Supply function equilibria in networks with transport constraints*

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Abstract

Transport constraints limit competition and arbitrageurs' possibilities of exploiting price differences between goods in neighbouring markets, especially when storage capacity is negligible. We analyse this in markets where strategic producers compete with supply functions, as in wholesale electricity markets. For networks with a radial structure, we show that existence of supply function equilibria (SFE) is ensured if demand shocks are sufficiently evenly distributed, and solve for SFE in symmetric radial networks with uniform multi-dimensional nodal demand shocks. An equilibrium offer in such networks is identical to an SFE offer in an isolated node where the symmetric number of firms has been scaled by a market integration factor, the expected number of nodes that are completely integrated with a node in the network. The analysis can be extended to mesh networks (as in electricity systems) although the resulting models do not simplify as in the radial case.

Key words: Oligopoly market, Divisible-good auction, Transmission network, Graph theory, Market integration, Wholesale electricity markets

JEL Classification D43, D44, C72, L91

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1 Introduction

Transport constraints limit trade, which makes production and consumption less efficient. Moreover, transport constraints reduce competition between agents situated in separated markets, which worsens market efficiency even further. Congestion is of particular importance for markets with negligible storage possibilities, such as wholesale electricity markets. Then demand and supply must be instantly balanced and temporary congestion in the network can result in large local price spikes. The same market can at times exhibit very little market power and, at other times, suffer from the exercise of a great deal of market power. Borenstein et al. [12] show that standard concentration measures such as the Herfindahl-Hirschman index (HHI) work poorly to assess the degree of competition in such markets. Thus competition authorities who need to predict market prices under various counterfactuals – what might happen if a merger or acquisition is accepted or transport capacity is expanded, need more detailed analytical tools.

We analyze the network's influence on an oligopoly market where producers sell a homogeneous commodity. We assume that firms compete with supply functions and that each node of the radial network has a local demand shock. The slope of the residual demand curve[1] is important in the calculation of a firm's optimal offer. However, this curve is uncertain due to the demand shocks. This uncertainty is characterized using Anderson and Philpott's [4] and Wilson's [42] market distribution function approach. We show that the optimal output of a producer is proportional to its mark-up and the expected slope of the residual demand curve that it is facing.[2] Thus a more elastic residual demand results in a more competitive offer.

We assume that transports along an arc are costless, as long as it is uncongested. We say that two nodes are completely integrated when they are connected via uncongested arcs. The flow through arcs with a strictly binding transport capacity is fixed on the margin. Thus on the margin, a firm's residual demand is only influenced by nodes that are completely integrated with the firm's node. For symmetric equilibria we find it useful to define a market integration factor, which equals the expected number of nodes that are completely integrated with any node in the network. We use our optimality conditions to solve for symmetric equilibria in two-node and star networks[3] with multi-dimensional uniformly distributed demand shocks. We show that an equilibrium offer in a node of such a network is identical to the equilibrium offer in an isolated node where the number of symmetric firms has been scaled by the market integration factor. Firms can influence the market integration factor with their offer curves, so it is endogenous. Still, in our symmetric equilibria, the market integration factor is found to be con-

[1] The residual demand at a specific price is given by demand at that price less competitors' sales as that price.

[2] Note that the output of the firm influences congestion in the network, which in its turn influences its residual demand curve. Thus with the slope of residual demand we here mean the slope of residual demand conditional on a fixed output.

[3] There are no strategic producers in the center node of the star network. Thus the network is symmetric from the producers' perspective.

stant for a given network with given transmission capacities and total production capacities. The factor does not depend on production costs nor on the number of symmetric firms.

We focus on characterising supply function equilibrium (SFE) in radial networks, but we also show how our optimality conditions can be generalized to consider meshed networks, although the resulting models do not simplify as in the radial case. Moreover, we describe how our conditions can be modified to calculate SFE in networks with discriminatory pricing and Cournot Nash equilibrium in networks with uncertain demand. Normally nodes represent a geographical location, and with transport we normally mean that the commodity is moved from one geographical location to another location. But nodes and transports could be interpreted in a more general sense. For example, a node could represent a point in time or a geographical location at a particular point in time. Moreover, storage at a geographical location can be represented by arcs that allow for transports of the commodity to a later point in time. The transport capacity of such arcs would then correspond to the local storage capacity.

The supply function equilibrium for single node markets was originally developed by Klemperer and Meyer [26]. This equilibrium represents a generalized form of competition in oligopoly markets, in-between the extremes of the Bertrand and Cournot equilibrium. The setting of the SFE is particularly well-suited for markets where producers submit offer curves to a uniform-price auction before demand has been realized, as in wholesale electricity markets [4][9][19][23]. This has also been confirmed qualitatively and quantitatively in several empirical studies of bidding in electricity markets.[4] But the SFE model is of more general interest. Klemperer and Meyer [26] argue that although most markets are not explicitly cleared by uniform-price auctions, firms typically face a uniform market price and they need predetermined decision rules for lower-level managers to deal with changing market conditions. Thus, in general, firms implicitly commit to supply functions. Vives [37] notes that competition in supply functions has also been used to model bidding for government procurement contracts, management consulting and airline pricing reservation systems.

Klemperer and Meyer's model has only one uncertain parameter, a demand shock. In equilibrium there is a one to one mapping between the price and shock. Thus each firm can choose its supply function such that its output is optimal for every realized shock. As noted by Anderson et al [6], this ex-post optimality feature is difficult to translate into a network with multi-dimensional demand shocks. They investigate a two-node transmission network with both independent and correlated demand at the nodes. Anderson et al derive formulae to represent the market distribution function for a producer when its network becomes interconnected to a previously separate grid under the assumption that the interconnection does not change competitors' offers.

[4]Empirical studies of the electricity market in Texas (ERCOT) show that offers of the two to three largest firms in this market match Klemperer and Meyer's first-order condition [24][35]. Further empirical support is provided by Wolak [43] who verifies that electricity producers in Australia choose their offers in order to maximize profits; at least observed data does not reject this hypotehesis.

By requiring that each firm's offer is optimal only in expectation, the recent paper by Wilson [41] takes a different approach, which enables him to extend Klemperer and Meyer's [26] model to consider the network's influence on bidding strategies.[5] This ex-ante optimality requirement is adopted in our paper, and so our work follows the same approach as [41]. Wilson, however, does not provide any second-order conditions in his paper, and so his analysis of SFE is missing a fundamental component.

Previous research has shown that second-order conditions are often violated in a network with strategic producers. The reason is that transport constraints can introduce nonsmoothness in a producer's residual demand curve which becomes discontinuously less price sensitive when net imports to its market are congested. Thus, in a market where imports are nearly congested it will be profitable for a producer to withhold production in order to push the price above the next breakpoint in its residual demand curve. This type of deviation will often rule out pure-strategy Nash equilibria.[6] Borenstein et al. [13] for example rule out Cournot NE when the transport capacity between two symmetric markets is sufficiently small and demand is certain. Downward et al. [15] analyse existence of Cournot equilibria in general networks with transport constraints. Neuhoff et al's [30] analysis of the northwestern European electricity market illustrates that non-existence of Cournot NE is a problem also in practice. We verify that monotonic solutions to our first-order conditions are Supply Function Equilibria (SFE) when the shock density is sufficiently evenly distributed, i.e. close to a uniform multi-dimensional distribution. In this case the demand shocks will smooth the residual demand curve, so that its breakpoints disappear in expectation. But existence of SFE cannot be taken for granted. Perfectly correlated shocks or steep slopes and discontinuities in the shock density will not smooth the residual demand curve sufficiently well, and then profitable deviations from the first-order solution will exist.[7]

Our paper also differs from Wilson [41] in the source of randomness. In his model, local demand is certain in all markets but one, and transmission capacities are uncertain. This simplifies the problem in an elegant way, especially for meshed networks. Nevertheless, even if our calculations are less straightforward, we find it important to also analyze the multi-market case with local net-demand shocks/variations and known transmission capacities. We believe that our model is of particular relevance for markets with long-lived bids, such as PJM[8], where

[5] Lin and Baldick [28] and Lin et al [29] also calculate first-order conditions for transmission networks with supply function offers, but their model is limited to cases with certain demand.

[6] But Escobar and Jofre [16] show that there is normally a mixed-strategy NE in those cases. Adler et al. [1] and Hu and Ralph [25] show that existence of pure-strategy Cournot NE depends on the assumptions made about the rationality of the players. Hobbs et al [20] bypasses the existence issue by using conjectural variations instead of a Nash equilibrium. Willems [40] analyse how a network operator's rule to allocate transmission capacity influences the Cournot NE. Wei and Smeers [39] calculate Cournot NE in transmission networks with regulated transmission prices.

[7] Note that a discontinuity in a node's shock density is fine as long as it occurs when transport capacities in all arcs to the node are binding.

[8] PJM is the largest existing deregulated wholesale electricity market. Originally PJM coor-

producers' offers are fixed during the whole day to meet a wide range of local demand outcomes. Also large local net-demand shocks can occur on short notice, especially in electricity networks with significant amounts of wind power, so our model is also relevant for markets with short lived bids.

2 The model

We shall consider markets for a single commodity that is traded over a network consisting of M nodes that are connected by N directed transport arcs. We assume that each pair of nodes are connected by at most one arc. The network is *connected*, so that there is at least one chain of arcs between every two nodes in the network. Thus we have that $N \geq M-1$. As is standard in graph theory, the topology of the network can be described by a *node-arc incidence* matrix $\mathbf{A}$ [11].[9] This matrix $\mathbf{A}$ has a row for every node and a column for every arc, and $ik-$th element a_{ik} defined as follows[10]:

$$a_{ik} = \begin{cases} -1, & \text{if arc } k \text{ is oriented away from node } i, \\ 1, & \text{if arc } k \text{ is oriented towards node } i, \\ 0, & \text{otherwise.} \end{cases}$$

Every arc starts in one node and ends in another node, so by definition we have that the rows of $\mathbf{A}$ add up to a row vector with zeros. Thus the rows are linearly dependent. It can be shown that the incidence matrix $\mathbf{A}$ of a connected network has rank $M-1$ [11].

The transported quantity in arc k is represented by the variable t_k which can be positive or negative, the latter indicating a flow in the opposite direction from the orientation of the arc. Thus the ith row of $\mathbf{At}$ represents the flow of the commodity into node i from the rest of the network. Transportation is assumed to be lossless and costless, but each arc k has a capacity K_k, so the vector $\mathbf{t}$ of arc flows satisfies

$$-\mathbf{K} \leq \mathbf{t} \leq \mathbf{K}. \tag{1}$$

At each node i there are n_i suppliers who play a simultaneous move, one shot game. Each supplier offers a nondecreasing differentiable supply function

$$Q_{ig}\left(p\right), \, g = 1, 2, \ldots, n_i,$$

that defines how much each firm is prepared to supply at price p. There can be many firms in each node, but for simplicity we assume that each firm is only active

dinated the movement of wholesale electricity in Pennsylvania, New Jersey and Maryland. Now PJM has expanded to also cover all or parts of Delaware, Illinois, Indiana, Kentucky, Michigan, North Carolina, Ohio, Tennessee, Virginia, West Virginia and the District of Columbia.

[9] This is different to Wilson [41] who describes the network with *power transfer distribution factors* (PTDFs).

[10] Many authors adopt a different convention in which $a_{ik} = 1$ if arc k is oriented away from node i.

in one node. Non-strategic net-demand at each node i is $D_i(p)+\varepsilon_i$,[11] where $D_i(p)$ is a nonincreasing function of p and ε_i is a random local shock having a known probability distribution with joint density $f(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_M)$. The demand shocks are realized after firms have committed to their offers. We denote the total deterministic net-supply in each node by $S_i(p_i) = \sum_{g=1}^{n_i} Q_{ig}(p_i) - D_i(p_i)$ and the vector with such components by $\mathbf{s}(\mathbf{p})$. We also introduce $S_{i,-g}(p_i) = \sum_{h=1,h\neq g}^{n_i} Q_{ih}(p_i) - D_i(p_i)$, which excludes the supply of firm g from the deterministic net-supply in node i.

We assume that there are many small price-taking traders active in the network. After the demand shocks have been realized, they buy in some nodes, transport the commodity through the network without violating its physical constraints, and then sell it in other nodes. The market is cleared when all profitable feasible trades have been exhausted. Equivalently it can be assumed that the network is cleared by a price-taking operator, i.e. it chooses demand and output in each node in order to maximize the stated[12] social welfare of market participants without violating the network's technical constraints. Similar to a uniform-price auction, we assume that accepted offers are paid the local clearing price at their node. In electric power networks this is called nodal pricing [14][21][34]. Hence, for each realization $\boldsymbol{\varepsilon}$ the market will be cleared by a set of prices that defines how much each supplier produces and what is transported through the network. The clearing prices ensure that net-demand equals net-imports in every node, i.e.

$$\mathbf{At} + \mathbf{s}(\mathbf{p}) = \boldsymbol{\varepsilon}. \tag{2}$$

We assume that each producer is risk-neutral and chooses its supply curve in order to maximize its expected profit. Ex-post, after demand shocks have been realized and prices and firm's output have been determined, the pay-off of firm g in node i is given by:

$$\Pi_{ig}(p, q) = pq - C_{ig}(q), \tag{3}$$

where p is the local price in node i, q is the output of the firm and $C_{ig}(q)$ is the firm's differentiable, convex and increasing production cost up to its capacity constraint $\overline{q}_{ig}$.

We let $\overline{p}$ be the highest realized price in the market. It is the monopoly price of a firm with output $\overline{q}_{ig}$. For networks with inelastic demand we let $\overline{p} > C'_{ig}(\overline{q}_{ig})$ be a reservation price that puts a cap on the monopoly price.

The residual demand curve of a firm is the market demand that is not met by other firms in the industry at a given price. The slope of this curve is important in the calculation of a firm's optimal offer. The demand shocks are additive, so they will not change the slope of a firm's residual demand, as long as the same set of arcs are congested in the cleared market. Thus similar to Wilson [41] we find it useful to group shock outcomes for which the same set of arcs are congested in the cleared market. If two market outcomes for different $\boldsymbol{\varepsilon}$ realizations have exactly the same arcs with $t_k = -K_k$ and the same arcs with $t_k = K_k$ then we say that they are in the same congestion state ω. For each congestion state, we

[11] Note that this is net-demand, so it is not necessarily non-negative. For example, fluctuating wind-power from small non-strategic firms can result in negative net-demand shocks.

[12] The operator acts as if submitted offers reflect true marginal costs.

denote by $L(\omega)$, $B(\omega)$, and $U(\omega)$ the sets of arcs where flows are at their lower bound (i.e. congested in the negative direction), between their bounds or at their upper bound, respectively.

2.1 Optimality conditions

In monopoly and Cournot markets without transport constraints the first-order condition is:

$$C'(q) = p - \frac{q}{-D'(p)},$$

i.e. the output is chosen such that the marginal cost equals the marginal revenue. Alternatively, the same condition can be written as:

$$q = (p - C'(q))(-D'(p)),$$

i.e. the optimal output is proportional to the firm's mark-up and (the absolute value of) the slope of the demand. The condition is similar in a market with supply function competition and no transport constraints. But in this case the slope of firm i's residual demand is also influenced by the slope of its competitors' supply, $Q'_{-1}(p)$, i.e. [26]

$$q_i = (p - C'_i(q_i))\left(Q'_{-1}(p) - D'(p)\right).$$

Another difference with supply functions is that firm i can choose its offer so that the output becomes optimal for every price. Thus even if there is an additive shock in the demand, firm i can still choose its supply function such that the resulting output is optimal for every shock realization. This property of the supply function equilibrium is referred to as *ex-post optimality*.

Solving for the equilibrium in our model is more complicated than for the standard SFE model, because there are many different vectors of shock realizations that can result in the same local price in a firm's node. Moreover, different transmission lines may be congested for the same price in a firm's node. Thus the slope of a firm's residual demand is typically not pinned down by its nodal price, so supply function equilibria in our setting are not ex-post optimal. Instead we will in the next section show that firm i's *ex-ante* optimal output is given by its mark-up times the expected slope of its residual demand conditional on the firms output. We need to condition the slope of the residual demand on firm i's output as congestion in the network depends on this output.

As in Anderson and Philpott [4], we use the market distribution function $\psi_{ig}(p,q)$ to characterize the uncertainty in the residual demand curve of firm g in node i. For given offers of the competitors this function returns the probability that an offer (p,q) from the firm is rejected.[13] The expected pay-off is given by the line-integral [4]:

[13] Note that the market distribution function is analogous to Wilson's [42] probability distribution of the market price, which returns acceptance probabilities for offers. The main contribution of Anderson and Philpott's analysis is that it provides a global second-order condition for optimality.

$$\int_{Q_{ig}(p)} \Pi_{ig}(p,q) d\psi_{ig}(p,q). \tag{4}$$

Thus, for any offer curve $Q_{ig}(p)$, the market distribution function contains all information of the residual demand that a firm needs to calculate its expected profit. It does not matter whether the rejection probability is driven by properties of the demand, competitors' offers or the network. As long as the firm's accepted offers are paid a (local) uniform-price, we can still apply Anderson and Philpott's optimality condition. We define

$$Z(p,q) = \frac{\partial \Pi_{ig}}{\partial q}\frac{\partial \psi_{ig}}{\partial p} - \frac{\partial \psi_{ig}}{\partial q}\frac{\partial \Pi_{ig}}{\partial p} = \left(p - C'_{ig}(q)\right)\frac{\partial \psi_{ig}}{\partial p} - q\frac{\partial \psi_{ig}}{\partial q}. \tag{5}$$

It can be shown that an offer curve $Q_{ig}(p)$ is globally optimal if it satisfies [4]:

$$\begin{cases} Z(p,q) \geq 0 & \text{if } q < Q_{ig}(p) \\ Z(p,q) = 0 & \text{if } q = Q_{ig}(p) \\ Z(p,q) \leq 0 & \text{if } q > Q_{ig}(p). \end{cases} \tag{6}$$

Intuitively we can explain the first-order condition $Z(p,q) = 0$ as follows. The same market distribution function (rejection probability) can be generated by different randomizations of the residual demand curve. In particular the same market distribution function can be generated by randomizing over crossing or non-crossing residual demand curves. Still, it follows from (4) that as long as the market distribution function is the same, expected profits and the optimal offer for firm g do not change. Thus, even if our randomization is different, we can simplify the derivation of first- and second-order conditions by considering a simpler equivalent case, where $\psi_{ig}(p,q)$ has been generated by a randomization over non-crossing residual demand curves forming the contours of $\psi_{ig}(p,q)$. In this case, it is obvious that the expected profit of firm g in node i is globally maximized if the payoff is globally optimized for each outcome of its residual demand curve, i.e. the offer is ex-post optimal. Thus it follows from the equivalence argument that the expected profit of firm g in node i is globally maximized if the payoff is globally optimized for each contour of $\psi_{ig}(p,q)$ irrespective of how the market distribution function was generated. A necessary condition for this is that the supply curve $Q_{ig}(p)$ crosses each contour of $\psi_{ig}(p,q)$ at a point where the latter is tangent to the firm's isoprofit line. This is illustrated in Figure 1.

Hence, the following conditions must be satisfied at every point along the optimal supply curve.

$$\left.\frac{dq}{dp}\right|_{\psi_{ig}(p,q)=const} = \left.\frac{dq}{dp}\right|_{\Pi_{ig}(p,q)=const}. \tag{7}$$

From (3) we have

$$\left.\frac{dq}{dp}\right|_{\Pi_{ig}(p,q)=const} = -\left.\frac{\frac{\partial \Pi_{ig}}{\partial p}}{\frac{\partial \Pi_{ig}}{\partial q}}\right|_{\Pi_{ig}(p,q)=const} = -\frac{q}{p - C'_{ig}(q)}. \tag{8}$$

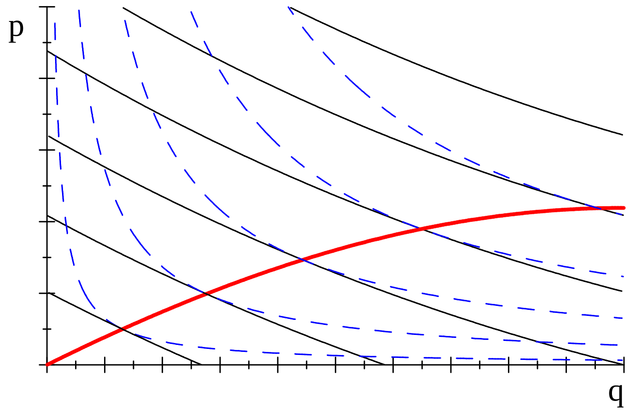

Figure 1: Contours of $\psi_{ig}(q,p)$ (thin) and isoprofit lines for $\pi_{ig}(q,p)$ (dashed). The optimal curve $q = Q_{ig}(p)$ (solid) passes through the points where these curves have the same slope.

Similarly,

$$\left.\frac{dq}{dp}\right|_{\psi_{ig}(p,q)=const} = -\left.\frac{\frac{\partial\psi_{ig}}{\partial p}}{\frac{\partial\psi_{ig}}{\partial q}}\right|_{\psi_{ig}(p,q)=const}. \tag{9}$$

Now (7), (8) and (9) together imply that

$$\frac{\frac{\partial\psi_{ig}}{\partial p}}{\frac{\partial\psi_{ig}}{\partial q}} = \frac{\frac{\partial\Pi_{ig}}{\partial p}}{\frac{\partial\Pi_{ig}}{\partial q}} = \frac{q}{p-C'_{ig}(q)}, \tag{10}$$

which is identical to the first-order condition given by (5) and $Z(p,q) = 0$. The global second-order condition in (6) ensures that profits increase to the right of each contour of $\psi_{ig}(p,q)$.

In order to apply the first- and second-order conditions we need to calculate the derivatives $\frac{\partial\psi_{ig}}{\partial p}$ and $\frac{\partial\psi_{ig}}{\partial q}$, which depend on competitors' offers, the joint density $f(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_M)$ and the properties of the network. In the next section we calculate these derivatives for radial networks.

3 Radial networks

We begin our analysis by focusing on radial networks (i.e. trees with M nodes and $N = M - 1$ arcs forming an acyclic connected graph). The generalization to meshed networks with $N > M - 1$ is presented in Section 4. In radial networks there is a unique transport route between any two nodes in the network. Thus network flows are straightforwardly determined by net-supply in the nodes, which simplifies the clearing process of the market.

We start this section by exploring some special properties of the node-arc incidence matrix $\mathbf{A}$ for a radial network. Let $\mathbf{A}_i$ be row i of matrix $\mathbf{A}$, and let

$\mathbf{A}_{-i}$ be matrix $\mathbf{A}$ with row i eliminated. For connected radial networks, it can be shown that $\mathbf{A}_{-i}$ is non-singular with determinant +1 or -1 [10]. We also have the following technical results.

Lemma 1 *Suppose* $\mathbf{A}$ *is the node-arc incidence matrix for a radial network with* M *nodes. If* $j < M$ *then* $\det \mathbf{A}_{-j} = -\det \mathbf{A}_{-(j+1)}$

Proof. Introduce a new matrix $\mathbf{Z}$ which is identical to $\mathbf{A}_{-j}$, except that row j of $\mathbf{A}_{-j}$, which is equal to $\mathbf{A}_{j+1}$, has been replaced by the sum of all rows in $\mathbf{A}_{-j}$. Such manipulations are allowed without changing the determinant [36], so $\det(\mathbf{Z}) = \det \mathbf{A}_{-j}$. Node-arc incidence matrices are such that the sum of all rows in $\mathbf{A}_{-j}$ is equal to $-\mathbf{A}_j$ (row j of $\mathbf{A}$). Thus $\mathbf{Z}$ is identical to $\mathbf{A}_{-(j+1)}$ except that elements have opposite signs in row j. In the calculation of the determinants we can expand them along row j of both $\mathbf{Z}$ and $\mathbf{A}_{-(j+1)}$, which gives the stated result [36]. ■

By applying Lemma 1 $|k-j|$ times we get the following result.

Corollary 2 *If* $\mathbf{A}$ *is the node-arc incidence matrix for a radial network then* $(-1)^j \det \mathbf{A}_{-j} = (-1)^k \det \mathbf{A}_{-k}$

Next we analyse the market clearing conditions for radial networks. We define $\boldsymbol{\rho}$ to be the vector of non-negative shadow prices (one for each arc) for flows in the positive direction. Similarly, we define $\boldsymbol{\sigma}$ to be the vector of non-negative shadow prices (one for each arc) for flows in the negative direction. Hence, the market clearing conditions for a radial network are[14]

$$
\begin{aligned}
&\mathbf{A}^\top \mathbf{p} = \boldsymbol{\rho} - \boldsymbol{\sigma} \\
&0 \leq \boldsymbol{\rho} \perp \mathbf{K} - \mathbf{t} \geq \mathbf{0} \\
&0 \leq \boldsymbol{\sigma} \perp \mathbf{K} + \mathbf{t} \geq \mathbf{0} \\
&\mathbf{A}\mathbf{t} + \mathbf{s}(\mathbf{p}) = \boldsymbol{\varepsilon}.
\end{aligned}
\tag{11}
$$

The first condition states that the shadow price for the arc gives the difference in nodal prices between the endpoints. The second and third set of conditions are called complementary slackness. They imply that the shadow price of an arc can only be strictly positive if the arc is congested in the direction associated with the shadow price. Hence, if two nodes are connected by a congested arc then the price at the importing end will be at least as large as the price in the exporting end, which ensures that there are no feasible profitable arbitrage trades in the radial network. Another implication of the complementary slackness conditions is that nodes connected by uncongested arcs will form a zone with the same market price. We say that such nodes are completely integrated. The fourth condition ensures that net-demand equals net-imports in every node.

Recall that for a given ω, $L(\omega)$, $B(\omega)$, and $U(\omega)$ are the sets of arcs where flows are at their lower bound (i.e. congested in the negative direction), between

[14] We derive these Karush-Kuhn-Tucker conditions formally for more general cases in Section 4.

their bounds or at their upper bound, respectively. Thus we realize that the complementary slackness conditions can be equivalently written as follows:

$$\begin{array}{llll} t_k = K_k, & \sigma_k = 0, & \rho_k \geq 0, & k \in U(\omega), \\ t_k \in (-K_k, K_k) & \sigma_k = 0, & \rho_k = 0, & k \in B(\omega), \\ t_k = -K_k, & \sigma_k \geq 0, & \rho_k = 0, & k \in L(\omega). \end{array}$$

Observe that given a congestion state ω and arc k, there is at most one variable t_k, ρ_k or σ_k that is not at a bound.

As in Wilson [41], we choose an arbitrary node i to be a "trading hub" with nodal price p. In the following we will express the other nodal prices $\mathbf{p}_{-i}$ in terms of p and the shadow prices. Let $\mathbf{1}_{M-1}$ be a column vector of $M-1$ ones and $\mathbf{0}_{M-1}$ be a column vector of $M-1$ zeros. We know that the columns of $\mathbf{A}^T$ sum to a column vector of zeros. Hence,

$$\begin{aligned} \left(\mathbf{A}_{-i}\right)^T \mathbf{1}_{M-1} + \mathbf{A}_i^T &= \mathbf{0}_{M-1} \\ \left(\left(\mathbf{A}_{-i}\right)^T\right)^{-1} \mathbf{A}_i^T &= -\mathbf{1}_{M-1}. \end{aligned}$$

Using this result, we can write the market clearing condition $\mathbf{A}^\top \mathbf{p} = \boldsymbol{\rho} - \boldsymbol{\sigma}$ as follows:

$$\begin{aligned} \left(\mathbf{A}_{-i}\right)^T \mathbf{p}_{-i} + p\mathbf{A}_i^T &= \boldsymbol{\rho} - \boldsymbol{\sigma} \\ \mathbf{p}_{-i} &= \left(\left(\mathbf{A}_{-i}\right)^T\right)^{-1} \left(\boldsymbol{\rho} - \boldsymbol{\sigma} - p\mathbf{A}_i^T\right) \\ \mathbf{p}_{-i} &= p\mathbf{1}_{M-1} + \left(\left(\mathbf{A}_{-i}\right)^T\right)^{-1} \left(\boldsymbol{\rho} - \boldsymbol{\sigma}\right). \end{aligned} \tag{12}$$

To simplify this further we introduce

$$\mathbf{E} = \left(\left(\mathbf{A}_{-i}\right)^T\right)^{-1}$$

We partition $\mathbf{t}$, $\mathbf{A}$, $\mathbf{E}$ and the shadow prices $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ into ($\mathbf{t}_L$, $\mathbf{t}_B$, $\mathbf{t}_U$), ($\mathbf{A}_L$, $\mathbf{A}_B$, $\mathbf{A}_U$), ($\mathbf{E}_L$, $\mathbf{E}_B$, $\mathbf{E}_U$), ($\boldsymbol{\sigma}_L$, $\mathbf{0}_B$, $\mathbf{0}_U$) and ($\mathbf{0}_L$, $\mathbf{0}_B$, $\boldsymbol{\rho}_U$) corresponding to flows at their upper bounds, strictly between their bounds, and at their lower bounds. Now (12) can be written as follows:

$$\mathbf{p}_{-i} = p\mathbf{1}_{M-1} + \mathbf{E}_{U(\omega)}\boldsymbol{\rho}_{U(\omega)} - \mathbf{E}_{L(\omega)}\boldsymbol{\sigma}_{L(\omega)}. \tag{13}$$

For any index set Υ of columns of $\mathbf{A}$ (or equivalently any set Υ of arcs) we will find it useful to define the volume that feasible flows and shadow prices associated with arcs in Υ can span. Thus we define the sets

$$\begin{aligned} \mathcal{T}(\Upsilon) &= \{\mathbf{t}_\Upsilon \mid -\mathbf{K}_\Upsilon \leq \mathbf{t}_\Upsilon \leq \mathbf{K}_\Upsilon\}, \\ \mathcal{U}(\Upsilon) &= \{\boldsymbol{\rho}_\Upsilon \mid \mathbf{0} \leq \boldsymbol{\rho}_\Upsilon\}, \\ \mathcal{L}(\Upsilon) &= \{\boldsymbol{\sigma}_\Upsilon \mid \mathbf{0} \leq \boldsymbol{\sigma}_\Upsilon\}, \end{aligned} \tag{14}$$

where $\mathbf{t}_\Upsilon$, $\boldsymbol{\rho}_\Upsilon$, $\boldsymbol{\sigma}_\Upsilon$, and $\mathbf{K}_\Upsilon$ are the vectors of components of $\mathbf{t}$, $\boldsymbol{\rho}$, $\boldsymbol{\sigma}$ and $\mathbf{K}$ corresponding to Υ. In particular we are interested in $\mathcal{S}(\omega) \subseteq \mathbb{R}^{M-1}$, which we define by

$$\mathcal{S}(\omega) = \mathcal{L}(L(\omega)) \times \mathcal{U}(U(\omega)) \times \mathcal{T}(B(\omega)). \tag{15}$$

As defined in (14), $\mathcal{L}(L(\omega))$ and $\mathcal{U}(U(\omega))$ are the volumes in $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ space spanned by the shadow prices of congested transmission lines for a congestion state ω. $\mathcal{T}(B(\omega))$ is the volume in $\mathbf{t}$ space that is spanned by flows in uncongested lines in the state. Hence, $\mathcal{S}(\omega)$ is the total volume in $\mathbf{t}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ space that is spanned for a congestion state ω.

3.1 Optimality conditions for radial networks

In order to apply the optimality conditions in (5) and (6), we need to derive the market distribution function $\psi_{ig}(q,p)$. It is the probability that an offer q at price p from firm g in node i is rejected. Thus the calculation of this function involves determining a market outcome for every realization of the vector $\boldsymbol{\varepsilon}$, and then integrating the density function f over the volume in ε-space that corresponds to firm g's offer not being fully accepted. In the general case this volume is complicated and it is even more complicated to differentiate $\psi_{ig}(q,p)$ (we need such derivatives in our optimality conditions) if one follows this direct approach. Like Wilson [41], we avoid this by transforming the problem into one where we instead integrate over the flows and shadow prices that arise in each congestion state. In the following we take supply functions $Q_{jh}(p)$ of the competitors as given and we want to calculate the best response of firm i in node g.

When calculating $\frac{\partial \psi_{i,g}(p,q)}{\partial p}$ we keep the output of firm g fixed while the price p at node i is free to change. Thus we set

$$Q_{jh}(p) = \begin{cases} q, & j=i,\, h=g \\ Q_{jh}(p), & \text{otherwise}, \end{cases} \tag{16}$$

i.e. firm g in node i submits a Cournot offer. For any price p in node i, and shadow prices $\boldsymbol{\rho}_{U(\omega)}$ and $\boldsymbol{\sigma}_{L(\omega)}$ we denote by $\mathbf{p}(p,\boldsymbol{\rho},\boldsymbol{\sigma})$ the vector of nodal prices defined by (13), and by $\mathbf{s}(\mathbf{p}(p,\boldsymbol{\rho},\boldsymbol{\sigma}))$ the corresponding vector of net-supply at the nodes including $Q_{jh}(p)$ defined by (16), where we choose to suppress the dependence on ω for notational convenience. We want to transform the volume in $\boldsymbol{\varepsilon}-$space into a corresponding volume in $\mathbf{t}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ space for variables that are not at a bound. To make this substitution of variables when computing the multi-dimensional integral, we need the following factor to represent the change in measure [8, p. 368].

$$J_p(\omega) = \left| \frac{\partial \boldsymbol{\varepsilon}}{\partial \left(\mathbf{t}_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}, \pi\right)} \right|, \tag{17}$$

the absolute value of the determinant of the Jacobian matrix representing the change of variables. Thus the rejection probability, i.e. the probability that the market clearing price π at node i is less than p can be calculated from

$$\psi_{i,g}(p,q) = \sum_{\omega} \int_{\pi=-\infty}^{p} \int_{\mathcal{S}(\omega)} f\left(\mathbf{At} + \mathbf{s}(\mathbf{p}(\pi,\boldsymbol{\rho},\boldsymbol{\sigma}))\right) J_p(\omega) \mathbf{dt}_{B(\omega)} \mathbf{d}\boldsymbol{\rho}_{U(\omega)} \mathbf{d}\boldsymbol{\sigma}_{L(\omega)} d\pi, \tag{18}$$

where $\mathcal{S}(\omega)$ is defined in (15). It is now straightforward to show that:

$$\frac{\partial\psi_{i,g}\left(p,q\right)}{\partial p}=\sum_{\omega}\int_{\mathcal{S}(\omega)}f\left(\mathbf{At}+\mathbf{s}\left(\mathbf{p}(p,\boldsymbol{\rho},\boldsymbol{\sigma})\right)\right)J_p(\omega)\mathbf{dt}_{B(\omega)}\mathbf{d}\boldsymbol{\rho}_{U(\omega)}\mathbf{d}\boldsymbol{\sigma}_{L(\omega)}. \tag{19}$$

When calculating $\frac{\partial\psi_{i,g}(p,q)}{\partial q}$ we keep p fixed in node i, while r, the output of firm g is free to change. Thus producer g makes a Bertrand offer at p. To compute this derivative, we define $\mathbf{s}\left(\mathbf{p}(p,\boldsymbol{\rho},\boldsymbol{\sigma}),r\right)$ to be the (vector) net-supply function with jth component

$$\begin{cases} r+\sum_{h=1,h\neq g}^{n_i}S_{ih}\left(p\right)-D_i(p), & j=i,\\ \sum_{h=1}^{n_j}S_{jh}\left(p_j\right)-D_j(p_j), & j\neq i.\end{cases} \tag{20}$$

In this case the substitution factor is given by:

$$J_q(\omega)=\left|\frac{\partial\boldsymbol{\varepsilon}}{\partial\left(\mathbf{t}_{B(\omega)},\boldsymbol{\rho}_{U(\omega)},\boldsymbol{\sigma}_{L(\omega)},r\right)}\right|. \tag{21}$$

Thus the rejection probability, i.e. the probability that the market clearing quantity r for generator g at node i is less than q can be calculated from

$$\psi_{i,g}\left(p,q\right)=\sum_{\omega}\int_{r=-\infty}^{q}\int_{\mathcal{S}(\omega)}f\left(\mathbf{At}+\mathbf{s}\left(\mathbf{p}(p,\boldsymbol{\rho},\boldsymbol{\sigma}),r\right)\right)J_q(\omega)\mathbf{dt}_{B(\omega)}\mathbf{d}\boldsymbol{\rho}_{U(\omega)}\mathbf{d}\boldsymbol{\sigma}_{L(\omega)}dr \tag{22}$$

and

$$\frac{\partial\psi_{i,g}\left(p,q\right)}{\partial q}=\sum_{\omega}\int_{\mathcal{S}(\omega)}f\left(\mathbf{At}+\mathbf{s}\left(\mathbf{p}(p,\boldsymbol{\rho}_U,\boldsymbol{\sigma}_L),q\right)\right)J_q(\omega)\mathbf{dt}_{B(\omega)}\mathbf{d}\boldsymbol{\rho}_{U(\omega)}\mathbf{d}\boldsymbol{\sigma}_{L(\omega)}. \tag{23}$$

Next we want to derive explicit expressions for $J_p(\omega)$ and $J_q(\omega)$. We start by introducing some new notation. For each state ω we partition the nodes into the sets $\Xi\left(\omega\right)$ and $F\left(\omega\right)$. $\Xi\left(\omega\right)$ includes all nodes that are completely integrated with node i, the arbitrarily picked trading hub, through some uncongested chain of arcs. The set $F\left(\omega\right)$ contains all other nodes in the network. Similarly we partition the shock vector into $\boldsymbol{\varepsilon}_{\Xi(\omega)}$ and $\boldsymbol{\varepsilon}_{F(\omega)}$.

Given a state ω, the price in node i and all nodes $j\in\Xi\left(\omega\right)$ is π (irrespective of $\boldsymbol{\rho}_{U(\omega)}$ and $\boldsymbol{\sigma}_{L(\omega)}$). Thus it follows from (2) and (16) that

$$\frac{\partial\varepsilon_j}{\partial\rho_k} = s_j'\left(p_j\right)\frac{\partial p_j}{\partial\rho_k}=0,\ \text{if } j\in\Xi\left(\omega\right)\ \text{ and } k\in U\left(\omega\right) \tag{24}$$

$$\frac{\partial\varepsilon_j}{\partial\sigma_k} = s_j'\left(p_j\right)\frac{\partial p_j}{\partial\sigma_k}=0,\ \text{if } j\in\Xi\left(\omega\right)\ \text{ and } k\in L\left(\omega\right) \tag{25}$$

$$\frac{\partial\varepsilon_j}{\partial\pi} = \begin{cases} S_j'\left(\pi\right) & \text{if } j\neq i\\ S_{j,-g}'\left(\pi\right) & \text{if } j=i.\end{cases} \tag{26}$$

The arcs are partitioned as follows. We let $\mathbf{t}_{\Xi(\omega)}$ be the flows in the uncongested arcs between nodes in the set $\Xi\left(\omega\right)$ and we let $\mathbf{t}_{F(\omega)}$ be the vector of flows in the other arcs. In particular, the vector $\mathbf{t}^F_{B(\omega)}$ denotes uncongested flows in the other arcs. Here $M_{\Xi(\omega)}$ is the number of nodes in $\Xi(\omega)$ and we note that they must be connected by $M_{\Xi(\omega)} - 1$ arcs. We use the node-arc incidence matrix $\mathbf{A}_{\Xi(\omega)}$ to describe the connected radial network with nodes in $\Xi(\omega)$ and arcs with flows $\mathbf{t}_{\Xi(\omega)}$ connecting nodes in this set. We let $\mathbf{A}_{F(\omega)}$ be a node-arc incidence matrix with $M - M_{\Xi(\omega)}$ rows/nodes and $M - M_{\Xi(\omega)}$ columns/arcs, describing the rest of the network. [15] Using this notation, the nodal flow balance in (2) can be written as follows:

$$\begin{aligned}
\mathbf{A}_{\Xi(\omega)}\mathbf{t}_{\Xi(\omega)} + \mathbf{s}_{\Xi(\omega)}(\mathbf{p}) &= \boldsymbol{\varepsilon}_{\Xi(\omega)} \\
\mathbf{A}_{F(\omega)}\mathbf{t}_{F(\omega)} + \mathbf{s}_{F(\omega)}(\mathbf{p}) &= \boldsymbol{\varepsilon}_{F(\omega)}.
\end{aligned}$$

Thus

$$\frac{\partial\left(\boldsymbol{\varepsilon}_{\Xi(\omega)}\right)_k}{\partial\left(\mathbf{t}_{\Xi(\omega)}\right)_j} = \left(\mathbf{A}_{\Xi(\omega)}\right)_{kj} \tag{27}$$

$$\frac{\partial\left(\boldsymbol{\varepsilon}_{\Xi(\omega)}\right)_k}{\partial\left(\mathbf{t}^F_{B(\omega)}\right)_j} = 0 \tag{28}$$

and

$$\frac{\partial\left(\boldsymbol{\varepsilon}_{F(\omega)}\right)_k}{\partial\left(\mathbf{t}_{F(\omega)}\right)_j} = \left(\mathbf{A}_{F(\omega)}\right)_{kj} \tag{29}$$

$$\frac{\partial\left(\boldsymbol{\varepsilon}_{F(\omega)}\right)_k}{\partial\left(\mathbf{t}_{\Xi(\omega)}\right)_j} = 0. \tag{30}$$

We can now show the following:

Lemma 3 $J_p(\omega) = \left(S'_{i,-g}(\pi) + \sum_{k\in\Xi(\omega)\setminus i} S'_k(\pi)\right) J_F(\omega)$, *where*

$$J_F(\omega) = \left|\frac{\partial\boldsymbol{\varepsilon}_{F(\omega)}}{\partial(\mathbf{t}^F_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)})}\right|. \tag{31}$$

Proof. From (24)-(30) we realize that

$$\frac{\partial\boldsymbol{\varepsilon}}{\partial\left(\mathbf{t}_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}, \pi\right)} = \begin{bmatrix} \mathbf{A}_{\Xi(\omega)} & \mathbf{0} & \frac{\partial\boldsymbol{\varepsilon}_{\Xi(\omega)}}{\partial\pi} \\ \mathbf{0} & \frac{\partial\boldsymbol{\varepsilon}}{\partial\left(\mathbf{t}^F_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}\right)} & \frac{\partial\boldsymbol{\varepsilon}_{F(\omega)}}{\partial\pi} \end{bmatrix} \tag{32}$$

Let

$$\mathbf{B} = \begin{bmatrix} \mathbf{A}_{\Xi(\omega)} & \mathbf{0} \\ \mathbf{0} & \frac{\partial\boldsymbol{\varepsilon}}{\partial\left(\mathbf{t}^F_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}\right)} \end{bmatrix}. \tag{33}$$

[15] Note that the remainder of the network has at least one arc that is lacking its start or end node. Also the remainder of the network is not necessarily connected.

When calculating $J_p(\omega)$, we will expand the determinant along the Mth column in (32) with entries $\frac{\partial \varepsilon_k}{\partial \pi}$ giving the net-supply slopes as shown in (26). It follows from the definition of the determinant that [36]:

$$J_p(\omega) = \left| (-1)^{i+M} S'_{i,-g}(\pi) \det(\mathbf{B}_{-i}) + \sum_{k \neq i} (-1)^{k+M} S'_k(\pi) \det(\mathbf{B}_{-k}) \right|.$$

$\mathbf{A}_{\Xi(\omega)}$ is the node arc incidence matrix of a connected radial network. This matrix has linearly dependent rows and has rank $M_{\Xi(\omega)} - 1$. Thus it follows from (33) that $\det(\mathbf{B}_{-k}) = 0$ if $k \in F(\omega)$. If $k \in \Xi(\omega)$ then $\mathbf{B}_{-k}$ is a block matrix with determinant $\left| \left(\mathbf{A}_{\Xi(\omega)}\right)_{-k} \right| J_F(\omega)$. Thus $J_p(\omega)$ can be written as

$$\begin{aligned} J_F(\omega) &\left| (-1)^{i+M} S'_{i,-g}(\pi) \det\left(\mathbf{A}_{\Xi(\omega)}\right)_{-i} + \textstyle\sum_{k \in \Xi(\omega) \setminus i} (-1)^{k+M} S'_k(\pi) \det\left(\mathbf{A}_{\Xi(\omega)}\right)_{-k} \right| \\ &= J_F(\omega) \left(S'_{i,-g}(\pi) + \textstyle\sum_{k \in \Xi(\omega) \setminus i} S'_k(\pi) \right) \left| (-1)^{M_{\Xi(\omega)}} \det\left(\mathbf{A}_{\Xi(\omega)}\right)_{-i} \right| \end{aligned}$$

by Corollary 2 and the monotonicity of net-supply functions. Now, since $\mathbf{A}_{\Xi(\omega)}$ is the node-arc incidence matrix of a connected radial network, it follows from Bapat [10] (p. 13) that $\left| (-1)^{M_{\Xi(\omega)}} \det\left(\mathbf{A}_{\Xi(\omega)}\right)_{-j} \right|$ is 1. ■

We now show that $J_F(\omega) = \left| \frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial(\mathbf{t}^F_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)})} \right|$ can be calculated from the following result.

Lemma 4 *Row k of the Jacobian matrix $\frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial\left(\mathbf{t}^F_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}\right)}$ can be constructed as follows for the state ω:*

$$\left(\frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial\left(\mathbf{t}^F_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}\right)} \right)_k = \left[\begin{array}{ccc} \left(\mathbf{A}_{B(\omega)}\right)_k & S'_k(p_k)\left(\mathbf{E}_{U(\omega)}\right)_k & -S'_k(p_k)\left(\mathbf{E}_{L(\omega)}\right)_k \end{array} \right] \tag{34}$$

for $k \in F(\omega)$.

Proof. We partition the columns of $\mathbf{A}_{F(\omega)}$ into $\mathbf{A}^F_{L(\omega)}$, $\mathbf{A}^F_{B(\omega)}$ and $\mathbf{A}^F_{U(\omega)}$, corresponding to flows $\mathbf{t}^F$ being at their lower bounds, strictly between their bounds, and at their upper bounds. Thus the trade, consumption and production balance in (2) can be written as follows

$$\mathbf{A}^F_{B(\omega)} \mathbf{t}^F_{B(\omega)} + \mathbf{A}^F_{U(\omega)} \mathbf{t}_{U(\omega)} + \mathbf{A}^F_{L(\omega)} \mathbf{t}_{L(\omega)} + \mathbf{s}_{F(\omega)}(\mathbf{p}) = \boldsymbol{\varepsilon}_{F(\omega)}. \tag{35}$$

Observe that (13) implies that

$$\frac{\partial \varepsilon^F_k}{\partial \rho_j} = \frac{\partial \varepsilon^F_k}{\partial p_k} \frac{\partial p_k}{\partial \rho_j} = S'_k(p_k)\left(\mathbf{E}_{U(\omega)}\right)_{kj}$$

and

$$\frac{\partial \varepsilon^F_k}{\partial \sigma_j} = \frac{\partial \varepsilon^F_k}{\partial p_k} \frac{\partial p_k}{\partial \sigma_j} = -S'_k(p_k)\left(\mathbf{E}_{L(\omega)}\right)_{kj}$$

which gives the result. ■

Lemma 5 *$J_q(\omega) = J_F(\omega)$.*

Proof. We have from (2) and (20) that

$$\frac{\partial \varepsilon_k}{\partial r} = \begin{cases} 0 & \text{if } k \neq i \\ 1 & \text{if } k = i. \end{cases}$$

Similar to (32) we have

$$\frac{\partial \boldsymbol{\varepsilon}}{\partial \left(\mathbf{t}_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}, r\right)} = \begin{bmatrix} \mathbf{A}_{\Xi(\omega)} & \mathbf{0} & \frac{\partial \boldsymbol{\varepsilon}_{\Xi(\omega)}}{\partial r} \\ \mathbf{0} & \frac{\partial \boldsymbol{\varepsilon}}{\partial\left(\mathbf{t}^F_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}\right)} & \frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial r} \end{bmatrix}. \tag{36}$$

Similar to the proof of Lemma 3, we expand the determinant $\left|\frac{\partial \boldsymbol{\varepsilon}}{\partial\left(\mathbf{t}_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}, r\right)}\right|$ along the Mth column, which has zeros in rows $k \neq i$ and a one in row i. We use the definition of $\mathbf{B}$ in (33), so it follows from the definition of the determinant that [36]:

$$\begin{aligned} J_q(\omega) &= \left|(-1)^{i+M} \det\left(\mathbf{B}_{-i}\right)\right| = \left|\det\left(\mathbf{B}_{-i}\right)\right| \\ &= \left|\left(\mathbf{A}_{\Xi(\omega)}\right)_{-i}\right| J_F(\omega) \end{aligned}$$

because $\mathbf{B}_{-i}$ is a block matrix with determinant $\left|\left(\mathbf{A}_{\Xi(\omega)}\right)_{-k}\right| J_F(\omega)$. $\mathbf{A}_{\Xi(\omega)}$ is a node-arc incidence matrix of a connected radial network. Thus it follows from Bapat [10] (p. 13) that $\left|\det\left(\mathbf{A}_{\Xi(\omega)}\right)_{-i}\right|$ is 1, which gives the stated result. ∎

The following can be shown from our results in Lemmas 3 and 5.

Proposition 6 *In a radial network, the optimal output q of firm g at price p in node i can be determined from the following Z function:*

$$Z = \left(p - C'_{ig}(q)\right) \sum_{\omega} \left(S'_{i,-g}(p) + \sum_{k \in \Xi(\omega) \setminus i} S'_k(p) \right) P(p, q, \omega) - q \sum_{\omega} P(p, q, \omega) \tag{37}$$

where

$$P(p, q, \omega) = \int_{\mathcal{S}(\omega)} f\left(\mathbf{A}\mathbf{t} + \mathbf{s}\left(\mathbf{p}(p, \boldsymbol{\rho}, \boldsymbol{\sigma})\right)\right) J_F(\omega) \mathbf{d}\mathbf{t}_{B(\omega)} \mathbf{d}\boldsymbol{\rho}_{U(\omega)} \mathbf{d}\boldsymbol{\sigma}_{L(\omega)}. \tag{38}$$

Proof. First we substitute results in Lemma 3 and Lemma 5 into (19) and (23). Next, we get (37) and (38) by substituting (19) and (23) into (5). ∎

Observe that $P(p, q, \omega)$ is a probability density in the sense that $P(p, q, \omega)\, dq$ is the probability that g is dispatched in the interval $(q, q + dq)$ and the congestion state is ω given that the clearing price is p. The term $\sum_{\omega} P(p, q, \omega)$ is then the probability that g is dispatched in the interval $(q, q + dq)$ given that the clearing price is p. The first-order optimality condition is given by $Z(p, q) = 0$, so it immediately follows from (37) that:

Corollary 7 *The optimal output q of firm g in node i at price p satisfies the first-order condition:*

$$q = \left(p - C'_{ig}(q)\right) \sum_{\omega} \left(S'_{i,-g}(p) + \sum_{k \in \Xi(\omega) \setminus i} S'_k(p) \right) P(\omega | p, q),$$

where

$$P(\varpi | p, q) := \frac{\int_{\mathcal{S}(\varpi)} f(\mathbf{At} + \mathbf{s}(\mathbf{p}(p, \boldsymbol{\rho}, \boldsymbol{\sigma}))) J_F(\varpi) \mathbf{dt}_{B(\bar{\omega})} \mathbf{d}\boldsymbol{\rho}_{U(\bar{\omega})} \mathbf{d}\boldsymbol{\sigma}_{L(\bar{\omega})}}{\sum_{\omega} \int_{\mathcal{S}(\omega)} f(\mathbf{At} + \mathbf{s}(\mathbf{p}(p, \boldsymbol{\rho}, \boldsymbol{\sigma}))) J_F(\omega) \mathbf{dt}_{B(\bar{\omega})} \mathbf{d}\boldsymbol{\rho}_{U(\bar{\omega})} \mathbf{d}\boldsymbol{\sigma}_{L(\bar{\omega})}}$$

is the conditional probability that the network is in state ϖ given that the price in node i is p and firm g has output q.

Recall that in a single-node network, the optimal output of a producer is proportional to its mark-up and the slope of the residual demand that it is facing [26]. In a network with multiple connected nodes, producer g in node i only faces the slope of the net-supply in nodes that are completely integrated with its own node. Thus according to Corollary 7, the slope of net-supply in each other node is scaled by the conditional probability that this node is completely integrated with node i. Hence, for multi-dimensional shocks, Klemperer and Meyer's condition generalizes to saying that the optimal output of a producer is proportional to its mark-up and the expected slope of the residual demand that it is facing. This first-order condition is consistent with Wilson's results [41]. We notice that in case all transmission-lines have unlimited capacities, we get the Klemperer and Meyer condition for a completely integrated network. The other extreme when all transmission-lines have zero capacity, yields the Klemperer-Meyer equation for the isolated node i.

Definition 8 *For firm g in node i we define the market integration factor by*

$$\mu_{ig}(p, q) = \sum_{\omega} M_{\Xi(\omega)} P(\omega | p, q).$$

Thus, the market integration factor is equal to the expected number of nodes (including node i itself) that are completely integrated with node i given that firm g has output q and node i has the market price p. As we will see in the next section, this factor is useful when characterizing SFE in symmetric networks.

3.2 Examples

By means of Corollary 7 we are able to construct a first-order condition for each firm in the radial network. The supply function equilibrium (SFE) can be solved from a system of such first-order conditions. The global second-order condition of an available first-order solution can be verified by (6). In this section we use these optimality conditions to derive SFE for two-node and star networks with symmetric firms.

3.2.1 Two node network

Consider a simple network with two nodes connected by one transmission-line from node 1 to node 2 with flow $t_1 \in [-K_1, K_1]$. We can for example pick node 1 as being the trading hub. Below we list the congestion states of the network and how we partition the nodes for each state:

State	t_1	ρ_1	σ_1	Ξ	$\digamma$
ω_1	$\in (-K_1, K_1)$	0	0	$\{1,2\}$	$\emptyset$
ω_2	K_1	$\in [0, \infty)$	0	$\{1\}$	$\{2\}$
ω_3	$-K_1$	0	$\in [0, \infty)$	$\{1\}$	$\{2\}$.

We have from (2) that

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \underbrace{\begin{bmatrix} S_1(p_1) \\ S_2(p_2) \end{bmatrix}}_{\mathbf{s}(\mathbf{p})} + \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\mathbf{A}} t_1. \tag{39}$$

It can be shown that:

Lemma 9 *In a two-node network, firm g's optimality condition in node* 1 *is given by:*

$$\begin{aligned} Z(q,p) = (p - C'_{ig}(q))(S'_j(p) + S'_{i,-g}(p))P(p,q,\omega_1) \\ +(p - C'_{ig}(q))S'_{i,-g}(p)\left(P(p,q,\omega_2) + P(p,q,\omega_3)\right) \\ -q(P(p,q,\omega_1) + P(p,q,\omega_2) + P(p,q,\omega_3)) = 0, \end{aligned} \tag{40}$$

where

$$\begin{aligned} P(p,q,\omega_1) &= \int_{-K_1}^{K_1} f(q + S_{1,-g}(p) - t_1, S_2(p) + t_1)\, dt_1 \\ P(p,q,\omega_2) &= \int_{S_2(p)+K_1}^{\infty} f(q + S_{1,-g}(p) - K_1, \varepsilon_2)\, d\varepsilon_2 \\ P(p,q,\omega_3) &= \int_{-\infty}^{S_2(p)-K_1} f(q + S_{1,-g}(p) + K_1, \varepsilon_2)\, d\varepsilon_2. \end{aligned} \tag{41}$$

Proof. We have from (39) that

$$\mathbf{A}_{-1} = 1 = (\mathbf{A}_{-1})^T = \left((\mathbf{A}_{-1})^T\right)^{-1} = \mathbf{E}.$$

We set $p_1 = \pi$, so it follows from (12) that

$$p_2 = \pi + \rho_1 - \sigma_1. \tag{42}$$

We also have:

State	$\mathbf{A}_B^{\digamma}$	$\mathbf{E}_U^{\digamma}$	$\mathbf{E}_L^{\digamma}$
ω_1	$\emptyset$	$\emptyset$	$\emptyset$
ω_2	$\emptyset$	1	$\emptyset$
ω_3	$\emptyset$	$\emptyset$	1

The network is completely integrated in state ω_1, so $\boldsymbol{\varepsilon}_{\digamma(\omega_1)}$ is empty. We only need the substitution factor $J_{\digamma}(\omega)$ for states ω_2 and ω_3. It follows from (34) and (42) that

$$\begin{aligned} J_{\digamma}(\omega_2) &= \left| \frac{\partial \boldsymbol{\varepsilon}_{\digamma(\omega_2)}}{\partial \left(\boldsymbol{\rho}_{\mathbf{U}(\boldsymbol{\omega_2})}\right)} \right| = S'_2(p_2) = S'_2(\pi + \rho_1) \\ J_{\digamma}(\omega_3) &= \left| \frac{\partial \boldsymbol{\varepsilon}_{\digamma(\omega_3)}}{\partial \left(\boldsymbol{\sigma}_{\mathbf{L}(\omega_3)}\right)} \right| = |-S'_2(p_2)| = S'_2(\pi - \sigma_1). \end{aligned}$$

(38) now yields:

$$P(p,q,\omega_1) = \int_{-K_1}^{K_1} f(\mathbf{A}t_1 + \mathbf{s}(p))\, dt_1 = \int_{-K_1}^{K_1} f(q + S_{1,-g}(p) - t_1, S_2(p) + t_1)\, dt_1,$$

$$\begin{gathered} P(p,q,\omega_2) = \int_0^\infty f(\mathbf{A}t_1 + \mathbf{s}(\mathbf{p}))\, J_F(\omega_2) d\rho_1 \\ = \int_0^\infty f(q + S_{1,-g}(p) - K_1, S_2(p+\rho_1) + K_1)\, S_2'(p+\rho_1)\, d\rho_1 \end{gathered}$$

and

$$\begin{gathered} P(p,q,\omega_3) = \int_0^\infty f(\mathbf{A}t_1 + \mathbf{s}(\mathbf{p}))\, J_F(\omega_3) d\sigma_1 \\ = \int_0^\infty f(q + S_{1,-g}(p) + K_1, S_2(p-\sigma_1) - K_1)\, S_2'(p-\sigma_1)\, d\sigma_1. \end{gathered}$$

This gives us (41) after the substitutions $\varepsilon_2 = S_2(p+\rho_1) + K_1$ and $\varepsilon_2 = S_2(p-\sigma_1) - K_1$, respectively, have been applied to the two integrals. The equation (40) follows from (37) and that the two nodes are only completely integrated in state ω_1. Figure 2 gives a geometric view of the probabilities in (41) for the special case when $S_{1,-g}(p) = 0$. ■

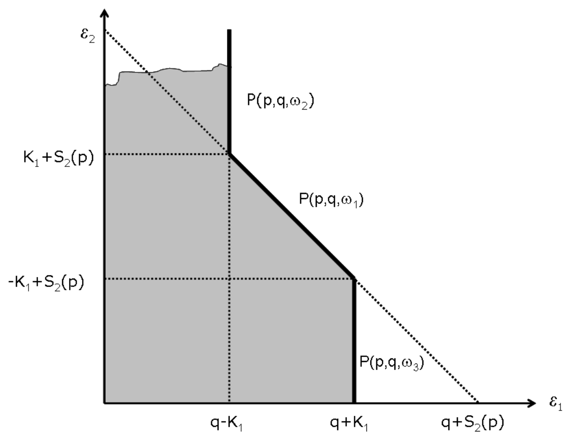

Figure 2: Computation of $P(p,q,\omega)$ when $S_{1,-g}(p) = 0$. The probability mass in the shaded area is equal to the rejection probability $\psi_{1,g}(q,p)$. The values of $P(p,q,\omega)$ are integrals along the right-hand boundary of this region as shown.

It follows from Corollary 7 that

$$Q_{1g} = (p - C'(Q_{1g}))\left(S_{1,-g}'(p) + P(\omega_1 | p, Q_{1g})\, S_2'(p)\right), \tag{43}$$

where $S_{1,-g}'(p)$ is the slope of the net-supply from competitors and demand in node 1, and $S_2'(p)$ is the slope of net-supply in node 2.

Below we consider symmetric NE for symmetric firms and symmetric shock densities. The existence of an equilibrium depends on the partial derivatives $f_i(\varepsilon_1, \varepsilon_2)$, $i = 1,2$, of the shock density which must be sufficiently small. It can be shown that symmetric solutions to (43) are equilibria under the following circumstances.

Proposition 10 *Consider a two-node network with n symmetric firms in each node, each firm having identical marginal costs that are either constant or strictly* *increasing. If demand is inelastic up to a price cap $\overline{p} > C'(\overline{q})$, and has a bounded shock density that satisfies $2n\overline{q}\left|f_i(\varepsilon_1, \varepsilon_2)\right| \leq (3n-2) f(\varepsilon_1, \varepsilon_2)$ when $(\varepsilon_1, \varepsilon_2) \in [-K_1, \overline{q}+K_1] \times [-K_1, \overline{q}+K_1] : \{\varepsilon_1 + \varepsilon_2 \leq 2\overline{q}\}$, then there exists a unique symmetric supply function equilibrium in the network, where each firm's monotonic offer, $Q(p)$, can be calculated from:*

$$Q'(p) = \frac{(P(p,Q,\omega_1)+P(p,Q,\omega_2)+P(p,Q,\omega_3))Q}{(p-C'(Q))((2n-1)P(p,Q,\omega_1)+(n-1)P(p,Q,\omega_2)+(n-1)P(p,Q,\omega_3))} \tag{44}$$

for $p \in (C'(0), \overline{p}]$ with the initial condition $Q(\overline{p}) = \overline{q}$.

Proof. The differential equation in the statement follows from (43), inelastic demand and symmetry of the network. Moreover, the equilibrium is symmetric, so that $S_2(p) = nQ(p)$ and $S_{1,-g}(p) = (n-1)Q(p)$. In case that production capacity binds at some price $p_b < \overline{p}$ then $Q(p)$ is inelastic in the range $(p_b, \overline{p})$, and it follows from (40) that $Z(q,p) < 0$ when $q < \overline{q}$ and $p \in (p_b, \overline{p})$, which would violate the second-order condition in (6). Thus the production capacity must bind at the price cap, which gives our initial condition.

We first show that the solution is unique. To simplify notation let

$$\alpha(p,Q) = P(p,Q,\omega_1) + P(p,Q,\omega_2) + P(p,Q,\omega_3), \tag{45}$$
$$\beta(p,Q) = (2n-1)P(p,Q,\omega_1) + (n-1)P(p,Q,\omega_2) + (n-1)P(p,Q,\omega_3) \tag{46}$$

It follows from the assumptions for $f(\varepsilon_1, \varepsilon_2)$ and our definitions of $P(p,Q,\omega_1)$, $P(p,Q,\omega_2)$ and $P(p,Q,\omega_3)$ in (41) that

$$\frac{\alpha(p,Q(p))}{\beta(p,Q(p))} > 0$$

is continuous in p and Lipschitz continuous in q. Consider a price $\tilde{p} \in (C'(0), \overline{p})$. We now want to show that $p - C'(Q(p))$ is bounded away from zero in the range $[\tilde{p}, \overline{p}]$. This is obvious for constant marginal costs, as we then have that $\tilde{p} - C'(Q(\tilde{p})) = \tilde{p} - C'(0) > 0$. For strictly increasing marginal costs we can use the following argument. It follows from Picard-Lindelöf's theorem and $\overline{p} > C'(\overline{q})$ that a unique solution to (44) must exist for some range $[p_0, \overline{p}]$. In this price range the mark-up, $p - C'(Q(p))$, is smallest at some price p^* where the supply function is at least as steep as the marginal cost curve, i.e. $Q'(p^*) \leq \frac{1}{C''(Q(p^*))}$. Thus it follows from (44) that

$$p^* - C'(Q(p^*)) \geq \frac{Q(p^*)C''(Q(p^*))\alpha(p^*,Q(p^*))}{\beta(p^*,Q(p^*))},$$

which is bounded away from zero whenever $Q(p^*)$ and $C''(Q(p^*))$ are bounded away from zero. In case $Q(p^*) = 0$ for some price $p^* > C'(0)$, it follows from (44) that $Q'(p) = 0$ for $p \in (\tilde{p}, p^*)$. Thus it follows from Picard-Lindelöf's theorem and the properties of (44) that a unique monotonic symmetric solution will exist for the price interval $[\tilde{p}, \overline{p}]$.

We now verify the global second order conditions. We know from (6) that the solution is an equilibrium if $Z(q,p) \geq 0$ when $q \leq Q(p)$ and $Z(q,p) \leq 0$ when $q \geq Q(p)$. We have from (40) that

$$Z(q,p) = (p - C'(q))\beta(p,q)Q'(p) - q\alpha(p,q).$$

As $\beta(p, Q(p)) \geq 0$ we can equivalently verify that

$$X(q,p) \equiv \frac{Z(q,p)}{\beta(p,q)} \equiv (p - C'(q))Q'(p) - \frac{\alpha(p,q)}{\beta(p,q)}q$$

is non-increasing with respect to q. This follows since $X_q(q,p) \leq 0$ implies that $Z_q(q,p) \leq 0$ whenever $Z(q,p) = 0$. Since $Z(Q(p), p) = 0$, we must have $Z(q,p) \geq 0$ when $q \leq Q(p)$ and $Z(q,p) \leq 0$ when $q \geq Q(p)$.

As $C'' \geq 0$, demand is inelastic, and $Q'(p) \geq 0$, the contribution from the first term of X_q (when we differentiate $C'(q)$) is non-positive and we can conclude that

$$\begin{aligned} X_q &\leq -\frac{d}{dq}\left(\frac{\alpha(p,q)}{\beta(p,q)}q\right) \\ &= -\frac{\beta(p,q)(\alpha(p,q) + q\alpha_q(p,q)) - q\alpha(p,q)\beta_q(p,q)}{\beta^2(p,q)} \end{aligned}$$

To show that $X_q \leq 0$, it suffices to show that

$$\beta(p,q)\alpha(p,q) + q\beta(p,q)\alpha_q(p,q) - q\alpha(p,q)\beta_q(p,q) \geq 0. \tag{47}$$

To show this observe that the assumption

$$2n\overline{q}\,|f_i(\varepsilon_1, \varepsilon_2)| \leq (3n-2)\, f(\varepsilon_1, \varepsilon_2)$$

implies from (41) that

$$\begin{aligned} 2nq\,|P_q(p,q,\omega_1)| &= 2nq\left|\int_{-K_1}^{K_1} \frac{\partial}{\partial q} f(q + S_{1,-g}(p) - t_1, S_2(p) + t_1)\, dt_1\right| \\ &\leq 2n\overline{q}\left|\int_{-K_1}^{K_1} \frac{\partial}{\partial q} f(q + S_{1,-g}(p) - t_1, S_2(p) + t_1)\, dt_1\right| \\ &\leq \int_{-K_1}^{K_1} 2n\overline{q}\,|f_1(q + S_{1,-g}(p) - t_1, S_2(p) + t_1)|\, dt_1 \\ &\leq (3n-2)\int_{-K_1}^{K_1} f(q + S_{1,-g}(p) - t_1, S_2(p) + t_1)\, dt_1 \\ &= (3n-2)\, P(p,q,\omega_1)\,. \end{aligned}$$

Similarly $2nq\,|P_q(p,q,\omega_3)| \leq (3n-2)\,P(p,q,\omega_3)$ and $2nq\,|P_q(p,q,\omega_2)| \leq (3n-2)\,P(p,q,\omega_2)$. It follows that

$$\begin{aligned} & q\beta(p,q)\alpha_q(p,q) - q\alpha(p,q)\beta_q(p,q) \\ =\ & qn\left(P(p,q,\omega_1)(P_q(p,q,\omega_2) + P_q(p,q,\omega_3)\right) - qnP_q(p,q,\omega_1)\left(P(p,q,\omega_2) + P(p,q,\omega_3)\right)) \\ \geq\ & qn\left(\frac{2-3n}{2nq}P(p,q,\omega_1)\left(P(p,q,\omega_2) + P(p,q,\omega_3)\right)\right) \\ & -qn\left(\frac{3n-2}{2nq}P(p,q,\omega_1)\left(P(p,q,\omega_2) + P(p,q,\omega_3)\right)\right) \\ =\ & (2-3n)P(p,q,\omega_1)\left(P(p,q,\omega_2) + P(p,q,\omega_3)\right) \end{aligned}$$

It follows from (45) and (46) that

$$\beta(p,q)\alpha(p,q) \geq (3n-2)P(p,q,\omega_1)\left(P(p,q,\omega_2)+P(p,q,\omega_3)\right).$$

Thus (47) is satisfied and thus $X_q \leq 0$, which is sufficient for an equilibrium. ∎

In the next step we will explicitly solve for symmetric SFE in the two-node network. To simplify the optimality conditions we consider the case where demand shocks follow a uniform multi-dimensional distribution.

Assumption 1: *Consider a network with two nodes connected by a transmission-line with capacity* K_1 and *with* n *symmetric firms in each node. Demand in each node is given by* $\varepsilon_i + D(p)$. *Without loss of generality we let* $D(C'(0)) = 0$. *We assume that shocks are uniformly distributed with a constant density,* $\frac{1}{V_1}$, *on the surface* $(\varepsilon_1, \varepsilon_2) \in [-K_1, n\overline{q} - D(\overline{p}) + K_1] \times [-K_1, n\overline{q} - D(\overline{p}) + K_1]$: $\{0 \leq \varepsilon_1 + \varepsilon_2 \leq 2n\overline{q} - 2D(\overline{p})\}$ *and zero elsewhere.*

Proposition 11 *Make Assumption 1, then the symmetric first-order condition for firm* i *in node* g *is given by:*

$$Q = (p - C'(Q))\left((\mu n - 1)Q' - \mu D'\right), \tag{48}$$

where the market integration factor is given by

$$\mu = 1 + P(\omega_1|p,q) = \frac{4K_1 + n\overline{q} - D(\overline{p})}{2K_1 + n\overline{q} - D(\overline{p})}. \tag{49}$$

For inelastic demand, solutions to (48) are SFE, and the inverse symmetric supply functions can be calculated from:

$$p(Q) = Q^{-1}(Q) = \frac{\overline{p}Q^{\mu n-1}}{\overline{q}^{\mu n-1}} + (\mu n - 1)Q^{\mu n-1}\int_Q^{\overline{q}} \frac{C'(u)\,du}{u^{\mu n}}. \tag{50}$$

For linear demand $\varepsilon_i + a - bp_i$, $a, b > 0$, *in each node and constant marginal costs* c, *solutions to (48) are SFE and given by:*

$$Q(p) = \frac{b(p-c)}{(\mu n-2)}\left((\mu n-1)\left(\frac{\overline{p}-c}{p-c}\right)^{(\mu n-2)/(\mu n-1)} - 1\right), \tag{51}$$

where $\overline{p} = \frac{\overline{q}}{\mu b}$.

Proof. It follows from the definitions of $P(p,q,\omega_1)$, $P(p,q,\omega_2)$ and $P(p,q,\omega_3)$ in (41) that under Assumption 1 we get:

$$\begin{aligned}
P(p,q,\omega_3) &= \int_{-\infty}^{S_2(p)-K_1} f\left(q + S_{1,-g}(p) + K_1, \varepsilon_2\right) d\varepsilon_2 = \int_{-K_1 - D(C'(0))}^{S_2(p)-K_1} \frac{d\varepsilon_2}{V_1} = \frac{S_2(p)}{V_1} \\
P(p,q,\omega_2) &= \int_{S_2(p)+K_1}^{\infty} f\left(q + S_{1,-g}(p) - K_1, \varepsilon_2\right) d\varepsilon_2 = \int_{S_2(p)+K_1}^{n\overline{q}-D(\overline{p})+K_1} \frac{d\varepsilon_2}{V_1} = \frac{n\overline{q}-D(\overline{p})-S_2(p)}{V_1} \\
P(p,q,\omega_1) &= \int_{-K_1}^{K_1} f\left(q + S_{1,-g}(p) - t_1, S_2(p) + t_1\right) dt_1 = \int_{-K_1}^{K_1} \frac{dt_1}{V_1} = \frac{2K_1}{V_1}.
\end{aligned} \tag{52}$$

Thus

$$P(\omega_1|p,q) = \frac{P(p,q,\omega_1)}{\sum_\omega P(p,q,\omega)} = \frac{2K_1}{n\overline{q} - D(\overline{p}) + 2K_1}. \tag{53}$$

Now, by substituting $S_2(p) = nQ(p) - D(p)$ and $S_{1,-g}(p) = (n-1)Q(p) - D(p)$ into (43) we get

$$Q = (p - C'(Q))\left((n-1)Q'(p) - D'(p) + P(\omega_1|p,q)(nQ'(p) - D'(p))\right), \tag{54}$$

which together with (53) gives (48) and (49). Next, we note the similarities with the first-order condition for single-node networks with m symmetric firms [26].[16]

$$Q = (p - C'(Q))(Q'(m-1) - D'). \tag{55}$$

By comparing (48) and (55) we can conclude that the first-order solution of a firm in a symmetric two-node network with n firms per node is the same as for a firm in an isolated node with μn symmetric firms and demand $\mu D' + \varepsilon$. Thus analytical solutions to (55) are also solutions to (48) when $m = \mu n$. For example, for single node networks, we know that explicit solutions can be derived for symmetric firms facing an inelastic demand and that these solutions are monotonic [5][22][33], which gives us (50). Moreover, we know that monotonic closed-form solutions exist for symmetric firms with identical constant marginal costs that face a linear demand [23], which gives us (51). It follows from Proposition 10 that both of these monotonic solutions constitute SFE. ■

Thus the equilibrium offer of a firm in the two-node network with n symmetric firms per node is identical to the equilibrium offer of a firm in an isolated node with μn symmetric firms, where μ is the expected number of nodes that are completely integrated with an arbitrary node in the network. It is interesting to note that μ is a constant when demand shocks are uniformly distributed. It does not depend on costs, the market price nor the number of firms. Fig. 3 illustrates how the total supply function in a node depends on μn if the total production capacity in each node is kept fixed.

From the single node case [22][23], we know that solutions to (50) and (51) behave as expected:

Corollary 12 *(i) Mark-ups are positive for a positive output, and (ii) for a given total production cost, mark-ups decrease with more elastic demand and with more firms in the market (or with more market integration).*

Proposition 10 ensures existence of equilibria when slopes in the shock density are sufficiently small. However, existence is problematic for steep slopes in the shock density and especially so when it has discontinuities. This is illustrated by the non-existence example below.

[16] Note that $P(\omega_1|p,q)$, which is calculated in (52), and accordingly the optimal output in (54) depend on the shock distribution, while ex-post optimal offers in a single node network do not depend on the shock distribution.

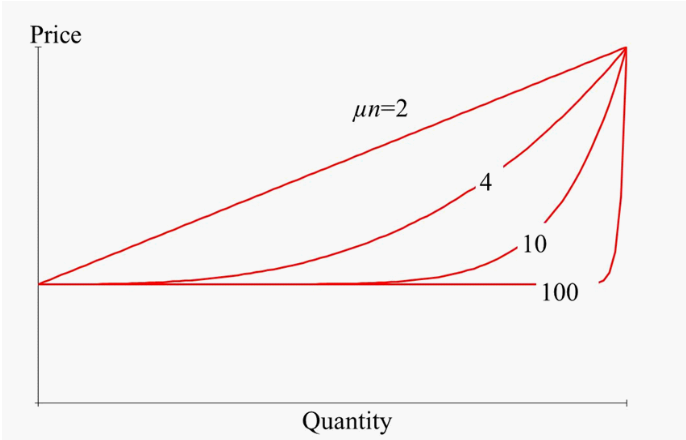

Figure 3: Total supply curve in one node with inelastic demand and constant marginal costs up to a fixed total production capacity, $n\overline{q}$. Here n is the number of firms per node and μ is the expected number of nodes that are completely integrated with a node in the network.

Example 13 ***Shock densities with discontinuities:*** *Assume that the support of the shock ε_i, $i \in \{1, 2\}$ is given by $[0, \overline{\varepsilon}]$. The density is differentiable inside the support set, but decreases discontinuously by $\Delta f(\varepsilon_2) < 0$ when $\varepsilon_1 = \overline{\varepsilon}$ and $\varepsilon_2 \in [0, \overline{\varepsilon}]$. Consider a potential symmetric NE of a duopoly market with one firm in each node with identical costs $C(q)$. Assume that the symmetric supply functions $Q(p)$ are monotonic, that demand is inelastic and that $\overline{q}+K_1 > \overline{\varepsilon} > 2K_1$. Thus unlike the distribution in Assumption 1, the demand shock can reach its discontinuity even if the transport capacity is non-binding. In the following we will show that firm 1 will have a profitable deviation from the potential symmetric pure-strategy NE. In particular we will consider the point (q_0, p_0) where $q_0 = Q(p_0) = \overline{\varepsilon} - K_1$. From (41) we have:*

$$P(p, q, \omega_3) = \int_{-\infty}^{Q(p)-K_1} f(q + K_1, t)dt$$

and accordingly

$$\lim_{q \uparrow \overline{\varepsilon}-K_1} P(p, q, \omega_3) = P(\omega_3, p, \overline{\varepsilon} - K_1) > \lim_{q \downarrow \overline{\varepsilon}-K_1} \int_{-\infty}^{Q(p)-K_1} f(q + K_1, t)dt = 0. \quad (56)$$

However, $P(p, q, \omega_1)$ and $P(p, q, \omega_2)$ are still continuous at the point (q_0, p_0). Recall

$$P(\omega_1, p_0, q_0) = \int_{Q(p)-K_1}^{Q(p)+K_1} f(q_0 + Q(p_0) - t, t)dt = \int_{\overline{\varepsilon}-2K_1}^{\overline{\varepsilon}} f(2q_0 - t, t)dt > 0$$

$$P(\omega_2, p_0, q_0) = \int_{K_1+Q(p)}^{\infty} f(q_0 - K_1, t)dt = 0.$$

We know from (6) that a necessary condition for the solution being an equilibrium *is that* $\lim_{q \uparrow \bar{\varepsilon}-K_1} Z(q, p_0) \geq 0$*, but together with (40) this would imply that*

$$\begin{aligned}
\lim_{q \downarrow \bar{\varepsilon}-K_1} Z(q, p_0) &= (p - C'(q_0))Q'(p_0)P(\omega_1, p_0, q_0) - q_0(\lim_{q \downarrow \bar{\varepsilon}-K_1} \sum_{\omega} P(\omega, p_0, q)) \\
&> (p - C'(q_0))Q'(p_0)P(\omega_1, p_0, q_0) - q_0(\lim_{q \uparrow \bar{\varepsilon}-K_1} \sum_{\omega} P(\omega, p_0, q)) \\
&\geq 0,
\end{aligned}$$

which would locally violate the local second-order condition in (6), and accordingly there is a profitable deviation from the symmetric solution $Q(p)$*.*

The next example illustrates that existence of SFE is also problematic if shocks are perfectly correlated. Wilson [41] outlines a solution of the first-order condition for this case, but he did not verify the second-order conditions for his setting. The example below illustrates why such a first-order solution is normally not an equilibrium in our setting.

Example 14 ***Perfectly correlated shocks:*** *Consider two nodes connected by one transmission-line. Demand shocks in the two nodes are perfectly correlated. This means that market prices are driven by a one-dimensional uncertainty and as in Klemperer and Meyer's [26] model of single markets, one would expect SFE in such a two-node network to be ex-post optimal. Wilson [41] provides a set of first-order conditions, from which a potentially ex-post optimal SFE can be derived. Following him we assume that the line is congested for prices above* p^* *and uncongested for lower prices. Assume that the first-order condition results in a well-behaved monotonic solution for each firm, where mark-ups are strictly positive at* p^**. We also assume that* $D_i' < 0$, $i \in \{1, 2\}$*, so that the slope of residual demand is always strictly negative. Consider a firm* g *in the importing node, which we without loss of generality denote node 1, with the first-order solution* $Q_{1g}(p)$*. We use (40) and consider the ratio*

$$\begin{aligned}
X_{1g}(q, p) := \frac{Z_{1g}(q,p)}{\sum_{\omega} P(p,q,\omega)} &= (p - C_{1g}'(q))(S_j'(p) + S_{1,-g}'(p))P(\omega_1|p, q) \\
&+ (p - C_{1g}'(q))S_{1,-g}'(p)\, P(\omega_2 \cup \omega_3|p, q) - q,
\end{aligned}$$

where $P(\omega_1|p, q) = \frac{P(p,q,\omega_1)}{P(p,q,\omega_1)+P(p,q,\omega_2)+P(p,q,\omega_3)}$ *is the conditional probability that the line is uncongested and* $P(\omega_2 \cup \omega_3|p, q) = \frac{P_2(p,q,\omega_3)+P_2(p,q,\omega_2)}{P(p,q,\omega_1)+P_2(p,q,\omega_2)+P_2(p,q,\omega_3)}$ *is the conditional probability that the line is congested. Consider a price* $p_0 < p^*$*. We know that* $X_{1g}(Q_{1g}(p_0), p_0) = Z_{1g}(Q_{1g}(p_0), p_0) = 0$*, that* $P(\omega_1|p_0, Q_{1g}(p_0)) = 1$ *and that* $P(\omega_2 \cup \omega_3|p_0, Q_{1g}(p_0)) = 0$*. Now, assume that firm* g *withholds a sufficient amount of power* $\Delta Q > 0$ *such that* $P(\omega_1|p_0, Q_{1g}(p_0) - \Delta Q) = 0$ *and* $P(\omega_2 \cup \omega_3|p_0, Q_{1g}(p_0) - \Delta Q) = 1$*. Less withholding is needed to congest the line for prices* p_0 *sufficiently close to* p^**. We have* $D_i' < 0$ *and* $Q_{ih}' \geq 0$*, so* $S_j'(p) > 0$*, which implies that* $\left.\frac{\partial X_{1g}(q,p_0)}{\partial q}\right|_{q=Q_{1g}(p_0)} > 0$ *and that* $\left.\frac{\partial Z_{1g}(q,p_0)}{\partial q}\right|_{q=Q_{1g}(p_0)} > 0$ *for* p_0 *sufficiently close to* p^**. Thus the second-order condition is not locally satisfied at*

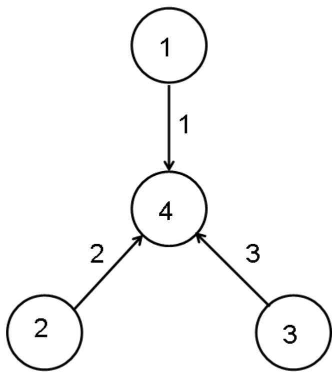

Figure 4: Star network example.

such points. The intuition is that a producer in an importing node always has an incentive to unilaterally deviate from the first-order solution by withholding power in order to congest the transmission-line at lower prices than p^*, which increases the price of the importing node.

3.2.2 Star network

In our second case we will consider a star network with four nodes and three radial lines with capacity K, as shown in Figure 4. We define all arcs to be directed towards the center node 4. Each arc has the same number as the starting node, i.e. $1, 2$ or 3.

Local net-imports must equal net-demand in every node, so

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{bmatrix} = \underbrace{\begin{bmatrix} S_1(p_1) \\ S_2(p_2) \\ S_3(p_3) \\ S_4(p_4) \end{bmatrix}}_{\mathbf{s}(\mathbf{p})} + \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}}_{\mathbf{t}}. \tag{57}$$

Thus

$$\mathbf{A}_{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

and

$$\mathbf{E} = \left((\mathbf{A}_{-1})^T \right)^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{58}$$

Each line t_i has three congestion states. In the uncongested state we have $\sigma_i = 0$, $\rho_i = 0$ and $t_i \in [-K_1, K_1]$. When the line is congested towards node 4 we have $t_i = K$, $\sigma_i = 0$, and $\rho_i \geq 0$ and when the line is congested away from node 4 we have $t_i = -K$, $\sigma_i \geq 0$, and $\rho_i = 0$. Altogether there are $3 \times 3 \times 3 = 27$ congestion states as shown in Table 1.

Table 1: The 27 congestion states of the star network.

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$P(\omega, p, q)$
ω_1	K	K	K	0
ω_2	K	K	$-K$	0
ω_3	K	K	$\in(-K,K)$	$\frac{K\left(S^2(\overline{p})-S^2(p)\right)}{V}$
ω_4	K	$-K$	$-K$	0
ω_5	K	$-K$	$\in(-K,K)$	$\frac{K(S(\overline{p})-S(p))^2}{V}$
ω_6	K	$\in(-K,K)$	$\in(-K,K)$	$\frac{8K^2(S(\overline{p})-S(p))}{V}$
ω_7	$-K$	K	K	0
ω_8	$-K$	K	$-K$	0
ω_9	$-K$	K	$\in(-K,K)$	$\frac{KS^2(p)}{V}$
ω_{10}	$-K$	$-K$	$-K$	0
ω_{11}	$-K$	$-K$	$\in(-K,K)$	$\frac{KS(p)(2S(\overline{p})-S(p))}{V}$
ω_{12}	$-K$	$\in(-K,K)$	$\in(-K,K)$	$\frac{8K^2S(p)}{V}$
ω_{13}	$\in(-K,K)$	K	K	$\frac{2KS^2(p)}{V}$
ω_{14}	$\in(-K,K)$	K	$-K$	$\frac{2KS(p)(S(\overline{p})-S(p))}{V}$
ω_{15}	$\in(-K,K)$	K	$\in(-K,K)$	$\frac{4K^2S(p)}{V}$
ω_{16}	$\in(-K,K)$	$-K$	$-K$	$\frac{2K(S(\overline{p})-S(p))^2}{V}$
ω_{17}	$\in(-K,K)$	$-K$	$\in(-K,K)$	$\frac{4K^2(S(\overline{p})-S(p))}{V}$
ω_{18}	$\in(-K,K)$	$\in(-K,K)$	$\in(-K,K)$	$\frac{8K^3}{V}$
ω_{19}	K	$-K$	K	0
ω_{20}	K	$\in(-K,K)$	K	$\frac{K\left(S^2(\overline{p})-S^2(p)\right)}{V}$
ω_{21}	K	$\in(-K,K)$	$-K$	$\frac{K(S(\overline{p})-S(p))^2}{V}$
ω_{22}	$-K$	$-K$	K	0
ω_{23}	$-K$	$\in(-K,K)$	K	$\frac{KS^2(p)}{V}$
ω_{24}	$-K$	$\in(-K,K)$	$-K$	$\frac{KS(p)(2S(\overline{p})-S(p))}{V}$
ω_{25}	$\in(-K,K)$	$-K$	K	$\frac{2KS(p)(S(\overline{p})-S(p))}{V}$
ω_{26}	$\in(-K,K)$	$\in(-K,K)$	K	$\frac{4K^2S(p)}{V}$
ω_{27}	$\in(-K,K)$	$\in(-K,K)$	$-K$	$\frac{4K^2(S(\overline{p})-S(p))}{V}$

Demand shocks are defined on the following region Θ:

$$\Theta = \left\{ \begin{array}{c} (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in \mathbb{R}^4 | -K \le \varepsilon_i \le n\overline{q} - D(\overline{p}) + K, -3K \le \varepsilon_4 \le 3K, \\ 0 \le \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 \le 3n\overline{q} - 3D(\overline{p}) \ \ \forall i \in \{1,2,3\}. \end{array} \right\}$$

and we let V_2 be the volume of this region.

Assumption 2. *Consider a star network with four nodes and three radial lines with capacity K directed towards the center node 4. There are n firms with identical costs $C(q)$ in each node $1-3$. There are no producers in node 4 (the center node) and demand is inelastic here, i.e. $S_4(p_4) \equiv 0$. Demand in nodes $i \in \{1,2,3\}$ is given by $\varepsilon_i + D(p)$. Without loss of generality we let $D(C'(0)) = 0$. Demand shocks are uniformly distributed such that:*

$$f(\boldsymbol{\varepsilon}) = \begin{cases} \frac{1}{V_2} & \text{if } \boldsymbol{\varepsilon} \in \Theta \\ 0 & \text{otherwise.} \end{cases}$$

Thus the shock density and network are symmetric with respect to nodes $1,2,3$. We can show the following under these circumstances:

Proposition 15 *Make Assumption 2, then the symmetric first-order condition for a firm in nodes $i \in \{1,2,3\}$ is given by:*

$$Q = (p - C'(Q))((\mu n - 1) Q' - \mu D'), \tag{59}$$

where the market integration factor is given by

$$\begin{aligned}\mu &= 1 + P(\omega_{15}|p,q) + P(\omega_{17}|p,q) + P(\omega_{26}|p,q) + P(\omega_{27}|p,q) + 2P(\omega_{18}|p,q) \\ &= \frac{3(n\overline{q}-D(\overline{p}))^2 + 12K(n\overline{q}-D(\overline{p})) + 12K^2}{3(n\overline{q}-D(\overline{p}))^2 + 8K(n\overline{q}-D(\overline{p})) + 4K^2}.\end{aligned} \tag{60}$$

For inelastic demand, solutions to (59) are SFE, and the unique inverse supply function of each firm in nodes $i \in \{1,2,3\}$ is given by (50). For linear demand $\varepsilon_i + a - bp_i$, $a, b > 0$, in each node and constant marginal costs c, solutions to (59) are SFE and given by (51).

Proof. As before we will use node 1 as a trading hub. The price in node 4 can be calculated from (12) and (58)

$$p_4 = p_1 + \rho_1 - \sigma_1 = \pi + \rho_1 - \sigma_1.$$

Similarly prices in the nodes $k \in \{2,3\}$ are given by:

$$p_k = p_1 + \rho_1 - \sigma_1 - \rho_k + \sigma_k = \pi + \rho_1 - \sigma_1 - \rho_k + \sigma_k.$$

In appendix we use (38) to calculate $P(p,q,\omega)$ for one state ω at a time. The results are summarized in Table 1. Viewing the network from node 1 we realize that it is symmetric with respect to nodes 2 and 3, and it is sufficient to make these calculations for the first eighteen states. Results for the last nine states follow from symmetry of the problem. Each competitor is assumed to submit a symmetric offer $Q(p)$, so $S_2(p) \equiv S_3(p) \equiv S(p) := nQ(p) - D(p)$. Adding the results in Table 1 yields:

$$\sum_{\omega} P(p, q, \omega) = \frac{6KS^2(\overline{p})}{V_2} + \frac{16K^2 S(\overline{p})}{V_2} + \frac{8K^3}{V_2}. \tag{61}$$

Node 1 is completely integrated with either node 2 or 3 in states ω_{15}, ω_{17}, ω_{26}, ω_{27} and completely integrated with both nodes in state ω_{18}. In the other states node 1 is either isolated or only completely integrated with node 4, which does not have any producers. We have

$$\begin{array}{c} P(p, q, \omega_{15}) + P(p, q, \omega_{17}) + P(p, q, \omega_{26}) + P(p, q, \omega_{27}) + 2P(p, q, \omega_{18}) \\ = \frac{4K^2 S(p)}{V_2} + \frac{4K^2 (S(\overline{p}) - S(p))}{V_2} + \frac{4K^2 S(p)}{V_2} + \frac{4K^2 (S(\overline{p}) - S(p))}{V_2} + \frac{16K^3}{V_2} = \frac{8K^2 S(\overline{p}) + 16K^3}{V_2}. \end{array} \tag{62}$$

Now we can calculate the expected number of nodes that are completely integrated with node 1.

$$\begin{array}{c} \mu = P(\omega_{15}|p, q) + P(\omega_{17}|p, q) + P(\omega_{26}|p, q) + P(\omega_{27}|p, q) + 2P(\omega_{18}|p, q) \\ = \frac{P(p,q,\omega_{15}) + P(p,q,\omega_{17}) + P(p,q,\omega_{26}) + P(p,q,\omega_{27}) + 2P(p,q,\omega_{18})}{\sum_{\omega} P(p,q,\omega)} = \frac{4KS(\overline{p}) + 8K^2}{3S^2(\overline{p}) + 8KS(\overline{p}) + 4K^2}, \end{array} \tag{63}$$

which yields (60) as $S(\overline{p}) := n\overline{q} - D(\overline{p})$. We now have from (6), (61) and (62) that

$$\begin{array}{c} Z(q, p) = (p - C'(q)) \left(S'_{1,-g}(p) \left(\frac{6KS^2(\overline{p})}{V_2} + \frac{16K^2 S(\overline{p})}{V_2} \right) + \frac{8K^2 S(\overline{p}) + 16K^3}{V_2} S'(p) \right) \\ -q \frac{2K}{V_2} \left[3S^2(\overline{p}) + 8KS(\overline{p}) + 4K^2 \right], \end{array}$$

which gives (59) for $Z(q,p) = 0$, as $S_2(p) \equiv S_3(p) \equiv S(p) := nQ(p) - D(p)$. We note that $\frac{\partial Z(q,p)}{\partial q} \le 0$, so if we find a monotonic stationary solution, then it is an equilibrium. The two explicit equilibrium expressions and monotonicity of these solutions can be established as in the proof of Proposition 11. ■

Figure 3 and Corollary 12 apply to the star network as well. It is only the market integration factor that depends on whether the network has two nodes or is star shaped.

4 Meshed network with potential flows

So far we have studied radial networks, where there is a unique path between every pair of nodes. Now we consider more complicated networks consisting of M nodes and N arcs, where $N \ge M$. This means that there will be at least one cycle in the network and there will be at least two paths between any two nodes in the cycle [11]. Thus we need to make assumptions of how the transport route is chosen for cases when there are multiple possible paths. Here we assume that flows are determined by physical laws that are valid for electricity and incompressible mediums with laminar (non-turbulent) flows. Such flows are sometimes called potential flows, because one can model them as being driven by the potentials $\boldsymbol{\phi}$ in the nodes. In case the commodity is a gas or liquid (e.g. oil), the potential is the pressure at the node. In a DC network it is the voltage that is the potential. For DC networks and laminar flows it can be shown that the electricity and flow choose paths that minimizes total losses. For AC networks it is standard to calculate electric power flows by means of a *DC-load flow approximation*, where $\boldsymbol{\phi}$ is the vector of voltage phase angles at the nodes [14].

In a potential flow model, the flow in the arc k is the result of the potential difference between its endpoints. Given a vector of potentials $\boldsymbol{\phi}$, we have

$$t_k = \frac{-\left(\mathbf{A}^\top \boldsymbol{\phi}\right)_k}{X_k} \tag{64}$$

where $-\left(\mathbf{A}^\top \boldsymbol{\phi}\right)_k$ is the potential difference and X_k is the impedance resisting the the flow through the arc. The impedance is determined by the geometrical and material properties of the line/pipe that transports the commodity. Thus it is an exogenous parameter and independent of the flow in the arc. In a DC network, the impedance is given by the resistance of the line. In a DC-load flow approximation of an AC network, X_k represents the reactance of the transmission line. The matrix $\mathbf{A}$ has rank $M-1$, so the potentials $\boldsymbol{\phi}$ are not uniquely defined by (64). Thus we can arbitrarily choose one node (say i) and set its potential ϕ_i arbitrarily. Normally, the potential of this *swing node* is set to zero. This corresponds to deleting row i from $\mathbf{A}$ to form the matrix $\mathbf{A}_{-i}$ with rank $M-1$ [36].

To simplify the analysis we rule out some unrealistic or unlikely cases: we assume that the impedance is positive and that the capacities of the arcs and impedance factors are such that for any feasible flow, the set of arcs with flows at a lower or upper bound contains no cycles. The latter assumption precludes certain degenerate solutions which can only arise if the values of the bounds and impedances for arcs forming a loop $\mathcal{L}$, satisfy equations of the form

$$\sum_{k \in \mathcal{L}} \delta_k X_k K_k = 0$$

where $\delta_k = 1$ if arc k is oriented in the direction that $\mathcal{L}$ is traversed and $\delta_k = -1$ otherwise. We can preclude instances having such solutions by perturbing the line capacities if necessary.

The market clearing conditions are less obvious in a meshed network. As shown by [14], they can be constructed as the optimality conditions of an economic dispatch problem (EDP) formulated as follows:

$$\begin{array}{llll} \text{EDP:} & \text{minimize} & \sum_{i=1}^{M} \sum_{g=1}^{n_i} \int_0^{q_{ig}} Q_{ig}^{-1}(x)dx - \sum_{i=1}^{M} \int_0^{y_i} D_i^{-1}(y)dy & \\ & \text{subject to} & \mathbf{At} + \mathbf{q} - \mathbf{y} = \boldsymbol{\varepsilon}, & [\mathbf{p}] \\ & & -\mathbf{K} \leq \mathbf{t} \leq \mathbf{K}, & [\boldsymbol{\sigma}, \boldsymbol{\rho}] \\ & & \mathbf{Xt} = -\mathbf{A}^\top \boldsymbol{\phi}. & [\boldsymbol{\lambda}], \end{array}$$

The shadow prices for the constraints are shown on the right-hand side in brackets. EDP seeks supply quantities $\mathbf{q}$, demand $\mathbf{y}$, and transported quantities $\mathbf{t}$ to maximize total producer and consumer welfare. The Karush-Kuhn-Tucker conditions of EDP are

$$\begin{aligned} \text{KKT:}\quad & \mathbf{A}^\top \mathbf{p} + \mathbf{X}^\top \boldsymbol{\lambda} = \boldsymbol{\rho} - \boldsymbol{\sigma} \\ & 0 \le \boldsymbol{\rho} \perp \mathbf{K} - \mathbf{t} \ge \mathbf{0} \\ & 0 \le \boldsymbol{\sigma} \perp \mathbf{K} + \mathbf{t} \ge \mathbf{0} \\ & \mathbf{A}\boldsymbol{\lambda} = \mathbf{0} \\ & \mathbf{At} + \mathbf{s}(\mathbf{p}) = \boldsymbol{\varepsilon} \\ & \mathbf{Xt} = -\mathbf{A}^\top \boldsymbol{\phi} \end{aligned}$$

In radial networks the columns of the matrix $\mathbf{A}$ correspond to network arcs defining a tree, and so they are linearly independent (see [36]). This means that $\mathbf{A}\boldsymbol{\lambda} = \mathbf{0}$ has a unique solution $\boldsymbol{\lambda} = \mathbf{0}$, which allows $\boldsymbol{\lambda}$ to be removed from the market clearing conditions. In this case the conditions become the same as those for radial networks in (11).

We now return to discuss the general case. The prices $\mathbf{p}$ that satisfy the KKT conditions in any congestion state ω must meet certain conditions. First observe that since $\mathbf{X}$ is diagonal and nonsingular,

$$\mathbf{X}^{-1}\mathbf{A}^\top \mathbf{p} + \boldsymbol{\lambda} = \mathbf{X}^{-1}(\boldsymbol{\rho} - \boldsymbol{\sigma}) \tag{65}$$

Multiplying by $\mathbf{A}$ and using the KKT condition that $\mathbf{A}\boldsymbol{\lambda} = \mathbf{0}$ yields

$$\mathbf{A}\mathbf{X}^{-1}\mathbf{A}^\top \mathbf{p} = \mathbf{A}\mathbf{X}^{-1}(\boldsymbol{\rho} - \boldsymbol{\sigma}) \tag{66}$$

In the context of power systems networks the matrix $\mathbf{A}\mathbf{X}^{-1}\mathbf{A}^\top$ is called a network admittance matrix, and when $\mathbf{X}$ is the identity it is a *Laplacian matrix*. The matrix $\mathbf{A}\mathbf{X}^{-1}\mathbf{A}^\top$ has rank $M-1$, so the price $\mathbf{p}$ is not uniquely determined by the choice of $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$. Recall that $\mathbf{A}_i$ is row i of matrix $\mathbf{A}$, and $\mathbf{A}_{-i}$ is matrix $\mathbf{A}$ with row i eliminated. As in section 3 we choose a node i, say, as trading hub and assign its price to be p. The prices in the other nodes for congestion state ω are then uniquely determined by

$$\mathbf{A}\mathbf{X}^{-1}\left((\mathbf{A}_{-i})^T \mathbf{p}_{-i} + p\,(\mathbf{A}_i)^T \right) = \mathbf{A}\mathbf{X}^{-1}(\boldsymbol{\rho} - \boldsymbol{\sigma}).$$

We can remove row i from this equation and multiply by $(\mathbf{A}_{-i}\mathbf{X}^{\mathbf{-1}}\mathbf{A}_{-i}^\top)^{-1}$, so that

$$\begin{aligned} \mathbf{p}_{-i} &= -(\mathbf{A}_{-i}\mathbf{X}^{\mathbf{-1}}(\mathbf{A}_{-i})^T)^{-1}\mathbf{A}_{-i}\mathbf{X}^{-1}(\mathbf{A}_i)^T p \qquad (67)\\ &\quad + (\mathbf{A}_{-i}\mathbf{X}^{\mathbf{-1}}(\mathbf{A}_{-i})^T)^{-1}\mathbf{A}_{-i}\mathbf{X}^{-1}(\boldsymbol{\rho} - \boldsymbol{\sigma}) \qquad (68)\\ &= p\mathbf{1}_{M-1} + (\mathbf{A}_{-i}\mathbf{X}^{\mathbf{-1}}(\mathbf{A}_{-i})^T)^{-1}\mathbf{A}_{-i}\mathbf{X}^{-1}(\boldsymbol{\rho} - \boldsymbol{\sigma}), \qquad (69) \end{aligned}$$

because

$$\begin{gathered} (\mathbf{A}_{-i})^T \mathbf{1}_{M-1} + (\mathbf{A}_i)^T = \mathbf{0}_{M-1} \\ \mathbf{A}_{-i}\mathbf{X}^{\mathbf{-1}}(\mathbf{A}_{-i})^T \mathbf{1}_{M-1} + \mathbf{A}_{-i}\mathbf{X}^{\mathbf{-1}}(\mathbf{A}_i)^T = \mathbf{0}_{M-1} \\ \mathbf{1}_{M-1} = -(\mathbf{A}_{-i}\mathbf{X}^{\mathbf{-1}}(\mathbf{A}_{-i})^T)^{-1}\mathbf{A}_{-i}\mathbf{X}^{-1}(\mathbf{A}_i)^T . \end{gathered} \tag{70}$$

Similar to the radial case, we introduce

$$\mathbf{E}(\omega) = (\mathbf{A}_{-i}\mathbf{X}^{\mathbf{-1}}(\mathbf{A}_{-i})^T)^{-1}\mathbf{A}_{-i}\mathbf{X}^{-1},$$

so that

$$\mathbf{p}_{-i} = p\mathbf{1}_{M-1} + \mathbf{E}(\omega)(\boldsymbol{\rho} - \boldsymbol{\sigma}). \tag{71}$$

Observe that when $\mathbf{A}_{-i}$ is nonsingular then $\mathbf{E}(\omega) = ((\mathbf{A}_{-i})^T)^{-1}$ which gives the expression we have in the radial case. More generally, $\mathbf{A}_{-i}$ will have $M-1$ rows and $N > M-1$ columns, and so it will not have an inverse. Define a matrix $\mathbf{H}$ with $N-(M-1)$ rows forming a basis for the null space of $\mathbf{A}$. For example, the rows of $\mathbf{H}$ can be the orientation vectors of a set of $N-(M-1)$ cycles in the network (see [36]). Since $\mathbf{A}\mathbf{H}^\top = \mathbf{0}$, it follows for any $\boldsymbol{\phi}$ that

$$\mathbf{H}\mathbf{A}^\top\boldsymbol{\phi} = \mathbf{0}.$$

Now the optimality conditions of the dispatch problem amount to:

$$\begin{aligned} \boldsymbol{\varepsilon} &= \mathbf{At} + \mathbf{s}(p\mathbf{1}_{M-1}+\mathbf{E}(\boldsymbol{\omega})(\boldsymbol{\rho} - \boldsymbol{\sigma})) \\ \mathbf{t} &\in [-K, K] \\ \mathbf{HXt} &= -\mathbf{H}\mathbf{A}^\top\boldsymbol{\phi} = 0 \end{aligned}$$

We seek M degrees of freedom in these equations that will specify a range over which to integrate $\boldsymbol{\varepsilon}$. When the integrand is $f(\mathbf{At} + \mathbf{s}(\pi\mathbf{1}_{M-1}+\mathbf{E}(\boldsymbol{\rho} - \boldsymbol{\sigma})))$, the variables of integration are π and $M-1$ variables from $\mathbf{t}$, $\boldsymbol{\rho}$, $\boldsymbol{\sigma}$. When the integrand is $f(\mathbf{A}_i\mathbf{t} + r, \mathbf{A}_{-i}\mathbf{t} + \mathbf{s}_{-i}(p\mathbf{1}_{M-1}+\mathbf{E}(\boldsymbol{\rho} - \boldsymbol{\sigma})))$, the variables of integration are r and $M-1$ variables from $\mathbf{t}$,$\boldsymbol{\rho}$,$\boldsymbol{\sigma}$.

If every $\mathbf{t} \in (-K, K)$ then $\boldsymbol{\rho} = \boldsymbol{\sigma} = \mathbf{0}$, and we have N variables and $N-(M-1)$ constraints from $\mathbf{HXt} = 0$, so we are left with $M-1$ variables to integrate with. For every component of $\mathbf{t}$ that is at a bound, we get a non-negative component of $\boldsymbol{\rho}$ or a component of $\boldsymbol{\sigma}$ that is free to leave its bound.

Let $\mathbf{Y} = \mathbf{HX}$. We partition this matrix into $\mathbf{Y}_L$, $\mathbf{Y}_B$ and $\mathbf{Y}_U$ corresponding to flows at the lower bound, between bounds and at the upper bound. We have $\mathbf{Yt} = \mathbf{0}$, so to integrate over a congestion state ω we fix constrained components ($\mathbf{t}_L = -\mathbf{K}$ and $\mathbf{t}_U = \mathbf{K}$) of $\mathbf{t}$ to get

$$\mathbf{Y}_B\mathbf{t}_B = -\mathbf{Y}_L\mathbf{t}_L - \mathbf{Y}_U\mathbf{t}_U$$

and free unconstrained components of $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ to get $\boldsymbol{\sigma}_L$ and $\boldsymbol{\rho}_U$. We integrate over

$$B(\omega) = \{\mathbf{t}_B : \mathbf{Y}_B\mathbf{t}_B = -\mathbf{Y}_L\mathbf{t}_L - \mathbf{Y}_U\mathbf{t}_U, \quad -\mathbf{K}_B \leq \mathbf{t}_B \leq \mathbf{K}_B\}$$

and $\boldsymbol{\sigma}_L \geq 0$ and $\boldsymbol{\rho}_U \geq 0$, which is an $M-1$ dimensional region.

This gives two different formulae for $\psi_{i,g}(p,q)$ in congestion state ω. These are:

$$\begin{aligned} \psi_{i,g}(p,q) &= \sum_{\omega} \int_{\pi=-\infty}^{p} \int_{S(\omega)} f(\mathbf{At} + \mathbf{s}(\pi\mathbf{1}_{M-1}+\mathbf{E}(\boldsymbol{\rho} - \boldsymbol{\sigma}))) \\ &\quad J_p(\omega)\mathbf{dt}_{B(\omega)}\mathbf{d}\boldsymbol{\rho}_{U(\omega)}\mathbf{d}\boldsymbol{\sigma}_{L(\omega)}d\pi, \end{aligned} \tag{72}$$

$$\psi_{i,g}(p,q) = \sum_{\omega}\int_{r=-\infty}^{q}\int_{S(\omega)} f\left(\mathbf{A}_i\mathbf{t}+r, \mathbf{A}_{-i}\mathbf{t}+\mathbf{s}_{-i}(p\mathbf{1}_{M-1}+\mathbf{E}(\boldsymbol{\rho}-\boldsymbol{\sigma}))\right) J_q(\omega)\mathbf{dt}_{B(\omega)}\mathbf{d}\boldsymbol{\rho}_{U(\omega)}\mathbf{d}\boldsymbol{\sigma}_{L(\omega)}dr, \quad (73)$$

where $S(\omega) = L(\omega)\times U(\omega)\times B(\omega)$. The expressions (72) and (73) can be differentiated and substituted into (5) and (6) to give optimality conditions in a meshed network. Unfortunately these are not as straightforward as the expressions obtained in the radial case in which each agent effectively faces a probability-weighted residual demand curve defined by Corollary 7. In the meshed case the residual demand curve in a congestion state ω involves combinations of the slopes of competitors' supply functions measured at different prices. In other words, nodes in a meshed market may be integrated in a congestion state in the sense that transport between their nodes is possible (with some adjustment in dispatch) but still experience different prices. This makes the computation of equilibrium a lot more challenging.

5 Alternative market designs and strategies

Finally, we want to briefly note that our expressions in Section 3.1 for how market distribution functions can be calculated in radial networks are not restricted to SFE in networks with nodal pricing. They can also be used to calculate Cournot NE in networks with additive demand shocks. We know from Anderson and Philpott [4] that the optimality condition of a vertical offer q from firm g in node i facing an uncertain residual demand is:

$$\int_0^{\overline{p}} Z(p,q)\,dp = 0$$

with the second-order condition that $\int_0^{\overline{p}} Z_q(p,q)\,dp \leq 0$. For radial networks $Z(p,q)$ can be calculated as in Proposition 6 if one sets $S_k'(p_k) = -D_k'(p_k)$.

Our approach is not limited to cases with local uniform-prices. As long as the network operator accepts feasible offers and bids in order to maximize stated social welfare, it is often straightforward to adjust our optimality conditions to networks with other auction formats. For example, consider networks with discriminatory pricing as in the electricity market of Britain. Anderson et al. [3] show that the optimality condition of a firm's offer in such an auction is given by [17]

$$Z = \frac{\partial \psi_{ig}}{\partial p}(p - C_{ig}'(q)) - 1 + \psi_{ig}(q,p),$$

and the same conditions as in (6). For a radial network, $\frac{\partial \psi_{ig}}{\partial p}$ and $\psi_{ig}(q,p)$ are given by (18) and (19), respectively. $J_p(\omega)$ can be calculated as in (17), and Lemma 3.

[17] Note that we have changed the sign of the Z function in Anderson et al [3] for pay-as-bid markets to keep it consistent with the Z function used in this paper.

6 Conclusions

We derive optimality conditions for firms offering supply functions into a radial transmission network with transport constraints and local demand shocks. We verify that monotonic solutions to the first-order conditions are Supply Function Equilibria (SFE) when the joint probability density of the local demand shocks is sufficiently evenly distributed, i.e. close to a uniform multi-dimensional distribution. But existence of SFE cannot be taken for granted. Perfectly correlated shocks or steep slopes and discontinuities in the shock density will not smooth the kink in the residual demand curve sufficiently well, and then profitable deviations from the first-order solution will exist.

In an isolated node, the optimal output of a producer is proportional to its mark-up and the slope of the residual demand that it is facing. We show that in a network with multi-dimensional shocks, this generalizes: the optimal output of a producer is proportional to its mark-up and the expected slope of the residual demand that it is facing. Thus the probability with which the producer's node is completely integrated with other nodes, i.e. connected to other nodes via uncongested arcs, is of great importance for the optimal offer.

For symmetric equilibria it is useful to define a market integration factor, which equals the expected number of nodes that are completely integrated with any node. We use our optimality conditions to solve for symmetric equilibria in two-node and star networks with multi-dimensional uniformly distributed demand shocks. We show that an equilibrium offer in a node of such a network is identical to the equilibrium offer in an isolated node where the number of symmetric firms per node has been scaled by the market integration factor. Firms can influence the market integration factor with their offer curves, so it is endogenous. Still, in our symmetric equilibria, the market integration factor is constant for a given network with given transmission capacities and total production capacities. The factor does not depend on production costs nor on the number of symmetric firms. We also show that the symmetric equilibria are well-behaved: (i) mark-ups are positive for a positive output, and (ii) for a given total production cost, mark-ups decrease with more elastic demand and with more firms in the market (or with more market integration).

We focus on characterising SFE in radial networks, but we also show how our optimality conditions can be generalized to consider meshed networks, albeit with a significant increase in complexity. We also present optimality conditions for SFE in networks with discriminatory pricing and Cournot NE in networks with uncertain demand. Typically each node in our network represents a geographical location, and typically a commodity is transported between two geographical locations. But nodes and transports in our network could be interpreted in a more general sense. For example, a node could represent a point in time or a geographical location at a particular point in time. Moreover, storage at a geographical location can be represented by an arc that can transport the commodity to a later point in time. The transport capacity of such an arc corresponds to the storage capacity at the geographical location. Thus, in principle our approach could be used to model producers' strategic behaviour in a network with local storage.

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Appendix

6.1 Calculations for star network

In the following we use $\mathbf{A}_B^{\digamma}(\omega)$, $\mathbf{E}_U^{\digamma}(\omega)$ and $\mathbf{E}_L^{\digamma}(\omega)$ to denote submatrices of $\mathbf{A}_B(\omega)$, $\mathbf{E}_U(\omega)$ and $\mathbf{E}_L(\omega)$ corresponding to nodes in the set $\digamma(\omega)$.

6.1.1 State ω_1

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	$\digamma$
ω_1	K	K	K	$\{1\}$	$\{2,3,4\}$

In this state we have from (57) and (58) that:

$$\mathbf{A}_B^{\digamma} = \emptyset \quad \mathbf{E}_U^{\digamma} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \quad \mathbf{E}_L^{\digamma} = \emptyset.$$

Thus it follows from (34) that

$$J_{\digamma}(\omega_1) = \left| \frac{\partial \boldsymbol{\varepsilon}_{\digamma}}{\partial\left(\mathbf{t}^{\digamma}_{\mathbf{B}(\boldsymbol{\omega})}, \boldsymbol{\rho}_{\mathbf{U}(\boldsymbol{\omega})}, \boldsymbol{\sigma}_{\mathbf{L}(\boldsymbol{\omega})}\right)} \right| = \left| \det \begin{bmatrix} S_2'(p_2) & -S_2'(p_2) & 0 \\ S_3'(p_3) & 0 & -S_3'(p_3) \\ S_4'(p_4) & 0 & 0 \end{bmatrix} \right| = 0,$$

because $S_4'(p_4) = 0$. Now, we have from (38) that

$$P(p, q, \omega_1) = 0.$$

6.1.2 State ω_2

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	$\digamma$
ω_2	K	K	$-K$	$\{1\}$	$\{2,3,4\}$

With similar calculations as for state ω_1, one gets

$$P(p, q, \omega_2) = 0.$$

6.1.3 State ω_3

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	$\digamma$
ω_3	K	K	$\in [-K, K]$	$\{1\}$	$\{2,3,4\}$

In this state we have from (57) and (58) that:

$$\mathbf{A}_B^{\digamma} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \mathbf{E}_U^{\digamma} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \mathbf{E}_L^{\digamma} = \emptyset.$$

Thus it follows from (34) that

$$J_F(\omega_3) = \left| \frac{\partial \boldsymbol{\varepsilon}_F}{\partial \left(\mathbf{t}^F_{\mathbf{B}(\boldsymbol{\omega})}, \boldsymbol{\rho}_{\mathbf{U}(\boldsymbol{\omega})}, \boldsymbol{\sigma}_{\mathbf{L}(\boldsymbol{\omega})} \right)} \right| = \left| \det \begin{bmatrix} 0 & S_2'(p_2) & -S_2'(p_2) \\ -1 & S_3'(p_3) & 0 \\ 1 & S_4'(p_4) & 0 \end{bmatrix} \right| = S_3'(p_3)\, S_2'(p_2),$$

because $S_4'(p_4) = 0$. Now, we have from (38) that

$$\begin{aligned} P(p,q,\omega_3) &= \int_{\rho_1=0}^{\overline{p}-p} \int_{\rho_2=0}^{p+\rho_1} \int_{t_3=-K}^{K} f(\mathbf{At} + \mathbf{s}(\mathbf{p}(\pi,\boldsymbol{\rho},\boldsymbol{\sigma})))\, S_3'(p_3)\, S_2'(p_2)\, dt_3 d\rho_2 d\rho_1 \\ &= \frac{2K}{V_2} \int_{\rho_1=0}^{\overline{p}-p} S_3'(p+\rho_1) \int_{\rho_2=0}^{p+\rho_1} S_2'(p+\rho_1-\rho_2)\, d\rho_2 d\rho_1 \\ &= \frac{2K}{V_2} \int_{\rho_1=0}^{\overline{p}-p} S_3'(p+\rho_1)\, S_2(p+\rho_1)\, d\rho_1 \\ &= \frac{K\left(S^2(\overline{p}) - S^2(p)\right)}{V_2}, \end{aligned}$$

where we have used that $S_2(p) = S_3(p) = S(p)$.

6.1.4 State ω_4

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	F
ω_4	K	$-K$	$-K$	$\{1\}$	$\{2,3,4\}$

With similar calculations as for state ω_1, one gets

$$P(p,q,\omega_4) = 0.$$

6.1.5 State ω_5

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	F
ω_5	K	$-K$	$\in [-K,K]$	$\{1\}$	$\{2,3,4\}$

In this state we have from (57) and (58) that:

$$\mathbf{A}^F_B = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \mathbf{E}^F_U = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{E}^F_L = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

Thus it follows from (34) that

$$J_F(\omega_5) = \left| \frac{\partial \boldsymbol{\varepsilon}_F}{\partial \left(\mathbf{t}^F_{\mathbf{B}(\boldsymbol{\omega})}, \boldsymbol{\rho}_{\mathbf{U}(\boldsymbol{\omega})}, \boldsymbol{\sigma}_{\mathbf{L}(\boldsymbol{\omega})} \right)} \right| = \left| \det \begin{bmatrix} 0 & S_2'(p_2) & S_2'(p_2) \\ -1 & S_3'(p_3) & 0 \\ 1 & S_4'(p_4) & 0 \end{bmatrix} \right| = S_2'(p_2)\, S_3'(p_3),$$

because $S_4'(p_4) = 0$. Now, we have from (38) that

$$\begin{aligned}
P(p,q,\omega_5) &= \int_{\rho_1=0}^{\overline{p}-p}\int_{\sigma_2=0}^{\overline{p}-p-\rho_1}\int_{t_3=-K}^{K} f(\mathbf{At}+\mathbf{s}(\mathbf{p}(\pi,\boldsymbol{\rho},\boldsymbol{\sigma})))\, S_2'(p_2)\, S_3'(p_3)\, dt_3 d\sigma_2 d\rho_1 \\
&= \frac{2K}{V_2}\int_{\rho_1=0}^{\overline{p}-p} S_3'(p+\rho_1)\int_{\sigma_2=0}^{\overline{p}-p-\rho_1} S_2'(p+\rho_1+\sigma_2)\, d\sigma_2 d\rho_1 \\
&= \frac{2K}{V_2}\int_{\rho_1=0}^{\overline{p}-p} S'(p+\rho_1)\left[S_2(\overline{p}) - S_2(p+\rho_1)\right] d\rho_1 \\
&= \frac{K}{V_2}\left[2S(\overline{p})\, S(p+\rho_1) - S^2(p+\rho_1)\right]_0^{\overline{p}-p} \\
&= \frac{K\left(S(\overline{p}) - S(p)\right)^2}{V_2},
\end{aligned}$$

where we have used that $S_2(p) = S_3(p) = S(p)$.

6.1.6 State ω_6

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	$\digamma$
ω_6	K	$\in[-K,K]$	$\in[-K,K]$	$\{1\}$	$\{2,3,4\}$

In this state we have from (57) and (58) that:

$$\mathbf{A}_B^{\digamma} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \quad \mathbf{E}_U^{\digamma} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{E}_L^{\digamma} = \emptyset.$$

Thus it follows from (34) that

$$J_{\digamma}(\omega_6) = \left|\frac{\partial \boldsymbol{\varepsilon}_{\digamma}}{\partial\left(\mathbf{t}^{\digamma}_{\mathbf{B}(\boldsymbol{\omega})}, \boldsymbol{\rho}_{\mathbf{U}(\boldsymbol{\omega})}, \boldsymbol{\sigma}_{\mathbf{L}(\boldsymbol{\omega})}\right)}\right| = \left|\det\begin{bmatrix} -1 & 0 & S_2'(p_2) \\ 0 & -1 & S_3'(p_3) \\ 1 & 1 & S_4'(p_4) \end{bmatrix}\right| = S_2'(p_2) + S_3'(p_3),$$

because $S_4'(p_4) = 0$. Now, we have from (38) that

$$\begin{aligned}
P(p,q,\omega_6) &= \int_{\rho_1=0}^{\overline{p}-p}\int_{t_3=-K}^{K}\int_{t_2=-K}^{K} f(\mathbf{At}+\mathbf{s}(\mathbf{p}(\pi,\boldsymbol{\rho},\boldsymbol{\sigma})))\,(S_2'(p_2) + S_3'(p_3))\, dt_2 dt_3 d\rho_1 \\
&= \frac{4K^2}{V_2}\int_{\rho_1=0}^{\overline{p}-p} 2S'(p+\rho_1)\, d\rho_1 \\
&= \frac{8K^2\left(S(\overline{p}) - S(p)\right)}{V_2},
\end{aligned}$$

where we have used that $S_2(p) = S_3(p) = S(p)$.

6.1.7 State ω_7

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	$\digamma$
ω_7	$-K$	K	K	$\{1\}$	$\{2,3,4\}$

With similar calculations as for state ω_1, one gets

$$P(p,q,\omega_7) = 0.$$

6.1.8 State ω_8

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	$\digamma$
ω_8	$-K$	K	$-K$	$\{1\}$	$\{2,3,4\}$

With similar calculations as for state ω_1, one gets

$$P(p,q,\omega_8) = 0.$$

6.1.9 State ω_9

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	$\digamma$
ω_9	$-K$	K	$\in [-K,K]$	$\{1\}$	$\{2,3,4\}$

In this state we have from (57) and (58) that:

$$\mathbf{A}_B^{\digamma} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \mathbf{E}_U^{\digamma} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{E}_L^{\digamma} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Thus it follows from (34) that

$$J_{\digamma}(\omega_9) = \left| \frac{\partial \boldsymbol{\varepsilon}_{\digamma}}{\partial \left(\mathbf{t}^{\digamma}_{\mathbf{B}(\boldsymbol{\omega})}, \boldsymbol{\rho}_{\mathbf{U}(\boldsymbol{\omega})}, \boldsymbol{\sigma}_{\mathbf{L}(\boldsymbol{\omega})} \right)} \right| = \left| \det \begin{bmatrix} 0 & -S_2'(p_2) & -S_2'(p_2) \\ -1 & 0 & -S_3'(p_3) \\ 1 & 0 & -S_4'(p_4) \end{bmatrix} \right| = S_3'(p_3)\, S_2'(p_2),$$

because $S_4'(p_4) = 0$. Now, we have from (38) that

$$\begin{aligned} P(p,q,\omega_9) &= \int_{\sigma_1=0}^{p} \int_{\rho_2=0}^{p-\sigma_1} \int_{t_3=-K}^{K} f\left(\mathbf{A}\mathbf{t} + \mathbf{s}\left(\mathbf{p}(\pi,\boldsymbol{\rho},\boldsymbol{\sigma})\right)\right) S_3'(p_3)\, S_2'(p_2)\, dt_3 d\rho_2 d\sigma_1 \\ &= \frac{2K}{V_2} \int_{\sigma_1=0}^{p} S'(p-\sigma_1) \int_{\rho_2=0}^{p-\sigma_1} S'(p-\sigma_1-\rho_2)\, d\rho_2 d\sigma_1 \\ &= \frac{K}{V_2} \int_{\sigma_1=0}^{p} 2S'(p-\sigma_1)\, S(p-\sigma_1)\, d\sigma_1 \\ &= \frac{KS^2(p)}{V_2}, \end{aligned}$$

where we have used that $S_2(p) = S_3(p) = S(p)$.

6.1.10 State ω_{10}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	$\digamma$
ω_{10}	$-K$	$-K$	$-K$	$\{1\}$	$\{2,3,4\}$

With similar calculations as for state ω_1, one gets

$$P(p,q,\omega_{10}) = 0.$$

6.1.11 State ω_{11}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	F
ω_{11}	$-K$	$-K$	$\in[-K,K]$	$\{1\}$	$\{2,3,4\}$

In this state we have from (57) and (58) that:

$$\mathbf{A}_B^F=\left[\begin{array}{c}0\\-1\\1\end{array}\right]\quad \mathbf{E}_U^F=\emptyset\quad \mathbf{E}_L^F=\left[\begin{array}{cc}1&-1\\1&0\\1&0\end{array}\right].$$

Thus it follows from (34) that

$$J_F(\omega_{11})=\left|\frac{\partial\boldsymbol{\varepsilon}_F}{\partial\left(\mathbf{t}^F_{\mathbf{B}(\boldsymbol{\omega})},\boldsymbol{\rho}_{\mathbf{U}(\boldsymbol{\omega})},\boldsymbol{\sigma}_{\mathbf{L}(\boldsymbol{\omega})}\right)}\right|=\left|\det\left[\begin{array}{ccc}0&-S_2'(p_2)&S_2'(p_2)\\-1&-S_3'(p_3)&0\\1&-S_4'(p_4)&0\end{array}\right]\right|=S_2'(p_2)\,S_3'(p_3)\,,$$

because $S_4'(p_4)=0$. Now, we have from (38) that

$$\begin{aligned}
P(p,q,\omega_{11}) &= \int_{\sigma_1=0}^{p}\int_{\sigma_2=0}^{\overline{p}-p+\sigma_1}\int_{t_3=-K}^{K} f\left(\mathbf{A}\mathbf{t}+\mathbf{s}\left(\mathbf{p}(\pi,\boldsymbol{\rho},\boldsymbol{\sigma})\right)\right)S_2'(p_2)\,S_3'(p_3)\,dt_3d\sigma_2d\sigma_1\\
&= \frac{2K}{V_2}\int_{\sigma_1=0}^{p}S'(p-\sigma_1)\int_{\sigma_2=0}^{\overline{p}-p+\sigma_1}S'(p-\sigma_1+\sigma_2)\,d\sigma_2d\sigma_1\\
&= \frac{2K}{V_2}\int_{\sigma_1=0}^{p}S'(p-\sigma_1)\left(S(\overline{p})-S(p-\sigma_1)\right)d\sigma_1\\
&= \frac{K\left(2S(\overline{p})\,S(p)-S^2(p)\right)}{V_2},
\end{aligned}$$

where we have used that $S_2(p)=S_3(p)=S(p)$.

6.1.12 State ω_{12}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	F
ω_{12}	$-K$	$\in[-K,K]$	$\in[-K,K]$	$\{1\}$	$\{2,3,4\}$

In this state we have from (57) and (58) that:

$$\mathbf{A}_B^F=\left[\begin{array}{cc}-1&0\\0&-1\\1&1\end{array}\right]\quad \mathbf{E}_U^F=\emptyset\quad \mathbf{E}_L^F=\left[\begin{array}{c}-1\\-1\\-1\end{array}\right].$$

Thus it follows from (34) that

$$J_F(\omega_{12})=\left|\frac{\partial\boldsymbol{\varepsilon}_F}{\partial\left(\mathbf{t}^F_{\mathbf{B}(\boldsymbol{\omega})},\boldsymbol{\rho}_{\mathbf{U}(\boldsymbol{\omega})},\boldsymbol{\sigma}_{\mathbf{L}(\boldsymbol{\omega})}\right)}\right|=\left|\det\left[\begin{array}{ccc}-1&0&S_2'(p_2)\\0&-1&S_3'(p_3)\\1&1&S_4'(p_4)\end{array}\right]\right|=S_3'(p_3)+S_2'(p_2)\,,$$

because $S_4'(p_4) = 0$. Now, we have from (38) that

$$\begin{aligned} P(p,q,\omega_{12}) &= \int_{\sigma_1=0}^{p}\int_{t_2=-K}^{K}\int_{t_3=-K}^{K} f(\mathbf{At}+\mathbf{s}(\mathbf{p}(\pi,\boldsymbol{\rho},\boldsymbol{\sigma})))\left(S_3'(p_3)+S_2'(p_2)\right) dt_3 dt_2 d\sigma_1 \\ &= \frac{4K^2}{V_2}\int_{\sigma_1=0}^{p} 2S'(p-\sigma_1)\, d\sigma_1 \\ &= \frac{8K^2 S(p)}{V_2}, \end{aligned}$$

where we have used that $S_2(p) = S_3(p) = S(p)$.

6.1.13 State ω_{13}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	F
ω_{13}	$\in [-K,K]$	K	K	$\{1,4\}$	$\{2,3\}$

In this state we have from (57) and (58) that:

$$\mathbf{A}_B^F = \emptyset \quad \mathbf{E}_U^F = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \mathbf{E}_L^F = \emptyset.$$

Thus it follows from (34) that

$$J_F(\omega_{13}) = \left| \frac{\partial \boldsymbol{\varepsilon}_F}{\partial\left(\mathbf{t}^F_{\mathbf{B}(\boldsymbol{\omega})}, \boldsymbol{\rho}_{\mathbf{U}(\boldsymbol{\omega})}, \boldsymbol{\sigma}_{\mathbf{L}(\boldsymbol{\omega})}\right)} \right| = \left| \det \begin{bmatrix} -S_2'(p_2) & 0 \\ 0 & -S_3'(p_3) \end{bmatrix} \right| = S_2'(p_2)\, S_3'(p_3).$$

Now, we have from (38) that

$$\begin{aligned} P(p,q,\omega_{13}) &= \int_{\rho_2=0}^{p}\int_{\rho_3=0}^{p}\int_{t_1=-K}^{K} f(\mathbf{At}+\mathbf{s}(\mathbf{p}(\pi,\boldsymbol{\rho},\boldsymbol{\sigma})))\, S_2'(p_2)\, S_3'(p_3)\, dt_1 d\rho_3 d\rho_2 \\ &= \frac{2K}{V_2}\int_{\rho_2=0}^{p} S'(p-\rho_2)\, d\rho_2 \int_{\rho_3=0}^{p} S'(p-\rho_3)\, d\rho_3 \\ &= \frac{2KS^2(p)}{V_2} = \frac{2KS^2(p)}{V_2}, \end{aligned}$$

where we have used that $S_2(p) = S_3(p) = S(p)$.

6.1.14 State ω_{14}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	F
ω_{14}	$\in [-K,K]$	K	$-K$	$\{1,4\}$	$\{2,3\}$

In this state we have from (57) and (58) that:

$$\mathbf{A}_B^F = \emptyset \quad \mathbf{E}_U^F = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \mathbf{E}_L^F = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Thus it follows from (34) that

$$J_F(\omega_{14}) = \left| \frac{\partial \boldsymbol{\varepsilon}_F}{\partial \left(\mathbf{t}^F_{\mathbf{B}(\boldsymbol{\omega})}, \boldsymbol{\rho}_{\mathbf{U}(\boldsymbol{\omega})}, \boldsymbol{\sigma}_{\mathbf{L}(\boldsymbol{\omega})} \right)} \right| = \left| \det \begin{bmatrix} -S_2'(p_2) & 0 \\ 0 & S_3'(p_3) \end{bmatrix} \right| = S_2'(p_2)\, S_3'(p_3)\,.$$

Now, we have from (38) that

$$\begin{aligned} P(p,q,\omega_{14}) &= \int_{\rho_2=0}^{p} \int_{\sigma_3=0}^{\overline{p}-p} \int_{t_1=-K}^{K} f\left(\mathbf{At} + \mathbf{s}\left(\mathbf{p}(\pi,\boldsymbol{\rho},\boldsymbol{\sigma})\right)\right) S_2'(p_2)\, S_3'(p_3)\, dt_1 d\sigma_3 d\rho_2 \\ &= \frac{2K}{V_2} \int_{\rho_2=0}^{p} S'(p-\rho_2)\, d\rho_2 \int_{\sigma_3=0}^{\overline{p}-p} S'(p+\sigma_3)\, d\sigma_3 \\ &= \frac{2KS(p)\left(S(\overline{p}) - S(p)\right)}{V_2} \\ &= \frac{2KS(p)\left(S(\overline{p}) - S(p)\right)}{V_2}, \end{aligned}$$

where we have used that $S_2(p) = S_3(p) = S(p)$.

6.1.15 State ω_{15}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	F
ω_{15}	$\in [-K,K]$	K	$\in [-K,K]$	$\{1,3,4\}$	$\{2\}$

In this state we have from (57) and (58) that:

$$\mathbf{A}^F_B = \emptyset \quad \mathbf{E}^F_U = [-1] \quad \mathbf{E}^F_L = \emptyset.$$

Thus it follows from (34) that

$$J_F(\omega_{15}) = \left| \frac{\partial \boldsymbol{\varepsilon}_F}{\partial \left(\mathbf{t}^F_{\mathbf{B}(\boldsymbol{\omega})}, \boldsymbol{\rho}_{\mathbf{U}(\boldsymbol{\omega})}, \boldsymbol{\sigma}_{\mathbf{L}(\boldsymbol{\omega})} \right)} \right| = \left| \left[-S_2'(p_2) \right] \right| = S_2'(p_2)\,.$$

Now, we have from (38) that

$$\begin{aligned} P(p,q,\omega_{15}) &= \int_{t_3=-K}^{K} \int_{\rho_2=0}^{p} \int_{t_1=-K}^{K} f\left(\mathbf{At} + \mathbf{s}\left(\mathbf{p}(\pi,\boldsymbol{\rho},\boldsymbol{\sigma})\right)\right) S_2'(p_2)\, dt_1 d\rho_2 dt_3 \\ &= \frac{4K^2}{V_2} \int_{\rho_2=0}^{p} S'(p-\rho_2)\, d\rho_2 \\ &= \frac{4K^2 S(p)}{V_2}, \end{aligned}$$

where we have used that $S_2(p) = S_3(p) = S(p)$.

6.1.16 State ω_{16}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	$\digamma$
ω_{16}	$\in[-K,K]$	$-K$	$-K$	$\{1,4\}$	$\{2,3\}$

In this state we have from (57) and (58) that:

$$\mathbf{A}_B^{\digamma}=\emptyset \quad \mathbf{E}_U^{\digamma}=\emptyset \quad \mathbf{E}_L^{\digamma}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}.$$

Thus it follows from (34) that

$$J_{\digamma}(\omega_{16})=\left|\frac{\partial\boldsymbol{\varepsilon}_{\digamma}}{\partial\left(\mathbf{t}^{\digamma}_{\mathbf{B}(\boldsymbol{\omega})},\boldsymbol{\rho}_{\mathbf{U}(\boldsymbol{\omega})},\boldsymbol{\sigma}_{\mathbf{L}(\boldsymbol{\omega})}\right)}\right|=\left|\det\begin{bmatrix}S_2'(p_2) & 0\\ 0 & S_3'(p_3)\end{bmatrix}\right|=S_2'(p_2)\,S_3'(p_3)\,.$$

Now, we have from (38) that

$$\begin{aligned}P(p,q,\omega_{16}) &= \int_{\sigma_2=0}^{\overline{p}-p}\int_{\sigma_3=0}^{\overline{p}-p}\int_{t_1=-K}^{K} f\left(\mathbf{At}+\mathbf{s}\left(\mathbf{p}(\pi,\boldsymbol{\rho},\boldsymbol{\sigma})\right)\right)S_2'(p_2)\,S_3'(p_3)\,dt_1d\sigma_3d\sigma_2\\ &= \frac{2K}{V_2}\int_{\sigma_2=0}^{\overline{p}-p}S'(p+\sigma_2)\,d\sigma_2\int_{\sigma_3=0}^{\overline{p}-p}S'(p+\sigma_3)\,d\sigma_3\\ &= \frac{2K\left(S(\overline{p})-S(p)\right)^2}{V_2},\end{aligned}$$

where we have used that $S_2(p)=S_3(p)=S(p)$.

6.1.17 State ω_{17}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	$\digamma$
ω_{17}	$\in[-K,K]$	$-K$	$\in[-K,K]$	$\{1,3,4\}$	$\{2\}$

In this state we have from (57) and (58) that:

$$\mathbf{A}_B^{\digamma}=\emptyset \quad \mathbf{E}_U^{\digamma}=\emptyset \quad \mathbf{E}_L^{\digamma}=[0]\,.$$

Thus it follows from (34) that

$$J_{\digamma}(\omega_{17})=0$$

and we have from (38) that

$$P(p,q,\omega_{17})=0.$$

6.1.18 State ω_{18}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	$\digamma$
ω_{18}	$\in[-K,K]$	$\in[-K,K]$	$\in[-K,K]$	$\{1,2,3,4\}$	$\emptyset$

In this state we have from (57) and (58) that:

$$\mathbf{A}_B^{\digamma}=\emptyset \quad \mathbf{E}_U^{\digamma}=\emptyset \quad \mathbf{E}_L^{\digamma}=\emptyset.$$

Now, we have from (38) that

$$P(p,q,\omega_{18})=\int_{t_3=-K}^{K}\int_{t_2=-K}^{K}\int_{t_1=-K}^{K} f\left(\mathbf{At}+\mathbf{s}\left(\mathbf{p}(\pi,\boldsymbol{\rho},\boldsymbol{\sigma})\right)\right)dt_1dt_2dt_3=\frac{8K^3}{V_2}.$$