side of (5.9).\(^{21}\)

In the Rawlsian limit of infinite inequality aversion, the equity concern dominates everything else. With infinite inequality aversion, the mechanism designer is only concerned about the base consumption $C(0)$, i.e. the payoff of people with $\theta = 0$, who are worst off in the economy. Optimal admission fees are then chosen to maximize the excess of revenues from public-good provision over costs and hence the resources that are available to raise $C(0)$ above $Y$. The second-best mechanism design problem then coincides with the problem of a profit-maximizing monopolist. Formally, one obtains:

**Proposition 5.1** Let $\{W_k\}$ be any sequence of increasing, concave, and twice continuously differentiable functions on $\mathbb{R}_+$ such that $\lim_{k \to \infty} -\frac{W_k''(v)}{W_k'(v)} = \infty$, uniformly in $v$. For any $k$, let $(Q_1^k, \ldots, Q_m^k, p_1^k, \ldots, p_m^k)$ be a vector of third-best public-good provision levels and admission fees for the welfare function $W_k$. Then the sequence $\{(Q_1^k, \ldots, Q_m^k, p_1^k, \ldots, p_m^k)\}$ has a limit point $(Q_1^\infty, \ldots, Q_m^\infty, p_1^\infty, \ldots, p_m^\infty)$. Moreover, any such limit point is a solution to the monopoly problem

$$\max_{(Q_1, \ldots, Q_m, p_1, \ldots, p_m)} \left[ \sum_{i=1}^m p_i (1 - F_i(\frac{p_i}{Q_i})) - K(Q) \right].$$  

(5.10)

For $m = 1$, a proof of this result is given in Hellwig (2003 b). The proof for arbitrary $m$ is practically the same and is therefore not given here. The important conclusion to be drawn is that inequality aversion may supersede participation constraints and the "government budget constraint" (3.14) as a rationale for charging fees for admission to excludable public goods. Even so, a third-best vector of admission fees will satisfy a weighted inverse-elasticities rule.\(^{22}\)

The role of renegotiation proofness as a necessary and sufficient condition for the relevance of third-best allocations is unaffected by the introduction of inequality aversion. With inequality aversion of the mechanism designer and stochastic independence of $\tilde{\theta}_1, \ldots, \tilde{\theta}_m$, mixed bundling is still preferred to the arbitrage free price schedule corresponding to a third-best allocation. Indeed for the case of infinite inequality aversion, the multiproduct monopoly analysis of McAfee et al. (1989) or Manelli and Vincent (2002) is directly applicable because

\(^{21}\)For a detailed account of this tradeoff, see Hellwig (2003 b).

\(^{22}\)These considerations suggest that the critique of the Ramsey-Boiteux approach to public-sector pricing and indirect taxation that was presented by Atkinson and Stiglitz (1976) must be modified once one allows for heterogeneity in tastes as well as inequality aversion of the mechanism designer. Even if one accepts the Atkinson-Stiglitz critique of the imposition of a government budget constraint (or the imposition of interim participation constraints), if inequality aversion is sufficiently large, one still obtains an inverse-elasticities characterization of optimal admission fees.
the mechanism designer is only interested in maximizing expected profits in order to raise the base consumption $C(0)$ of those people who are worst off in the economy.

A Appendix: Proofs

Proof of Lemma 3.1. The "if" part of the lemma is an instance of the first welfare theorem. To prove the "only if" part, let $A$ be a renegotiation proof allocation. For $i = 1, \ldots, m$, let $\hat{\theta}_i^A$ be the unique solution to the equation

$$1 - F_i(\hat{\theta}_i^A) = \int_{[0,1]^m \times \Omega} \chi_i(\theta, \omega) \ dF(\theta)\ d\nu(\omega). \quad (A.1)$$

Consider the net-trade allocation $(z_c(\ldots), z_1(\ldots), \ldots, z_m(\ldots))$ satisfying

$$z_i(\theta, \omega) = -\chi_i^A(\theta, \omega) \quad \text{if} \quad \theta_i < \hat{\theta}_i^A, \quad (A.2)$$

$$z_i(\theta, \omega) = 1 - \chi_i^A(\theta, \omega) \quad \text{if} \quad \theta_i \geq \hat{\theta}_i^A, \quad (A.3)$$

for $i = 1, \ldots, m$, and

$$z_c(\theta, \omega) = -\sum_{i=1}^m z_i(\theta, \omega)\theta_i^A Q_i^A. \quad (A.4)$$

One easily verifies that, for the given vector $Q^A$ of public-good provision levels, the price system $(1, p_1^A, \ldots, p_m^A)$ and the allocation $(c^A(\ldots), \chi_1^A(\ldots), \ldots, \chi_m^A(\ldots)) + (z_c(\ldots), z_1(\ldots), \ldots, z_m(\ldots))$ correspond to a competitive equilibrium of the exchange economy with trade in the private good and in admission tickets for the public goods, with initial endowments given by $(c(\ldots), \chi_1(\ldots), \ldots, \chi_m(\ldots))$. Feasibility and net-trade incentive compatibility of the net-trade allocation $(z_c(\ldots), z_1(\ldots), \ldots, z_m(\ldots))$ follow immediately, as does the dominance condition (3.4). Reallocation proofness of $A$ therefore implies that (3.6) does not hold. Given that (3.4) does hold, it follows that

$$z_c(\theta, \omega) + \sum_{i=1}^m z_i(\theta, \omega)\theta_i^A Q_i^A = 0 \quad (A.5)$$

for almost all $(\theta, \omega) \in [0,1]^m \times \Omega$. From (A.4), one has

$$z_c(\theta, \omega) + \sum_{i=1}^m z_i(\theta, \omega)\theta_i^A Q_i^A = \sum_{i=1}^m z_i(\theta, \omega)(\theta_i - \hat{\theta}_i^A) Q_i^A. \quad (A.6)$$

By (A.2) and (A.3), each of the summands on the right-hand side of (A.6) is nonnegative, so (A.5) implies that, for each $i$ and almost all $(\theta, \omega) \in [0,1]^m \times \Omega$, one has

$$z_i(\theta, \omega)(\theta_i - \hat{\theta}_i^A) Q_i^A = 0, \quad (A.7)$$

26
hence, by (A.2) and (A.3),

$$[\max(\theta_i - \hat{\theta}_i^A, 0) - \chi_i^A(\theta, \omega)(\theta_i - \hat{\theta}_i^A)]Q_i^A = 0. \quad (A.8)$$

Now (A.8) implies \( \chi_i(\theta, \omega) = 0 \) if \( \theta_iQ_i^A < \hat{\theta}_i^A Q_i^A \) and \( \chi_i(\theta, \omega) = 1 \) if \( \theta_iQ_i^A > \hat{\theta}_i^A Q_i^A \). Upon setting, \( p_i^A = \hat{\theta}_i^A Q_i^A \), one finds that the claim of the lemma is established. \( \blacksquare \)

**Proof of Lemma 3.2.** The "if" part of the lemma is trivial. To prove the "only if" part, let \( A \) be a renegotiation proof and incentive compatible allocation, and let \( v^A(.) \) be the associated expected-payoff function as given by (2.14). By (2.14), one has \( v^A(0) = C^A(0) \). Moreover, Lemmas 2.2 and 3.1, and (2.11) imply that for almost any \( \theta \in [0, 1]^m \), the function \( v(.) \) has first partial derivatives satisfying

$$v_i^A(\theta) = 0 \quad \text{if} \quad \theta_i Q_i^A < p_i^A \quad (A.9)$$

and

$$v_i^A(\theta) = Q_i^A \quad \text{if} \quad \theta_i Q_i^A > p_i^A, \quad (A.10)$$

where \( p_1^A, ..., p_m^A \) are the prices given by Lemma 3.1. By integration, one then obtains (3.12), so (2.15), implies that (3.9) and (3.10) hold for all \( \theta \in [0, 1]^m \). From (3.12) and (2.14), one also obtains

$$C^A(\theta) = C^A(0) + \sum_{i=1}^m \max(\theta_i Q_i^A - p_i^A, 0) - \sum_{i=1}^m \pi_i^A(\theta)\theta_i Q_i^A. \quad (A.11)$$

so (3.11) follows from (3.9) and (3.10). \( \blacksquare \)

The proof of Proposition 3.3 is trivial and is left to the reader.

**Proof of Proposition 3.4.** By Proposition 3.3, \( Q_1, ..., Q_m \) and \( p_1, ..., p_m \) maximize (3.16) subject to (3.17). For some \( \lambda \geq 0 \) therefore, \( Q_1, ..., Q_m \) and \( p_1, ..., p_m \) maximize the Lagrangian expression

$$\sum_{i=1}^m \int_{\theta_i(p_i, Q_i)}^1 (\theta_i Q_i - p_i) \, dF_i(\theta_i) + \lambda \left( \sum_{i=1}^m p_i(1 - F_i(\hat{\theta}_i(p_i, Q_i))) - K(Q) \right). \quad (A.12)$$

Given that \( p_i = \hat{\theta}_i(p_i, Q_i)Q_i \) for all \( i \), the problem of maximizing (A.12) with respect to \( Q_1, ..., Q_m \) and \( p_1, ..., p_m \) is equivalent to the problem of maximizing

$$\sum_{i=1}^m \int_{\hat{\theta}_i}^1 (\theta_i - \hat{\theta}_i)Q_i \, dF_i(\theta_i) + \lambda \left( \sum_{i=1}^m \hat{\theta}_i Q_i(1 - F_i(\hat{\theta}_i)) - K(Q) \right) \quad (A.13)$$

with respect to \( Q_1, ..., Q_m \) and \( \hat{\theta}_1, ..., \hat{\theta}_m \).

To prove that \( Q_i > 0 \), it suffices to observe that, for \( Q_i = 0 \), (A.13) is independent of \( \hat{\theta}_i \), and, for \( \hat{\theta}_i < 1 \), at \( Q_i = 0 \), (A.13) is strictly increasing in \( Q_i \).
Any pair \((Q_i, \hat{\theta}_i)\) with \(Q_i = 0\) is therefore dominated by the pair \((\varepsilon, \frac{1}{2})\) provided that \(\varepsilon > 0\) is sufficiently small.

Given that public-good provision levels must be positive, the first-order conditions for maximizing (A.13) are given as

\[
\int_{\hat{\theta}_i}^1 (\theta_i - \hat{\theta}_i) dF_i(\theta_i) + \lambda \hat{\theta}_i (1 - F_i(\hat{\theta}_i)) - \lambda K_i(Q) = 0 \tag{A.14}
\]

for \(Q_i\) and

\[
-\int_{\hat{\theta}_i}^1 Q_i dF_i(\theta_i) + \lambda Q_i (1 - F_i(\hat{\theta}_i)) - \lambda \hat{\theta}_i f_i(\hat{\theta}_i) \leq 0 \tag{A.15}
\]

for \(\hat{\theta}_i\) with a strict inequality only if \(\hat{\theta}_i = 0\). With \(Q_i > 0\), (A.15) simplifies to:

\[
(\lambda - 1)(1 - F_i(\hat{\theta}_i)) - \lambda \hat{\theta}_i f_i(\hat{\theta}_i) \leq 0, \tag{A.16}
\]

with a strict inequality only if \(\hat{\theta}_i = 0\).

The Lagrange multiplier must exceed one. For \(\lambda \leq 1\), (A.16) would imply \(\hat{\theta}_i = 0\), hence \(p_i = 0\) for all \(i\), and, by the constraint (3.17), \(K(Q) = 0\), which is impossible if \(Q_i > 0\) for all \(i\). Therefore one cannot have \(\lambda \leq 1\). For \(\lambda > 1\), (A.16) implies \(1 > \hat{\theta}_i > 0\), hence \(Q_i > p_i > 0\) for all \(i\). Now (3.19) follows from (A.16) by substituting for \(\hat{\theta}_i = p_i/Q_i > 0\). (3.18) follows from (A.15) by a rearrangement of terms.

**Proof of Proposition 4.1.** By contradiction, suppose that the first statement of Proposition 4.1 is false. Then there exist \(Q >> 0\) and \(P^*\) such that \(P^*\) is arbitrage free and, given \(Q\), \(P^*\) maximizes (4.5) over the set of price schedules inducing individually rational and feasible allocations.

For any price schedule \(P\), an allocation induced by \(P\) is individually rational if \(P(\emptyset) = 0\). By (2.20), the allocation is also feasible if

\[
\int_{[0,1]^m} [v(\theta|P) - \sum_{i=1}^m \theta_i v_i(\theta|P)] f(\theta) d\theta \leq Y - K(Q). \tag{A.17}
\]

Through integration by parts, as in McAfee et al. (1989), (A.17) is seen to be equivalent to the inequality

\[
\int_{[0,1]^m} v(\theta|P)[(m + 1)f(\theta) + \Theta \cdot \nabla f(\theta)] d\theta - \sum_{i=1}^m \int_{[0,1]^{m-1}} v(1, \Theta_{-i}|P) f(1, \Theta_{-i}) d\Theta_{-i} \leq Y - K(Q). \tag{A.18}
\]

Thus \(P^*\) maximizes (4.5) over the set of price schedules \(P\) satisfying \(P(\emptyset) = 0\) and (A.18).
For some $\lambda \geq 0$ and some $\mu$ therefore, $P^*$ maximizes the Lagrangian expression

$$
\int_{[0,1]^m} v(\theta | P) f(\theta) d\theta - \lambda \int_{[0,1]^m} v(\theta | P) [(m + 1) f(\theta) + \theta \cdot \nabla f(\theta)] d\theta + \lambda \sum_{i=1}^m \int_{[0,1]^{m-1}} v(1, \theta_{-i} | P) f(1, \theta_{-i}) d\theta_{-i} + \lambda (Y - K(\mathcal{Q})) + \mu P(\emptyset). \quad (A.19)
$$

Given that any price schedule $P$ is characterized by the finite list of numbers $P(\emptyset), P(\{1\}), ..., P(\{m\}), ..., P(M)$, it follows that for any nonempty set $J \subset M$, the first-order condition

$$
\frac{\partial}{\partial P(J)} \int_{[0,1]^m} v(\theta | P) f(\theta) d\theta - \lambda \frac{\partial}{\partial P(J)} \int_{[0,1]^m} v(\theta | P) [(m + 1) f(\theta) + \theta \cdot \nabla f(\theta)] d\theta + \lambda \frac{\partial}{\partial P(J)} \sum_{i=1}^m \int_{[0,1]^{m-1}} v(1, \theta_{-i} | P) f(1, \theta_{-i}) d\theta_{-i} \leq 0,
$$

with equality unless $P^*(J) = 0$,

must be satisfied at $P = P^*$. By an argument of Manelli and Vincent (2002),\(^{23}\) (A.20) can be rewritten as

$$
- \int_{A_J(P^*)} f(\theta) d\theta + \lambda \int_{A_J(P^*)} [(m + 1) f(\theta) + \theta \cdot \nabla f(\theta)] d\theta - \lambda \sum_{i \in J} \int_{B^*_i(P^*)} f(1, \theta_{-i}) d\theta_{-i} \leq 0,
$$

with equality unless $P^*(J) = 0$,

where, for any price schedule $P$, $A_J(P) = \{ \theta \in [0,1]^m | q_P(J; \theta) > 0 \}$ and, for any $i \in J$, $B^*_i(P) = \{ \theta_{-i} \in [0,1]^{m-1} | q_P(J; 1, \theta_{-i}) > 0 \}$. Given that $f$ takes the form $f(\theta) = \prod_{j=1}^m f_j(\theta_j)$, (A.21) can be rewritten as:

$$
(\lambda - 1) \int_{A_J(P^*)} \prod_{j=1}^m f_j(\theta_j) d\theta + \lambda \sum_{i \in J} \int_{A_J(P^*)} (f_i(\theta_i) + \theta_i f_i'(\theta_i)) \prod_{j \neq i} f_j(\theta_j) d\theta - \lambda \sum_{i \in J} \int_{B^*_i(P^*)} f_i(1) \prod_{j \neq i} f_j(\theta_j) d\theta_{-i} \leq 0,
$$

with equality unless $P^*(J) = 0$.

I first show that, because $P^*$ is arbitrage free, (A.22) implies the Ramsey-Boiteux first-order condition (3.19). For any $j$, I define $\mathcal{J}(j) := \{ J \subset M | j \in J \}$

\(^{23}\)The idea behind the argument is that the derivatives of the integrals in (A.20) with respect to $P(J)$ can be equated with the integrals of the derivatives of the integrands on the interiors of the sets $A_J(P)$ and $B^*_i(P)$. The boundaries of these sets do not matter because they have Lebesgue measure zero.
as the set of all sets \( J \) that contain public good \( j \). Summation of (A.22) over \( J \in \mathcal{J}(j) \) yields

\[
(\lambda - 1) \int_{\bigcup_{j \in \mathcal{J}(j)\backslash \{j\}}} \prod_{j=1}^{m} f_j(\theta_j) d\theta + \lambda \sum_{i=1}^{m} \int_{\bigcup_{j \in \mathcal{J}(j)\backslash \{j\}}} (f_i(\theta_i) + \theta_j f'_i(\theta_i)) \prod_{j \neq i} f_j(\theta_j) d\theta
\]

\[\begin{aligned}
- \lambda \sum_{i \in M} \int_{\bigcup_{j \in \mathcal{J}(j)\backslash \{j\}}} B_j^{(P^*)}(f_i(1) \prod_{j \neq i} f_j(\theta_j)) d\theta \leq 0, \\
\text{with equality unless } P^*(J) = 0 \text{ for some } J \in \mathcal{J}(j).
\end{aligned}\]

Because \( P^* \) is arbitrage free, one has \( \bigcup_{j \in \mathcal{J}(j)} A_j(P^*) = \{ \theta \in [0,1]^m | \theta_j \geq \hat{\theta}_j \} \), \( \bigcup_{j \in \mathcal{J}(j)} B_j^{(P^*)} = [0,1]^{m-1} \), and, for \( i \neq j \), \( \bigcup_{j \in \mathcal{J}(j)\backslash \{j\}} B_j^{(P^*)} = \{ \theta_{-i} \in [0,1]^{m-1} | \theta_j \geq \hat{\theta}_j \} \), where \( \hat{\theta}_j = \frac{P^j(i)}{Q_j} \). Thus (A.23) can be rewritten as

\[
(\lambda - 1)(1 - F_j(\hat{\theta}_j)) + \lambda \int_{\hat{\theta}_j}^{1} [f_j(\theta_j) + \theta_j f'_j(\theta_j)] d\theta_j
\]

\[\begin{aligned}
+ \sum_{k=1 \atop k \neq j}^{m} (1 - F_j(\hat{\theta}_j)) \int_{0}^{1} [f_k(\theta_k) + \theta_j f'_k(\theta_k)] d\theta_k - \lambda f_j(1) - \sum_{i=1 \atop i \neq j}^{m} (1 - F_j(\hat{\theta}_j)) f_i(1) \leq 0, \\
\text{with equality unless } \hat{\theta}_j = 0.
\end{aligned}\]

Upon computing the integrals and cancelling terms involving \( f_j(1) \) or \( f_i(1) \), one can rewrite this condition as

\[
(\lambda - 1)(1 - F_j(\hat{\theta}_j)) - \hat{\theta}_j f_j(\hat{\theta}_j) \leq 0, \quad \text{(A.24)}
\]

\[\text{with equality unless } \hat{\theta}_j = 0.\]

This is identical with condition (A.16) in the proof of Proposition 3.4. The same argument as was given there shows that one must have \( \lambda > 1 \) and \( 1 > \hat{\theta}_i > 0 \), and hence

\[
(\lambda - 1)(1 - F_j(\hat{\theta}_j)) - \hat{\theta}_j f_j(\hat{\theta}_j) = 0 \quad \text{(A.25)}
\]

for all \( j \).

Because \( P^* \) is arbitrage free, for \( j = 1, \ldots, m \), one also has \( A_{(j)}(P^*) = [\hat{\theta}_j, 1] \times \prod_{i \neq j} [0, \hat{\theta}_i] \) and \( B_{(j)}^{(P^*)} = \prod_{i \neq j} [0, \hat{\theta}_i] \), where again \( \hat{\theta}_j = \frac{P^*(\{j\})}{Q_j} \) and
\( \hat{\theta}_i = \frac{P^\ast(i)}{q_i} \). Since \( \hat{\theta}_j > 0 \) implies \( P^\ast([j]) > 0 \), for \( J = \{j\} \), (A.22) becomes

\[
-(1 - \lambda)(1 - F_j(\hat{\theta}_j) \prod_{i=1 \atop i \neq j}^m F_i(\hat{\theta}_i) + \lambda \int_{\hat{\theta}_j}^1 [f_j(\theta_j) + \theta_j f_j'(\theta_j)] d\theta_i \prod_{i=1 \atop i \neq j}^m F_i(\hat{\theta}_i)

+ \lambda(1 - F_j(\hat{\theta}_j)) \sum_{k=1 \atop k \neq j}^m \int_{[0, \hat{\theta}_k]} [f_k(\theta_k) + \theta_k f_k'(\theta_k)] d\theta_k \prod_{i=1 \atop i \neq j,k}^m F_i(\hat{\theta}_k)

- \lambda f_j(1) \prod_{i=1 \atop i \neq j}^m F_i(\hat{\theta}_i) = 0. \tag{A.26}
\]

Upon computing the integrals in (A.26), cancelling terms involving \( f_j(1) \) and dividing by \( \prod_{i \neq j} F_i(\hat{\theta}_i) \), one further obtains

\[
(\lambda - 1)(1 - F_j(\hat{\theta}_j)) - \lambda \hat{\theta}_j f_j(\hat{\theta}_j) + \lambda(1 - F_j(\hat{\theta}_j)) \sum_{k=1 \atop k \neq j}^m \frac{\hat{\theta}_k f_k(\hat{\theta}_k)}{F_k(\hat{\theta}_k)} = 0. \tag{A.27}
\]

By (A.24), it follows that

\[
\lambda(1 - F_j(\hat{\theta}_j)) \sum_{k=1 \atop k \neq j}^m \frac{\hat{\theta}_k f_k(\hat{\theta}_k)}{F_k(\hat{\theta}_k)} = 0,
\]

which is impossible because \( \hat{\theta}_j < 1 \) and \( \hat{\theta}_k > 0 \) for all \( k \). The assumption that the first statement of the proposition is false has thus led to a contradiction.

Turning to the second statement of the proposition, the argument just given implies that at the arbitrage free price schedule \( P^\ast \) which induces a third-best allocation, the derivative of the Lagrangian (A.19) with respect to the singleton prices \( P([j]) \) is strictly positive. As for the bundle \( M \), because \( P^\ast \) is arbitrage free, one has \( A_M(P^\ast) = \prod_{j=1}^m [\hat{\theta}_j, 1] \) and, for \( i = 1, \ldots, m \), \( B_{iM}(P^\ast) = \prod_{j \neq i} [\hat{\theta}_j, 1] \), so at \( P = P^\ast \), the derivative of the Lagrangian (A.19) with respect to the price \( P(M) \) takes the form

\[
-(1 - \lambda) \prod_{j=1}^m (1 - F_j(\hat{\theta}_j)) + \lambda \sum_{i=1}^m \prod_{j=1 \atop j \neq i}^m (1 - F_j(\hat{\theta}_j)) \int_{\hat{\theta}_i}^1 [f_i(\theta_i) + \theta_i f_i'(\theta_i)] d\theta_i

- \lambda \sum_{i=1}^m f_i(1) \prod_{j=1 \atop j \neq i}^m (1 - F_j(\hat{\theta}_j)),
\]

which simplifies to

\[
(\lambda - 1) \prod_{j=1}^m (1 - F_j(\hat{\theta}_j)) - \lambda \sum_{i=1}^m \prod_{j=1 \atop j \neq i}^m (1 - F_j(\hat{\theta}_j)) \hat{\theta}_i f_i(\hat{\theta}_i). \tag{A.28}
\]
Upon using (A.24) to substitute for $\lambda \hat{f}_i(\hat{\theta}_j)$, $i = 1, ..., m$, one finds that (A.28) is equal to

\[
(\lambda - 1)(1 - m) \prod_{j=1}^{m} (1 - F_j(\hat{\theta}_j)),
\]

which is strictly negative because $m > 1$. The dominating price schedule $\hat{P}$ in the vicinity of $P^*$ may thus be chosen with $\hat{P}(M) < P^*(M)$ as well as $\hat{P}(\{j\}) > P^*(\{j\})$ for $j = 1, ..., m$. ■

**Proof of Proposition 4.2.** Suppose that the proposition is false. For $m = 2$, let $Q >> 0$ and $P^*$ be such that $P^*$ is an optimal price schedule given $Q$ and $P^*(\{1, 2\}) \geq P^*(\{1\}) + P^*(\{2\})$. Optimality of $P^*$ implies that, for some $\lambda \geq 0$ and some $\mu$, $P^*$ satisfies the first-order condition (A.22). Feasibility implies that $P^*(\{1, 2\}) > 0$.

If $P^*(\{1, 2\}) \geq P^*(\{1\}) + P^*(\{2\})$, one has $A_{\{1, 2\}}(P^*) = [\hat{\theta}_1, 1] \times [\hat{\theta}_2, 1]$ where, for $i = 1, 2$, $\hat{\theta}_i := P^*(\{1, 2\}) - P^*(\{i\})$. For $J = \{1, 2\}$, with $P^*(\{1, 2\}) > 0$, (A.22) then yields:

\[
(\lambda - 1)(1 - F_1(\hat{\theta}_1))(1 - F_2(\hat{\theta}_2)) - \lambda \hat{\theta}_1 f_1(\hat{\theta}_1)(1 - F_2(\hat{\theta}_2)) - \lambda \hat{\theta}_2 f_2(\hat{\theta}_2)(1 - F_1(\hat{\theta}_1)) = 0,
\]

or

\[
(\lambda - 1) = \lambda \sum_{i=1,2} \hat{\theta}_i f_i(\hat{\theta}_i) - \frac{\hat{\theta}_1 f_1(\hat{\theta}_1)}{1 - F_1(\hat{\theta}_1)}. \quad (A.29)
\]

If $P^*(\{1, 2\}) \geq P^*(\{1\}) + P^*(\{2\})$, one also has

\[
A_{\{1\}}(P^*) = \{ (\theta_1, \theta_2) | \theta_2 \leq \hat{\theta}_2 \text{ and } \theta_1 \geq \hat{\theta}_1(\theta_2) \},
\]

where $\hat{\theta}_1(\theta_2) := P^*(\{1\}) + \max(\theta_2 - P^*(\{2\}), 0)$. For $J = \{1\}$ therefore, (A.22) implies

\[
(\lambda - 1) \int_{0}^{\hat{\theta}_2} (1 - F_1(\hat{\theta}_1(\theta_2))) f_2(\theta_2) d\theta_2 - \lambda \int_{0}^{\hat{\theta}_2} \hat{\theta}_1(\theta_2) f_1(\hat{\theta}_1(\theta_2)) f_2(\theta_2) d\theta_2 \leq 0
\]

or

\[
\int_{0}^{\hat{\theta}_2} \left( (\lambda - 1) - \frac{\hat{\theta}_1(\theta_2) f_1(\hat{\theta}_1(\theta_2))}{(1 - F_1(\hat{\theta}_1(\theta_2)))} \right) (1 - F_1(\hat{\theta}_1(\theta_2))) f_2(\theta_2) d\theta_2 \leq 0. \quad (A.30)
\]

For $\theta_2 \leq \hat{\theta}_2$, one has

\[
\hat{\theta}_1(\theta_2) \leq P^*(\{1\}) + \max(\theta_2 - P^*(\{2\}), 0) = P^*(\{1\}) + \max(P^*(\{1, 2\}) - P^*(\{1\}) - P^*(\{2\}), 0) = P^*(\{1, 2\}) - P^*(\{2\}) = \hat{\theta}_1.
\]

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Given the assumed monotonicity of the functions \( \theta_i \rightarrow \frac{\theta_i f_i(\theta_i)}{1 - F_i(\theta_i)} \), it follows that (A.30) implies

\[
(\lambda - 1) \leq \frac{\tilde{\theta}_1 f_1(\tilde{\theta}_1)}{1 - F_1(\tilde{\theta}_1)}. 
\]

(A.31)

By a precisely symmetric argument for the set \{2\}, one also has

\[
(\lambda - 1) \leq \frac{\tilde{\theta}_2 f_2(\tilde{\theta}_2)}{1 - F_2(\tilde{\theta}_2)}. 
\]

(A.32)

Upon combining (A.31) and (A.32) with (120), one obtains

\[
(\lambda - 1) \geq 2(\lambda - 1),
\]

which implies \( \lambda \leq 1 \). By (A.31), it follows that \( \tilde{\theta}_i = 0 \) for \( i = 1, 2 \), hence \( P^*(\{1, 2\}) = 0 \), which is impossible if the cost \( K(Q) > 0 \) is to be covered. The assumption that \( P^*(\{1, 2\}) \geq P^*(\{1\}) + P^*(\{2\}) \) thus leads to a contradiction and must be false.

B Appendix: Weak Renegotiation Proofness

In this appendix, I consider the possibility that the mechanism designer prepares bundles of admission tickets to the different public goods in such a way that participants are unable to unbundle them. In this case, an allocation will be an array

\[
A = (Q^A, c^A(\theta, \omega), \{\chi^A_J(\theta, \omega)\}_{J \subseteq M})
\]

such that \( Q^A = (Q^A_1, \ldots, Q^A_m) \) is the vector of public-good provision levels, \( c^A(\theta, \omega) \) is a function which stipulates for each \((\theta, \omega) \in [0, 1]^{m+1}\) a level \( c^A(\theta, \omega) \) of private-good consumption for consumer \( h \) in the state \( x \) if \((\tilde{\theta}^h(x), \tilde{\omega}^h(x)) = (\theta, \omega)\), and, for each subset \( J \) of the set \( M \) of public goods, \( \chi^A_J(\theta, \omega) \) is a function which stipulates for each \((\theta, \omega) \in [0, 1]^{m+1}\) whether a consumer \( h \) gets a ticket to the bundle \( J \) if \((\tilde{\theta}^h(x), \tilde{\omega}^h(x)) = (\theta, \omega)\) or whether he does not get such a ticket. In the first case, \( \chi^A_J(\theta, \omega) \) takes the value one, in the second, the value zero. Assuming that the mechanism designer specifies exactly \(^{24}\) one bundle per consumer, one also has \( \sum_{J \subseteq M} \chi^A_J(\theta, \omega) = 1 \).

For any \((\theta, \omega) \in [0, 1]^{m+1}\), the allocation \( A = (Q^A, c^A(\theta, \omega), \{\chi^A_J(\theta, \omega)\}_{J \subseteq M}) \) provides consumer \( h \) with the payoff

\[
c^A(\theta, \omega) + \sum_{J \subseteq M} \chi^A_J(\theta, \omega) \sum_{j \in J} \theta_j Q^A_j
\]

if \((\tilde{\theta}^h(x), \tilde{\omega}^h(x)) = (\theta, \omega)\).

\(^{24}\)In this formalism, the empty set is one of the bundles that can be assigned.
A net-trade allocation is now defined as an array

\[
(z_c(\ldots), \{z^i(\ldots)\}_{J \in M})
\]  

(B.3)
such that for any \((\theta, \omega) \in [0, 1]^{m+1}\), \(z_c(\theta, \omega)\) and \(z^i_J(\theta, \omega), J \subset M\), are the net additions to private-good consumption and admission tickets to bundle \(J\) which are stipulated for consumer \(h\) if \((\theta^h(x), \omega^h(x)) = (\theta, \omega)\). Given an initial allocation \(A\), a net-trade allocation is said to be feasible if \(z^A_c(\theta, \omega) + z^A_J(\theta, \omega) \in \{0, 1\}\) for all \(J\). \(\sum_{J \subset M} [\chi^A_J(\theta, \omega) + z^A_J(\theta, \omega)] = 1\) and, moreover,

\[
\int_{[0,1]^{m+1}} z^A_c(\theta, \omega)f(\theta)d\theta d\nu(\omega) = 0
\]  

(B.4)
for \(i = c, \{1\}, \{2\}, \ldots, M\). Given the initial allocation \(A\), the net-trade allocation is incentive compatible if

\[
z^A_c(\theta, \omega) + \sum_{J \subset M} z^A_J(\theta, \omega) \sum_{j \in J} \theta_j Q^A_j \geq z^A_c(\theta', \omega') + \sum_{J \subset M} z^A_J(\theta', \omega') \sum_{j \in J} \theta_j Q^A_j
\]  

(B.5)
for all \((\theta, \omega)\) and \((\theta', \omega')\) in \([0, 1]^{m+1}\) for which \(\chi^A_J(\theta, \omega) + z^A_J(\theta', \omega') \in \{0, 1\}\) and \(\sum_{J \subset M} [\chi^A_J(\theta, \omega) + z^A_J(\theta', \omega')] = 1\).

An allocation \(A\) is said to be weakly renegotiation proof if, starting from \(A\), there is no feasible and incentive compatible net-trade allocation which provides a Pareto improvement in the sense that

\[
z^A_c(\theta, \omega) + \sum_{J \subset M} z^A_J(\theta, \omega) \sum_{j \in J} \theta_j Q^A_j \geq 0
\]  

(B.6)
for all \((\theta, \omega) \in [0, 1]^{m+1}\) and, moreover,

\[
\int_{[0,1]^{m+1}} [z^A_c(\theta, \omega) + \sum_{J \subset M} z^A_J(\theta, \omega) \sum_{j \in J} \theta_j Q^A_j] f(\theta)d\theta d\nu(\omega) > 0
\]  

(B.7)

The following lemmas provide analogues of Lemmas 3.1 and 3.2 for this weaker concept of renegotiation proofness.

**Lemma B.1** An allocation \(A\) is weakly renegotiation proof if and only if there exists a price schedule \(P^A(\cdot)\) such that, for almost all \((\theta, \omega) \in [0, 1]^{m+1}\), the vector \(\{\chi^A_J(\theta, \omega)\}_{J \subset M}\) is a solution to the problem

\[
\max_{\chi_J, J \subset M} \chi_J \left[ \sum_{j \in J} \theta_j Q^A_j - P^A(J) \right]
\]  

(B.8)
under the constraints that \(\chi_J \in \{0, 1\}\) for all \(J \subset M\) and \(\sum_{J \subset M} \chi_J = 1\).

**Proof Sketch.** The argument is the same as for Lemma 3.1. The "if" part of the lemma is again an instance of the first welfare theorem. As for the "only
if” part, one easily verifies that, for any allocation $A$ and any price schedule $P$, the set of solutions to problem (B.8) is nonempty. Moreover, the solution correspondence is upper hemi-continuous in $P$. Given that the measure $F \times \nu$ is atomless, it follows that the aggregate excess demand correspondence which is induced by the maximizer correspondence for problem (B.8) and the endowment specification in $A$ is upper hemi-continuous and convex-valued. Moreover, one easily verifies that, if $P$ is required to take values in $[0, m]$, the aggregate excess demand correspondence satisfies a suitable boundary condition. For any initial allocation $A$, a standard fixed-point argument therefore yields the existence of a competitive-equilibrium price schedule $P^A$. If $\{(\chi_j^A(\theta, \omega))_{J \subset M}\}$ fails to be a solution to problem (B.8), the associated competitive-equilibrium net-trade allocation is feasible and incentive compatible and provides Pareto improvement over $A$. Hence if $A$ is weakly renegotiation proof, $\{(\chi_j^A(\theta, \omega))_{J \subset M}\}$ must be a solution to problem (B.8).}

**Lemma B.2** An allocation $A$ is weakly renegotiation proof and incentive compatible if and only if there exists a price schedule $P^A(\cdot)$ such that, for all $\theta \in [0, 1]^m$, the purchase probabilities

$$q^A(J; \theta) := \int_{[0,1]} \chi_j^A(\theta, \omega) d\nu(\omega)$$

maximize

$$\sum_{J \subset M} q_J \left[ \sum_{j \in J} \theta_j Q_j^A - P^A(J) \right]$$

under the constraints that $q_J \geq 0$ for all $J \subset M$ and $\sum_{J \subset M} q_J = 1$, and moreover, there exists $C$ such that

$$C^A(\theta) := \int_{[0,1]} c^A(\theta, \omega) d\nu(\omega) = C - \sum_{J \subset M} q^A(J; \theta) P(J)$$

and

$$v^A(\theta) = C + \max_{J \subset M} \left[ \sum_{j \in J} \theta_j Q_j^A - P^A(J) \right]$$

for all $\theta \in [0, 1]^m$.

**Proof.** As in the proof of Lemma 3.2, the "if" part of the lemma is trivial. As for the "only if" part, I first note that, if $A$ is weakly renegotiation proof and incentive compatible, then Lemma B.1 and (B.9) imply that there exists a price schedule $P^A(\cdot)$ such that the vector $\{q^A(J; \theta)\}_{J \subset M}$ of purchase probabilities maximizes (B.10) under the constraints that $q_J \geq 0$ for all $J \subset M$ and $\sum_{J \subset M} q_J = 1$. 

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Conditional on $\tilde{\theta} = \theta$, the allocation $A$ generates the expected payoff

$$
v^A(\theta) = \int_{[0,1]} c^A(\theta, \omega) + \sum_{J \subseteq M} \chi_j^A(\theta, \omega) \sum_{j \in J} \theta_j Q^A_j d\nu(\omega) \tag{B.13}
$$

Further, let $v^A(\theta) = C^A(\theta) + \sum_{J \subseteq M} q^A(J; \theta) \sum_{j \in J} \theta_j Q^A_j.

(B.14)

Trivially, (B.14) implies $v^A(0) = C^A(0)$. Moreover, Lemma 2.2 implies that for almost any $\theta \in [0,1]^m$, the function $v(\cdot)$ has first partial derivatives satisfying

$$
v^A_i(\theta) = \sum_{J \subseteq M} \delta_{iJ} q^A(J; \theta) Q^A_i,
$$

where again $\delta_{iJ} = 1$ if $i \in J$ and $\delta_{iJ} = 0$ if $i \notin J$.

Define $\lambda_0 = 0$ and, for $k = 1, \ldots$ let $\lambda_k, J_k$ be such that, for $\lambda \in [\lambda_{k-1}, \lambda_k]$,

$$
J_k \in \arg \max_j \left[ \sum_{j \in J} \lambda \theta_j Q^A_j - P^A(J) \right]. \tag{B.16}
$$

Further, let $\bar{k}$ be such that $\lambda_{\bar{k}} < 1$ and $\lambda_{\bar{k} + 1} \geq 1$. From (B.14) - (B.16), one obtains

$$
v^A(\theta) - v^A(\lambda_k \theta) = \sum_{J \subseteq M} q^A(J; \theta) \sum_{j \in J} [\theta_j Q^A_j - \lambda_k \theta_j Q^A_j]
$$

$$
= \sum_{J \subseteq M} q^A(J; \theta) \left( \sum_{j \in J} \theta_j Q^A_j - P^A(J) \right) - \sum_{J \subseteq M} q^A(J; \theta) \left( \sum_{j \in J} \lambda_k \theta_j Q^A_j - P^A(J) \right)
$$

$$
= \sum_{j \in J_{\bar{k} + 1}} \theta_j Q^A_j - P^A(J_{\bar{k} + 1}) - \left( \sum_{j \in J_{\bar{k} + 1}} \lambda_k \theta_j Q^A_j - P^A(J_{\bar{k} + 1}) \right)
$$

$$
= \sum_{j \in J_{\bar{k} + 1}} \theta_j Q^A_j - P^A(J_{\bar{k} + 1}) - \left( \sum_{j \in J_{\bar{k} + 1}} \lambda_k \theta_j Q^A_j - P^A(J_{\bar{k} + 1}) \right),
$$

the last equation following from the maximization property of the vectors $\{q^A(J; \lambda \theta)\}_{J \subseteq M}$ and the sets $J_{\bar{k} + 1}$ and $J_{\bar{k}}$. By a precisely analogous argument, one also obtains

$$
v^A(\lambda_{k-1} \theta) - v^A(\lambda_{k-1} \theta) = \sum_{j \in J_k} \lambda_{k-1} \theta_j Q^A_j - P^A(J_k) - \left( \sum_{j \in J_{k-1}} \lambda_{k-1} \theta_j Q^A_j - P^A(J_{k-1}) \right)
$$

for $k = \bar{k}, \bar{k} - 1, \ldots, 1$. Upon adding these equations, one concludes that

$$
v^A(\theta) - v^A(0) = \sum_{j \in J_{\bar{k} + 1}} \theta_j Q^A_j - P^A(J_{\bar{k} + 1}) - (-P^A(J_0)).
$$

$$
= \sum_{j \in J_{\bar{k} + 1}} \theta_j Q^A_j - P^A(J_{\bar{k} + 1}) + \sum_{J \subseteq M} q^A(J; 0) P^A(J),
$$

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or, equivalently,

\[ v^A(\theta) = C^A(0) + \sum_{J \subseteq M} q^A(J; 0)P^A(J) + \max_{J \subseteq M} \left[ \sum_{J \subseteq J} \theta_J q^A_J - P^A(J) \right]. \]

Upon setting \( \bar{C} := C^A(0) + \sum_{J \subseteq M} q^A(J; 0)P^A(J) \), one obtains (B.12). From (B.14), one then also obtains (B.11).

In view of Lemma B.2, the problem of finding an allocation that maximizes welfare over the set of allocations that are feasible, incentive compatible, weakly renegotiation proof and individually rational is equivalent to the problem of finding an optimal price schedule.
References


