Strategic supply function competition with private information

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Introduction

- Competition in supply functions has been used to model several markets,
  - In particular the spot market for electricity
  - Also managament consulting, airline reservation systems, ..
- The Cournot framework is a contender
- The supply function models considered typically do not allow for private information
  - Exceptions in empirical work: Hortaçsu and Puller (2006) and Kuhn and Machado (2004).

Relevance of private information

- On costs:
  - Plant availability
  - Hydro availability
  - Terms of supply contracts for energy inputs or imports
    - E.g. constraints in take-or-pay contracts for gas
    - E.g. price of transmission rights in electricity imports
  - Internalization of costs of emission rights (depending of private assessement of future rights allocations)
- On contract positions of firms:
  - Hortaçsu and Puller (2006) in the Texas balancing market (the day-ahead market is resolved with bilateral contracts).
  - Information on costs available from information sellers and also because the balancing market takes place very close to the generation moment.
- On retail sales:
  - Kühn and Machado (2004) study vertically integrated firms in the Spanish pool.

- Aim of the paper:
  - Study supply function competition when firms have private information about costs and compare it with Cournot competition.
- Modeling strategy:
  - Linear-quadratic model coupled with an affine information structure that yields a linear Bayesian Supply Function Equilibrium (and a linear Bayesian Cournot equilibrium)

- Linear SF model is widely used in electricity markets
  - e.g. Green (1996, 1999), Baldick, Grant, and Kahn (2004), and Rudkevich (2005)
- Typically there are a plethora of equilibria in SF models
  - Under some circumstances there is unique equilibrium (Klemperer and Meyer (1989) and Green and Newbery (1992)).
  - But in a linear-quadratic model nonlinear equilibria (in the range between the least competitive Cournot one and the most competitive) are unstable (Baldick and Hogan (2006))
- Linear SFE are tractable (in particular with private information)

- The Cournot framework has been used in a variety of studies
  - E.g. Borenstein, and Bushnell (1999) for the US
- Advantages:
  - Robustness
  - Provides upper bound to the exercise of (non-collusive) market power
  - Capacity constraints are easily incorporated
- Problem:
  - It tends to predict prices that are too high given realistic estimates of the demand elasticity.
  - However, including vertical relations and contracts in a Cournot setting provides good estimates
    (see Bushnell, Mansur, and Saravia (2005)).

Results

- There is a unique LBSFE
- The equilibrium is privately revealing
  - The private information of a firm and the price provide a sufficient statistic of the joint information in the market.
- Supply functions are upward sloping provided that the informative role of price does not overwhelm its traditional capacity as index of scarcity.
  - This happens when costs shocks are not very correlated and/or information precision not too low

- An increase in the correlation of cost parameters or in the noise in private signals makes supply functions steeper and increases price-cost margins.
- Ignoring private cost information with supply function competition may therefore overestimate the slope of supply.
- This is not the case with Cournot competition, where the margin is not affected by the information parameters.

- The welfare evaluation of the LBSFE is in marked contrast with the Cournot equilibrium in the presence of private information:
  - At LBSFE there is only a deadweight loss due to market power but not due to private information.
    - Cournot overestimates the welfare loss with respect to an actual SF mechanism on two counts: excessive market power and lack of information aggregation
  - In a large market, at a LBSFE there is no efficiency loss while there is with Cournot competition

- In both the SF and Cournot cases the order of magnitude of the distorsion because of strategic behavior is 1/n in prices and $1/n^2$ in the deadweight loss where n is the number of firms (and size of the market)
- Convergence to the limit as the market grows happens at the rate $1/\sqrt{n}$

The model

- Consider a market for a homogenous product with n consumers, each having the net benefit function

  $U(x) - px$ with $U(x) = \alpha x - \beta x^2/2$,

  where α, β are positive parameters and x the consumption level
- Inverse demand: $P(X) = a - b\,(X/n)$
- X is total output

- n firms in the market
- Firm *i* produces according to a quadratic cost function:

$$C(x_i;\theta_i) = \theta_i x_i + \frac{\lambda}{2} x_i^2$$

θ_i is a random parameter and $\lambda > 0$

Total surplus: $TS = n\,U(X/n) - \Sigma_i\, C(x_i;q_i)$

Assumptions:

- θ_i is normally distributed (with mean $\bar{\theta}$ > 0).
- Parameters are correlated, with correlation coefficient $\rho \in [0,1]$.
- Firm *i* receives a signal $s_i = \theta_i + \varepsilon_i$
- Signals are of the same precision $\varepsilon_i \sim N(0, \sigma_\varepsilon^2)$.

- In the electricity example:
  - random cost shock may be linked to plant availability because of technical issues or transport problems.
  - The common component in the shock may be related to the prices of energy in international markets to which the supply contracts of firms are linked.

- Before uncertainty is realized, all firms face the same prospects.

- $(\theta_1, \ldots, \theta_n)$ is jointly normally distributed with $E\theta_i = \bar{\theta}$, Var $\theta_i = \sigma_\theta^2$, Cov $(\theta_i, \theta_j) = \rho\, \sigma_\theta^2$, for $j \neq i$, $0 \leq \rho \leq 1$.

Thus, $$\tilde{\theta}_n \equiv (\sum_{i=1}^{n} \theta_i)/n \sim N\left(\bar{\theta}, (1 + (n-1)\rho)\sigma_\theta^2/n\right)$$

and cov($\tilde{\theta}_n$, θ_i) = Var $\tilde{\theta}_n$.

- For $\rho = 1$ the θ parameters are perfectly correlated: *common value* model.

- When signals are perfect, $\sigma^2_{\varepsilon_i} = 0$ for all i, and $0 < \rho < 1$: *private values* model.

- When $\rho = 0$, parameters are independent: *independent values* model.

- It is not difficult to see that

$$E(\theta_i|s_i) = \xi s_i + (1- \xi)\bar{\theta}$$

$$E(s_j|s_i) = E(\theta_j|s_i) = \xi\rho s_i + (1- \xi\rho)\bar{\theta} .$$

- When signals are perfect, $\xi = 1$ and $E(\theta_i|s_i) = s_i$, and $E(\theta|s_i) = \rho s_i + (1-\rho)$.

- When they are not informative, $\xi = 0$ and $E(\theta_i|s_i)= E(\theta_j|s_i) = \bar{\theta}$.

- Firms compete in supply functions.

- We will restrict attention to Linear Bayesian Supply Function Equilibria (LBSFE).

- Strategy for firm *i* is a price contingent schedule $X_i(s_i, .)$.

- Given the strategies of firms $X_j(s_j, .)$, $j = 1, …, n$, for given realizations of signals market clearing implies that

$$p = P_n(\sum_{j=1}^{n} X_j(s_j, p))$$

- Suppose there is a unique market clearing price $\hat{p}(X_1(s_1,\cdot),...,X_n(s_n,\cdot))$ for any realizations of the signals.

- Profits for firm i, for any given realization of the signals, are given by

$$\pi_i\left(X_1\left(s_1,\cdot\right),\ldots,X_n\left(s_n,\cdot\right)\right)=pX_i\left(s_i,p\right)-C\left(X_i\left(s_i,p\right)\right)$$

where

$$p=\hat{p}\left(X_1\left(s_1,\cdot\right),\ldots,X_n\left(s_n,\cdot\right)\right)$$

- This defines a game in supply functions and we want to characterize a Bayesian symmetric linear supply function equilibrium.

- Posit a candidate symmetric equilibrium for the game with n firms:

$$X_n(s_i, p) = b_n - a_n s_i + c_n p$$

- Average output is given by $\tilde{x}_n = b_n - a_n \tilde{s}_n + c_n p$, where

$$\tilde{s}_n = \left(\sum_i s_i\right)/n = \tilde{\theta}_n + \left(\sum_i \varepsilon_i\right)/n$$

- Substituting in the inverse demand we obtain

$$p = \left(1+\beta c_n\right)^{-1}\left(\alpha - \beta b_n + \beta a_n \tilde{s}_n\right)$$

- Given the strategies of rivals $X_n(s_j, .)$, $j \neq i$, firm i faces a residual inverse demand

$$p = \alpha - \frac{\beta}{n}\sum_{j\neq i} X_n\left(s_j, p\right) - \frac{\beta}{n}x_i = \alpha - \frac{\beta}{n}\left(n-1\right)\left(b_n + c_n p_n\right) + \frac{\beta}{n}a_n \sum_{j\neq i} s_j - \frac{\beta}{n}x_i$$

- It follows that,

$$p = I_i - \frac{\beta}{n}\left(1 + \beta \frac{n-1}{n} c_n\right)^{-1} x_i$$

where

$$I_i = \left(1 + \beta \frac{n-1}{n} c_n\right)^{-1} \left(\alpha - \frac{\beta}{n}(n-1) b_n + \frac{\beta}{n} a_n \sum_{j \neq i} s_j\right)$$

- Firm *i* chooses x_i to maximize

$$E(\pi_i | s_i, p) = x_i \left(p - E(\theta_i | s_i, p) \right) - \frac{\lambda}{2} x_i^2 = x_i \left(I_i - \frac{\beta}{n} \left(1 + \beta \frac{n-1}{n} c_n \right)^{-1} x - E(\theta_i | s_i, p) \right) - \frac{\lambda}{2} x_i^2$$

- The F.O.C. is

$$I_i - E(\theta_i | s_i, I_i) - 2 \frac{\beta}{n} \left(1 + \beta \frac{n-1}{n} c_n \right)^{-1} x_i - \lambda x_i = 0$$

or

$$p - E(\theta_i | s_i,\ p) = \left(\frac{\beta}{n + \beta(n-1) c_n} + \lambda \right) x_i$$

- Proposition 1: In the n-firm market with $\rho < 1$ there is a unique symmetric Bayesian linear supply function equilibrium. It is given by

$$X_n(s_i, p) = b_n - a_n s_i + c_n p$$

where

$$a_n = \frac{(1-\rho)\sigma_\theta^2}{\left(\sigma_\varepsilon^2 + (1-\rho)\sigma_\theta^2\right)}\left(\frac{\beta}{n+\beta(n-1)c_n} + \lambda\right)^{-1}$$

$$b_n = \left(1 + \frac{\sigma_\varepsilon^2 \rho}{(1-\rho)K}\right)^{-1}\left(\frac{\sigma_\varepsilon^2 \rho \alpha}{\beta(1-\rho)K} - \frac{\sigma_\varepsilon^2 \overline{\theta}}{nK}\left(\frac{\beta}{n+\beta(n-1)c_n} + \lambda\right)^{-1}\right)$$

and c_n is the largest solution to the quadratic equation

$$\lambda\beta(n-1)\left(1+\frac{\rho\sigma_\varepsilon^2}{K(1-\rho)}\right)c_n^2+\left((\beta+\lambda n)\left(1+\frac{\rho\sigma_\varepsilon^2}{K(1-\rho)}\right)+\frac{\lambda(n-1)\rho\sigma_\varepsilon^2}{K(1-\rho)}-\beta(n-1)\right)c_n$$

$$+(\beta+\lambda n)\frac{\rho\sigma_\varepsilon^2}{\beta K(1-\rho)}-n=0$$

where

$$K=\frac{\left(\sigma_\varepsilon^2+\left(1+(n-1)\rho\right)\sigma_\theta^2\right)}{n}$$

- In equilibrium we have that $1+\beta c_n>0$

- Proof: The price equation

$$p_n = \left(1 + \beta c_n\right)^{-1}\left(\alpha - \beta b_n + \beta a_n \tilde{s}_n\right)$$

can be rearranged to define

$$h_i \equiv \frac{p_n\left(1 + \beta c_n\right) - \alpha + \beta b_n}{\beta a_n} n - s_i = \sum_{j \neq i} s_j$$

- The pair (s_i, p) is informationally equivalent to the pair (s_i, h_i) , hence $E(\theta_i | s_i, p_n) = E(\theta_i | s_i, h_i)$

- Because of the assumed information structure we have

$$\begin{pmatrix}\theta_i \\ s_i \\ h_i\end{pmatrix} \sim N\left[\begin{pmatrix}\overline{\theta} \\ \overline{\theta} \\ (n-1)\overline{\theta}\end{pmatrix}, \begin{pmatrix}\sigma_\theta^2 & \sigma_\theta^2 & (n-1)\rho\sigma_\theta^2 \\ \sigma_\theta^2 & \sigma_\theta^2+\sigma_\varepsilon^2 & (n-1)\rho\sigma_\theta^2 \\ (n-1)\rho\sigma_\theta^2 & (n-1)\rho\sigma_\theta^2 & \Lambda\end{pmatrix}\right]$$

, where $\Lambda = (n-1)(\sigma_\theta^2 + \sigma_\varepsilon^2) + (n-1)(n-2)\rho\sigma_\theta^2$

We obtain

$$E\left[\theta_i \middle| s_i, h_i\right] = E\left[\theta_i \middle| s_i, \frac{h_i}{n-1}\right] = \frac{\sigma_\varepsilon^2}{\sigma_\theta^2\left(1+(n-1)\rho\right)+\sigma_\varepsilon^2}\bar{\theta} +$$

$$\frac{\sigma_\theta^2\left[\sigma_\theta^2(1-\rho)\left(1+(n-1)\rho\right)+\sigma_\varepsilon^2\right]}{\left[\sigma_\theta^2(1-\rho)+\sigma_\varepsilon^2\right]\quad\left[\sigma_\theta^2\left(1+(n-1)\rho\right)+\sigma_\varepsilon^2\right]}s_i + \frac{\sigma_\theta^2\sigma_\varepsilon^2\rho}{\left[\sigma_\theta^2(1-\rho)+\sigma_\varepsilon^2\right]\,\left[\sigma_\theta^2\left(1+(n-1)\rho\right)+\sigma_\varepsilon^2\right]}h_i.$$

Using the F.O.C. and the expression for we obtain the following

$$-\frac{\sigma_\varepsilon^2\left(\sigma_\varepsilon^2+(1-\rho)\sigma_\theta^2\right)\bar{\theta}+\left(\sigma_\theta^2\sigma_\varepsilon^2 n\rho\left(\beta b_n-\alpha\right)/\beta a_n\right)}{\left(\sigma_\varepsilon^2+(1-\rho)\sigma_\theta^2\right)\left(\sigma_\varepsilon^2+\left(1+(n-1)\rho\right)\sigma_\theta^2\right)} - \frac{(1-\rho)\sigma_\theta^2\left(\sigma_\varepsilon^2+\left(1+(n-1)\rho\right)\sigma_\theta^2\right)}{\left(\sigma_\varepsilon^2+(1-\rho)\sigma_\theta^2\right)\left(\sigma_\varepsilon^2+\left(1+(n-1)\rho\right)\sigma_\theta^2\right)}s_i$$

$$+\left(1-\frac{\sigma_\theta^2\sigma_\varepsilon^2 n\rho\left(\left(1+\beta c_n\right)/\beta a_n\right)}{\left(\sigma_\varepsilon^2+(1-\rho)\sigma_\theta^2\right)\left(\sigma_\varepsilon^2+\left(1+(n-1)\rho\right)\sigma_\theta^2\right)}\right)p = \left(\frac{\beta}{n+\beta(n-1)c_n}+\lambda\right)\left(b_n - a_n s_i + c_n p\right)$$

- We can use the method of undetermined coefficients and find a_n, b_n, c_n by solving the following system of equations

$$\begin{cases} \dfrac{(1-\rho)\sigma_\theta^2}{\left(\sigma_\varepsilon^2 + (1-\rho)\sigma_\theta^2\right)} = \left(\dfrac{\beta}{n+\beta(n-1)c_n} + \lambda\right) a_n \\ -\dfrac{\sigma_\varepsilon^2 \overline{\theta}}{nK} - \dfrac{\sigma_\varepsilon^2 \rho(\beta b_n - \alpha)}{\beta(1-\rho)K}\left(\dfrac{\beta}{n+\beta(n-1)c_n} + \lambda\right) = \left(\dfrac{\beta}{n+\beta(n-1)c_n} + \lambda\right) b_n \\ \left(1 - \dfrac{\sigma_\varepsilon^2 \rho(1+\beta c_n)}{\beta(1-\rho)K}\left(\dfrac{\beta}{n+\beta(n-1)c_n} + \lambda\right)\right) = \left(\dfrac{\beta}{n+\beta(n-1)c_n} + \lambda\right) c_n \end{cases}$$

- The largest root of the quadratic equation defining c is the only one compatible with the second order condition. *q.e.d*.

- The price p_n reveals the aggregate information $\tilde{s}_n$
- The equilibrium is *privately revealing* (i.e. for firm i (s_i, p) or $(s_i, \tilde{s}_n)$ is a sufficient statistic of the joint information in the market).
- The incentives to collect information are preserved because for firm i the signal still helps in estimating the parameter even though the price reveals the aggregate statistic.

- Increasing the noise in the private signal or the correlation of the random cost parameters ρ makes the slope of supply steeper (decreases c).
  - If this result generalizes to asymmetric settings it may help explain the fact that in the Texas balancing market small firms use steeper supply functions than those predicted by theory (Hortaçsu and Puller (2006)).
    - Small firms may have signals of worse quality because of economies of scale in information gathering

- The slope of supply *c* may be negative if costs shocks are correlated and signals not perfect.
- The price serves a dual role as index of scarcity and as conveyor of information:
  - A high price has a direct effect to increase the competitive supply of a firm, but also conveys news that costs are high.
- If $\rho\sigma_{\varepsilon}^{2} = 0$ then the price conveys no information on costs and $c > 0$. As it increases then the slope c decreases because of the informational component of the price and turns negative at some point.

- There are particular parameter combinations for which the scarcity and informational effects balance and firms set a zero weight ($c = 0$) on the price. (Cournot model)
- However, for reasonable parameter values supply will be upward sloping, the scarcity effect dominating the information effect:
  - in the electricity example: low correlation of plants outages and/or good private precision on them

- A conjecture to be checked is that the slope of supply becomes steeper also when decreasing the number of firms *n* (i.e. *c* increases with *n*).
- A consequence is that the margin over expected marginal cost will tend to be increasing in σ_{ε}^{2}, ρ and decreasing in *n*. Indeed, from the F.O.C. we have that

$$p-\left(E\left(\theta_i \middle| s_i,\ p\right)+\lambda x_i\right)=\left(\frac{1}{n\beta^{-1}+\left(n-1\right)c_n}\right)x_i$$

where the slope of residual demand is

$$n\beta^{-1} + (n-1)c_n$$

A similar relation holds for the margin over average expected marginal cost :

$$E[MC_n] = \frac{1}{n}\sum_{i=1}^{n}\left(E(\theta_i | s_i, p) + \lambda x_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(\theta_i | s_i, p) + \lambda \tilde{x}_n$$

$$\frac{p - E[MC_n]}{p} = \frac{1}{\left(n + \beta(n-1)c_n\right)\eta_n}$$

where $\eta_n = p/(\beta\tilde{x}_n)$ is the elasticity of demand.

- Remark: When $\rho = 1$ the FRREE is not implementable. Indeed, if $\rho = 1$ and $\sigma_{\varepsilon}^{2} < \infty$ (common value) there is no linear equilibrium.

- Remark: The replica market considered can be the outcome of free entry in a market parameterized by the number of consumers m.

  (The free entry number of firms n*(m) is of the same order as m.)

Welfare analysis

- Full (shared) information competitive equilibria are Pareto optimal and characterized by the equality of price and expected marginal cost (with full information):

$$p = E\left(\theta_i \middle| s_i, \tilde{s}_n\right) + \lambda x_i$$

- This allocation is implemented by a price-taking LBSFE) which yields as F.O.C.:

$$p = E\left(\theta_i \,\middle|\, s_i, p\right) + \lambda x_i$$

where $p = E\left(\theta_i \,\middle|\, s_i, p\right) + \lambda x_i$ and the coefficients given by the system of equations

$$\begin{cases} \dfrac{(1-\rho)\sigma_\theta^2}{\left(\sigma_\varepsilon^2 + (1-\rho)\sigma_\theta^2\right)} = \lambda a_n \\ -\dfrac{\sigma_\varepsilon^2 \overline{\theta}}{nK} - \dfrac{\sigma_\varepsilon^2 \rho\left(\beta b_n - \alpha\right)}{\beta(1-\rho)K}\lambda = \lambda b_n \\ \left(1 - \dfrac{\sigma_\varepsilon^2 \rho\left(1 + \beta c_n\right)}{\beta(1-\rho)K}\lambda\right) = \lambda c_n \end{cases}$$

- It follows that

$$\hat{a}_n = \frac{(1-\rho)\sigma_\theta^2}{\lambda\left(\sigma_\varepsilon^2 + (1-\rho)\sigma_\theta^2\right)}$$

$$\hat{b}_n = \left(1 + \frac{\sigma_\varepsilon^2\rho}{(1-\rho)K}\right)^{-1}\left(\frac{\sigma_\varepsilon^2\rho\alpha}{\beta(1-\rho)K} - \frac{\sigma_\varepsilon^2\overline{\theta}}{nK}\lambda^{-1}\right)$$

$$\hat{c}_n = \frac{\lambda^{-1}(1-\rho)K - \beta^{-1}\rho\sigma_\varepsilon^2}{\left(\sigma_\varepsilon^2\rho + (1-\rho)K\right)}$$

where $$K = \frac{\left(\sigma_\varepsilon^2 + \left(1 + (n-1)\rho\right)\sigma_\theta^2\right)}{n}$$

Firms are more cautious while responding to their private signals when they have market power. From the S.O.C. Of the case with market power we have that:

$$a_n = \frac{\tau_\varepsilon}{\tau_\varepsilon + \tau_\theta/(1-\rho)} \frac{1}{\left(\lambda + \beta/\left(n+\beta(n-1)\right)c_n\right)} < \frac{\tau_\varepsilon}{\left(\tau_\varepsilon + \tau_\theta/(1-\rho)\right)} \frac{1}{\lambda} = a$$

The competitive limit and convergence properties

- The continuum economy:
  inverse demand $p = \alpha - \beta\tilde{x}$ where $\tilde{x}$ is average output.

- Firms are indexed in the unit interval (endowed with the Lebesgue measure).

- We can derive the relationship of θ_i, s_i, and the average parameter $\tilde{\theta} = \int \theta_j dj$.

- Average parameter $\tilde{\theta} = \int \theta_j dj$ is normally distributed with mean $\bar{\theta}$ and variance $\rho\,\sigma_\theta^2$

$E(\theta_i|\tilde{\theta}) = \tilde{\theta}$,

$E(\tilde{\theta}|\theta_i) = E(\theta_j|\theta_i) = \rho\,\theta_i + (1-\rho)\,\bar{\theta}$,

$E(\tilde{\theta}|s_i) = E(\theta_j|s_i)$

and $E(\theta_i|\tilde{\theta}, s_i) = (1-d)\tilde{\theta} + ds_i$

, where $d = [\sigma_\theta^2(1-\rho)]/[\sigma_\theta^2(1-\rho) + \sigma_\varepsilon^2]$.

- If signals are perfect, then $d = 1$ and $E(\theta_i | \tilde{\theta}, s_i) = s_i$.

- If signals are useless or correlation perfect ($\rho = 1$), then $d = 0$ and $E(\theta_i | \tilde{\theta}, s_i) = \tilde{\theta}$.

- If both signals and correlation are perfect, then $E(\theta_i | \tilde{\theta}, s_i) = \tilde{\theta} = s_i$ (a.s.).

- Proposition 2. Let $\rho \in [0,1)$. In the continuum economy with inverse demand , there is a unique LBSFE. It is given by

$$X(s_i, p) = b - as_i + cp$$

,where

$$a = \frac{(1-\rho)\sigma_\theta^2}{\lambda\left(\sigma_\varepsilon^2 + (1-\rho)\sigma_\theta^2\right)} = \frac{1}{\lambda\left(1+\left(\sigma_\varepsilon^2 / \left((1-\rho)\sigma_\theta^2\right)\right)\right)}$$

$$b = \frac{\alpha}{\beta}\frac{\sigma_\varepsilon^2}{\left(\sigma_\varepsilon^2 + (1-\rho)\sigma_\theta^2\right)} = \frac{\alpha}{\beta}(1-\lambda a)$$

$$c = \frac{\lambda^{-1}(1-\rho)\sigma_\theta^2 - \beta^{-1}\sigma_\varepsilon^2}{\left(\sigma_\varepsilon^2 + (1-\rho)\sigma_\theta^2\right)} = \frac{1}{\beta}\left(a(\beta+\lambda)-1\right)$$

Moreover, the equilibrium price is given by

$$p = (1 + \beta c)^{-1}\left(\alpha - \beta b + \beta a \tilde{\theta}\right)$$

Proof:

In the continuum economy the F.O.C. is given by

$$p = E\left(\theta_i \middle| s_i, \ p\right) + \lambda x_i$$

Assuming linear strategies and using the inverse demand function and our convention we obtain an expression for the price

$$p = (1 + \beta c)^{-1}\left(\alpha - \beta b + \beta a \tilde{\theta}\right)$$

- Given joint normality of the stochastic variables we obtain

$$\begin{pmatrix} \theta_i \\ s_i \\ p \end{pmatrix} \sim N\left[\begin{pmatrix} \overline{\theta} \\ \overline{\theta} \\ C + D\overline{\theta} \end{pmatrix}, \begin{pmatrix} \sigma_\theta^2 & \sigma_\theta^2 & D\rho\sigma_\theta^2 \\ \sigma_\theta^2 & \sigma_\theta^2 + \sigma_\varepsilon^2 & D\rho\sigma_\theta^2 \\ D\rho\sigma_\theta^2 & D\rho\sigma_\theta^2 & D^2\rho\sigma_\theta^2 \end{pmatrix}\right]$$

, where $C = (1 + \beta c)^{-1}(\alpha - \beta b)$ and $D = (1 + \beta c)^{-1}(\beta a)$

- Using the projection theorem for normal random variables we obtain:

$$E\left(\theta_i \middle| s_i, p\right) = -\frac{C\sigma_\varepsilon^2}{D\left((1-\rho)\sigma_\theta^2 + \sigma_\varepsilon^2\right)} + \frac{(1-\rho)\sigma_\theta^2}{(1-\rho)\sigma_\theta^2 + \sigma_\varepsilon^2} s_i + \frac{\sigma_\varepsilon^2}{D\left((1-\rho)\sigma_\theta^2 + \sigma_\varepsilon^2\right)} p$$

- Plugging in the F.O.C. of the limit economy we obtain:

$$-\frac{\sigma_\varepsilon^2}{\left(\sigma_\varepsilon^2 + (1-\rho)\sigma_\theta^2\right)} \frac{\beta b - \alpha}{\beta a} - \frac{(1-\rho)\sigma_\theta^2}{\left(\sigma_\varepsilon^2 + (1-\rho)\sigma_\theta^2\right)} s_i + \left[1 - \frac{\sigma_\varepsilon^2}{\left(\sigma_\varepsilon^2 + (1-\rho)\sigma_\theta^2\right)} \frac{(1+\beta c)}{\beta a}\right] p =$$
$$= \lambda\left(b - a s_i + c p\right)$$

Using the method of undetermined coefficients, we have the following system of equations

$$\begin{cases} a = \dfrac{(1-\rho)\sigma_\theta^2}{\lambda\left(\sigma_\varepsilon^2 + (1-\rho)\sigma_\theta^2\right)} \\ -\dfrac{\sigma_\varepsilon^2}{\left(\sigma_\varepsilon^2 + (1-\rho)\sigma_\theta^2\right)} \dfrac{\beta b - \alpha}{\beta a} = \lambda b \\ 1 - \dfrac{\sigma_\varepsilon^2}{\left(\sigma_\varepsilon^2 + (1-\rho)\sigma_\theta^2\right)} \dfrac{(1+\beta c)}{\beta a} = \lambda c \end{cases}$$

The solution to the above system gives the result.

- When signals are perfect ($\sigma_\varepsilon^2 = 0$), we have that $a = c = \lambda^{-1}$, $b = 0$, $x_i = \lambda^{-1}(p - \theta_i)$ and

$$p = \frac{\alpha + \beta\lambda^{-1}\tilde{\theta}}{1 + \beta\lambda^{-1}} = \frac{\alpha\lambda + \beta\tilde{\theta}}{\lambda + \beta}$$

- The equilibrium is just the usual complete information competitive equilibrium (it is independent of ρ) and Pareto optimal.
- Equilibrium is also efficient when $\sigma_\varepsilon^2 > 0$: it is price-taking and firms act with a sufficient statistic for the shared information in the economy.

- If $\rho = 0$ or $\sigma_\varepsilon^2 = 0$ then the price conveys no information on costs and $c = 1/\lambda$

- As increases the slope c decreases because of the informational component of price, when $\beta/\lambda = \sigma_\varepsilon^2 / \left((1-\rho)\sigma_\theta^2\right)$ we have that $c = 0$, and for larger values of $\sigma_\varepsilon^2 / \left((1-\rho)\sigma_\theta^2\right)$ it becomes negative.

Convergence to price-taking behavior

Definitions:

- A sequence (of real numbers) b_n is of the *order* n^{υ}, with υ a real number, whenever $n^{-\upsilon} b_n \xrightarrow[n]{} k$ for some nonzero constant *k*.
- The sequence of random variables $\{y_n\}$ converges in *mean square* to zero at the rate $1/\sqrt{n^r}$ if $E\left[(y_n)^2\right]$ converges to zero at the rate $1/n^r$
- Given that $E\left[(y_n)^2\right] = \left(E[y_n]\right)^2 + \text{var}[y_n]$, a sequence $\{y_n\}$ such that $E[y_n] = 0$ and $\text{var}[y_n]$ is of the order of $1/n$, converges to zero at the rate $1/\sqrt{n}$.

A more refined measure of convergence speed for a given convergence rate: *asymptotic variance*.

Suppose that $\{y_n\}$ is such that $E[y_n] = 0$ and $E\left[(y_n)^2\right] = var[y_n]$ converges to 0 at the rate $1/n^r$ for some $r > 0$.

Then the asymptotic variance is given by the constant AV = $\lim_{n\to\infty} n^r \, var(y_n)$.

A higher asymptotic variance means that the speed of convergence is slower.

It is worth noting that if the sequence $\{y_n\}$ is normally distributed then $\sqrt{n^r}(y_n)$ converges in distribution to N(0, AV).

Convergence to price-taking behavior

- As the market grows large the market price p_n (at the LBSFE) converges in mean square to the price-taking Bayesian price p_n^c at the rate of $1/n$.

- The difference between (per capita) expected deadweight loss at the LBSFE and at the Bayesian price-taking equilibrium (ETS_n^c - ETS_n)/n is of the order of $1/n^2$.

Convergence to price-taking behavior

Proposition 4. Let $\rho \in [0,1)$. As *n* tends to infinity the symmetric LBSFE of the *n*-replica market converges to the limit equilibrium:

(i) $p_n - p \underset{n}{\rightarrow} 0$ in mean square at the rate of $1/\sqrt{n}$

(ii) $\sqrt{n}(p_n - p)$ converges in distribution to

$$N\left(0, \left(\frac{\beta}{\beta+\lambda}\right)^2 \left((1-\rho)\sigma_\theta^2 + \sigma_\varepsilon^2\right)\right)$$

Cournot competition

- Now firm i sets a quantity contingent on its information $\{s_i\}$.
- Firm has no other source of information and, in particular, does not condition on the price.
- Expected conditional profits of firm i are:

$$E\left(\pi_i \middle| s_i\right) = x_i\left(P_n\left(\sum_{j\neq i} X_j(s_j) + x_i\right) - E\left(\theta_i \middle| s_i\right)\right) - \frac{\lambda}{2}x_i^2$$

- From the F.O.C. of the optimization of a firm we obtain

$$p-\left(E\left(\theta_i \mid s_i\right)+\lambda x_i\right)=\left(\frac{\beta}{n}\right)x_i$$

- A similar relation holds for the margin over average expected marginal cost:

$$\frac{p-E\left[MC_n\right]}{p}=\frac{1}{n\eta_n}$$

- Proposition 5. There is a unique Bayesian Cournot equilibrium and a unique Bayesian price-taking equilibrium. They are symmetric and affine in the signals.
- Letting $\xi \equiv \tau_\varepsilon / (\tau_\theta + \tau_\varepsilon)$ the strategies of the firms are given (respectively) by:

$$X_n(s_i) = b_n (a - \bar{\theta}) - a_n(s_i - \bar{\theta}),$$

where $a_n = \dfrac{\xi}{\frac{2\beta}{n} + \lambda + \beta \frac{n-1}{n} \rho\xi}$, and $b_n = \dfrac{1}{\lambda + \beta\left(\frac{1+n}{n}\right)}$

$$X_n^c(s_i) = b_n^c(a - \bar{\theta}) - a_n^c(s_i - \bar{\theta})$$

where $$a_n^c = \frac{\xi}{\frac{\beta}{n} + \lambda + \beta \frac{n-1}{n} \rho \xi},$$

and $$b_n^c = \frac{1}{\lambda + \beta}.$$

Remark: From the F.O.C. of profit maximization it is immediate that in equilibrium expected profits for firm *i* are given by $E\pi_n = (\frac{\lambda}{2} + \frac{\beta}{n})\, E\,(X_n(s_i))^2$.

- <u>Remark:</u> In the case of independent values (i.e. $\rho = 0$ and $\tau_e = \infty$) the formulae are valid for a general distribution of the uncertainty.

- When $\rho = 0$: $a_n^{Cournot} < a_n^{SF}$ whenever supply functions are upward sloping.

- Proposition 6: As the market grows large the market price p_n (at the Bayesian Cournot equilibrium) converges in mean square to the price-taking Bayesian price p_n^c at the rate of 1/n. (That is, $E(p_n - p_n^c)^2$ tends to 0 at the rate of $1/n^2$.)

- The difference between (per capita) expected deadweight loss at the market outcome and at the Bayesian price-taking equilibrium $(ETS_n^c - ETS_n)/n$ is of the order of $1/n^2$.

- Proposition 7. As the economy is replicated, the Bayesian Cournot equilibrium converges to the equilibrium in the continuum limit economy:

  $X(s_i) = b(\alpha - \bar{\theta}) - a(s_i - \bar{\theta})$, where $a = \xi / (\lambda + \beta\rho\xi)$ and $b = 1/(\lambda + \beta)$.

- The Bayesian Cournot price p_n converges (in mean square) to $p = \alpha - \beta\,(b(\alpha - \bar{\theta}) - a(\tilde{\theta} - \bar{\theta}))$ at the rate of $1/\sqrt{n}$ and $\sqrt{n}\,(p_n - p)$ converges in distribution to

  $$N(0, \beta^2 a^2((1-\rho)\,\sigma_\theta^2 + \sigma_\varepsilon^2)).$$

- Convergence is slower, according to the asymptotic variance $\beta^2 a^2((1-\rho)\sigma_\theta^2 + \sigma_\varepsilon^2))$. with larger σ_θ^2 or β, and faster with larger ρ or λ.

- A larger ρ means that we are closer to a common value environment.

- Effect of an increase in σ_ε^2 is ambiguous