First-order (Conditional) Risk Aversion, Background Risk and Risk Diversification

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Intuition

First-order risk aversion is not very intuitive.

This is like having only one Chimay at the Grande Place de Mons!

Second-order (or 2 Chimay) is more acceptable for EU fans because it introduces some variance…

For more than two, I refer you to Eeckhoudt and Schlesinger.

Motivation

The concepts of second-order and first-order risk aversion were coined by Segal and Spivak (1990).

For an actuarially fair random variable $\tilde{\varepsilon}$, second-order risk aversion means that the risk premium the agent is willing to pay to avoid $k\tilde{\varepsilon}$ is proportional to k^2 as $k \to 0$.

Under first-order risk aversion, the risk premium is proportional to *k*.
With EU model, however:

$$\pi(k) \approx -\frac{k^2}{2}\sigma_{\tilde{\varepsilon}}^2 \frac{U''(x)}{U'(x)}$$

$$\frac{\partial^2 \pi(k)}{\partial k^2} = -\frac{\sigma_{\tilde{\varepsilon}}^2}{2}\frac{U''(x)}{U'(x)} \neq 0; \quad \left.\frac{\partial \pi}{\partial k}\right|_{k=0} = 0.$$

The reason differentiable EU is only second order is that the derivative is taken around the certainty line (k = 0 means no risk), and, in small neighborhoods, differentiable functions behave like linear functions (in the present context, expected value, hence risk neutrality).

However, in many applications, first-order risk aversion implies that small risks matter. Because expected utility theory is limited to second-order risk aversion, it cannot take into account many real world results or phenomena.

Loomes and Segal (1994) extend this notion to preferences about uninsured events, such as independent additive background risks.

The conditional risk premium is the amount of money the decision maker is willing to pay to avoid $\tilde{\varepsilon}$ in the presence of $\tilde{y}$.

Utility functions in the von Neumann-Morgenstern expected utility class still exhibit only second-order conditional risk aversion, with independent additive background risks.

In this paper, we extend the concepts of orders of conditional risk aversion to orders of conditional dependent risk aversion, for which $\tilde{\varepsilon}$ and the background risk $\tilde{y}$ are dependent and $\tilde{y}$ may enter the agent's utility function arbitrarily $U\left(w_0 + k\tilde{\varepsilon} + \tilde{y}\right), U\left(\tilde{w}\tilde{y}\right), U\left(\tilde{w}, \tilde{y}\right)$.

We propose conditions on the stochastic structure between $\tilde{\varepsilon}$ and $\tilde{y}$ that guarantee first-order conditional dependent risk aversion for expected utility agents with a certain type of risk preference, i.e., with correlation aversion, $U_{12} \neq 0$, a concept developed by Louis Eeckhoudt in many collaborations.

Different economic examples are discussed in the last part of the presentation. One of them is the link between market incompleteness, asset pricing and the equity premium puzzle.

The model without background risk

For an agent with utility function u, $E\tilde{y}$, $k\tilde{\varepsilon}$ and non-random initial wealth w, the risk premium $\pi(k)$ must satisfy the following condition:

$$u\left(w+Ek\tilde{\varepsilon}-\pi(k),E\tilde{y}\right)=Eu\left(w+k\tilde{\varepsilon},E\tilde{y}\right).$$

Segal and Spivak (1990) give the following definitions of first and second-order risk aversion:

Definition 2.1 *(Segal and Spivak, 1990) The agent's attitude towards risk at w is of first order if for every* $\tilde{\varepsilon}$ *with* $E\tilde{\varepsilon}=0$, $\pi'(0)\neq 0$.

Definition 2.2 *(Segal and Spivak, 1990) The agent's attitude towards risk at w is of second order if for every* $\tilde{\varepsilon}$ *with* $E\tilde{\varepsilon}=0$, $\pi'(0)=0$ *but* $\pi''(0)\neq 0$.

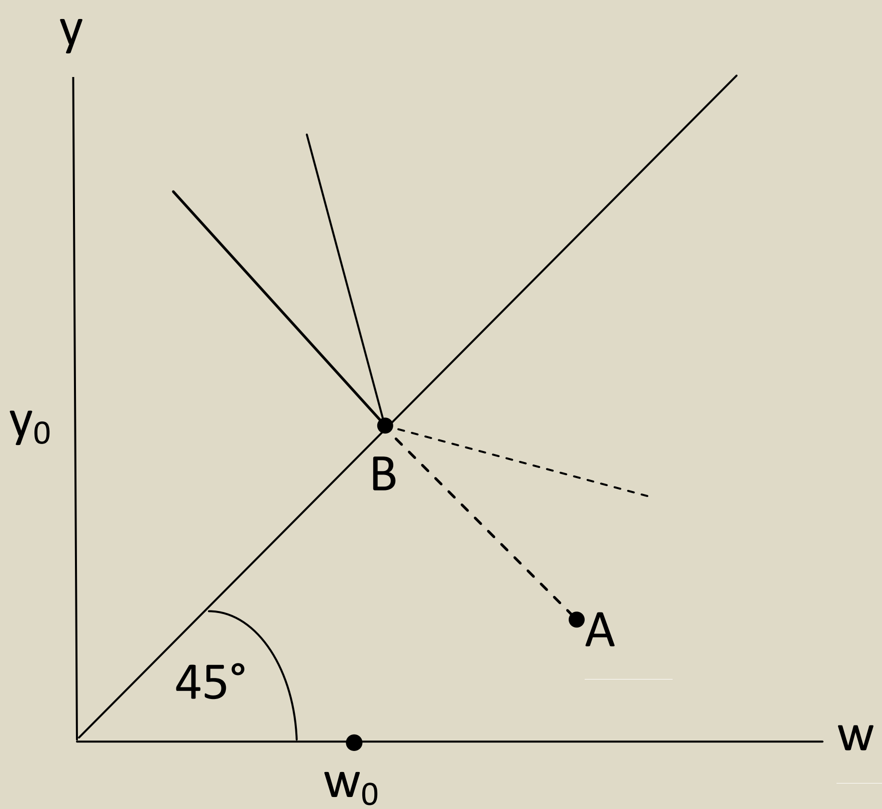
(a) *If a risk averse von Neumann-Morgenstern utility function u is not differentiable at w but has well-defined and distinct left and right derivatives at w, then the agent exhibits first-order risk aversion at w.*

$$\left.\frac{\partial \pi}{\partial k}\right|_{k=0^+} = \left[1 - \frac{U'_+(w)}{U'_-(w)}\right] \int_{\varepsilon>0} \varepsilon dF(\varepsilon)$$

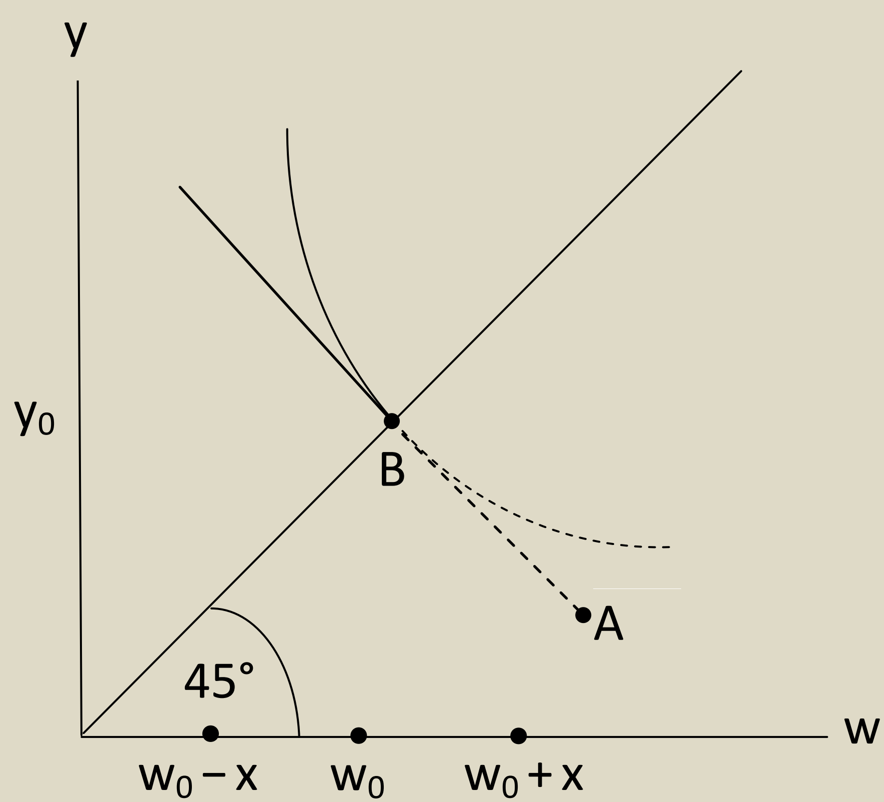
If not differentiable at w and risk averse, $U'_+(w) < U'_-(w)$ and $\frac{\partial \pi}{\partial k} \neq 0$.

(b) *If a risk averse von Neumann-Morgenstern utility function u is twice differentiable at w with $u_{11} \neq 0$, then the agent exhibits only second-order risk aversion at w.*

Suppose $w = w_0$.



Order 1



Order 2

In our approach, we suppose that *w* is random with, for example, $\left(w_0 - x, w_0 + x\right)$.

Suppose that risk *ke* is added to risk *w* and the two are not independent. For example, suppose that $w = \left(w_0 - x, H; w_0 + x, T\right)$ and $ke = \left(-k, H; k, T\right)$ (same *H* and *T*).

When we take the derivative of the risk premium, we take derivatives of the utility function at two different points: $w_0 - x$ and $w_0 + x$. Since these derivatives are typically different, the derivative of the risk premium with respect to *k* will not be zero but a function of the difference between the two derivatives of the utility function.

Order of conditional risk aversion

For an agent with utility function *u* and initial wealth *w*, the conditional risk premium $\pi_c(k)$ must satisfy the following condition:

$$Eu\left(w + Ek\tilde{\varepsilon} - \pi_c(k), \tilde{y}\right) = Eu\left(w + k\tilde{\varepsilon}, \tilde{y}\right)$$

where $\tilde{\varepsilon}$ and $\tilde{y}$ are independent.

Definition 3.1 *(Loomes and Segal, 1994) The agent's attitude towards risk at w is first-order conditional risk aversion if for every* $\tilde{\varepsilon}$ *with* $E\tilde{\varepsilon} = 0,\ \pi_c^{\prime}(0) \neq 0.$

Definition 3.2 *(Loomes and Segal, 1994) The agent's attitude towards risk at w is second-order conditional risk aversion if for every* $\tilde{\varepsilon}$ *with* $E\tilde{\varepsilon} = 0,\ \pi_c^{\prime}(0) = 0$ *but* $\pi_c^{\prime\prime}(0) \neq 0.$

Order of conditional dependent risk aversion

For an agent with utility function *u* and initial wealth *w*, the conditional dependent risk premium $\pi_{cd}(k)$ must satisfy the following condition:

$$Eu\left(w + Ek\tilde{\varepsilon} - \pi_{cd}(k), \tilde{y}\right) = Eu\left(w + k\tilde{\varepsilon}, \tilde{y}\right)$$

where $\tilde{\varepsilon}$ and $\tilde{y}$ are not necessarily independent.

Definition 4.1 *The agent's attitude towards risk at w is first-order conditional dependent risk aversion if for every* $\tilde{\varepsilon}$, $\pi_{cd}(k) - \pi_c(k) = O(k)$.

Definition 4.2 *The agent's attitude towards risk at w is* second-*order conditional dependent risk aversion if for every* $\tilde{\varepsilon}$, $\pi_{cd}(k) - \pi_c(k) = O(k^2)$.

Expectation dependence

Wright (1987) introduced the concept of expectation dependence in the economic literature. In the following definition, we use this definition of expectation dependence $\left(ED\left(y\right)\right)$.

Definition 4.3 *If*

$$ED\left(y\right)=\left[E\tilde{\varepsilon}-E\left(\tilde{\varepsilon}\middle|\tilde{y}\leq y\right)\right]\geq 0 \textit{ for all } y,$$

and there is at least some y_0 for which a strong inequality holds,

then $\tilde{\varepsilon}$ is positive expectation dependent on $\tilde{y}$. Similarly, $\tilde{\varepsilon}$ is negative expectation dependent on $\tilde{y}$ if the equation above holds with the inequality sign reversed. The family of all distributions F satisfying the equation will be denoted by H_1 and the family of all negative expectation dependent distributions will be denoted by I_1.

Lemma 4.4

$$\pi_{cd}(k) = -k \frac{\int_{-\infty}^{\infty} ED(y) u_{12}(w,\tilde{y}) F_y(y) dy}{Eu_1(w,\tilde{y})} + O(k^2).$$

If $ED(y) = 0$ *for all y,* $\pi_{cd}(k) = O(k^2)$.

If $u_{12} = 0$ *for all w,y,* $\pi_{cd}(k) = O(k^2)$.

Proposition 4.5

(i) *If* $\tilde{\varepsilon}$ *is positive expectation dependent on* $\tilde{y}$ *and* $u_{12} < 0$*, then the agent's attitude towards risk is first-order conditional dependent risk aversion and* $\pi_{cd}(k) - \pi_c(k) = |O(k)|$*;*

(ii) *If* $\tilde{\varepsilon}$ *is negative expectation dependent on* $\tilde{y}$ *and* $u_{12} > 0$*, then the agent's attitude towards risk is first-order conditional dependent risk aversion and* $\pi_{cd}(k) - \pi_c(k) = |O(k)|$*;*

(iii) *If* $\tilde{\varepsilon}$ *is positive expectation dependent on* $\tilde{y}$ *and* $u_{12} > 0$*, then the agent's attitude towards risk is first-order conditional dependent risk aversion and* $\pi_{cd}(k) - \pi_c(k) = -|O(k)|$*;*

(iv) *If* $\tilde{\varepsilon}$ *is negative expectation dependent on* $\tilde{y}$ *and* $u_{12} < 0$*, then the agent's attitude towards risk is first-order conditional dependent risk aversion and* $\pi_{cd}(k) - \pi_c(k) = -|O(k)|$*.*

Example 1

Consider the additive background risk case:

$$u(x,y) = U(x+y).$$

Example 2

Consider the multiplicative background risk case:

$$u(x,y) = U(xy).$$

In both cases, we need $U_{11} \neq 0$ instead of $u_{12} \neq 0$.

First-order conditional dependent risk aversion and N^th-order expectation dependent background risk

Proposition 5.3

(i) *If* $(\tilde{\varepsilon},\tilde{y}) \in H_N$ *and* $(-1)^m u_{12^{(m-1)}} \le 0$ *for* $m = 1,2,\ldots,N+1$*, then the agent's attitude towards risk is first-order conditional dependent risk aversion and* $\pi_{cd}(k) - \pi_c(k) = |O(k)|$*;*

(ii) *If* $(\tilde{\varepsilon},\tilde{y}) \in I_N$ *and* $(-1)^m u_{12^{(m-1)}} \ge 0$ *for* $m = 1,2,\ldots,N+1$*, then the agent's attitude towards risk is first-order conditional dependent risk aversion and* $\pi_{cd}(k) - \pi_c(k) = |O(k)|$*;*

(iii) *If* $(\tilde{\varepsilon},\tilde{y}) \in H_N$ *and* $(-1)^m u_{12^{(m-1)}} \geq 0$ *for* $m = 1,2,...,N+1$, *then the agent's attitude towards risk is first-order conditional dependent risk aversion and* $\pi_{cd}(k) - \pi_c(k) = -|O(k)|$;

(iv) *If* $(\tilde{\varepsilon},\tilde{y}) \in I_N$ *and* $(-1)^m u_{12^{(m-1)}} \leq 0$ *for* $m = 1,2,...,N+1$, *then the agent's attitude towards risk is first-order conditional dependent risk aversion and* $\pi_{cd}(k) - \pi_c(k) = -|O(k)|$.

Applications: The importance of background risk in risk diversification and portfolio choice

The effect of introducing a background risk on equilibrium asset price

Let P represent the price of the risky asset and β denote the demand for additional units of $\tilde{x}$. We assume the agent faces the following optimization program:

$$\beta^* \in \arg\max_{\beta} Eu\left(w + \tilde{x} + \beta\left(\tilde{x} - P\right)\right).$$

Gollier and Schlesinger (2002) show that the equilibrium asset price (at $\beta^* = 0$) is

$$P^* = \frac{E\left[\tilde{x}u'\left(w + \tilde{x}\right)\right]}{Eu'\left(w + \tilde{x}\right)}.$$

One way to explain the equity premium puzzle in the theoretical model is to recognize that there are other sources of risk on final wealth than the riskiness of assets returns.

To capture the effects of these types of risks, we introduce a labor income risk, $\tilde{y}$, which cannot be fully insured. This yields the following organization program:

$$\beta^* \in \arg\max_{\beta} Eu\left(w + \tilde{x} + \beta\left(\tilde{x} - P\right) + \tilde{y}\right)$$

and the modified equilibrium asset price:

$$P^{**} = \frac{E\left[\tilde{x}u'\left(w + \tilde{x} + \tilde{y}\right)\right]}{Eu'\left(w + \tilde{x} + \tilde{y}\right)}$$

We want to compare *P*** with *P**.

Proposition 5.4

Define $\tilde{x} = \tilde{x} + k\tilde{\varepsilon}$ *with* $E\tilde{\varepsilon} = 0$. *Suppose* $\tilde{x} > 0$ *and* $\tilde{y} > 0$ *almost surely.*

(i) *If* $\tilde{\varepsilon}$ *and* $\tilde{y}$ *are independent, then* $P^{**} - P^{*} = O\left(k^2\right)$.

This is the result of Constantinides and Duffie (JPE, 1996) and Krueger and Lusing (JET, 2010).

(ii) *If* $\tilde{\varepsilon}$ *is positive expectation dependent on* $\tilde{y}$ *and relative prudence coefficient is larger than 2* $\left(-x\frac{u^{///}(x)}{u^{//}(x)} \geq 2 \text{ for } \forall x \right)$, *then* $P^{**} - P^{*} = O(k)$.

Background risk and risk classification

Common wisdom suggests that diversification is a good way to reduce risk.

Consider a set of *n* lotteries whose net gains are characterized by $\tilde{\varepsilon}_1, \tilde{\varepsilon}_2, \ldots, \tilde{\varepsilon}_n$ which are assumed to be independent and identically distributed. Define the sample mean $\tilde{\varepsilon} = 1/n \sum_{i=1}^{n} \tilde{\varepsilon}_i$ then, when *w* is not random,

$$Eu\left(w + E\tilde{\varepsilon} - \pi_c\left(1/n\right), \tilde{y}\right) = Eu\left(w + \tilde{\varepsilon}, \tilde{y}\right),$$

where $\tilde{\varepsilon}$ and $\tilde{y}$ are independent, and

$$Eu\left(w + E\tilde{\varepsilon} - \pi_{cd}\left(1/n\right), \tilde{y}\right) = Eu\left(w + \tilde{\varepsilon}, \tilde{y}\right),$$

where $\tilde{\varepsilon}$ and $\tilde{y}$ are not necessary independent.

From

$$\pi_c(k) = \pi_c(0) + \pi_c'(0)k + O(k^2) = O(k^2),$$

we know that

$$\pi_c(1/n) = O(1/n^2).$$

When

$$n \to \infty,\ \pi_c(1/n) \to 0$$

because diversification is an efficient way to reduce risk.

Proposition 6.1

(i) *If* $\left(\tilde{\varepsilon},\tilde{y}\right)\in H_N$ *and* $\left(-1\right)^m u_{12^{(m-1)}}\leq 0$ *for* $m=1,2,...,N+1$*, then* $\pi_{cd}\left(1/n\right)=\left|O\left(1/n\right)\right|$;

(ii) *If* $\left(\tilde{\varepsilon},\tilde{y}\right)\in I_N$ *and* $\left(-1\right)^m u_{12^{(m-1)}}\geq 0$ *for* $m=1,2,...,N+1$*, then* $\pi_{cd}\left(1/n\right)=\left|O\left(1/n\right)\right|$;

(iii) *If* $\left(\tilde{\varepsilon},\tilde{y}\right)\in H_N$ *and* $\left(-1\right)^m u_{12^{(m-1)}}\geq 0$ *for* $m=1,2,...,N+1$*, then* $\pi_{cd}\left(1/n\right)=-\left|O\left(1/n\right)\right|$;

(iv) *If* $\left(\tilde{\varepsilon},\tilde{y}\right)\in I_N$ *and* $\left(-1\right)^m u_{12^{(m-1)}}\leq 0$ *for* $m=1,2,...,N+1$*, then* $\pi_{cd}\left(1/n\right)=-\left|O\left(1/n\right)\right|$.

Suppose that the government undertakes an investment with returns represented by $\tilde{B}$, which are independent of $\tilde{A}$. Let $\overline{B} = E\tilde{B}$ and $\tilde{X} = \tilde{B} - \overline{B}$.

Consider a specific taxpayer and denote his fraction of this investment by s with $0 \le s \le 1$. Suppose that each taxpayer has the same tax rate and that there are n taxpayers, then $s = 1/n$.

Arrow and Lind (1970) show that:

$$EU\left(\tilde{A} + \frac{\overline{B}}{n} + r(n)\right) = EU\left(\tilde{A} + \frac{\overline{B} + \tilde{X}}{n}\right)$$

where $r(n)$ is the risk premium of the representative individual.

Proposition 6.1 allows us to investigate the case where $\tilde{A}$ and $\tilde{B}$ are dependent Since the previous equation can be rewritten as:

$$EU\left(\tilde{A}+\frac{\overline{B}}{n}+r(n)\right)=EU\left(\tilde{A}+\frac{\tilde{B}}{n}\right),$$

from Proposition 6.1, we obtain:

Proposition 6.2

(i) *If* $\left(\tilde{B},\tilde{A}\right)\in H_N$ *and* $\left(-1\right)^m U^{(m)}\leq 0$ *for* $m=1,2,...,N+1$, *then* $r(n)=-\left|O(1/n)\right|$;

(ii) *If* $\left(\tilde{B},\tilde{A}\right)\in I_N$ *and* $\left(-1\right)^m U^{(m)}\leq 0$ *for* $m=1,2,...,N+1$, *then* $r(n)=\left|O(1/n)\right|$.

Stock market participation puzzle

Our results can offer a new explanation for the stock market participation puzzle by adding first-order risk aversion to the standard expected utility framework and by proposing expectation dependence, which is a more general definition of dependence than covariance.

Other examples

- Naïve diversified portfolio model
- Insurance supply
- Lottery supply

Conclusion

In this study, we have extended the concept of first-order conditional risk aversion to first-order conditional dependent risk aversion. We have shown that first-order conditional dependent risk aversion can appear in the framework of the expected utility function hypothesis and may explain the equity premium puzzle.

Recent studies show that background risk is significant to explain portfolio choices. The decision of a household to participate in the stock market is a function of many random factors such as labour income, housing risk, private business income, and health.

Our results can be used to explain these empirical results in this literature.