Fair intergenerational sharing of a natural resource

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May 15, 2009

Abstract

Overlapping generations are extracting a natural resource over an infinite future. We examine fair allocation of resource and compensations among generations. Fairness is defined by core lower bounds and solidarity upper bounds. The core lower bounds require that all coalition of generations obtains at least what it could achieve by itself. The solidarity upper bounds require that no coalition of generations enjoy a higher welfare that it would achieve nobody else extract the resource. We show that, upon existence, the allocation that satisfies the two fairness criteria is unique. It assigns to each generation its marginal contribution to the preceding generation. We then describe its dynamics.

1 Introduction

Sustainable development is defined as "development that meets the needs of the present without compromising the ability of future generations to meet their own needs" (Our Common Future, also known as the Brundtland Report). In an economy with natural resources, this definition of sustainable development would require literally that present generations abstain to extract any resource. Indeed, as long as the resource is scarce in the precise sense that no all generations can meet their own resource needs, meeting the needs of present generations

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compromises the ability of future generations to meet their own needs. Therefore natural resource scarcity implies that sustainable development as defined above is impossible.

The aim of the paper is to posit a formal definition of sustainable development in natural resource economy, that is a fair intergenerational sharing of natural resources. One way to reconcile the above definition of sustainable development with scarce natural resource extraction is to consider the welfare equivalent of resource needs. Indeed meeting future generation needs require that the first generation reduce their extraction and, therefore, consume less than their needs which reduces their welfare. Yet they might enjoy as much welfare than if they would consume their needs if future generations transfer part of their welfare from resource extraction. Present generations welfare is then preserved through compensations from future generations for not extracting to much resource. This principle describes our first fairness criteria, the so-called core lower bounds. It requires that the welfare of any generation or group of generation is not lower than it could achieve by itself by meeting its own need given the resource constraints.

On the other hand, the compensation paid by future generations to present generations from not extracting their needs should not be too high. Otherwise it would compromise the welfare of future generations. In particular, it should not assign to generations a higher welfare than these generations would achieve by meeting their own needs given the resource stock available. This is the spirit to the second fairness principle we introduce, the so-called the solidarity upper bounds. It requires that no generation or group of generations enjoy a welfare higher than the welfare it would achieve if nobody else extract the resource (or in the absence of other generations). Since water is scarce in the sense that not all generations can meet their own needs, no generation should en up with strictly more than their welfare from meeting their needs.

We show that, upon existence, there exists a unique extraction path and vector of compensation that satisfy the two fairness criteria: the core lower bounds and the aspiration welfare upper bounds. It assigns to each generation a welfare equals to its marginal contribution to the preceding generation. We then describe the dynamic of the fair extraction path and the compensations. Compensations are increasing over time at least for first generations which questions their feasibility: some generations might not be able to produce enough goods from the resource stock to pay-back previous generations. As a consequence, the fair allocation might not exist. We provide a special case where it does exist. Finally, we show that if there is no technical progress on resource productivity then generations' welfare are decreasing over time.

Our paper links together two streams of the literature that deals with the management of natural resources in a normative way. On the first hand, axiomatic theory of justice has been recently applied to compare welfare among generations by Bossert et al. (2007), Roemer and Suzumura (2007) and Asheim (2007). On the other hand, dynamic programming methods have been used to solve the social planner's problem featured by a representative infinitely lived individual maximizing the sum of a discounted flow of utilities. Pioneer works have been proposed by Dasgupta and Heal (1974) and Solow (1974) for exhaustible resources and were extended in many directions. Among them, the vintage structure of the population was exploited by Marini and Scaramozzino (1995), notably. Nevertheless, as argued by Calvo and Obstfeld (1988), intertemporal planning with overlapping generations reduces to a standard problem with a heterogeneous but infinitely lived agents. By combining the two approaches, we are able to ground our fairness axioms on the physical law of motion of the ressource and on the production structure. From these axioms, we are then in position to analyse the fairness properties of extraction paths and wealfare intergenerational sharing.

The paper proceeds as follows: section 2 introduces the model while sections 3 and 4 define the fairness principles. In section 5, we characterize the fair allocation. We describe its dynamics and discuss its existence in section 6. Section 7 concludes.

2 Model

A natural resource is exploited by successive overlapping generations at $t \in \mathbb{N}^+$. Let k_0 be the initial stock of resource and ρ its regeneration rate with $\rho \geq 1$ (the case $\rho = 1$ corresponds to an exhaustible resource). Denote by x_t the amount of resource extracted during period t (date t). The remaining resource stock at $t+1$ for every t is given by the following resource dynamic equation:

$$k_{t+1} = \rho(k_t - x_t) \tag{1}$$

Each generation t lives two periods. An individual is young in the first and old in the second period. Generation t exploits the resource when young as an input to produce consumption units. It is endowed with a production function f_t. We assume f_t strictly concave and increasing up to a maximal extraction level $\hat{x}_t$ for every t, formally $f_t'(x_t) > 0$ for every $x_t < \hat{x}_t$, $f_t'(\hat{x}_t) = 0$, $f_t'(x_t) < 0$ for every $x_t > \hat{x}_t$, $f_t''(x_t) < 0$ for every x_t and t.[1] The maximal extraction level $\hat{x}_t$ is called generation t's demand or satiated consumption. Importantly, we also assume that $f_t(0) = 0$ and $f_t'(0) = +\infty$ for every $t \in \mathbb{N}^+$. Co-existing generations might perform transfers among them. A generation t might share its production when young with old people from the preceding generation. Let us denote m_t the consumption units transferred by the generation t when young to the generation t_1 for every t. It allows generation t to consume $f_t(x_t) - m_t$ when young and m_{t+1} when old. Without loss of generality denote m_0 the first transfer given to the first generation 0 to the generation born in -1 whose welfare is not considered here (m_0 can be normalized to zero). Let γ be the individual discount rate, i.e. the value in terms of the intertemporal utility at time t of a marginal increase in the instantaneous utility at time $t+1$. We assume $0 < \gamma < 1$. Generation t's consumption from resource exploitation, hereafter called "utility", view at date t with x_t units extracted and transfers m_t and m_{t+1} is:

$$u_t = f_t(x_t) - m_t + \gamma m_{t+1}. \tag{2}$$

We assume that resource is scarce in the sense that all generations cannot extract their demand $\hat{x}_t$. More precisely, $\exists \tilde{t} \in \mathbb{N}^{++}$ such that if all generations $t < \tilde{t}$ extract $\hat{x}_t$, the resource available for generation $\tilde{t}$ (i.e. at date $\tilde{t}$) is strictly lower than generation $\tilde{t}$'s demand $\hat{x}_{\tilde{t}}$, formally $\rho^{\tilde{t}} k_0 - \sum_{t=0}^{\tilde{t}-1} \rho^{\tilde{t}-t} \hat{x}_t < \hat{x}_t$.

In this set-up with scarce resource and transferable utility, the selfish outcome under autarky is inefficient (Pareto dominated). Under autarky, it is optimal for each generation to extract the resource up to satiation. The first generations t up to $\tilde{t}$ extract their demand $\hat{x}_t$, therefore enjoying $f_t(\hat{x}_t)$ consumption units or utility at time t. Generation $\tilde{t}$ extracts the remaining resource $\rho^{\tilde{t}} k_0 - \sum_{t=0}^{\tilde{t}-1} \rho^{\tilde{t}-t} \hat{x}_t$, thereby exhausting the resource and leaving nothing

[1] Negative returns above $\hat{x}_t$ can be due to production costs that exceed the benefit from resource extraction, e.g. bottleneck effects on complementary inputs (e.g. labor or capital) that renders the resource unproductive but still costly to extract.

to future generation who therefore obtain $f_t(0)$ for every $t > \tilde{t}$. Given that f_t is concave with $f_t'(0) = +\infty$ for every t, total production from resource extraction up to a date higher than $\tilde{t}$ can be increased if at least one generation l before $\tilde{t}$ his extraction to let some resource for future generations after $\tilde{t}$. The increased production can be shared among generations through the transfers m_t to make every generation better-off at least weakly and strongly for some of them.

We examine coordinated extractions and transfers among generations. Generations agree on an allocation $\{x_t, m_t\}_{t=0,...,+\infty}$ which assigns resource extraction levels and intergenerational transfers for every generations $t = 0, 1, 2, ..., +\infty$. The allocation $\{x_t, m_t\}$ must satisfy the following feasibility conditions for every $t \in \mathbb{N}^+$:

$$0 \leq x_t \leq k_t, \tag{3}$$

$$0 \leq m_t \leq f_t(x_t). \tag{4}$$

The first above feasibility condition (3) insures that the (non-negative) amount of resource extracted does not exceed the stock available at date t. The second feasibility condition (4) insures that the (non-negative) transfer to the old of the previous generation is lower than the consumption produced at date t.

3 Core lower bounds

Our first fairness criteria refers to a fictitious cooperative game. Suppose that all generations could meet at the same date to agree on an allocation. A core allocation of the fictitious cooperative game is such that any coalition of generation obtains at least what it could obtain on its own, i.e. by coordinating extraction and performing transfers among its members. It satisfies the core lower bounds defined as the highest welfare that a coalition can achieve on its own.

In the fictitious cooperative game, generations can share the benefit from resource extraction without constraints: transfers can be performed among generations that are not contemporaneous in reality. Time impacts transfers only through an "exchange rate" for utility among generations. This rate is given by the individual discount rate γ which means that if generation t transfers 1 unit of consumption to generation $t + i$, the later receives only γ^i units. Consequently, transfers among coexisting generation are treated in a symmetric fashion: a transfer of 1 unit of consumption from the young born at time t to the old born at time $t-1$ is equally valuated by both generations in their respective inter-temporal utility functions. Importantly non-contemporaneous generations might benefit from coordinated extraction and share this benefit through transfers. In cooperative game theory words, non-consecutive coalitions can create value[2]. Of course, in the fictitious cooperative game, the sequence of extraction remains fixed: generations cannot exchange the timing of their extraction.

A coalition of generations is a non-empty subset of $\mathbb{N}^+$. Given two coalitions S and T, we write $S < T$ if $i < j$ for all $i \in S$ and all $j \in T$. Given a coalition S, we denote by $minS$ and $maxS$ respectively the first and last generation in S. Let $Pi = \{1, \ldots, i\}$ denote the set of predecessors of generation i and $P^0 i = Pi \backslash \{i\}$ denote the set of strict predecessors of generation i. Similarly, let $Fi = \{i, i+1, \ldots, n\}$ denote the set of followers of generation i and let $F^0 i = Fi \backslash \{i\}$ denote the set of strict followers of generation i. We often omit set brackets for sets and write i instead of $\{i\}$ or $v(i,j)$ instead of $v(\{i,j\})$. A coalition S is consecutive (or connected) if for all $i, j \in S$ and all $k \in N$, $i < k < j$ implies $k \in S$.

We need to define the highest welfare that a coalition can achieve on its own in the fictitious cooperative game. It is a cooperative game with externalities: The welfare of a coalition S depends on extraction strategy by generations outside S through the stock of resource available to S. We assume that the outsiders behave non-cooperatively by extracting the resource under autarky. Let $v(S)$ be the "value" function which assigns to any arbitrary coalition S its highest welfare. Consider first a coalition of consecutive generations $S = \{minS, ..., maxS\}$. The welfare that S can achieve depends on the stock of resource available for the first generation $minS$. We consider the worst credible[3] scenario for S which is that the generations preceding the coalition have extracted their satiated level whenever possible. Therefore the stock of resource available for the first generation $minS$ of a coalition S is

$$k_{minS}^{ncS} \equiv \min\{\rho^{minS} k_0 - \sum_{t=0}^{minS-1} \rho^{minS-t} \hat{x}_t, 0\}.$$

[2]The cooperative game associated to the problem is not a consecutive game à la Greenberg and Weber (2000).

[3]Extracting more than $\hat{x}_t$ is not credible for a generation t since it reduces utility.

Let $x_S = (x_i)_{i \in S}$ be the resource allocation assigned to members of S. The welfare $v(S)$ valued at date 0 that the consecutive coalition S can achieve by its own is:

$$v(S) = \max_{x_S} \sum_{t \in S} \gamma^t f_t(x_t),$$
$$s.t. \left| \begin{array}{l} k_{t+1} = \rho \left(k_t - x_t \right), \\ k_t \geq x_t \geq 0, k_t \geq 0, \\ k_{minS} = k_{minS}^{ncS} \end{array} \right. \tag{5}$$

The constraints to the maximization program are respectively the resource dynamic, the feasibility and initial resource stock constraints. In particular, for singletons $S = \{i\}$, we have

$$v(i) = f_i(\min\{\hat{x}_i, k_i^{nc}\}).$$

For any arbitrary coalition S, we need to decompose S into its unique partition into consecutive components $\mathcal{C}(S) = \{T_l\}_{l=1}^{L}$, where $T_1 < T_2 < ... < T_L$. Since the generations in-between two consecutive sub-coalitions $T_{l-1}, T_l \in \mathcal{C}(S)$ extract up to be satiated, given the resource stock k_{maxT_l} left by the last generation in T_l, the resource stock available for T_l for $l = 2, ..., L$ is

$$k_{minT_l}^{ncS} \equiv \min \left\{ \rho^{(minT_l - maxT_{l-1}+1)} k_{maxT_{l-1}+1} - \sum_{t=maxT_{l-1}+1}^{minT_l - 1} \rho^{(minT_l - t)} \hat{x}_t, 0 \right\}.$$

The welfare $v(S)$ valued at date 0 that S can achieve by its own is thus:

$$v(S) = \max_{x_S} \sum_{t \in S} \gamma^t f_t(x_t),$$
$$s.t. \left| \begin{array}{l} k_{t+1} = \rho \left(k_t - x_t \right), \\ k_t \geq x_t \geq 0, k_t \geq 0, \\ k_{minT_l} = k_{minT_l}^{ncS} \text{ for } l = 1, ..., L \end{array} \right. \tag{6}$$

In contrast to the case of consecutive coalitions, the last initial resource stock constraints are defined for each consecutive component of S. Let denote x_S^S the solution to (6) for any coalition S.

An important property of the value function defined in (6) is its superadditivity. Consider any disjoint coalitions $T, S \subset \mathbb{N}^+$. Since the resource allocation (x_T^T, x_S^S) can be implemented by coalition $T \cup S$, we have:

$$v(S \cup T) \geq v(S) + v(T).$$

An allocation $\{x_t, m_t\}$ satisfies the core lower bounds if and only if for all coalitions $S \subset \mathbb{N}^+$

$$\sum_{t \in S} \gamma^t \left(f_t(x_t) - m_t + \gamma m_{t+1}\right) \geq v(S). \tag{7}$$

4 Aspiration welfare upper bound

Our second criteria is a solidarity principle inspired by Moulin (1990). In the absence of the other generations, a generation t would be endowed with $\rho^t k_0$ units of the resource, which is the "natural" stock. It could enjoy the benefit of extracting this resource stock up to be satiated. Call this benefit valued at date 0 the generation t's *aspiration welfare*. Denote it $W(t) = \gamma^t f_t(\min\{\rho^t k_0, \hat{x}_t\})$. Since the resource is scarce in the precise sense that $\rho^t k_0 - \sum_{j=0}^{t-1} \rho^{j-t} \hat{x}_j < \hat{x}_t$ for every t, it is impossible to assign to every generation its aspiration welfare[4]. In Moulin (1990)'s terms, the sustainable resource exploitation problem exhibits negative group externalities. Because no particular agent bears any distinguished responsibility for these externalities, it is natural to ask that everyone takes up a share of them: no one should end up above his aspiration welfare. This argument generalizes to coalitions in a very natural way. The aspiration welfare of an arbitrary coalition S is the highest welfare it could achieve in the absence of the other generations[5].

In contrast to the core lower bound $v(S)$, coalition S inherits from an untouched resource when it computes the aspiration welfare. Formally, coalition S has access to $\rho^{minS} k_0 > k^{ncS}_{minS}$. For connected coalitions, it is the solution to the following program:

$$w(S) = \max_{x_S} \sum_{t \in S} \gamma^t f_t(x_t),$$

$$s.t. \left| \begin{array}{l} k_{t+1} = \rho\left(k_t - x_t\right), \\ k_t \geq x_t \geq 0, k_t \geq 0, \\ k_{minS} = \rho^{minS} k_0. \end{array} \right. \tag{8}$$

The constraints to the maximization program are respectively the resource dynamic, the feasibility and initial resource stock constraints.

[4] Indeed, for any $t > \tilde{t}$ (where $\tilde{t}$ is defined above) and consecutive coalitions $t \in S$, we have $\sum_{t \in S} w(t) > v(S)$, that is the sum of the generations' aspiration welfare exceed the maximal welfare from resource exploitation.

[5] Similarly to the core lower bounds we allow for transfers among non contemporaneous generations in S.

A disconnected coalitions S that leaves some resource stock following a connected sub-coalition T_l to supply the next sub-coalition T_{l+1} experience no extraction from outsiders. Therefore, the resource stock entering T_{l+1} is $\rho^{(minT_{l+1}-maxT_l)}k_{maxT_l}$. The aspiration welfare of an arbitrary coalition S is thus

$$w(S) = \max_{x_S} \sum_{t\in S} \gamma^t f_t(x_t),$$

$$s.t. \left| \begin{array}{l} k_{t+1} = \rho\,(k_t - x_t)\,, \\ k_t \geq x_t \geq 0, k_t \geq 0, \\ k_{minT_l} = \rho^{(minT_{l+1}-maxT_l)}k_{maxT_l} \text{ for } l = 1, ..., L. \end{array} \right. \tag{9}$$

The constraints to the maximization program are respectively the resource dynamic, the feasibility and initial resource stock constraints. The main difference between the programs (6) and (9) are the initial resource stock constraints which are reduces by generation outside S autarky extraction in (6) but not in (9).

An allocation $\{x_t, m_t\}$ satisfies the core lower bounds if and only if for all coalitions $S \subset \mathbb{N}^+$

$$\sum_{t\in S} \gamma^t\,(f_t(x_t) - m_t + \gamma m_{t+1}) \leq w(S). \tag{10}$$

5 A unique fair allocation

Consider the efficient resource allocation $\{x^*\}$ solution to the maximization program defined by $v(\mathbb{N}^+)$. Formally, $\{x_t^*\}$ maximizes $\sum_{t=0}^{\infty} \gamma^t f_t(x_t)$ subject to the initial resource stock constraint k_0, the resource dynamic constraint $k_{t+1} = \rho(k_t - x_t)$ and the feasibility constraints $k_t \geq x_t \geq 0$ for $t = 0, 1, 2, ...$ The concavity of f_t ensures that $\{x^*\}$ is unique.

A transfer scheme $\{m\}$ define a distribution of the welfare from intergenerational resource extraction. We focus to the transfer scheme that leads to the downstream welfare distribution introduced by Ambec and Sprumont (2002). Denoted $\{m^*\}$, it is the unique transfer scheme that assigns to each generation its marginal contribution to the preceding generation. Formally, $\{x_t^*, m_t^*\}$ assigns $u_t^* = f_t(x_t^*) - m_t^* + \gamma m_{t+1}^*$ to every generation $t \in \mathbb{N}^+$ with:

$$\gamma^t u_t^* = v(Pt) - v(P^0 t).$$

Proposition 1 *If $m_t^* \leq f_t(x_t^*)$ for every $t \in \mathbb{N}^+$, $\{x_t^*, m_t^*\}$ is the unique allocation that satisfies the core lower bounds and the aspiration welfare upper bounds.*

Proof

First, we prove that if an allocation $\{x_t\}$ satisfies the core lower bounds $\{x_t\} = \{x_t^*\}$. The core lower bounds imply for every $j \in \mathbb{N}^+$:

$$\sum_{t=0}^{j} \gamma^t (f_t(x_t) - m_t + \gamma m_{t+1}) \geq v(Pj).$$

Since $\sum_{t=0}^{j} \gamma^t (f_t(x_t) - m_t + \gamma m_{t+1})$, the above inequality for $j \longrightarrow \infty$ leads to

$$\sum_{t=0}^{\infty} \gamma^t f_t(x_t) + \lim_{j \longrightarrow \infty} \gamma^{j+1} m_{j+1} \geq v(\mathbb{N}^+). \tag{11}$$

Since $\gamma < 1$ then $\lim_{j \longrightarrow \infty} \gamma^{j+1} = 0$ and, since the feasibility constraint (4) bounds upward m_{j+1}, $\lim_{j \longrightarrow \infty} \gamma^{j+1} m_{j+1} = 0$. Therefore (11) implies

$$\sum_{t=0}^{\infty} \gamma^t f_t(x_t) \geq v(\mathbb{N}^+),$$

which combined with the definition of $v(\mathbb{N}^+)$ implies $\{x_t\} = \{x_t^*\}$.

Second, it is easy to see that if a welfare distribution $\{m_t\}$ satisfies both the core lower bounds and the aspiration upper bounds, then $\{m_t\} = \{m_t^*\}$. This is due to the fact that for coalitions starting from 0 up to any generation t, we have $v(Pt) = w(P^0 t)$. Given m_0, since $v(0) = w(0)$, we must have $m_1 = m_1^*$. Let $m_t = m_t^*$ for all $t \leq j+1$. The core constraints and the aspiration upper bounds force $\sum_{t=0}^{j} \gamma^t (f_t(x_t^*) - m_t + \gamma m_{t+1}) = v(Pj)$, hence $\gamma^j (f_j(x_j^*) - m_j + \gamma m_{j+1}) = v(Pj) - \sum_{t=0}^{j-1} \gamma^t (f_t(x_t^*) - m_t + \gamma m_{t+1})$. Thus by $m_t = m_t^*$ for all $t \leq j+1$, then $\sum_{t=0}^{j-1} \gamma^t (f_t(x_t^*) - m_t + \gamma m_{t+1}) = \sum_{t=0}^{j-1} \gamma^t (f_t(x_t^*) - m_t^* + \gamma m_{t+1}^*) = v(P^0 j)$, we therefore obtain $\gamma^j (f_j(x_j^*) - m_j + \gamma m_{j+1}) = v(Pj) - v(P^0 j)$ the desired conclusion.

Next we show that $\{x_t^*, m_t^*\}$ satisfies the core lower bounds, that is $\sum_{t \in S} \gamma^t u_t^* \geq v(S)$ for any coalition S where $u_t^* \equiv f_t(x_t^*) - m_t^* + \gamma m_{t+1}^*$.

Before we proceed, we note the following: for all t, we have $v(P^0 t) + \gamma^t b_t(\hat{x}_t) \geq v(Pt)$. Thus for all generation t,

$$\gamma^t b_t(\hat{x}_t) \geq v(Pt) - v(P^0 t). \tag{12}$$

Suppose first that S is a consecutive coalition. Since $PS = P^0 S \cup S$, by superadditivity of v, $v(PS) \geq v(P^0 S) + v(S)$ and $\sum_{t \in S} \gamma^t u_t^* = v(PS) - v(P^0 S)$, which imply $\sum_{t \in S} \gamma^t u_t^* \geq v(S)$.

Second, consider any coalition S. Take the last generation in S who obtains some resource $l(S) = \max_t\{t \in S : x_t^S > 0\}$. If $l(S)$ does not exists then $v(S) = 0 \leq \sum_{t\in S} \gamma^t u_t^*$. Let $\bar{S} = Pl(S)\backslash P^0 \min S$ be the coalition of all generations from $minS$ to $l(S)$. Since $\bar{S}$ is connected, $\sum_{t\in\bar{S}}^t \gamma u_t^* = v(P\bar{S}) - v(P^0\bar{S}) \geq v(\bar{S})$. Adding $\sum_{t\in\bar{S}\backslash S} \gamma^t u_t^*$ both sides in the last inequality yields:

$$\sum_{t\in S}^{t} \gamma^t u_t^* \geq v(\bar{S}) - \sum_{t\in\bar{S}\backslash S}^{t} \gamma^t u_t^*. \tag{13}$$

Since generations in-between connected coalitions in S up to $l(S)$ extracts their satiated level, the allocation $(x^S_{S\cap Pl(S)}, \hat{x}_{S\backslash\bar{S}})$ can be implemented in $\bar{S}$ which implies,

$$v(\bar{S}) \geq v(S \cap Pl(S)) + \sum_{t\in\bar{S}\backslash S} f_t(\hat{x}_t). \tag{14}$$

Since there is no more resource to be shared in S after $l(S)$, $f_t(x_t^S) = f_t(0) = 0$ for any $t \in S\backslash Pl(S)$ and, therefore, which implies $v(S) = v(S \cap Pl(S))$. Combine (13) and (14) to obtain

$$\sum_{t\in S} \gamma^t u_t^* \geq v(S) + \sum_{t\in\bar{S}\backslash S} \gamma^t \left(f_t(\hat{x}_t) - u_t^*\right)$$

From (12) we know that $\gamma^t f_t(\hat{x}_t) \geq \gamma^t u_t^*$ for all t. Hence, $\sum_{t\in S} \gamma^t u_t^* \geq v(S)$ which shows that $\{m_t^*\}$ satisfies the core lower bounds.

Lastly, we show that $\{x_t^*, m_t^*\}$ satisfies the aspiration welfare upper bounds. The proof uses the following lemma which is proved in Appendix.

Lemma 1 If $S \subseteq T \subseteq N$ and $T < i$, then $w(S \cup i) - w(S) \geq w(T \cup i) - w(T)$.

Then for any coalition S we obtain

$$\sum_{i\in S} \gamma^t u_i^* = \sum_{i\in S} (w(Pi) - w(P^0 i)) \leq \sum_{i\in S} (w(Pi \cap S) - w(P^0 i \cap S)) = w(S),$$

where the inequality follows from Lemma 1 and the last equality follows from the fact that all terms cancel out except $w(P \max S \cap S) = w(S)$ and $-w(P^0 \min S \cap S) = w(\emptyset) = 0$. $\square$

6 Description of the fair allocation

Let us now describe the unique allocation, denoted $\{x_t^*, m_t^*\}$, that satisfies the core lower bounds and the aspiration welfare upper bounds. To proceed, we need some additional assumptions on the time dependency of the production function. We will notably focus on the time-invariant case such that $f_t(x) = f_{t+1}(x)$, that can be interpreted as the case with no technical progress. We will then give some intuitions how the fair allocation is modified when specific technical progresses are introduced.

Proposition 1 states that the fair path of extraction $\{x_t^*\}$ is the efficient one. It therefore can be studied independently of the fair path of transfers $\{m_t^*\}$. In the specific case $f_t(x) = f_{t+1}(x)$, which implies that $\hat{x}_t = \hat{x}_{t+1}$, $\{x_t^*\}$ is the solution of the following problem:

$$\max_{\{x_t\}} \sum_{t=0}^{\infty} \gamma^t f(x_t),$$

$$s.t. \left| \begin{array}{l} k_{t+1} = \rho(k_t - x_t), \\ x_t \geq 0, k_t \geq 0, \\ k_0 > 0 \text{ given.} \end{array} \right. \tag{15}$$

The following Proposition characterizes the solution of problem (15).

Proposition 2 *If $f_t(x) = f(x)$ for every t, the fair path of extraction $\{x_t^*\}$ and the stock of resource are:*

i) monotonically increasing if $\gamma\rho > 1$ with an asymptotical constant extraction path $x_\infty^* = \hat{x}$ and $k_\infty = \frac{\rho}{\rho-1}\hat{x}$,

ii) monotonically decreasing with a stock asymptotically exhausted if $\gamma\rho < 1$,

iii) constant for every t if $\gamma\rho = 1$ with a constant extraction path $x_t^* = \frac{\rho-1}{\rho}k_0$ for every t.

Proof

To begin, remark that an x_t is optimal if and only if it belongs to $[0, \hat{x}]$. Suppose by contradiction that $\tilde{x}_t$ is optimal and is such that $\tilde{x}_t > \hat{x}$. Then, there exists $\varepsilon > 0$ such that $f((1-\varepsilon)\tilde{x}_t) > f(\tilde{x}_t)$ and $\rho(k_t - (1-\varepsilon)x_t) > \rho(k_t - \tilde{x}_t)$. Hence $\tilde{x}_t$ is not optimal.
The first order condition of problem (15) is:

$$f'(x_{t-1}) - \gamma\rho f'(x_t) = 0, \tag{16}$$

for all $t \in \mathbb{N}^{++}$, while the transversality condition is:

$$\lim_{t\to+\infty} \gamma^t f'(x_t)\, k_{t+1} = 0. \tag{17}$$

Hence $\{x_t^*\}$ solves (16), the resource constraint and (17). Since $x_t \geq x_{t-1} \Leftrightarrow f'(x_t) \leq f'(x_{t-1})$, use (16) to conclude that: $x_t^* \geq x_{t-1}^* \Leftrightarrow \gamma\rho \geq 1$. Thus,there are three distinct cases depending on the value of $\gamma\rho$.

Case 1: $\gamma\rho > 1$. The optimal trajectory x_t^* converge to $\hat{x}$. It remains to determine x_0^*. There are three families of candidates that are represented in the following phase diagram (see Figure 1).

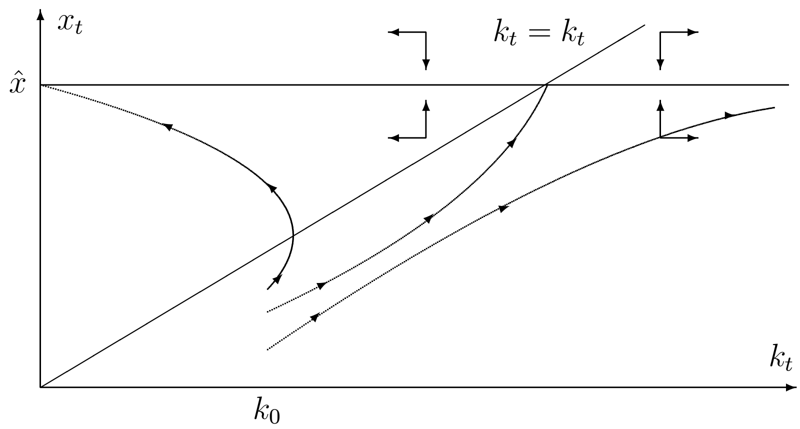


Figure 1

The first family of candidates are such that k_t converge to 0. After a while, this convergence is monotonic. With equation $k_{t+1} = \rho(k_t - x_t)$ this implies that x_t converge to 0, which is a contradiction. These trajectories are not optimal. The second family of candidates are such that k_t converge to $+\infty$. These trajectories do not satisfy the transversality condition. Indeed, at the optimum, one has:

$$\frac{\gamma^t f'(x_t)\, k_{t+1}}{\gamma^{t-1} f'(x_{t-1})\, k_t} = \frac{k_{t+1}}{\rho k_t} = 1 - \frac{x_t}{k_t},$$

where the first equality is due to (16) and the second to the resource constraint. Therefore,

$$\lim_{t \to +\infty} \frac{\gamma^t f'(x_t)\, k_{t+1}}{\gamma^{t-1} f'(x_{t-1})\, k_t} = 1 \text{ and } \lim_{t \to +\infty} \gamma^t f'(x_t)\, k_{t+1} \to +\infty.$$

The third candidate is the saddle-point solution for which k_t converges to $\frac{\rho}{\rho-1}\hat{x}$. This solution satisfies the transversality condition. Along the trajectory the resource stock is monotonically increasing.

Case 2: $\gamma\rho < 1$. Because of condition $\lim_{x\to 0} f'(x) = +\infty$, the optimal trajectory x_t^* converge to 0. To determine x_0^*, one should study two families of candidates that are represented in the following phase diagram (see Figure 2).

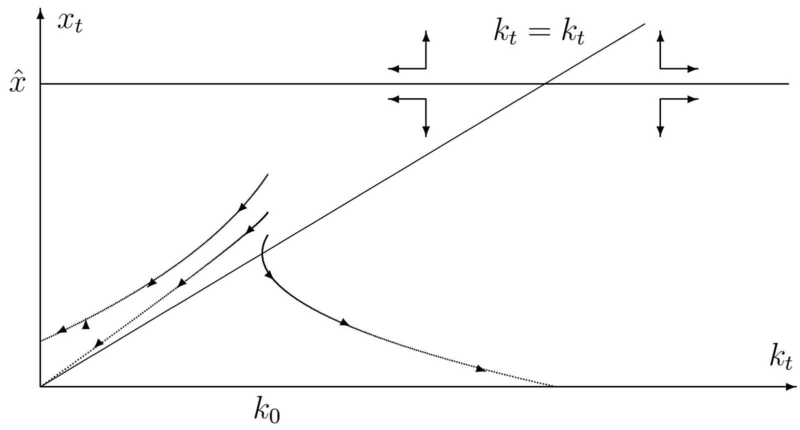


Figure 2

The first family of candidates are such that k_t converge to 0. Among them, only one is such that x_t^* converges to 0, while the others exhibits a sequence of x_t that converges to positive values, and are thus impossible. It remains to check that the good trajectory satisfies the transversality condition. At the optimum, since k_t converge to 0, one has:

$$\frac{\gamma^t f'(x_t) k_{t+1}}{\gamma^{t-1} f'(x_{t-1}) k_t} = \frac{k_{t+1}}{\rho k_t} < \frac{1}{\rho} < 1,$$

from which we deduce that: $\lim_{t \to +\infty} \gamma^t f'(x_t) k_{t+1} = 0$. Along this path, the stock of the resource monotonically decreases and is asymptotically exhausted.

The second family of candidates are such that k_t converge to $+\infty$. As in Case 1, these trajectories do not satisfy the transversality condition.

Case 3: $\gamma\rho = 1$. In this particular case, any constant solution solves (16). Let x^* be the optimal solution. Given the objective: $\max_{x_t} \sum_{t=0}^{\infty} \gamma^t f(x_t)$, the closest x^* to $\hat{x}$, the better. To compute x^* rewrite the resources dynamics such that:

$$k_{t+1} = \rho^{t+1} \left[k_0 - x^* \frac{1 - \gamma^{t+1}}{1 - \gamma} \right],$$

and replace it in (17) to obtain: $\lim_{t \to +\infty} f'(x^*) \rho [k_0 - x^* / (1 - \gamma)] = 0$. The optimal solution is thus: $x^* = (1 - \gamma) k_0$ if $(1 - \gamma) k_0 < \hat{x}$ and $x^* = \hat{x}$ otherwise. The later solution is eliminated by assumption of resource scarcity. In the former, the stock of resource is constant. □

Remark that these results can be immediately extended to specific technical progress. Let us suppose for instance that: $f_t(x_t) = A_t f(x_t) = A_0 \eta^t f(.)$ with $1 \leq \eta < 1/\gamma$. The problem now writes: $\max_{x_t} \sum_{t=0}^{\infty} (\gamma\eta)^t f(x_t)$, subject to the same constraint. The problem is thus the same as (15) except that we now compare $\gamma\eta$ with ρ.

Another way to introduce technical progress would be to suppose that $f_t(x_t) = f(A_t x_t)$ with $A_t = A_0 \eta^t$ and $\eta \geq 1$. The first order condition (16) should then be replaced by: $f'(A_{t-1}x_{t-1}) - \gamma\rho\eta f'(A_t x_t) = 0$. Defining: $\breve{x}_t = A_t x_t$ and $\breve{k}_t = A_t k_t$, the optimal solution can thus be found solving:

$$\begin{cases} \breve{k}_{t+1} - \rho\eta\left(\breve{k}_t - \breve{x}_t\right) = 0 \\ \\ f'(\breve{x}_{t-1}) - \gamma\rho\eta f'(\breve{x}_t) = 0 \end{cases}$$

which is the same as the one previously studied provided that ρ is replaced by $\rho\eta$.

Let us now turn to the characterization of the fair path of transfers $\{m_t^*\}$. From Proposition 1, we have for all $t \in \mathbb{N}^+$:

$$m_{t+1}^* = \frac{\sum_{i=0}^{t} \gamma^i f_i\left(x_i^{P_t}\right) - \sum_{i=0}^{t} \gamma^i f_i\left(x_i^*\right)}{\gamma^{t+1}}, \tag{18}$$

where $x_i^{P_t}$ is the solution of $\max_{x_i} \sum_{i=0}^{t} \gamma^i f_i(x_i)$ subject to the resource and non negativity constraints. As it has been discussed above, $\lim_{t\to+\infty} x_i^{P_t} = x_i^*$. Hence, by the definition of the maximum, $m_{t+1}^* \geq 0$. However, we have seen that fair allocation exist if and only if $m_{t+1}^* \leq f_{t+1}\left(x_{t+1}^*\right)$ for all $t \in \mathbb{N}^+$. We want to stress that this condition is very restrictive and is not satisfied in many cases. Indeed, the fair transfers are likely to increase over time: each generation have to compensate the previous one for not exploiting to the resource in an autarkic way and also for having compensated her previous generation. Hence, as shown in the following Lemma, fair transfers increase, at least for an initial interval of time.

Proposition 3 *For all $t \leq \tilde{t} - 2$, $m_{t+2}^* \geq m_{t+1}^*$.*

Proof

Using (18), $m_{t+2}^* \geq m_{t+1}^*$ if and only if:

$$\sum_{i=0}^{t} \gamma^i \left[f_i\left(x_i^{P_{t+1}}\right) - \gamma f_i\left(x_i^{P_t}\right)\right] + \gamma^{t+1} f_{t+1}\left(x_{t+1}^{P_{t+1}}\right) \geq (1-\gamma) \sum_{i=0}^{t} \gamma^i f_i\left(x_i^*\right) + \gamma^{t+1} f_{t+1}\left(x_{t+1}^*\right)$$

Remember that $\tilde{t}$ relies on the scarcity of the resource and gives the date at which the resource is depleted under an autarkic exploitation. Hence, for all $t \leq \tilde{t} - 2$, the resource is abundant and the optimal exploitation is kept at the generations' satiation point: i.e. $x_i^{P_{t+1}} = \hat{x}_i$. The previous inequality hence rewrites:

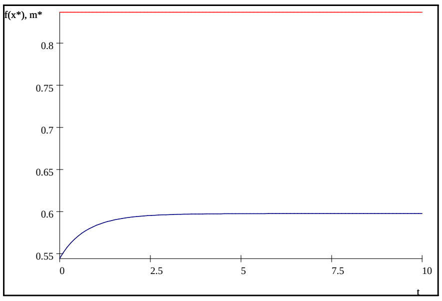
$$(1-\gamma)\sum_{i=0}^{t}\gamma^i f_i\left(\hat{x}_i\right) + \gamma^{t+1} f_{t+1}\left(\hat{x}_{t+1}\right) \geq (1-\gamma)\sum_{i=0}^{t}\gamma^i f_i\left(x_i^*\right) + \gamma^{t+1} f_{t+1}\left(x_{t+1}^*\right),$$

which, given that $\hat{x}_t \geq x_t^*$ for every t from Proposition 2, is obviously satisfied. $\square$

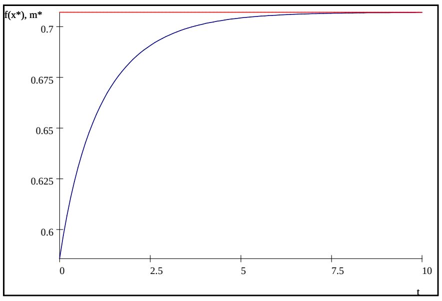
Let us illustrate the existence problem driven by the increase of transfers over time by a simple numerical application. Using Proposition 2, a specific case can be indeed easily derived. Suppose that $\gamma\rho = 1$ and that $f_t\left(x_t\right) = \sqrt{x_t}$ (implying that $\hat{x} \rightarrow +\infty$). Thus, $x_i^{P_t} = \left(1-\gamma\right) k_0 / \left(1-\gamma^{t+1}\right)$ for all i, and:

$$m_{t+1}^* = \frac{\sqrt{\frac{k_0}{(1-\gamma)}}\left(\sqrt{1-\gamma^{t+1}} - \left(1-\gamma^{t+1}\right)\right)}{\gamma^{t+1}},$$

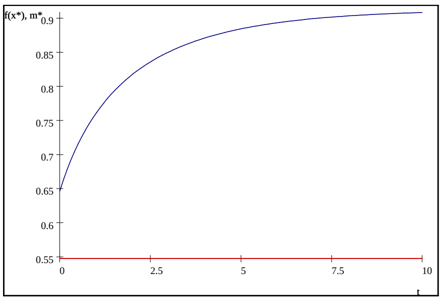
which can be shown to be an increasing function of time. Moreover, since $x_t^* = \left(1-\gamma\right) k_0$ for all t, the feasibility condition $m_{t+1}^* \leq f_{t+1}\left(x_{t+1}^*\right)$ rewrites: $\sqrt{1-\gamma^{t+1}} \leq \left(1-\gamma^{t+2}\right)$, which is always satisfied for low enough γ and never satisfied for large enough γ. For instance, m_{t+1}^* and $f_{t+1}\left(x_{t+1}^*\right)$ are plotted as (continuous) functions of time in the figures below for various values of γ. The increasing dashed curve represents m_{t+1}^* while the solid line is the constant $f_{t+1}\left(x_{t+1}^*\right)$. We see that for $\gamma = 0.3$ and $\gamma = 0.5$, the condition is satisfied, while it is not for $\gamma = 0.7$. To interpret this remember that a larger γ implies (in this very specific case) a lower regeneration rate for the resource.



$\gamma = 1/\rho = 0.3$



$\gamma = 1/\rho = 0.5$



$\gamma = 1/\rho = 0.7$

Figures 3

To conclude this characterization of the fair allocation let us discuss the dynamics of the utilities of each generation u_t^*. The following Proposition give a sufficient condition under which the utilities decrease over time.

Proposition 4 *For all $t \geq 2$, $u_t^* \leq u_1^*$ if $f_t\left(x_t^{P_t}\right) \leq f_1\left(\hat{x}_1\right)$.*

Proof

The Proof of Proposition 1 implies that $u_1^* = f_1\left(\hat{x}_1\right)$. As a consequence, $u_t^* \leq u_1^* \Leftrightarrow \gamma m_{t+1}^* - m_t \leq f_1\left(\hat{x}_1\right) - f_t\left(x_t^*\right)$, which using (18) implies that $u_t^* \leq u_1^*$ if and only if:

$$\sum_{i=0}^{t-1} \gamma^i f_i\left(x_i^{P_t}\right) - \sum_{i=0}^{t-1} \gamma^i f_i\left(x_i^{P_{t-1}}\right) \leq \gamma^t\left[f_1\left(\hat{x}_1\right) - f_t\left(x_t^{P_t}\right)\right].$$

Using the definition of a maximum, observe that the left-hand-side of the inequality is negative, which is sufficient to conclude. $\square$

A direct implication is that a technical progress is a necessary condition for the fair allocation to keep the utilities at least constant. Indeed, if $f_t(x) = f_{t+1}(x)$, then the maximal production level decided by the first generation cannot be overcome.

7 Conclusion

In this paper, we proposed a fair allocation of a scarce resource over an infinite sequence of overlapping generations. When it satisfies two fairness criteria, the core lower bounds and the solidarity upper bounds, the allocation is unique. The exploitation of the resource is efficient and there is no generation left without resource. First generations are compensated for through a transfer scheme that assigns to each generation its marginal contribution to the preceding generation. Such a scheme is likely to induce an increase of transfer over time that may cause the infeasibility of the allocation. Finally, a technical progress is necessary for the utilities of generation not to decrease. A remaining issue is the stability of the fair allocation, which is related to our last result. If the utilities decrease over time while the resource stock increases, future generation have a incentive to deviate. This important question is left for future research.

A Proof of Lemma 1

The proof is adapted from Ambec and Ehlers (2008). Let y_S^S denote the solution of the program defined by $w(S)$ in (9) for any arbitrary coalition $S \subset \mathbb{N}^+$. As a first step in the proof of this lemma, let us show that if $\emptyset \neq S \subset T \subset N$, then $y_S^S \geq y_S^T$. Clearly, it suffices to establish that $y_S^S \geq y_S^{S\cup t}$ whenever $\emptyset \neq S \neq N$ and $t \in N \backslash S$. Write $y_S^S = x_S$ and $y_S^{S\cup t} = y_S$. All agents under consideration in the argument belong to S. By definition of x and y, $\sum_{i\in S} y_i \leq \sum_{i\in S} x_i$. Let $i_1 \leq ... \leq i_L$ be those i such that $x_i \neq y_i$ (if none exists, there is nothing to prove). We claim that $y_{i_1} < x_{i_1}$. Suppose, by contradiction, that the opposite (necessarily strict) inequality is true. Let j be the smallest successor of i_1 such that $y_j < x_j$ (which necessarily exists). Moreover, $y_j < \hat{x}_j$ since $x_j \leq \hat{x}_j$. Define $y_{i_1}^\varepsilon = y_{i_1} - \varepsilon$, $y_j^\varepsilon = y_j + \rho^{j-i_1}\varepsilon$, $y_i^\varepsilon = y_i$ for $i \neq i_1, j$. Since $f_j'(y_j) > f_j'(x_j)$ and $f_{i_1}'(x_{i_1}) > f_{i_1}'(y_{i_1})$, choosing $\varepsilon > 0$ small enough (in particular such that $y_j + \rho^{j-i_1}\epsilon < \hat{x}_j$) ensures that $\sum_{i\in S} \gamma^i f_i(y_i^\varepsilon) > \sum_{i\in S} \gamma^i f_i(y_i)$ while y_S^ε meets the same constraints as y_S, a contradiction. Because $y_{i_1} - x_{i_1} < 0$, it now follows that $y_{i_l} - x_{i_l} < 0$ successively for $l = 2, ..., L$.

Moving to the second step, let $S \subset T \subset N$ and $T < i$. Define $x_i' = y_i^{T\cup i}$ and $x_j' = y_j^{T\cup i} + y_j^S - y_j^T$ for $j \in S$. By our first step, $y_j^{T\cup i} \leq y_j^T \leq y_j^S$ for all $j \in S$. Therefore $0 \leq y_j^{T\cup i} \leq x_j' \leq y_j^S$ for all $j \in S$ and the consumption plan x' for $S \cup i$ satisfies the same constraints as $y_{S\cup i}^{S\cup i}$. Hence, $w(S\cup i) \geq \sum_{j\in S\cup i} \gamma^j f_j(x_j')$ and

$$w(S \cup i) - w(S) \geq \gamma^i f_i(x_i') + \sum_{j\in S} \gamma^j [f_j(x_j') - f_j(y_j^S)]. \tag{19}$$

On the other hand, since $y_j^{T\cup i} \leq y_j^T$ for all $j \in T\backslash S$,

$$w(T \cup i) - w(T) \leq \gamma^i f_i(x_i') + \sum_{j\in S} \gamma^j [f_j(y_j^{T\cup i}) - f_j(y_j^T)]. \tag{20}$$

Since $x_j' - y_j^S = y_j^{T\cup i} - y_j^T$ and $y_j^{T\cup i} \leq x_j'$ for all $j \in S$, it follows from (19), (20), and the concavity of f_j on its increasing part that $w(T\cup i) - w(T) \leq w(S\cup i) - w(S)$. This completes the proof of the lemma. $\square$

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