Optimal Taxation and Public Good Provision in a Two-Class Economy

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Abstract

This paper combines a problem of optimal income taxation with a decision on public good provision. The economy consists of two groups of individuals with private information on two characteristics, earning ability and the valuation of a public good. It is shown that the level of redistribution has to be cut if tax revenues are needed to finance public good provision. Consequently, adjustments of the transfer system serve to discourage the more productive from exaggerating the desirability of public good provision. Similarly, an understatement of the less productive has to be prevented. One either ends up in a situation in which public good provision and redistribution are less than optimal or in a situation where both are higher than optimal.

Keywords: Income Taxation, Public Good Provision, Revelation of Preferences, Two-dimensional Heterogeneity.

1 Introduction

Consider asking a politician, who is known to be a representative of the upper class, about his views on using public money for a certain project. For concreteness, imagine an opera building or the construction of a highway. In addition, assume that this politician lives in a “welfare state”, which allocates a lot of resources to a transfer system and the public provision of goods. The politician is hence confronted with a choice between two alternative uses of public money. So, he will support public good provision because his clientele does not benefit from redistribution. In particular, he will do

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so whether or not the average upper class person really values the opera a lot or really benefits from improved infrastructure.

This story illustrates the problem analyzed in this paper, namely the interplay of income taxation, redistribution and decisions on public good provision. To this end, the typical framework of the literature on optimal income taxation\(^1\) is extended. The standard approach either leaves the level of public good provision unexplained or assumes that a utilitarian planner, who happens to know the average valuation of public goods in the economy, determines the level of public good provision according to some version of the Samuelson Rule. The present paper, in contrast, allows for private information of individuals regarding their valuation of public goods. It attempts to characterize an optimal tax constitution, which reflects that decisions on public spending are based on statements of self-interested agents on the desirability of public good provision.

The analysis is based on a concept of coalition-proof income taxation, which has been introduced in Bierbrauer (2005a). The contribution of the present paper is to derive the optimal coalition-proof income tax for the a two-class economy. This economy – which consists of two groups of individuals, those with a high and those with a low level of earning ability – is a special case that has received some attention in the literature on optimal taxation.\(^2\) Compared to this literature, the present paper enriches the set of class characteristics. It is assumed, that members of the same class have a common valuation of a public project, which is only known to them. I.e. a member of the less productive class is not informed about the valuation of the more productive and vice versa. Hence, from an ex ante perspective, the valuation of public projects constitutes an unknown class characteristic. Coalition-proofness applied to this setting, boils down to the requirement that collective behavior on the class level must not manipulate the decision on public good provision.

The main result of the analysis is as follows: As soon as willingness to pay for public goods is dependent on earning ability, incentives for collective action have to be taken into account. I.e. the optimal nonlinear income tax, as derived by Stiglitz (1982, 1987) or Boadway and Keen (1993), is not robust to the introduction of unknown class characteristics. The reason is the incentive problem hinted at initially. If a transfer system and the public provision of goods compete for the same funds, then the more productive tend to exaggerate when asked about their valuation of public goods. The logic is simple, they want to benefit from the spending of their tax payments. Analogously, the less productive tend to understate their valuation if public good provision necessitates a cut of transfers.\(^3\) The need to take incentives

\(^1\)This literature starts with Mirrlees (1971). See Hellwig (2004a) for a recent treatment.

\(^2\)See e.g. Mirrlees (1975), Stiglitz (1982), Boadway and Keen (1993) or Gaube (2005).

\(^3\)In contrast, if the decision on public good provision and the level of transfers are independent, then the less productive support public good provision more enthusiastically.
for collective action into account implies a complementarity between the level of redistribution and the decision on public good provision: To prevent the more productive class from exaggerating, public good provision has to be accompanied by an increased level of redistribution. Similarly, understatement of the less productive is prevented by a reduced level of redistribution if there is no public good provision.

This paper is related to different strands of the literature. It combines the screening problem that is underlying the theory of optimal income taxation with the classical free-rider problem in public good provision.\(^4\) This requires two allow for a two-dimensional heterogeneity of individuals.\(^5\) Here, I follow a modelling strategy of Armstrong and Rochet (1999) and Cremer et al. (2001). The complexity of two-dimensional heterogeneity is mitigated by the assumption that all interesting variables can only take two values: A “yes or no”-decision on public good provision has to be taken. Earning abilities as well as valuations of the public good are either high or low.

The notion of coalition-proof income taxation, discussed at length in Bierbrauer (2005a), has been inspired by the literature on mechanism design problems under a threat of collusion among agents.\(^6\) Applied to the very simple setup of a two-class economy, a coalition proof income tax admits an interpretation in terms of political economics. Either group of individuals can be assumed to speak with one voice and thereby to choose the collective announcement that is most attractive from a class perspective. Hence, behavior is as if there happened to exist two ideological parties, one for each class, with private information on the preferences of their constituencies.\(^7\) Moreover, the decision on public good provision has to be based on party statements on the desirability of provision.

The remainder of the paper is organized as follows. The model is stated...
in section 2. In Section 3, incentives for collective action are clarified. They follow from a comparative static property of the optimal nonlinear income tax: the need to finance a public project hurts the poor class relatively more. Intuitively, the optimal nonlinear income tax is an incentive compatible scheme of redistribution. Binding incentive constraints prevent extracting higher payments from the “rich” if a public good has to be financed. Hence, public good provision necessitates a cut of redistribution, implying that the more productive tend to exaggerate when asked about the desirability of public good provision. Section 4 derives properties of the optimal, coalition-proof tax constitution. Truth-telling of the more productive can be ensured only via an excessive level of redistribution in case of public good provision. By the same logic, in situations without public good provision redistribution has to be reduced below its optimal level. Hence, one either ends up in a situation in which public good provision and redistribution are less than optimal or in a situation where both are higher than optimal. The last section contains concluding remarks on the notion of unknown class characteristics. All proofs can be found in an appendix.

2 The Model

2.1 Individuals, Preferences and Class Characteristics

The economy consists of a continuum of agents $j \in I := [0, 1]$. An individual is characterized by a pair of characteristics $(w^j, \theta^j)$, where $w^j$ is a productivity parameter and $\theta^j$ is a taste parameter that captures $j$’s valuation of a public project $Q \in \{0, 1\}$. It is assumed that any individual $j$ belongs to one of two types (classes) $t \in \{1, 2\}$. A type $t$ individual is characterized by a tuple $(w^t, \theta^t)$. That is, for each $j$, $(w^j, \theta^j)$ either equals $(w^1, \theta^1)$ or $(w^2, \theta^2)$. Type 2 individuals are more productive, $0 < w^1 < w^2$. The utility function of a type $t$ individual is given as

$$\bar{U}_t = \theta_t Q + U[C, Y, w_t] .$$

(1)

$C$ denotes consumption of private goods and $Y = Lw_t$ denotes effective labour or income. The function $U$ is referred to as the private utility component. To simplify the analysis, this function is assumed additively separable:

$$U = u(C) - v \left( \frac{Y}{w_t} \right) .$$

The functions $u$ and $v$ are strictly increasing and twice continuously differentiable. Moreover, $u$ is concave, $v$ is convex and those functions satisfy the following boundary condition which ensures interior solutions to optimization problems: for all $w_t$ and all $C > 0$, there exists $Y(C, w_t) > 0$, such that

$$u'(C) - \frac{1}{w_t} v' \left( \frac{Y(C, w_t)}{w_t} \right) = 0 .$$
In addition, preferences satisfy the single crossing condition
\[ \forall (Y, C) : \frac{1}{w_1} w' \left( \frac{Y}{w_1} \right) > \frac{1}{w_2} w' \left( \frac{Y}{w_2} \right), \]
according to which at any point in the \( Y-C \) plane, the indifference curve of a less productive individual is steeper.

### 2.2 Information Structure

Individuals have private information on their type realization. I.e. an individual \( j \) privately observes whether \((w^j, \theta^j) = (w_1, \theta_1)\) or \((w_2, \theta_2)\).

The following is assumed to be commonly known: The proportions of individuals belonging to either type are equal. The earning ability levels \( w_1 \) and \( w_2 \) are known parameters. That is, if one knows an individual \( j \)'s type, then one can infer \( j \)'s earning ability. In this sense, the level of earning ability is a commonly known class characteristic.

The taste parameters \( \theta_1 \) and \( \theta_2 \), in contrast, are unknown class characteristics. In terms of common knowledge, only the following information is available: either taste parameter indicates a high or a low valuation of the public project,

\[ \forall t : \theta_t \in \{ \theta_L, \theta_H \} , \text{ where } 0 \leq \theta_L < \theta_H. \]

Whether \( \theta_t \) equals \( \theta_L \) or \( \theta_H \) is known only to individuals of type \( t \).

Hence, from an ex ante perspective, two different aspects of the information structure can be distinguished. On the individual level, there is assignment uncertainty; it is not known which individuals \( j \in I \) are of type 1 and which are of type 2. In addition, there is aggregate uncertainty in the following sense: The economy can be in four different states depending on the preference parameters of type 1 and type 2 individuals, i.e. depending on the value of the vector \((\theta_1, \theta_2)\), where

\[ (\theta_1, \theta_2) \in \{ (\theta_L, \theta_L), (\theta_L, \theta_H), (\theta_H, \theta_L), (\theta_H, \theta_H) \} . \]

It proves helpful in subsequent sections to have the following notation available. From an ex ante perspective, the feasible set \( \Gamma \) for individual characteristics \((w^j, \theta^j)\) is given by

\[ \Gamma = \{ (w_1, \theta_L), (w_1, \theta_H), (w_2, \theta_L), (w_2, \theta_H) \} . \]

Denote by \( \Delta(\Gamma) \) the set of probability measures on \( \Gamma \). The setup of a two-class economy implies that aggregate uncertainty can be represented by a feasible set \( D \) for distributions of characteristics in the economy. This feasible set is a subset of \( \Delta(\Gamma) \), which looks as follows.

\[ D = \{ d_{LL}, d_{LH}, d_{HL}, d_{HH} \} , \]
where $d_{LL}$ is a measure on $\Gamma$ such that $d_{LL}(w_1, \theta_L) = 1/2$, $d_{LL}(w_1, \theta_H) = 0$ and $d_{LL}(w_2, \theta_L) = 1/2$, $d_{LL}(w_2, \theta_H) = 0$. Similarly, $d_{LH}$ is a measure on $\Gamma$ with $d_{LH}(w_1, \theta_L) = 1/2$, $d_{LH}(w_1, \theta_H) = 0$ and $d_{LH}(w_2, \theta_L) = 0$, $d_{LH}(w_2, \theta_H) = 1/2$. $d_{HL}$ and $d_{HH}$ are analogously defined. Denote by $\Gamma_r(d)$ the subset of $\Gamma$ which is given strictly positive mass under a distribution $d \in D$.

### 2.3 Coalition-proof Income Taxation

Admissible allocations are implementable using an income tax mechanism which satisfies a condition of coalition-proofness. This set is defined in the remainder of this paragraph.

An income tax is interpreted as an anonymous, direct revelation game. It specifies for each possible distribution of characteristics, i.e. for each $d \in D$, a level of public good provision $Q(d)$ and assigns to each characteristic $(w, \theta) \in \Gamma$ a combination of consumption and income, denoted $(C, Y)(w, \theta, d)$.\(^8\) This menu of income-consumption-pairs has to fulfill an individual incentive constraint: for any $d \in D$, an individual with characteristics $(w, \theta)$ prefers the consumption-income pair dedicated to her over the one dedicated to individuals with characteristics $(\hat{w}, \hat{\theta})$.\(^9\) Formally an income tax mechanism is defined as follows.

**Definition 1** A collection of mappings $Q : D \rightarrow \{0, 1\}$, $(C, Y) : D \times \Gamma \rightarrow \mathbb{R}^2_+$ is called an allocation rule. An income tax mechanism is an allocation rule for which the following individual incentive compatibility (I-IC) constraints, \(\forall d \in D, \forall (w, \theta) \in \Gamma \) and \(\forall (\hat{w}, \hat{\theta}) \in \Gamma\),

\[
U[(C, Y)(w, \theta, d), w] \geq U[(C, Y)(\hat{w}, \hat{\theta}, d), w],
\]

and the budget constraints, \(\forall d \in D, \forall (w, \theta) \in \Gamma_r(d)\),

\[
Y(w_1, \theta_1, d) - C(w_1, \theta_1, d) + Y(w_2, \theta_2, d) - C(w_2, \theta_2, d) \geq kQ(d),
\]

are satisfied. $k$ denotes the cost of public good provision.

The I-IC constraints provide incentives for truth-telling on the individual level. As the economy under consideration is large, those constraints are

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\(^8\)A complete description of the game requires to specify what happens if the announced distribution of characteristics does not indicate a feasible state of the economy. That is, if the announced distribution belongs to $\Delta(\Gamma)$ but not to $D$. Out of equilibrium payoffs have to be such that the incentive structure is preserved, i.e. such that truth-telling remains a weakly dominant strategy. This is, for instance, achieved by choosing $Q = 0$ and a degenerate consumption-income menu which contains only one $C$-$Y$-combination.

\(^9\)This requirement of self-selection implies that there exists, for each $d$, a nonlinear income tax schedule $T(Y)$ by which the menu $\{(C, Y)(w, \theta, d)\}_{(w, \theta) \in \Gamma}$ can be decentralized. This explains the use of the term income tax mechanism. Details can be found in Hammond (1979); Guesnerie (1995) or Bierbrauer (2005a).
stated for a given level of $d$. This reflects the fact, that, in a large economy, no single individual is able to influence the announced distribution of characteristics. In particular, no individual has an impact on public good provision.

In contrast, from the perspective of the tax setting institution $d$ cannot be taken as given. It learns the distribution of characteristics as a by-product of interviewing all individuals in the economy about their characteristics. This explains why truth-telling has to be ensured for all $(w, \theta) \in \Gamma$ and not only for those which belong to $\Gamma_r(d)$.

As the actual distribution $d$ is not known, there may be scope for collective action. If a subset of agents, with positive measure, jointly deviates, this affects the tax setting institution’s perception of the distribution of characteristics in the economy. E.g. if $\theta_2 = \theta_L$ and all type 2 individuals tell the truth, then, if all type 1 individuals report $(w_1, \theta_L)$ instead of $(w_1, \theta_H)$ the announced distribution switches from $d_{LL}$ to $d_{HL}$. In the following the concept of a coalition-proof income tax mechanism is defined. Such a mechanism ensures that truth-telling is an equilibrium outcome even under the threat of coalitional manipulations.

Some additional notation is needed. Denote by $S$ the set of measurable subsets of the set of agents, $I = [0, 1]$, with positive length. A typical element is denoted $S$. Denote the true profile of characteristics in $S$ by $\gamma_S := \{(w_j, \theta_j)\}_{j \in S}$. Denote the reported profile by $\hat{\gamma}_S := \{(\hat{w}_j, \hat{\theta}_j)\}_{j \in S}$. Denote the announced distribution induced by $\hat{\gamma}_S$, if the true state of the economy is $d \in D$ and all individuals not in $S$ report truthfully, by $\delta(\hat{\gamma}_S, d)$.

**Definition 2** A coalition $S$ is said to manipulate an income tax mechanism under the following conditions. There exist $d \in D$, and $\hat{\gamma}_S \neq \gamma_S$ such that:

i) The induced distribution of reports is feasible: $\delta(\hat{\gamma}_S, d) \in D$.

ii) All coalition members are strictly better off when choosing $\hat{\gamma}_S$. $\forall j \in S$:

$$\theta^j Q(\delta(\hat{\gamma}_S, d)) + U[(C, Y)(\hat{w}_j, \hat{\theta}_j, \delta(\hat{\gamma}_S, d)), w_j]$$

$$> \theta^j Q(d) + U[(C, Y)(w_j, \theta_j, \delta(\hat{\gamma}_S, d)), w_j].$$

iii) No coalition member wants to depart – unilaterally – from coalitional behavior. Given the $I$-$IC$-constraints, this requires $\forall j \in S$:

$$\theta^j Q(\delta(\hat{\gamma}_S, d)) + U[(C, Y)(\hat{w}_j, \hat{\theta}_j, \delta(\hat{\gamma}_S, d)), w_j]$$

$$= \theta^j Q(\delta(\hat{\gamma}_S, d)) + U[(C, Y)(w_j, \theta_j, \delta(\hat{\gamma}_S, d)), w_j].$$

An income tax mechanism is said to be coalition-proof if there exists no manipulating coalition $S$. 

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According to this definition, a coalition considers a collective deviation in response to truth-telling of all other individuals. The scope for manipulation is limited by the requirement that it must not be detectable. I.e. the relevant coalitional plans are only those for which it does not become apparent that a manipulation has occurred.\footnote{Note that even if a manipulation becomes apparent, the manipulating individuals are not yet identified. The above definition hence implicitly relies on the assumption, that the mechanism may punish all individuals harshly in response to an obvious collective lie. This in turn implies, that no coalition will consider such a collective plan.} In addition, coalition members have to agree unanimously on a deviation and may not use side payments to reach such an agreement. Finally, the I-IC-constraints, i.e. the incentives coalition members face individually, outside the coalition’s control, may not undermine the coalitional plan.

There are two different interpretations of the implicit assumptions on coalition formation. The first deals with the ability of individuals to communicate before the mechanism is played. Suppose such communication is possible and resolves the uncertainty among individuals about the actual state of the economy. This gives the rise to a so called implementation problem under complete information, in which the tax setting institution is the only uninformed party.\footnote{See Moore (1992) for a survey of this literature. Further discussion of pre-play communication among agents can be found in Palfrey (1992). The above definition presumes that in so far as individuals face pure coordination problems they are able to solve them by cheap talk. The latter assumption is discussed further in Farrell and Rabin (1996).} The above definition then requires that truth-telling is a best response for each coalition.

An alternative interpretation relies on a similarity to the notion of blocking used in cooperative game theory. Accordingly, an outcome is blocked, if there exists a coalition with a collective deviation, under which all members are better off. This interpretation does not require to be specific about coalition formation. It just says that an income tax has to be immunized against the threat of such manipulations.

### 2.4 Admissible allocation rules

In the specific setting of a two-class economy the requirement of coalition-proofness can be simplified. This paragraph establishes that incentive concerns can be separated: Incentives on the individual level are needed to ensure that – presuming a revelation of taste parameters – individuals are willing to reveal their earning ability. Incentives for collective action need to be addressed on the class level only. Given the revelation of earning ability, individuals belonging to the same class have to be prevented from a collective lie on their taste parameter.

**Definition 3** A utility allocation $\tilde{U}^A : \mathcal{D} \times \Gamma_\tau(\mathcal{D}) \rightarrow \mathbb{R}$ is a mapping which
assigns, for every state of the economy $d \in \mathcal{D}$, a utility level $\tilde{U}^A((w, \theta), d)$ to every pair of characteristics $(w, \theta) \in \Gamma_r(d)$. A utility allocation is said to be implementable if there exists a coalition-proof income tax mechanism such that $\forall d \in \mathcal{D}$ and for all $(w, \theta) \in \Gamma_r(d)$,

$$\tilde{U}^A((w, \theta), d) = \theta Q(d) + U[(C, Y)(w, \theta, d), w].$$

From a welfare perspective the essential property of any allocation rule is the induced utility allocation. The characterization of admissible utility allocations uses the following concepts:

**Definition 4** Let $x \in \{L, H\}$ be an indicator for the taste parameter of type 1 individuals, i.e. if $x = L$, then $\theta_1 = \theta_L$ and if $x = H$, then $\theta_1 = \theta_H$. Similarly, $y \in \{L, H\}$ is an indicator for the taste parameter of type 2 individuals. Hence, the actual state of the economy is given by $d_{xy} \in \mathcal{D}$.

i) An allocation rule is said to have the collective revelation of taste on the class level (C-RT-C)-property if $\forall x \in \{L, H\}, \forall \hat{x} \in \{L, H\}, \forall y \in \{L, H\}$ and $\forall \hat{y} \in \{L, H\}$:

$$\theta_x Q(d_{xy}) + U[(C, Y)(w_1, \theta_x, d_{xy}), w_1] \geq \theta_x Q(d_{\hat{y}y}) + U[(C, Y)(w_1, \theta_{\hat{y}}, d_{xy}), w_1],$$

$$\theta_y Q(d_{xy}) + U[(C, Y)(w_2, \theta_y, d_{xy}), w_2] \geq \theta_y Q(d_{x\hat{y}}) + U[(C, Y)(w_2, \theta_y, d_{xy}), w_2].$$

ii) An allocation rule is said to have the individual revelation of earning ability (I-REA)-property if $\forall x \in \{L, H\}, \forall y \in \{L, H\}$

$$U[(C, Y)(w_1, \theta_x, d_{xy}), w_1] \geq U(C, Y)(w_2, \theta_x, d_{xy}), w_1] ,$$

$$U[(C, Y)(w_2, \theta_y, d_{xy}), w_2] \geq U[(C, Y)(w_1, \theta_y, d_{xy}), w_2].$$

iii) An allocation rule is said to have the no discrimination of taste in terms of consumption and income (NDT-CY)-property if $\forall d \in \mathcal{D}$, $\forall w \in \{w_1, w_2\}, \forall \theta \in \{\theta_L, \theta_H\}$ and $\forall \theta' \in \{\theta_L, \theta_H\}$,

$$(C, Y)(w, \theta, d) = (C, Y)(w, \theta', d).$$

The NDT-CY property requires that, for a given distribution of characteristics in the economy, the allocation of private goods is independent of
taste parameters. The I-REA-property states – for the case of a two-class-economy – the incentive constraints which are typically analyzed in the literature on optimal income taxation.\textsuperscript{12} According to the C-RT-C-property manipulations of coalitions consisting only of individuals with the same type and which misreport only the taste parameter, are ruled out. Obviously, this condition is necessary for coalition-proofness. The following proposition states that, in conjunction with the NDT-CY and the I-REA-property, it is also sufficient.

**Proposition 1** Suppose that pooling of earning ability can be ruled out, i.e. \( \forall d \in D, (C, Y)(w_1, \theta_1, d) \neq (C, Y)(w_2, \theta_2, d) \).\textsuperscript{13} Then, a utility allocation is implementable if and only if it is implementable by an allocation rule which satisfies the NDT-CY, the I-REA and the C-RT-C-property.

Proposition 1 justifies the restriction to allocation rules with the NDT-CY-property. This implies in particular that a more concise notation can be used. In the following the consumption and income for an individual of type \( t \), given that the state of the world is \( d \), is written as

\[
(C_t(d), Y_t(d)) := (C, Y)(w_t, \theta_L, d) = (C, Y)(w_t, \theta_H, d).
\]

The I-REA-property is hence written in the following as, \( \forall d \in D, \forall t \in \{1, 2\}, \forall t' \in \{1, 2\}, U[C_t(d), Y_t(d), w_t] \geq U[C_{t'}(d), Y_{t'}(d), w_t]. \)

The budget constraints become, \( \forall d \in D, Q(d) \in \{0, 1\} \) and

\[
Y_1(d) - C_1(d) + Y_2(d) - C_2(d) \geq kQ(d) .
\]

The private utility level of a type \( t \) individual, in economy \( d \), is denoted

\[
V_t(d) := U[C_t(d), Y_t(d), w].
\]

The C-RT-C-constraints can be written as, \( \forall x \in \{L, H\}, \forall \hat{x} \in \{L, H\}, \forall y \in \{L, H\} \text{ and } \forall \hat{y} \in \{L, H\} : \)

\[
\theta_x Q(d_{xy}) + V_1(d_{xy}) \geq \theta_{\hat{x}} Q(d_{\hat{xy}}) + V_1(d_{\hat{xy}}),
\]

\[
\theta_y Q(d_{xy}) + V_2(d_{xy}) \geq \theta_{\hat{y}} Q(d_{\hat{xy}}) + V_2(d_{\hat{xy}}).
\]

\textsuperscript{12}These constraints are also called self-selection constraints, see e.g. Stiglitz (1982, 1987) or Broadway and Keen (1993).

\textsuperscript{13}Absence of pooling is required only to make the presentation more accessible. In subsequent sections, optimal tax mechanisms are characterized without imposing this assumption. It will turn out, that an optimum does not involve pooling.
The set of admissible allocation rules is represented in the remainder of the paper by the collections

\[ \left\{ Q(d), Y_1(d), C_1(d), Y_2(d), C_2(d) \right\}_{d \in \mathcal{D}}, \]

which satisfy the inequalities (2), (3) and (4).

### 2.5 Welfare Maximization

Utilitarian welfare as a function of the actual distribution of characteristics \( d_{xy} \) is denoted \( W(d_{xy}) := (\theta_x + \theta_y)Q(d_{xy}) + V_1(d_{xy}) + V_2(d_{xy}) \). The optimal coalition-proof income tax mechanism maximizes utilitarian welfare from an ex ante perspective. I.e. it is assumed that the tax setting planner perceives the actual distribution \( d \) of characteristics in the economy as the realization of a random variable \( \bar{d} \). Moreover, the planner has some prior beliefs on the likelihood of the different states of the economy. Those prior beliefs are denoted \( p_{LL} := \text{prob}(\bar{d} = d_{LL}), p_{LH} := \text{prob}(\bar{d} = d_{LH}), \) etc. Expected welfare from the planner’s perspective is accordingly given by

\[ EW = p_{LL}W(d_{LL}) + p_{LH}W(d_{LH}) + p_{HL}W(d_{HL}) + p_{HH}W(d_{HH}). \]

**Definition 5** The *optimal coalition-proof income tax mechanism* solves the problem of choosing \( \{Q(d), Y_1(d), C_1(d), Y_2(d), C_2(d)\}_{d \in \mathcal{D}}, \) subject to the inequalities in (2), (3) and (4), in order to maximize \( EW \).

This optimization problem differs from the one typically analyzed in the literature on optimal income taxation by the presence of the C-RT-C-restrictions, i.e by the inequalities in (4). Implicitly, an optimal income tax presumes knowledge of the distribution \( d \) of characteristics in the economy. Formally, the optimal income tax solves the relaxed problem, for which the C-RT-C-constraints are neglected.

**Definition 6** The *optimal income tax mechanism* solves the problem of choosing \( \{Q(d), Y_1(d), C_1(d), Y_2(d), C_2(d)\}_{d \in \mathcal{D}}, \) subject to the inequalities in (2) and (3), in order to maximize \( W(d) \), for every \( d \in \mathcal{D} \).

### 3 Properties of the Optimal Income Tax

This section presents some properties of the optimal income tax. Those results serve as an input to the subsequent section which characterizes the optimal coalition-proof income tax.
3.1 Exogenous revenue requirement

I first consider the problem of optimal nonlinear taxation with an exogenous revenue requirement. A utilitarian planner faces the problem of redistribution under I-REA constraints and the restriction that tax revenues of at least \( k \) have to be extracted. In particular, there is no decision on public good provision. Making use of the assumption that the private utility component is additively separable, this optimal income tax problem reads as

\[
\begin{align*}
\text{max}_{C_1,Y_1,C_2,Y_2} & \quad u(C_1) - v \left( \frac{Y_1}{w_1} \right) + u(C_2) - v \left( \frac{Y_2}{w_2} \right) \\
\text{s.t.} & \quad Y_1 - C_1 + Y_2 - C_2 \geq k , \\
& \quad u(C_1) - v \left( \frac{Y_1}{w_1} \right) \geq u(C_2) - v \left( \frac{Y_2}{w_2} \right) , \\
& \quad u(C_2) - v \left( \frac{Y_2}{w_2} \right) \geq u(C_1) - v \left( \frac{Y_1}{w_1} \right) .
\end{align*}
\]

(5)

A shorthand notation for the utility level, realized by individuals of type \( t \), at a solution \((Y_1^*(k), C_1^*(k), Y_2^*(k), C_2^*(k))\) to problem (5) is defined:

\[
R_t(k) := u(C_t^*(k)) - v \left( \frac{Y_t^*(k)}{w_t} \right) .
\]

The following result is known from the literature (see e.g. Stiglitz (1982)).

**Lemma 1** At a solution to problem (5) the budget constraint and only the I-REA for the more productive individuals are binding, implying that there is a distortion at the bottom and no distortion at the top:

\[
MRS_{1}^* := \frac{1}{w_2} v' \left( \frac{Y_2^*(k)}{w_2} \right) u'' \left( \frac{C_1^*(k)}{w_1} \right) < 1 \quad \text{and} \quad MRS_{2}^* := \frac{1}{w_2} v' \left( \frac{Y_2^*(k)}{w_2} \right) u'' \left( \frac{C_2^*(k)}{w_2} \right) = 1 .
\]

Those properties of the optimum can be used to determine how the utility levels of individuals vary with changes in the revenue requirement \( k \). The following Lemma shows, that both types realize a lower utility level if the revenue requirement \( k \) goes up and moreover that this utility loss is larger for the less productive. This statement relies on a technical assumption:

\[
\forall x \geq 0 : \quad \frac{1}{w_1} v'' \left( \frac{x}{w_1} \right) \geq \frac{1}{w_2} v'' \left( \frac{x}{w_2} \right) .
\]

(6)

Note that a sufficient condition for assumption (6) is \( v''' \geq 0 \).

**Lemma 2** Let \( v(\cdot) \) satisfy assumption (6). Let \( k' > k \). Then:

\[
R_1(k) - R_1(k') > R_2(k) - R_2(k') > 0 .
\]
The intuition behind this observation may be put as follows: Consider the planner’s optimum and suppose the revenue requirement is slightly increased. The more productive cannot be forced to cover the resulting small budget deficit as this would violate their I-REA-constraint. To the contrary, less able individuals can be made worse off without violating any constraint. Consequently, the planner has to reduce the extent of redistribution. The next result shows what the I-REA-constrained Pareto-frontier looks like in a neighborhood of the solution to problem (5). To this end the following optimization problem is considered.

$$\max_{C_1,Y_1,C_2,Y_2} u(C_1) - v\left(\frac{Y_1}{w_1}\right)$$

s.t. 
$$Y_1 - C_1 + Y_2 - C_2 \geq k \quad (BC) ,$$
$$u(C_1) - v\left(\frac{Y_1}{w_1}\right) \leq \bar{U}_2 \quad (I-REA_2) ,$$
$$u(C_2) - v\left(\frac{Y_2}{w_2}\right) = \tilde{U}_2 .$$

Lemma 3 Let \( v(\cdot) \) satisfy assumption (6). A solution to problem (7) has the following properties.

i) For all \( \bar{U}_2 \), (BC) is binding and one has no distortion at the top.

ii) The value function of this problem \( U_1^*(\bar{U}_2) \) is continuous, globally strictly concave and has a unique maximum at \( \tilde{U}_2^*(k) \).

iii) There is a maximal value \( \tilde{U}_2^*(k) \) such that for \( \bar{U}_2 < \tilde{U}_2^*(k) \) the (I-REA_2) is binding, implying a distortion at the bottom. For \( \bar{U}_2 > \tilde{U}_2^*(k) \), (I-REA_2) is not binding and there is no distortion at the bottom.

iv) For \( \bar{U}_2 = R_2(k) \) – i.e. at the utilitarian solution – \( U_1^*(\bar{U}_2) \) is strictly decreasing in \( \bar{U}_2 \). I.e. the (I-REA_2)-constrained utilitarian optimum still leaves room to make type 1 individuals better off.

Lemma 3 shows that there is a well defined range of parameters such that the planner indeed faces a tradeoff between the utility of the “rich” and the utility of the “poor”. Moreover the I-REA-constrained utilitarian optimum does not lie at the boundary of the region where the tradeoff prevails.

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14 This is not trivial as there is a region where both types can be made better off if \( \bar{U}_2 \) is increased. In that region, type 1 individuals do not benefit from a lower utility level of type 2 individuals: the potential utility gain, which stems from the fact that less resources are needed to generate a utility level of \( \bar{U}_2 \), is overcompensated by the utility loss of type 1 from a more severe distortion at the bottom needed to maintain incentive compatibility. See the appendix for a mathematical formulation of those statements.
Figure 1: The graph illustrates Lemmas 2 and 3. It shows the I-REA-constrained Pareto-frontier in a neighborhood of the utilitarian optimum for different revenue requirements, $k'$ and $k$, with $k' > k$. In particular, the utility difference $R_t(k) - R_t(k')$ is larger for $t = 1$.

3.2 The decision on public good provision

A utilitarian planner, who faces only I-REA-constraints will decide on public good provision according to the following rule: The public project is installed if and only if the implied aggregate utility gain compensates for the aggregate loss of private utility. Hence, $Q = 1$, if and only if

$$\theta_1 + \theta_2 + R_1(k) + R_2(k) \geq R_1(0) + R_2(0).$$

For the remainder of the paper, the following assumptions are imposed on the parameters of the model: At the I-REA-constrained utilitarian optimum, public good provision is desired if $d = d_{HH}$ but not desired if $d = d_{LL}$.

Formally,

$$2\theta_H \geq R_1(0) + R_2(0) - \left(R_1(k) + R_2(k)\right) \geq 2\theta_L. \quad (8)$$

Moreover, for $d = d_{LH}$ and $d = d_{HL}$, whether the planner chooses $Q = 1$, depends on the relation between $\theta_L + \theta_H$ and $R_1(0)+R_2(0)-(R_1(k)+R_2(k))$.

\footnote{If one of these assumptions was violated, then the a utilitarian planner would choose the same allocation in every state of the world. Hence, those assumptions are necessary in order to give a meaning to the distinction of different elements of $D$.}
4 Coalition-Proof Income Taxation

4.1 Possible Scenarios

Before the optimal coalition-proof income tax mechanism is derived, different parameter constellations of the model are distinguished. In particular, this allows to clarify under which circumstances incentives for collective action impose an effective burden on the welfare maximum. More formally, it is determined in which cases a solution to the relaxed problem of maximizing utilitarian welfare subject to I-REA-constraints only, is compatible with the C-RT-C-requirement. It turns out that one may safely ignore C-RT-C-constraints only if a condition is fulfilled, which can be interpreted, as saying that “willingness to pay for the public good” is independent of earning ability.

The different scenarios are defined with reference to the relaxed optimal income tax problem. A scenario is characterized by the views of more and less productive individuals, respectively, on the desirability of public good provision, at a solution to the relaxed problem.

In conjunction with the fact that the private utility loss due to public good provision is always higher for less able individuals, as shown in Lemma 2, assumption (8) implies that less able individuals oppose public good provision if

\[ \theta_1 < \theta_L \]

Similarly, the more productive desire provision if

\[ \theta_2 = \theta_H, \]

\[ \theta_H + R_1(k) > R_1(0), \quad \theta_L + R_2(k) > R_2(0). \]

(9)

The inequalities in (9) reduce the set of possible parameter constellations. The following scenarios may arise.

Scenario 1: Less productive individuals desire provision only if \( \theta_1 = \theta_H \), the more productive always desire provision, i.e. even if \( \theta_2 = \theta_L \):

\[ \theta_H + R_1(k) > R_1(0), \quad \theta_L + R_2(k) > R_2(0). \]

(10)

Using assumption (8) and the inequalities in (9) and (10) one verifies that a solution to the relaxed problem involves the following decisions on public good provision,

\[ Q(d_{LL}) = 0, \quad Q(d_{LH}) = Q(d_{HL}) = Q(d_{HH}) = 1. \]

(11)

However, the solution to the relaxed problem violates the C-RT-C for type 2 individuals in state \( d = d_{LL} \). Type 2 individuals will not admit a low valuation if this implies non-provision of the public good.

Scenario 2: Less productive individuals prefer non-provision even with \( \theta_1 = \theta_H \), the more productive individuals desire provision only if \( \theta_2 = \theta_H \), Under this constellation, a solution of the the relaxed problem implies

\[ Q(d_{LL}) = Q(d_{LH}) = Q(d_{HL}) = 0, \quad Q(d_{HH}) = 1. \]

(12)
Now, the solution to the relaxed problem violates the $C$-$RT$-$C$ for type 1 individuals in state $d = d_{HH}$. They will not report a high valuation if this implies $Q = 1$.

**Scenario 3:** The less able desire non-provision even if $\theta_1 = \theta_H$. Type 2 individuals desire provision even with $\theta_2 = \theta_L$. Under this parameter constellation, a solution to the relaxed problem either uses provision rule (11) or provision rule (12). Again, the solution to the relaxed problem is not compatible with $C$-$RT$-$C$. Now both problem arise: Type 1 individuals will not admit a high valuation and type 2 individuals will not admit a low valuation.

**Scenario 4:** Less productive individuals desire provision if $\theta_1 = \theta_H$ and more productive individuals prefer non-provision if $\theta_2 = \theta_L$. Making use of the inequalities in (9), this implies that, \(\forall t\),

\[
\theta_H \geq R_t(k) - R_t(0) \geq \theta_L.
\]

Hence, at a solution to the relaxed problem, individuals of any type prefer $Q = 1$ if $\theta_t = \theta_H$ and prefer $Q = 0$ if $\theta_t = \theta_L$. Scenario 4 hence gives rise to the statement that, at a solution to the relaxed problem, “willingness to pay for the public good” is independent of earning ability.

Under scenario 4, a solution to the relaxed problem uses one of the provision rules (11) or (12). Consequently, for $d = d_{LH}$ and $d = d_{HL}$, there are conflicting interests on public good provision. However, this conflict does not cause incentive problems on the class level. If (13) holds, then both provision rules have the property, that individuals of the class which opposes the decision on provision, do not gain from a joint deviation.

**Proposition 2** Suppose that (13) holds. Then, the solution to the relaxed problem has the $C$-$RT$-$C$-property.

This result shows that there exists indeed a parameter constellation, in which a tax setting planner, gets the information on taste parameter for free as the $C$-$Rt$-$C$-constraints do not affect the chosen policy. The next section describes the outcomes under scenarios 1-3. That is, for the cases in which willingness to pay for public goods is not independent of earning ability.

### 4.2 Optimal Coalition-Proof Income Taxation

**Scenario 1.** Under scenario 1, the optimal provision rule, for the relaxed problem, is given by (11). I.e the public good is installed as soon as one class of individuals has a high taste parameter. Recall that $C$-$RT$-$C$ fails for the outcome of the relaxed problem as the more productive want to induce public good provision even if $\theta_2 = \theta_L$.

This paragraph aims at a characterization of the optimal allocation rule, if,
in addition to the I-REA-constraints of the relaxed problem, the C-RT-C-requirement is imposed. To this end, first, provision rule (11) is taken as given and the following exercise is solved: Given (11), solve for the optimal choice of \( \{Y_1(d), C_1(d), Y_2(d), C_2(d)\}_{d \in D} \), subject to the I-REA-constraints in (2), the budget constraints in (3) and the C-RT-C-constraints in (4). In a second step the question is addressed, whether (11) really is the appropriate provision rule given that C-RT-C-constraints are imposed.

With an exogenous provision rule, the objective function for the problem of welfare maximization simplifies to

\[
EW_{ex} := p_{LL}[V_1(d_{LL}) + V_2(d_{LL})] + p_{LH}[V_1(d_{LH}) + V_2(d_{LH})] + p_{HL}[V_1(d_{HL}) + V_2(d_{HL})] + p_{HH}[V_1(d_{HH}) + V_2(d_{HH})]
\]

Moreover, taking provision rule (11) as given, the C-RT-C-constraints can be written as:

\[
V_1(d_{LL}) = V_1(d_{HH}) , \quad \theta_L \leq V_1(d_{LL}) - V_1(d_{HL}) \leq \theta_H ,
\]

\[
V_2(d_{HL}) = V_2(d_{HH}) , \quad \theta_L \leq V_2(d_{LL}) - V_2(d_{HL}) \leq \theta_H .
\]

The problem of implementing provision rule (11) is called Problem (P1) in the following. It is defined as: Choose \( \{Y_1(d), C_1(d), Y_2(d), C_2(d)\}_{d \in D} \) in order to maximize \( EW_{ex} \) subject to the constraints in (2), (3) and (14).

To characterize the solution of Problem (P1), it proves helpful first to solve a simpler optimization problem, referred to as Problem (P2). (P2) takes only the following subset of the constraints in (14) into account:

\[
V_1(d_{LL}) = V_1(d_{HH}) ,
\]

\[
V_2(d_{HL}) = V_2(d_{HH}) , \quad \theta_L \leq V_2(d_{LL}) - V_1(d_{LH}) .
\]

Formally, (P2) is defined as: Choose \( \{Y_1(d), C_1(d), Y_2(d), C_2(d)\}_{d \in D} \) in order to maximize \( EW_{ex} \) subject to the constraints in (2), (3) and (15).

**Proposition 3** Let \( v(\cdot) \) satisfy assumption (6). Let the parameters \( \theta_L \) and \( \theta_H \) be such that scenario 1 arises, i.e. the inequalities (9) and (10) hold true. There exists an upper bound \( \bar{\theta}_L \) such that for \( \theta_L \leq \bar{\theta}_L \) the following statements are true: A solution of Problem (P2) has the following properties:

\[
V_1(d_{LL}) < R_1(0) \text{ and } V_2(d_{LL}) > R_2(0) ;
\]

\[
V_1(d_{LH}), V_1(d_{HL}), V_1(d_{HH}) > R_1(k) \text{ and }
\]

\[
V_2(d_{LH}), V_2(d_{HL}), V_2(d_{HH}) < R_2(k) .
\]

Moreover, for all \( d \in D \), the budget constraint is binding, there is no distortion at the top and a distortion at the bottom. In addition, the constraint
\[ V_2(d_{LL}) - V_2(d_{LH}) \geq \theta_L \] is binding. Finally, a solution to Problem (P2) is also a solution to Problem (P1).

A solution to Problem (P2) basically has to fix the problem that, at a solution to the relaxed problem, more able individuals suffer so little from an increased revenue requirement, that they want the public good to be installed even if \( \theta_2 = \theta_L \). Consequently, to prevent collective deviations from truth-telling by type 2 individuals, the planner has to deviate from the relaxed optimum. In particular, the deviation has to be such that, from the perspective of type 2 individuals, the utility difference between provision and non-provision goes up. This requires to increase the level of redistribution above the relaxed optimum in states with public good provision and to reduce the level of redistribution in states with non-provision.

![Figure 2](image.png)

**Figure 2**: The figure illustrates the solution to Problem (P2). As the difference \( R_2(0) - R_2(k) \) is too small to satisfy C-RT-C for class 2, the planner has to deviate to points \( A \) and \( B \).

There are two reasons why proposition 3 requires \( \theta_L \) not to exceed some upper bound \( \bar{\theta}_L \). The first is to ensure that the I-REA-constraints for type 2 individuals are still binding at a solution to Problem (P2).\(^{16}\) The second

\(^{16}\)However, figure 1 suggests that proposition 3 does not rely on binding I-REA-constraints of type 2 individuals. This follows directly from Lemma 3. Moving along the (I-REA\(_2\))-constrained Pareto frontier, one enters the region where type 2 individuals strictly prefer the allocation dedicated to them. As there is no discontinuity involved, the
and more important reason is the requirement that a solution to Problem (P2) solves Problem (P1) as well. (P2) neglects C-RT-C-constraints of the less productive individuals, which ensure that they desire provision with $\theta_1 = \theta_H$ but prefer non-provision if $\theta_1 = \theta_L$, 
\[ \theta_H \geq V_1(d_{LL}) - V_1(d_{HL}) \geq \theta_L. \]
The correction of redistribution (relative to the relaxed optimum), which arises at a solution of Problem (P2) implies that the utility difference $V_1(d_{LL}) - V_1(d_{HL})$ shrinks. If the parameter $\theta_L$ is small, then there is enough room for such an adjustment. That is, the incentive corrections required to solve (P2), do not induce the less productive to prefer $Q = 1$, just in order to prevent the reduction of transfers that accompanies $Q = 0$. A solution of (P2) hence only solves (P1), if the adjustments of the transfer system – needed to fix collective incentives for type 2 individuals – do not imply a violation of the previously slack C-RT-C constraint of type 1 individuals.

Henceforth, incentive problems are called *modest* if a solution to (P2) also solves (P1). In contrast, incentive problems are called *severe*, if a solution to (P2) does not solve (P1). Here *severity* refers to the fact that the attempt to restore C-RT-C for one class of individuals, renders collective manipulations attractive for the other class.

I now return to the question whether a utilitarian planner, who faces C-RT-C constraints in addition to I-REA-constraints, indeed wants to implement provision rule (11). Possibly, the welfare burden of having to adjust the transfer system if (11) is chosen, is such that a different provision rule turns out to be superior. E.g. one alternative provision scheme would be to install the public good in every state of the world, $Q(d) = 1$ for all $d \in D$. While this provision rule has the disadvantage that resources are used to cover the cost of provision even if $d = d_{LL}$, there is no need to ask individuals about their taste parameters. Hence, there is no need to deviate from the private utility levels $R_1(k)$ and $R_2(k)$, respectively.

In case of a *modest* incentive problem, it depends on the planner’s prior whether or not provision rule (11) is chosen. Suppose $p_{LL}$ is very small, then the provision rule $Q \equiv 1$ seems attractive as the situation, in which a deviation from the relaxed optimum is required, is very unlikely. However, if the parameters of the model are such that the adjustment of the transfer system, required under provision rule (11), are negligible, then one may find priors such that this provision rule remains the optimal one.

In contrast, under a *severe* incentive problem, provision rule (11) will not be chosen. To see this suppose that the C-RT-C-constraints
\[ \theta_L \leq V_1(d_{LL}) - V_1(d_{HL}) \quad \text{and} \quad \theta_L \leq V_2(d_{LL}) - V_2(d_{LH}) \]

planner may as well adjust redistribution such that no I-REA-constraint is binding.
are both binding. The two binding incentive constraints imply that all indi-
viduals are indifferent between public good provision and non-provision if 
\( d = d_{LL} \). I.e. for given private utility levels all individuals are indifferent 
between provision rules provision rule (11) and \( Q \equiv 1 \). However, \( Q \equiv 1 \) 
avoids any departure from the relaxed optimum, implying that utilitarian 
welfare is higher in every state of the world.

To sum up, if provision rule (11) – or any other rule that makes the decision 
on provision dependent on the preferences of type 2 individuals – is chosen 
for implementation, the planner has to accept the necessity of excessive re-
distribution if the public good is installed and suboptimal redistribution if 
not. This may imply that the planner prefers a different provision rule, in 
order to limit the deviations from optimal, \( I\text{-REA} \)-constrained redistribu-

**Scenarios 2 and 3.** The analysis of scenario 2 is analogous to scenario 1. 
The collective incentive problem now stems from the less able class, which 
tends to understate preferences, in order to prevent a reduction of income 
transfers. To ensure truth-telling of type 1 individuals, the by now familiar 
pattern of incentive corrections is used: Public good provision is made more 
attractive in the eyes of the less able by an increase of redistribution above 
the relaxed optimum. Similarly, a situation without provision is made less 
attractive by a reduction of redistribution. Under a severe incentive problem 
those corrections imply that more productive individuals, even with a high 
valuation, do not desire public good provision. In this case, provision rule 
\( Q \equiv 0 \), yields a higher level of expected utilitarian welfare as compared to 
provision rule (12), which is part of the relaxed optimum 

Under Scenario 3, more productive individuals desire provision of the public 
project even if \( \theta_2 = \theta_L \) and the less able oppose provision even with \( \theta_1 = \theta_H \). 
That is, the incentive problems which were driving the analysis of scenario 
1, as well as those underlying scenario 2 are present. Consequently, a case 
of modest incentive problems is not possible under scenario 3, implying that 
the provision rule which would be chosen by an informed planner (either 
(11) or (12)) becomes undesirable.

5 Concluding Remarks

A typical feature of contemporary welfare states is their engagement in the 
provision of certain goods or services such as public infrastructure, national 
defense or a public school system. This raises the question to which of those 
cases the crude variable \( Q \) of this paper applies. \( Q \) stands for a a costly 
“yes-or-no”-decision to be made based on the valuations of individuals in 
the economy. In this sense \( Q \) might be related to any field of public activ-
ity, be it the construction of public school buildings, theaters, highways or equipment for the army.\footnote{Besley and Coate (1991) and Blomquist and Christiansen (1995) have shown that public provision of private goods may relax the limits to redistribution. I.e. it follows the same logic as do deviations from the Samuelson Rule in the analysis of Boadway and Keen (1993). This suggests a possible modification of this paper’s analysis dealing with the public provision of private goods and assuming that preferences for private goods are private information.}

Whatever $Q$ stands for, this paper assumes that there are, for some reason, differences in opinion. This suggests the following critique: The expenditure side of public policy is treated as a mere matter of taste. This is not appropriate if different views on a public project are themselves a consequence of differences in earning capacities. E.g. more and less productive individuals may, in principle, have the same views on a public school system. However, one has the option to send her children to a private school while the other one has not. Consequently, differences in the willingness to pay for a public school building are due to differences in productivity levels. In that case, there is no need to bother about the revelation of taste parameters.

In this paper’s analysis a taste for the public project is an unknown class characteristic and, from an ex ante perspective, any pattern of correlation between taste and productivity is allowed. One easily finds examples where this treatment seems plausible, e.g. regarding public engagement in cultural activities or national defense. For a lot of interesting problems however both views are restrictive. The valuation of a public project is probably neither a deterministic function of productivity parameters nor entirely unrelated to them. An obvious consequence of this remark would be to drop the separability assumption on private and public sources of utility. This, more complex case, is left to future research.

\section{Appendix}

**Proof of Proposition 1:**

An allocation rule is said to have the \textit{no discrimination of taste in terms of utility (NDT-U)-property} if $\forall d \in D, \forall w \in \{w_1, w_2\}, \forall \theta \in \{\theta_L, \theta_H\}$ and $\forall \theta' \in \{\theta_L, \theta_H\}$,

$$U(C(w, \theta, d), Y(w, \theta, d), w) = U(C(w, \theta', d), Y(w, \theta', d), w).$$

Note that, because preferences satisfy the separability property stated in equation (1), the \textit{NDT-U}-property is an implication of \textit{I-IC}.

\textit{Claim 1.} An allocation rule is an income tax mechanism if and only if it has the \textit{NDT-U} and the \textit{I-REA}-property.
The only if-part is trivial. To prove the if-part, suppose an allocation rule, such that the \( NDT-U \) and the \( I-REA \) property hold, is not \( I-IC \). Then there exist \((w, \theta)\) and \((\hat{w}, \hat{\theta})\) and \(d\) such that

\[
U((C,Y)(w, \theta, d), w) < U((C,Y)(\hat{w}, \hat{\theta}, d), w) .
\]

Using \( NDT-U \) and \( I-REA \) one has:

\[
U((C,Y)(\hat{w}, \hat{\theta}, d), w) = U((C,Y)(\hat{w}, \theta, d), w)
\leq U((C,Y)(w, \theta, d), w) .
\]

Hence, a contradiction.

Claim 2. Suppose there are no pooling outcomes. Then, an income tax mechanism is coalition-proof with respect to manipulations by coalitions, that contain individuals of both types.

Suppose to the contrary, that there exist \(d \in D\), a coalition \(S \in S\) which contains individuals of both types, and \(\delta(\hat{\gamma}_S, d) \in D\) such that for all \(j \in S\):

\[
\theta^j Q(\delta) + U[(C,Y)(w^j, \theta^j, \delta), w^j]
= \theta^j Q(\delta) + U[(C,Y)((\hat{w})^j, \hat{\theta}^j, \delta), w^j]
> \theta^j Q(d) + U[(C,Y)(w^j, \theta^j, d), w^j] .
\]

Evaluating this condition for both types and using the \( NDT-U \)-property implies that, for all \(t \in \{1, 2\}\)

\[
U[(C,Y)(w_1, \theta_L, \delta), w_t] = U[(C,Y)(w_1, \theta_H, \delta), w_t]
= U[(C,Y)(w_2, \theta_L, \delta), w_t] = U[(C,Y)(w_2, \theta_H, \delta), w_t] .
\]

Due to the single crossing property, those equalities hold \(\forall t\) only if

\[
(C,Y)(w_1, \theta_L, \delta)) = (C,Y)(w_1, \theta_H, \delta))
= (C,Y)(w_2, \theta_L, \delta)) = (C,Y)(w_2, \theta_H, \delta)) .
\]

Hence, contradicting the assumption, that there is no pooling.

Claim 3. Suppose an allocation rule has the \( C-RT-C \)-property. Then, it is coalition-proof with respect to manipulations by coalitions, that contain individuals of only one type.
Suppose all individuals of type $t'$ reveal their characteristics truthfully. Then all individuals of type $t$, $t \neq t'$, have to reveal their earning ability truthfully. Otherwise the announced distribution is not compatible with the commonly known fact that half of the population has earning ability $w_{t'}$ and half of the population has $w_t$. Moreover all individuals of type $t$ have to report the same taste parameter. Otherwise the announced distribution does not belong to $D$. Hence, any conceivable manipulation must involve a revelation of earning ability and a collective misreport of taste on the class level. Those manipulations are ruled, by the $C$-$RT$-$C$-property.

Claims 1 to 3 can be summarized as follows: Suppose there are no pooling outcomes. Then an allocation rule is a coalition-proof income tax mechanism if and only if it has the $NDT$-$U$, the $I$-$REA$ and the $C$-$RT$-$C$-property. The following Claim remains to be established

Claim 4. If pooling can be excluded, a utility allocation is implementable if and only if it is implementable by an allocation rule which satisfies the $NDT$-$CY$, the $I$-$REA$ and the $C$-$RT$-$C$-property.

As the $NDT$-$CY$-property implies the $NDT$-$U$-property, the if-part is trivial. To prove the only if-part, consider an implementable utility allocation and the underlying coalition-proof income tax mechanism $Q(\cdot), C(\cdot), Y(\cdot)$. Construct an allocation rule $Q'(\cdot), C'(\cdot), Y'(\cdot)$, which has the $NDT$-$CY$-property and coincides with $Q(\cdot), C(\cdot), Y(\cdot)$ “on the equilibrium path” as follows: $\forall d \in D$, $Q'(d) = Q(d)$ and $\forall (w_t, \theta) \in \Gamma_r(d)$, $C'(w_t, \theta, d) = C(w_t, \theta, d)$ and $Y'(w_t, \theta, d) = Y(w_t, \theta, d)$. For $\theta' \neq \theta$, $C'(w_t, \theta', d) = C'(w_t, \theta, d)$ and $Y'(w_t, \theta', d) = Y'(w_t, \theta, d)$. By construction, $Q'(\cdot), C'(\cdot), Y'(\cdot)$ inherits the $NDT$-$U$, the $I$-$REA$ and the $C$-$RT$-$C$-property from $Q(\cdot), C(\cdot), Y(\cdot)$.

Proof of Lemma 2:

Claim 1.

$$\frac{dY'_1(k)}{dk} > 0 \quad ; \quad \frac{dY'_2(k)}{dk} > 0 \quad ; \quad \frac{dC'_2(k)}{dk} < 0$$

These comparative statics are derived as follows: knowing that at a solution to problem (5) the $I$-$REA$-constraint for type 2 as well as the budget constraint are binding allows to setup the Lagrangean for the planner’s problem. The first order conditions imply the following system of equations:

$$\frac{u'(C'_2(k))}{u(C'_2(k))} = \frac{(1 - MRS'_1) + (1 - MRS'_2)}{MRS'_1 - MRS'_2},$$

(16)
where \( MRS^* := \frac{\frac{1}{w_2} v' \left( \frac{Y'_1(k)}{w_2} \right)}{u'(C'_1(k))} \):

\[
Y'_1(k) - C'_1(k) + Y'_2(k) - C'_2(k) = k, \quad (17)
\]

\[
u'(C'_2(k)) = \frac{1}{w_2} v' \left( \frac{Y'_2(k)}{w_2} \right), \quad (18)
\]

\[
u(C'_1(k)) - v \left( \frac{Y'_1(k)}{w_2} \right) = u(C'_2(k)) - v \left( \frac{Y'_2(k)}{w_2} \right), \quad (19)
\]

Differentiating these equations with respect to \( k \) yields a system of equations that can be used to solve for the derivatives of \( Y'_1(k), C'_1(k), Y'_2(k) \) and \( C'_2(k) \) with respect to \( k \). In particular, equation (16) implies:

\[
\frac{dY'_1(k)}{dk} = \alpha \frac{dC'_1(k)}{dk} + \beta \frac{dC'_2(k)}{dk}, \quad (20)
\]

where \( \alpha := \frac{u''(C'_1(k)) \left[ 2u'(C'_2(k)) + \frac{1}{w_2} v' \left( \frac{Y'_2(k)}{w_2} \right) - \frac{1}{w_1} v' \left( \frac{Y'_1(k)}{w_1} \right) \right]}{\frac{1}{w_1} v'' \left( \frac{Y'_1(k)}{w_1} \right) \left[ u'(C'_1(k)) + u'(C'_2(k)) \right] - \frac{1}{w_2} v'' \left( \frac{Y'_2(k)}{w_2} \right) \left[ u'(C'_1(k)) - u'(C'_2(k)) \right]}
\]

and \( \beta := \frac{u''(C'_2(k)) \left[ 2u'(C'_1(k)) - \frac{1}{w_2} v' \left( \frac{Y'_2(k)}{w_2} \right) - \frac{1}{w_1} v' \left( \frac{Y'_1(k)}{w_1} \right) \right]}{\frac{1}{w_1} v'' \left( \frac{Y'_1(k)}{w_1} \right) \left[ u'(C'_1(k)) + u'(C'_2(k)) \right] - \frac{1}{w_2} v'' \left( \frac{Y'_2(k)}{w_2} \right) \left[ u'(C'_1(k)) - u'(C'_2(k)) \right]}
\]

Note that by assumption (6) the common denominator of \( \alpha \) and \( \beta \) is strictly positive. The numerator of \( \beta \) is negative because of the distortion at the bottom and the single crossing property, which imply that:

\[
\frac{1}{w_2} v' \left( \frac{Y'_1(k)}{w_2} \right) < \frac{1}{w_1} v' \left( \frac{Y'_1(k)}{w_1} \right) < 1
\]

To see that the numerator of \( \alpha \) is negative as well, note that equation (16) implies:

\[
2u'(C'_2(k)) + \frac{1}{w_2} v' \left( \frac{Y'_2(k)}{w_2} \right) - \frac{1}{w_1} v' \left( \frac{Y'_1(k)}{w_1} \right) = \frac{\left[ \frac{1}{w_1} v' \left( \frac{Y'_1(k)}{w_1} \right) \right]^2 - \left[ \frac{1}{w_2} v' \left( \frac{Y'_2(k)}{w_2} \right) \right]^2}{2u'(C'_1(k)) - \frac{1}{w_2} v' \left( \frac{Y'_2(k)}{w_2} \right) - \frac{1}{w_1} v' \left( \frac{Y'_1(k)}{w_1} \right)} > 0.
\]
From equation (17):
\[
\frac{dY^*_1}{dk} \cdot \frac{dC^*_1}{dk} + \frac{dY^*_2}{dk} \cdot \frac{dC^*_2}{dk} = 1 .
\] (21)

From equation (18):
\[
\frac{dY^*_2}{dk} = \gamma \frac{dC^*_2}{dk} ,
\] (22)

where:
\[
\gamma := \frac{u''(C^*_2(k))}{\frac{1}{w^2}v''\left(\frac{Y^*_2(k)}{w^2}\right)} \leq 0 .
\]

Differentiating equation (19) and using equation (18) allows to derive the following:
\[
\frac{dY^*_2}{dk} \cdot \frac{dC^*_2}{dk} = \left[\frac{1}{w^2}v''\left(\frac{Y^*_2(k)}{w^2}\right)\right] \frac{dY^*_1}{dk} - \left[\frac{1}{w^2}v''\left(\frac{Y^*_2(k)}{w^2}\right)\right] \frac{dC^*_1}{dk} .
\]

Using this expression to substitute for \(\frac{dY^*_2}{dk} \cdot \frac{dC^*_2}{dk}\) in (21) gives the following equation:
\[
\frac{dY^*_1}{dk} = \delta \frac{dC^*_1}{dk} + \epsilon ,
\] (23)

where:
\[
\delta := \frac{u'(C^*_1(k)) + \frac{1}{w^2}v''\left(\frac{Y^*_1(k)}{w^2}\right) + \frac{1}{w^2}v''\left(\frac{Y^*_2(k)}{w^2}\right)}{\frac{1}{w^2}v''\left(\frac{Y^*_1(k)}{w^2}\right) + \frac{1}{w^2}v''\left(\frac{Y^*_2(k)}{w^2}\right)} \geq 1 ,
\]
\[
\epsilon := \frac{\frac{1}{w^2}v''\left(\frac{Y^*_1(k)}{w^2}\right) + \frac{1}{w^2}v''\left(\frac{Y^*_2(k)}{w^2}\right)}{\frac{1}{w^2}v''\left(\frac{Y^*_1(k)}{w^2}\right) + \frac{1}{w^2}v''\left(\frac{Y^*_2(k)}{w^2}\right)} \in \, ]0,1[ .
\]

After having done these preparations, the comparative static effects can be analyzed using equations (20),(21),(22) and (23). The resulting expressions are the following:
\[
\frac{dC^*_2}{dk} = \frac{(\alpha - \delta) + \epsilon(1 - \delta)}{(\alpha - \delta)(\gamma - 1) + \beta(1 - \delta)} < 0 ,
\]
\[
\frac{dY^*_2}{dk} = \gamma \frac{(\alpha - \delta) + \epsilon(1 - \delta)}{(\alpha - \delta)(\gamma - 1) + \beta(1 - \delta)} > 0 ,
\]
\[
\frac{dY_1^*(k)}{dk} = -\frac{\beta\delta}{\alpha - \delta} \frac{(\alpha - \delta) + \epsilon(1 - \delta)}{(\alpha - \delta)(\gamma - 1) + \beta(1 - \delta)} + \frac{\alpha\epsilon}{\alpha - \delta} > 0,
\]
\[
\frac{dC_1^*(k)}{dk} = \frac{\epsilon(\gamma - 1) - \beta(\alpha - \delta)(\gamma - 1) + \beta(1 - \delta)}{\alpha - \delta}.\]

Note that the sign of \(\frac{dC_1^*(k)}{dk}\) is ambiguous.

**Claim 2.**

\[
0 > \frac{d}{dk} \left[ u(C_1^*(k)) - v\left(\frac{Y_1^*(k)}{w_1}\right) \right] > \frac{d}{dk} \left[ u(C_1^*(k)) - v\left(\frac{Y_2^*(k)}{w_2}\right) \right].
\]

To see this, recall that at a solution of problem (5) the I-REA-constraint of the more productive type is binding (see equation (19)). Hence, it must be the case that:

\[
\frac{d}{dk} \left[ u(C_2^*(k)) - v\left(\frac{Y_2^*(k)}{w_2}\right) \right] = \frac{d}{dk} \left[ u(C_1^*(k)) - v\left(\frac{Y_1^*(k)}{w_1}\right) \right].
\]

Due to the convexity of \(v(\cdot)\) and \(\frac{dY_1^*(k)}{dk} > 0\) one also has:

\[
\frac{d}{dk} \left[ u(C_1^*(k)) - v\left(\frac{Y_1^*(k)}{w_2}\right) \right] > \frac{d}{dk} \left[ u(C_1^*(k)) - v\left(\frac{Y_1^*(k)}{w_1}\right) \right].
\]

To see that also the first inequality holds, note that using (18) one has:

\[
\frac{d}{dk} \left[ u(C_2^*(k)) - v\left(\frac{Y_2^*(k)}{w_2}\right) \right] = u'(C_2^*(k)) \left[ \frac{dC_2^*(k)}{dk} - \frac{dY_2^*(k)}{dk} \right].
\]

This expression is strictly negative by **Claim 1.**

**Proof of Lemma 3:** Consider the Lagrangean of problem (7):

\[
L = u(C_1) - v\left(\frac{Y_1}{w_1}\right) - \mu[k + C_1 + C_2 - Y_1 - Y_2] - \lambda[u(C_1) - v\left(\frac{Y_1}{w_1}\right) - \bar{U}_2] - \nu[\bar{U}_2 - u(C_2) - v\left(\frac{Y_2}{w_2}\right)].
\]

Deriving first order conditions one easily verifies that (BC) has to be binding, that there is no distortion at the top and a distortion at the bottom if and only if (I-REA_2) is binding. Denote the solution of problem (7) by \((\bar{Y}_1, \bar{C}_1, \bar{Y}_2, \bar{C}_2)\). Denote the values of the multipliers at the optimum by \(\bar{\mu}, \bar{\lambda}, \bar{\nu}\). Denote the solution of problem (7) by \((\bar{Y}_1, \bar{C}_1, \bar{Y}_2, \bar{C}_2)\). Denote the values of the multipliers at the optimum by \(\bar{\mu}, \bar{\lambda}, \bar{\nu}\).
\( \bar{\lambda}(\bar{U}_2) \) and \( \bar{\nu}(\bar{U}_2) \). Write

\[
\overline{MRS}_1 := \frac{\frac{1}{w_1}' \left( \frac{\bar{Y}_1}{w_1} \right)}{u'(C_1)} \quad \text{and} \quad \overline{MRS} := \frac{\frac{1}{w_2}' \left( \frac{\bar{Y}_2}{w_2} \right)}{u'(C_1)} .
\]

Suppose \( (I-REA_2) \) is binding. The first order conditions imply:

\[
\bar{\lambda}(\bar{U}_2) = 1 - \frac{\overline{MRS}_1}{1 - \overline{MRS}} \quad \text{and} \quad \bar{\nu}(\bar{U}_2) = \frac{u'(C_1)}{u'(C_2)} \frac{\overline{MRS}_1 - \overline{MRS}}{1 - \overline{MRS}} . \tag{24}
\]

Those multipliers are used in the following to study how the level of utility type 1 individuals realize depends on \( \bar{U}_2 \). The following property of the multipliers is used:

\[
\frac{dU^*_1(\bar{U}_2)}{d\bar{U}_2} = \bar{\lambda}(\bar{U}_2) - \bar{\nu}(\bar{U}_2) .
\]

\( \bar{\lambda}(\bar{U}_2) > 0 \) captures the effect that a lower level of \( \bar{U}_2 \) tends to reduce \( U^*_1 \) due to a worsening of incentive problems. The expression \(-\bar{\nu}(\bar{U}_2) < 0\) shows that a lower level of \( \bar{U}_2 \) allows to increase \( U^*_1 \) as less resources are needed to equip type 2 individuals with a utility level of \( \bar{U}_2 \). Moreover, using the comparative static results mentioned in footnote 18 and assumption (6) one easily verifies that the function \( \bar{\lambda}(\bar{U}_2) \) decreases in \( \bar{U}_2 \), i.e. the marginal burden of incentive problems is higher if type 2 individuals receive a low level of utility \( \bar{U}_2 \). The function \( \bar{\nu}(\bar{U}_2) \) increases in \( \bar{U}_2 \), i.e. the marginal utility gain type 1 individuals can obtain if type 2 individuals are made slightly worse off decreases in \( \bar{U}_2 \). Put differently, as long as \( (I-REA_2) \) is binding the function \( U^*_1(\bar{U}_2) \) is strictly concave and one has:

\[
\frac{d^2U^*_1(\bar{U}_2)}{d(\bar{U}_2)^2} = \bar{\lambda}(\bar{U}_2) - \bar{\nu}'(\bar{U}_2) < 0 .
\]

The observation that \( \bar{\lambda}(\bar{U}_2) \) decreases in \( \bar{U}_2 \) has already been stated. Hence, there is a critical value of \( \bar{U}_2 \), in the following referred to as \( \hat{\bar{U}}_2(k) \), such that if this critical value is exceeded \( (I-REA_2) \) is not binding anymore.\(^{20}\)

Deriving the first order conditions assuming that \( (I-REA_2) \) is not binding constraints and the no distortion at the top property if \( (I-REA_2) \) is binding. Otherwise it is characterized by the remaining two constraints, the no distortion at the top and the bottom property. Similarly as for the proof of Lemma 2 the comparative statics with respect to \( \bar{U}_2 \) can be derived. The details are skipped. Statements on the properties of the multipliers are based on this exercise.

\(^{19}\)The existence of a value of \( \bar{U}_2 \) such that \( (I-REA_2) \) is binding follows from the utilitarian solution characterized in Lemma 1

\(^{20}\)The existence of a value of \( \bar{U}_2 \) such that \( (I-REA_2) \) is not binding can e.g. be established by the so called laissez faire solution, where individuals of type \( t \) choose \( (Y_t, C_t) \) to maximize utility under the constraint \( Y_t = C_t + k \).
reveals:
\[
\tilde{\lambda}(\tilde{U}_2) = 0 \quad \text{and} \quad \tilde{\nu}(\tilde{U}_2) = \frac{u'(\tilde{C}_1)}{u'(\tilde{C}_2)}.
\] (25)

Again, using the comparative static results mentioned in footnote 18 shows that \(\tilde{\nu}(\tilde{U}_2)\) increases in \(\tilde{U}_2\). Put differently, as long as \((\text{I-REA}_2)\) is not binding the function \(U^*_1(\tilde{U}_2)\) is strictly concave and one has:
\[
\frac{d^2U^*_1(\tilde{U}_2)}{d(\tilde{U}_2)^2} = -\tilde{\nu}'(\tilde{U}_2) < 0.
\]

Moreover, one can show that the function \(\tilde{\lambda}(\tilde{U}_2) - \tilde{\nu}(\tilde{U}_2)\) is continuous at \(\hat{U}_2(k)\) and is hence globally strictly concave. It has to be shown that \(U^*_1\) has a maximum.\(^{21}\) This follows from the existence of a solution of the following problem:

\[
\max_{C_1,Y_1,C_2,Y_2} \quad u(C_1) - v \left( \frac{Y_1}{w_1} \right)
\]

s.t.
\[
Y_1 - C_1 + Y_2 - C_2 \geq k,
\]
\[
u(C_2) - v \left( \frac{Y_2}{w_2} \right) \geq u(C_1) - v \left( \frac{Y_1}{w_1} \right).
\]

Denote by \(\hat{U}_2^*(k)\) the utility level that results for type 2 individuals at a solution to problem (26). Moreover, deriving the first order conditions of problem (26) reveals:
\[
\tilde{\lambda}(\hat{U}_2^*(k)) - \tilde{\nu}(\hat{U}_2^*(k)) = 0.
\]

Finally, use the first order condition (16) of the utilitarian problem to substitute for \(u'(\tilde{C}_1)/u'(\tilde{C}_2)\) in the formula for \(\tilde{\nu}(\tilde{U}_2)\) (see (24)). One gets:
\[
\tilde{\lambda}(R_2(k)) - \tilde{\nu}(R_2(k)) = -1.
\]

\[\blacksquare\]

**Proof of Proposition 2:** Suppose that \(\theta_L + \theta_H + R_1(k) + R_2(k) > R_1(0) + R_2(0)\), implying that a solution to the relaxed problem uses provision rule (11).\(^{22}\) Consider classes \(t\) and \(t'\) with \(t \neq t'\).

1. Suppose \(\theta_t = \theta_L\). Let class \(t'\) report \(\theta_L\). Then, by definition of scenario

\[^{21}\text{Uniqueness then follows from the concavity of } U_1^*(\tilde{U}_2).\]

\[^{22}\text{If the reverse inequality holds, provision according to (12) is optimal. The proof is analogous.}\]
4, class \( t \) members would be worse off by reporting \( \theta_H \) instead of \( \theta_L \) as this would imply \( Q = 1 \) and a revenue requirement of \( k \). Let \( t' \) report \( \theta_H \). Then, given provision rule (11), class \( t \) cannot affect the chosen allocation.

2. Suppose \( \theta_t = \theta_H \). Let \( t' \) report \( \theta_L \). Then, by definition of scenario 4, class \( t \) members would be worse off by reporting \( \theta_L \) instead of \( \theta_H \) as this would imply \( Q = 0 \) and a revenue requirement of 0. Let \( t' \) report \( \theta_H \). Then, given provision rule (11), class \( t \) cannot affect the chosen allocation.

Proof of Proposition 3: First, the solution to Problem (P2) is characterized. An upper bound on \( \theta_L \) is needed, to ensure that the I-REA-constraint of type 2 individuals will be binding at a solution to Problem (P2). The following assumption provides an upper bound on \( \theta_L \), which is sufficient: Consider the indifference curve of the more productive type defined by the utility level \( \tilde{U}_2 = R_2(k) + \theta_L \). Assume:

\[ \tilde{U}_2(0) > \tilde{U}_2, \]  

(27)

where \( \tilde{U}_2(0) \) is the critical value of \( \tilde{U}_2 \) defined in the statement of Lemma 3. Recall that by definition of scenario 1, \( \theta_L + R_2(k) > R_2(0) \) implying that \( \tilde{U}_2 > R_2(0) \). The inequality in (27) ensures that even at the maximal utility level type 2 individuals could potentially realize – obtained if the planner adjusts payoffs only in states involving \( Q = 0 \) to guarantee C-RT-C –, there is still a level of redistribution that causes (I-REA) to be binding.

Claim 1. Let \( \theta_L \) be such that assumption (27) holds. Then, a solution to Problem (P2) satisfies:

\[ V_1(d_{LL}) < R_1(0) \text{ and } V_2(d_{LL}) > R_2(0); \]

\[ V_1(d_{LH}), V_1(d_{HL}), V_1(d_{HH}) > R_1(k) \text{ and } \]

\[ V_2(d_{LH}), V_2(d_{HL}), V_2(d_{HH}) < R_2(k). \]

Moreover, at a solution to Problem (P2), \( \forall d \in D \), the budget and the (I-REA) constraints are binding, implying no distortion at the top.

i) Let \( V_2(d_{LL}) \) and \( V_2(d_{HL}) \) be given at their optimal level. Then, for all \( x \in \{L, H\} \) the optimal level of \( V_1(d_{xL}) \) is determined as the value
function of the following problem:

\[
\begin{align*}
\text{max} & \quad u(C_1) - v\left(\frac{Y_1}{w_1}\right) \\
\text{s.t.} & \quad u(C_2) - v\left(\frac{Y_2}{w_2}\right) = V_2(d_{xL}) , \\
& \quad u(C_2) - v\left(\frac{Y_2}{w_2}\right) \geq u(C_1) - v\left(\frac{Y_1}{w_2}\right) , \\
& \quad Y_1 - C_1 + Y_2 - C_2 \geq kQ(d_{xL}) .
\end{align*}
\]

(28)

Let \( \theta_1 = \theta_L \). Due to assumption (27), at a solution to \( P2 \) one has \( V_2(d_{LL}) \leq \tilde{U}_2 \) as there cannot be a deviation from the informed planner’s optimum larger than dictated by incentive problems. Consequently, by Lemma 3, (I-REA_2) and the budget constraint have to be binding for \( d = d_{LL} \).

Note that \( V_2(d_{LL}) \leq \tilde{U}_2 \) implies according to the incentive constraint for party 2 that \( V_2(d_{LL}) \geq R_2(0) \) and \( V_1(d_{LL}) \leq R_1(0) \). Suppose to the contrary that \( V_2(d_{LL}) < R_2(0) \) and \( V_1(d_{LL}) > R_1(0) \). Then, using the monotonicity properties established in Lemma 3 it was possible to increase \( V_2(d_{LL}) \) without violating the constraint \( V_2(d_{LL}) - V_2(d_{LL}) \geq \theta_L \) thereby increasing utilitarian welfare.

Let \( \theta_1 = \theta_H \). It will be shown below that \( V_2(d_{HL}) \leq R_2(k) \). By Lemma 3, this implies that (I-REA_2) and the budget constraint are binding and that \( V_1(d_{HL}) \geq R_1(k) \).

ii) Let \( V_2(d_{HL}), V_2(d_{HH}) \) and \( V_2(d_{LL}) \) be given at their optimal level. Then the optimal level of \( V_1(d_{HL}) \) and \( V_1(d_{HH}) \) is determined as the value function of the following problem: Choose

\[
\left\{ \left( Y_1(d_{xH}), C_1(d_{xH}), Y_2(d_{xH}), C_2(d_{xH}) \right) \right\}_{x \in \{L,H\}}
\]

in order to maximize \( V_1(d_{HH}) \) subject to the constraints \( V_1(d_{LL}) = \).

---

\(^{23}\text{To see this formally, suppose to the contrary that the optimum involves } V_2(d_{LL}) > \tilde{U}_2. \text{ Then the planner could choose instead the allocation } (Y_1^*(k), C_1^*(k), Y_2^*(k), C_2^*(k)), \text{ for all } d \in D \text{ with } Q(d) = 1. \text{ For } d \text{ with } Q(d) = 0 \text{ the planner could choose the solution of (28) which arises for the case } V_2(d_{LL}) = \tilde{U}_2. \text{ Due to the monotonicity properties established in Lemma 3 this would increase utilitarian welfare.}\)
\[ V_1(d_{HH}) \text{ and, for all } x \in \{L, H\}: \]
\[ u(C_2(d_{xH}) - v \left( \frac{Y_2(d_{xH})}{w_2} \right) = V_2(d_{xH}), \]
\[ Y_1(d_{xH}) - C_1(d_{xH}) + Y_2(d_{xH}) - C_2(d_{xH}) \geq k, \]
\[ u(C_1(d_{xH})) - v \left( \frac{Y_1(d_{xH})}{w_2} \right) \leq V_2(d_{xH}). \]

From setting up the Lagrangean of this problem and deriving first order conditions one can easily verify that both budget constraints are binding and that the no distortion at the top property holds. It remains to be shown that both (I-REA2)-constraints are binding. Suppose the (I-REA2)-constraint for \(\theta_1 = \theta_L\) was not binding. Then the multiplier associated with this constraint has to be equal to zero and the first order conditions imply that there is no distortion at the bottom. However as \(V_2(d_{LL}) \leq \tilde{U}_2\), Lemma 3 implies: a system characterizing \((Y_1(d_{LH}), C_1(d_{LH}), Y_2(d_{LH}), C_2(d_{LH}))\) by a budget equation, a given utility level for type 2 individuals smaller than \(\tilde{U}_2(k)\) and no distortion at the top requires a distortion at the bottom to be consistent with the (I-REA2)-constraint. Hence, a contradiction. Using Lemma 3, the observation \(V_2(d_{LL}) \leq \tilde{U}_2(k)\) made in step i) implies that \(V_1(d_{LH})\) exceeds \(R_1(k)\). Using that \(V_1(d_{LH}) = V_1(d_{HH})\) allows to conclude that:

\[ V_1(d_{LH}) = V_1(d_{HH}) \geq R_1(k), (29) \]

This inequality implies that, at an optimum, one also has

\[ V_2(d_{HL}) = V_2(d_{HH}) \leq R_2(k), (30) \]

Suppose to the contrary that \(V_2(d_{HH}) > R_2(k)\). Using the inequality in (29) yields

\[ V_1(d_{HH}) + V_2(d_{HH}) > R_1(k) + R_2(k), \]

thereby contradicting that \(R_1(k) + R_2(k)\) is the highest level of welfare that is achievable at a solution to the relaxed problem.

The observation that \(V_2(d_{HH}) \leq R_2(k)\) implies that for \(d = d_{HH}\) the (I-REA2)-constraint has to be binding. This follows from the same argument that was used to show that it is binding for \(d = d_{LH}\).
iii) It remains to be shown that at an optimum

\[ V_2(d_{LL}) - V_2(d_{LH}) = \theta_L, \]

\[ V_1(d_{LL}) \neq R_1(0) \text{ and } V_2(d_{LL}) \neq R_2(0); \]

\[ V_1(d_{LH}), V_1(d_{HL}), V_1(d_{HH}) \neq R_1(k) \text{ and} \]

\[ V_2(d_{LH}), V_2(d_{HL}), V_2(d_{HH}) \neq R_1(k). \]

This follows from setting up the Lagrangean of Problem \((P2)\) and deriving first order conditions using the above results on the pattern of binding incentive constraints. In particular, if the constraint \(V_2(d_{LL}) - V_2(d_{LH}) \geq \theta_L\) was not binding, then the first order conditions result in the solution of the relaxed problem, which is known to violate this constraint. This implies that the multipliers associated with the incentive constraints \(V_1(d_{LH}) = V_1(d_{HH})\) and \(V_2(d_{HL}) = V_2(d_{HH})\) are different from zero as well. The presence of those multipliers in the first order conditions shows that for all \((\theta_1, \theta_2)\) the resulting allocation differs from the one chosen by the informed planner.

Claim 2. If \(\theta_L\) is sufficiently low, then a solution to Problem \((P2)\), satisfies \(\theta_H \geq V_1(d_{LL}) - V_1(d_{HL}) \geq \theta_L\), is not violated. I.e. the solution to Problem \((P2)\) is compatible with the C-RT-C-constraints for type 1 individuals, which enter the definition of Problem \((P1)\) but not the definition of Problem \((P2)\). In addition, the neglected C-RT-C-constraints for type 2 individuals, \(\theta_H \geq V_2(d_{LL}) - V_2(d_{LH})\), is not violated because the solution to Problem \((P2)\) satisfies \(V_2(d_{LL}) - V_2(d_{LH}) = \theta_L\).

According to the assumption \(\theta_H + R_1(k) > R_1(0)\) and Claim 1 at a solution to Problem \((P2)\) one has

\[ \theta_H > R_1(0) - R_1(k) > V_1(d_{LL}) - V_1(d_{HL}). \]

The assumptions outlined in the following are sufficient to ensure that an optimum has as well the property that the increased level of redistribution if the public good is provided, does not induce type 1 individuals to prefer public good provision in case of a low valuation. By Claim 1, \(V_2(d_{LL}) \leq \hat{U}_2\) implying that there is a corresponding utility level \(\hat{U}_1\) such that \(V_1(d_{LL}) \geq \hat{U}_1\).\(^{24}\) Analogously to \(\hat{U}_2\) define \(\hat{U}_2 := R_2(0) - \theta_L\). That is, \(\hat{U}_2\) gives the minimal level of \(V_2(d_{HH})\) the more productive individuals could potentially

\(^{24}\)More precisely, \(\hat{U}_1\) is the value function of the problem to maximize \(u(C_1) - v(Y_1/w_1)\) subject to \(\text{I-REA-constraints, a budget constraint for the case that } k = 0\) and the requirement that type 2 individuals realize a utility level of \(\hat{U}_2\).
realize at a solution to Problem \((P2)\). Again, there is a corresponding utility level \(\hat{U}_1\) such that \(V_1(d_{HL}) \leq \hat{V}_1\). Consequently, at a solution of Problem \((P2)\) it must be the case that \(V_1(d_{LL}) - V_1(d_{HL}) \geq \hat{U}_1 - \hat{U}_1\). Hence,

\[
\hat{U}_1 - \hat{U}_1 \geq \theta_L,
\]

is a sufficient condition such that the \(C-RT-C\)-constraint for class 1 is not binding. This inequality holds if \(\theta_L\) does not exceed \(R_2(0) - R_2(k)\) by too much. To sum up, \(\theta_L\) is sufficiently low if assumptions (27) and (31) are satisfied.

\[\hat{U}_1 - \hat{U}_1 \geq \theta_L, \tag{31}\]

References


