The Cake-Eating problem: Non-linear sharing rules

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Conference In Honor of Louis Eeckhoudt

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Outline of the talk

1 The model
Outline of the talk

1. The model
2. The stories
Outline of the talk

1. The model
2. The stories
3. The results
The Model
A program P with identical utility

\[
\max_x \sum_{i=1}^{n} a_i v(x_i)
\]

s.t. \( p'x = y \). (1)

- \( v : \mathbb{R}_+ \rightarrow \mathbb{R} \) strictly increasing and concave, satisfies "Inada conditions" and is the same for each attribute;
- The goods are ranked such that the "kernel prices" \( \frac{p_i}{a_i} \) are decreasing with \( i \)
The aim of the paper

The FOC

\[
\frac{v'(x_i^*)}{v'(x_j^*)} = \frac{p_i a_j}{p_j a_i} = \pi_{ij} \quad \forall i, j
\]

\[x_i^* < x_j^* \iff \pi_{ij} > 1 \Leftrightarrow i < j\] (2)

- Exploring integrability conditions

- How is the shape of the demand of the least demanded good related to the properties of the utility function?
Stories

- Individual wealth sharing:

  Arrow Debreu securities, Standard Portfolio (tax evasion n=2)
Individual wealth sharing:

Investor who allocates wealth over assets carrying different risk

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  - Individual deciding her optimal insurance coverage (n=2)
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- Group sharing problem(same utility but unequal weights)
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- Group sharing risks
Applications to individual decision-making

1- Arrow Debreu contingency claims

\[ y = \text{initial wealth}, \quad v = \text{state independent utility} \]

\[ x_1(y, p; a) = \text{demand of the contingent claim with "kernel price" } p_1 a. \]

2- Intertemporal consumption choice

Ingredients: Initial wealth \( y \), interest rate \( r \), intertemporal separable utility \( v(x_1) + \beta v(x_2) \) with discount factor \( \beta \).

\[ x_1 = 1 + \beta + x_2 \]

Peluso & Trannoy (Conference In Honor of Louis Eeckhoudt)
1- Arrow Debreu contingency claims

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- Initial wealth, \( y \)
- State independent utility \( v(x_1) + \beta v(x_2) \) with discount factor \( \beta \)
- Time weights prices \( p_1 \) and \( p_2 \)
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**Ingredients:** Initial wealth $y$, interest rate $r$, intertemporal separable utility $v(x_1) + \beta v(x_2)$ with discount factor $\beta \leq 1$. Then
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Intra-household allocation: No prices, Samuelson’s household welfare function.

$$\max_{x_1,x_2} av(x_1) + (1 - a)v(x_2)$$

s.t. \quad \begin{align*} p_1x_1 + p_2x_2 &= y \\ z_1(\theta) + z_2(\theta) &= y(\theta) = x_1(\theta) + x_2(\theta) \end{align*}$$
Group decision-making

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- **Risk-sharing:** \( \theta \in \Theta \) states of the world, risk: \( F: \Theta \rightarrow [0, 1] \), while \( v(x) \) are the identical vNM utility of the two individuals.

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\max_{x_1, x_2} \ a \int_{\Theta} v(x_1(\theta))dF(\theta) + (1 - a) \int_{\Theta} v(x_2(\theta))dF(\theta), \quad \text{with } a \in (0, \frac{1}{2}]
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s.t. \( z_1(\theta) + z_2(\theta) = y(\theta) = x_1(\theta) + x_2(\theta), \ \forall \theta \in \Theta; \quad x_1 \geq 0; \quad x_2 \geq 0. \)

Borch (1960): the consumption in each state of the world only depends on the total wealth in that state. Wealth is not transferable from one state to another.

Solving the risk-sharing problem then reduces to solve the intra-household allocation for any feasible \( y \).
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We normalize \( a = 1/2, \ p_1 = p > 1 \) and \( p_2 = 1 \). Then
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Let $h(x, p)$ be the demand of good 2 as a function of good 1 and $p$. 
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Let $h(x, p)$ be the demand of good 2 as a function of good 1 and $p$.

$h(x, p) = g(x, p) - px$, where $g(x, p)$ is the inverse function of $x(y, p)$ wrt $y$ using the fact that the two goods are normal.
Integrability conditions

Proposition

A function $x(y, \pi)$, strictly increasing with $y$ and decreasing with $p$ is a solution of program $P$ for all $y \in \mathbb{R}_+$ and for all $p > 1$, iff there exist a positive function $A(x)$ such that:

$$\frac{h_x(x, p)}{h_p(x, p)} = A(x)p$$

(3)

Then $A$ represents the Arrow-Pratt absolute risk aversion coefficient, that is $v'(x) = \exp \int_0^x A(s)ds$. 
Integrability conditions: examples

- \( x_1^*(y, p) = \frac{1}{2p} y^\gamma \), for \( \gamma < 1 \) does not satisfy the integrability conditions.

- If \( h(x, p) = (1 + x)^p - 1 \), we get \( \frac{h_x}{h_p} = \frac{p}{(1 + x) \ln(1 + x)} \). Then \( h \) is the solution of \( \mathbf{P} \) with the log-integral utility function
  \[
  v(x) = \int_0^x \frac{1}{\ln(1+s)} \, ds
  \]

- If \( h(x, p) = \ln(1 + e^x - p) - \ln p \), we get \( \frac{h_x}{h_p} = \frac{e^x}{1 + e^x} p \), solution of \( \mathbf{P} \) under the linex utility function \( v(x) = x - e^{-x} \).
Integrability without prices
The "group" case

- Group decision-making set-up: prices are fixed (eventually equal to 1) and weights are fixed
Integrability without prices

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- Does identical utility impose more restrictions on the class of non-linear sharing functions generated by $P$ beyond $x_1 < x_2$ for all $y$?
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- Answer: No for $n = 2$, Yes for $n > 2$ but $\ldots$

Proposition
For all $f(y)$ and $a \in (0, 1/2)$, there exists a continuous differentiable utility function $v$ such that, for all $y \in \mathbb{R}^+$, from Program (1) we get $x_1(y; a) = f(y)$.
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The sharing function

- A *sharing function* $f$ maps wealth $y$ into the quantity consumed or invested in one good $x_1 = f(y)$
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From $p_1 x_1 + p_2 x_2 = y$ we know $x_1 = x_2 \implies x_1 = \frac{y}{p_1 + p_2}$
Three classes of diverging sharing functions

Type 1: Class $\mathcal{M}$, or "Moving Away" sharing functions
Type 2: Class $\mathcal{P}$, or "progressive" sharing functions
Type 3: Class $C$, or "concave"
2.d Remark

The classes are nested

\[ C \subset P \subset M \]
A characterization result for the first good

Proposition

- Suppose that \( x_1^*(y; \cdot) \) is twice continuously differentiable. Then:
  
  i) \( \nu \in \text{DARA} \iff x_1^*(y; \cdot) \in M \) for all \( \pi \geq 1 \)
  
  ii) \( \nu \in \text{DRRA} \iff x_1^*(y; \cdot) \in P \) for all \( \pi \geq 1 \)
  
  iii) \( \nu \in \text{CT} \iff x_1^*(y; \cdot) \in C \) for all \( \pi \geq 1 \)

Where

- DARA = Decreasing Absolute Risk Aversion
- DRRA = Decreasing Relative Risk Aversion
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Proposition (3bis)

Let $P$ represent an intertemporal consumption choice, with $n = T$ periods and initial wealth $y$. Let us consider the associated dynamic programming problem where at time $t$ the consumer chooses the optimal consumption pattern $c_t, c_{t+1}, ..., c_T$ of the remaining $T - t$ periods as a function of the current wealth $y_t$. Then the conditions of the previous proposition apply to the sharing function linking the current consumption $c_t$ to the current wealth $y_t$ for each period $t = 1, ..., T - 1$. 
Among CT utility functions, an interesting and general family: *linHARA* utility functions, obtained by adding a linear term to HARA utility functions.
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- The *linex* $v(x) = \alpha x - e^{-\beta x}$ is well known in the risk literature.
- *linpower* $v(x) = \frac{k}{1-a} x^{1-a} + bx$, with parameters $a > 1$, $b$ and $k > 0$.

(the corresponding $h(x, p) = x \left[ \frac{pk}{k - (\lambda - 1)bx^a} \right]^{\frac{1}{a}}$ is bounded)

\[ x < \left( \frac{k}{(p-1)b} \right)^{\frac{1}{a}} \]
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- linlog utility function $v(x) = \alpha x + \beta \log x$. 
Applications: Individual Choice
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- Results:
  - $v \in DARA \iff x_2^* - x_1^*$ is increasing with $y$
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- $y =$ initial wealth, $v =$ state independent utility
- $x_1^*(y, p; a) =$ demand for the contingent claim with the highest "kernel price" $\frac{p_1}{a}$.

Results:
- $v \in DARA \iff x_2^* - x_1^*$ is increasing with $y$
- $v \in DRRA \iff \frac{p_1 x_1^*}{y}$ is decreasing with $y$
Applications: Individual Choice

1- Arrow Debreu contingency claims

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- \( y \) = initial wealth, \( v \) = state independent utility
- \( x_1^*(y, p; a) \) = demand for the contingent claim with the highest "kernel price" \( \frac{p_1}{a} \).

Results:
- \( \nu \in DARA \iff x_2^* - x_1^* \) is increasing with \( y \)
- \( \nu \in DRRA \iff \frac{p_1 x_1^*}{y} \) is decreasing with \( y \)
- \( \nu \in CT \iff x_1^* \) is concave in \( y \) (the marginal share of the less demanded attribute decreases with wealth)
Individual choice

Insurance

- Initial wealth $Y$; risk of a loss $-X$ in state 1 with probability $a$. 

Insurance contract where $0 < C < X$. The premium $\beta C$ is proportional to the coverage, with $\beta < 1$. 

Results: $v_2(\cdot)(\cdot)$ is increasing with $\cdot$, proportion of uninsured wealth is increasing with $\cdot$, uninsured wealth is concave with $\cdot$.
Individual choice

Insurance

- Initial wealth $Y$; risk of a loss $-X$ in state 1 with probability $a$.
- Insurance contract where $0 \leq C \leq X$. 

Results:

$\nu_2(DRA)(z_1)$ is increasing with $y$

$\nu_2(DRA)$ proportion of uninsured wealth is increasing with $y$

$\nu_2(CT)(uinsured)$ wealth is concave with $y$. 

Peluso & Trannoy (Conference In Honor of Louis Eeckhoudt)
Initial wealth $Y$; risk of a loss $-X$ in state 1 with probability $a$.

Insurance contract where $0 \leq C \leq X$.

The premium $\beta C$ is proportional to the coverage, with $\beta < 1$. 
Individual choice

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- Uninsured loss $z_1 = x_2 - x_1$
Individual choice

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- Initial wealth $Y$; risk of a loss $-X$ in state 1 with probability $a$.
- Insurance contract where $0 \leq C \leq X$.
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Uninsured loss $z_1 = x_2 - x_1$

Results:

$\nu \in DARA \iff z_1^* \text{ is increasing with } y$
Individual choice

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- $v \in DRRA \iff$ proportion of uninsured wealth is increasing with $y$
Individual choice

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3.4 Individual Choice
4- Intertemporal Consumption

- **Given the model**

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The initial condition $\lambda = \frac{p_1(1-a)}{p_2a} \geq 1$ becomes $\beta \geq \frac{1}{1+r}$. The marginal opportunity cost of saving is lower than the intertemporal MRS $\implies$ lower consumption in the first period.
3.4 Individual Choice
4- Intertemporal Consumption

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- Results:
  - $\nu \in DARA \iff$ saving increasing with $y$
### 3.4 Individual Choice

#### 4- Intertemporal Consumption

- **Given the model**

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- **Results:**
  - \( v \in DARA \iff \) saving increasing with \( y \)
  - \( v \in DRRA \iff \) decreasing average propensity to consume with wealth (Keynes)
3.4 Individual Choice

4- Intertemporal Consumption

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- **Results:**
  - $v \in DARA \iff$ saving increasing with $y$
  - $v \in DRRA \iff$ decreasing average propensity to consume with wealth (Keynes)
  - $v \in CT \iff x_1^*$ is concave with $y$
Samuelson’s household welfare function, with balance of power among the members given by $a$. 
Samuelson’s household welfare function, with balance of power among the members given by \( a \).

If individual 1 is the "weaker" individual \( (a \leq \frac{1}{2}) \) then \( x_1^*(y, a) \leq \frac{1}{2} y \).
Samuelson’s household welfare function, with balance of power among the members given by $a$.

If individual 1 is the "weaker" individual ($a \leq \frac{1}{2}$) then $x_1^*(y, a) \leq \frac{1}{2} y$.

Immediate interpretation of the Proposition 1, for the risk-sharing too.