What is a divisible good auction?

- Auction of large quantities, where bidders can be thought of as bidding for a “share” of the quantity offered for sale rather than the actual number of units
- “Share” of the good is a continuous choice variable
- Bidders submit whole demand (bid) functions
- Government securities, electricity, IPOs, emission permits

Objectives of this paper

- Wilson’s model is too restrictive to fit the data: Need a model yielding equilibria in step functions
- Investigate the extent to which the restrictions of the model with continuous bid functions matter
- Structural estimation using proper link between the primitives of the model and the data
- Provide a method for evaluating the performance of an auction mechanism based on data on individual bids

Literature

- Theory:
- Reduced form estimation:
  - Umlauf (1993), Nyborg and Sundaresan (1996)
- Structural approach:
Why a different model?

- Step functions observed since upper bound on number of bidpoints imposed by auctioneers. (Exogenous restriction of the strategy set)

- Moreover: bidders never approach this bound, and hence intentionally submit step functions.

- How to explain the data generating process if the equilibrium of the assumed underlying model is in continuously differentiable functions?

- No rationing in the traditional model. (with step functions it occurs with probability one)

Basic divisible good auction model w/ private info and values

- Marginal valuation function for bidder $i$: $v_i(q, s_i)$ where $s_i \sim F_i(s_i)$; $s_i \in \mathbb{R}$

- Total quantity normalized: $Q = 1$

- Submit a bid function: $y_i(p | s_i)$ specifying a share $y_i \in [0, 1]$ that type $s_i$ requests at price $p$

- Auctioneer aggregates the bids to determine the aggregate bid function $\sum_i y_i(p | s_i)$

- Market clearing price is determined, and payments and allocations are made according to auction rules

Bidder’s problem
Uniform price auctions: Wilson’s Approach

- BNE (bidders strategies restricted to be continuously differentiable functions) characterized by:
  \[ v_i(y_i(p|s_i), s_i) = p + \text{Nonnegative Shading Term} \]

- Notice: a UPA in which bidders bid truthfully their marginal valuation schedules constitutes an upper bound on the revenue of any equilibrium of a UPA characterized above - \( p \) is the bid for share \( y_i(p|s_i) \)

K-step equilibrium characterization

- Theorem (Characterization): In a UPA with private values, and rationing pro-rata on the margin, in any \( K \)-step equilibrium the quantity requested at \( k \)th bidpoint has to satisfy:
  \[ v_i(q_k, s_i) = \mathbb{E} \left[ p | p_k > p > p_{k+1} \right] + \frac{q_k}{\Pr(p_k > p > p_{k+1})} \frac{\partial \mathbb{E}[P; p_k > p > p_{k+1}]}{\partial q_k} \]

- Bidder is (almost) like an oligopolist facing a random (residual) demand who has to commit to one quantity
  \[ MC = \mathbb{E}(MR) = \mathbb{E} \left[ P(q) + qP'(q) \right] \]

K-step equilibrium of a UPA with private values

- BNE: \( y_i(p|s_i, t_i) : S \times T \rightarrow \mathcal{Y}_i^{K_i} \)
  \( \mathcal{Y}_i^{K_i} \) set of left-continuous step functions with at most \( K \) steps

- \( K_i \) can differ across bidders

- Cost \( c(K, t) \) where \( t \sim G(t|s) \) is private info

- Rationing rule: pro-rata on the margin
  If excess demand, marginal bidders with bids exactly at the market clearing price will be rationed proportionally. (Bids above market clearing price are given priority.)

Intuition
Major differences from the model with continuous bids

- Different optimality condition, which is used for empirical identification

- Ex post revenue in a UPA is NOT bounded by the revenue from a UPA in which each bidder bids his true marginal valuation schedule:

\[ v_i(q_k, s_i) = E [p | p_k > p > p_{k+1}] + \frac{q_k}{\Pr (p_k > p > p_{k+1})} \frac{\partial E [p_k | p_k > p > p_{k+1}]}{\partial q_k} \]

\[ v_i(q_k, s_i) = p_k + \text{Nonnegative Shading Term} \]

Estimation Strategy

- optimality condition approach - obtain consistent estimates of all pieces to obtain an estimate of marginal valuation at the submitted bid

\[ v_i(q_k, s_i) = E [p | p_k > p > p_{k+1}] + \frac{q_k}{\Pr (p_k > p > p_{k+1})} \frac{\partial E [p_k | p_k > p > p_{k+1}]}{\partial q_k} \]

- Suppose we observe \( \{ y(p | s_1, t_1), ..., y(p | s_N, t_N) \} \) for each auction and data are generated by a symmetric K-step equilibrium behavior.

- A bid function \( y(p | s_1, t_1) \) is the same bid function any bidder would submit had he drawn the type \( (s_1, t_1) \).

- Estimate \( E [p | p_k > p > p_{k+1}] \) by resampling bid functions

Resampling Procedure

- fix a bidder

- draw \( N - 1 \) bid functions (with replacement)

- construct the residual supply and obtain the corresponding market clearing price

Distribution of the market clearing price
Auctions of Czech T-bills: Rules

- uniform price auctions
- rationing rule: pro-rata on-the-margin
- maximum number of bidpoints allowed is 10
- noncompetitive bids allowed (explained later)
- face value: 1,000,000 CZK
- auctions conducted every Wednesday
- auction plan announced quarterly

Bidders

- bidders (mostly banks) must be registered with the auctioneer before the auction
- banking or broker license within EU required for registration
- one bidder can acquire at most 50% of the supply in any given auction
- minimum buying limit for a calendar year - never binding, satisfied much earlier

Private values?

- major reason for interest in the T-bill auctions: reserve requirements on risky investment
- banks buy T-bills for their portfolios to obtain some return on their cash reserves
- secondary market for T-bills virtually nonexistent
- banks have private information about their liquidity positions and investment opportunities
- if there is a common value component, it is (almost) perfectly known
- Independent private values? - I test for affiliation

Data

- on average 13 active bidders
- each bidder submits on average 2.3 bidpoints (maximum is 9)
- bid in terms of the annual yield varies from 4.99% to 5.65%
- market clearing yield varies from 5.22% to 5.54%
- for estimation using resampling we need larger number of bidders - I will assume 4 neighboring auctions are repetitions of the same experiment and group the bids from those together (test for potential problems)
bidders seem to be asymmetric and can be split into two groups according to size

<table>
<thead>
<tr>
<th></th>
<th>Mean Large</th>
<th>Mean Small</th>
</tr>
</thead>
<tbody>
<tr>
<td>Active Bidders in an Auction</td>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td>Number of Submitted Bidpoints</td>
<td>2.88</td>
<td>1.59</td>
</tr>
<tr>
<td>Price Bids(^a)</td>
<td>5.30</td>
<td>5.30</td>
</tr>
<tr>
<td>Quantity Bids(^b)</td>
<td>0.077</td>
<td>0.02</td>
</tr>
</tbody>
</table>

\(^a\) In terms of the annual yield of T-bills
\(^b\) As a share of total quantity offered for sale

Commitment to buy \(q\) at the market clearing price (supply reduction)

- not used by regular bidders
- auctioneer reserves the right to buy some or all of the T-bills offered for sale for his own portfolio
- actual noncompetitive bid ranges from 0 to 75% of T-bills offered (on average 36%)

For estimation purposes, I will treat the bids on behalf of the auctioneer as a separate bidder group (might be problematic if signals affiliated - I test for this)

Example of the estimation results for individual bidders

<table>
<thead>
<tr>
<th>Auction</th>
<th>Actual yield</th>
<th>Highest yield(^a)</th>
<th>Lowest yield(^b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>94(^a)</td>
<td>5.26</td>
<td>5.31</td>
<td>5.30</td>
</tr>
<tr>
<td>95(^a)</td>
<td>5.28</td>
<td>5.34</td>
<td>5.34</td>
</tr>
<tr>
<td>107</td>
<td>5.41</td>
<td>5.40</td>
<td>5.40</td>
</tr>
<tr>
<td>108</td>
<td>5.40</td>
<td>5.40</td>
<td>5.38</td>
</tr>
<tr>
<td>Mean (all)</td>
<td>5.31</td>
<td>5.32</td>
<td>5.30</td>
</tr>
</tbody>
</table>

\(^a\) Ex post revenue higher than under truthful bidding
\(^b\) Achieved by bidding the upper envelope of marginal valuations

Suppose we run a uniform price auction and bidders submit their marginal valuation schedules as bids

- use the upper envelope of marginal valuations
Truthful bidding

- Results: in 7 out of 28 auctions, ex post revenue is higher than the revenue achieved under truthful bidding.

- Therefore using the indirect comparison to argue that discriminatory auction performs better in terms of revenue may not be correct.

Performance of the mechanism

- ideal mechanism (1\textsuperscript{st} best world): 1) extracts all value given an allocation, 2) implements the efficient allocation

- if efficiency $\approx 1$ and bidders’ interim profits $\approx 0$, then the mechanism performs well

- use estimated distribution of the market clearing price and marginal valuation to estimate bidders’ interim profits

- estimate efficiency: \( \frac{\text{realized surplus}}{\text{efficient surplus}} \)

Table: Interim profit of bidders per T-bill for sale

<table>
<thead>
<tr>
<th>Auction</th>
<th>Int. Profit (^b)</th>
<th>Average</th>
<th>Max</th>
<th>Min</th>
<th>Total</th>
<th>Efficiency (^c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>94(^a)</td>
<td></td>
<td>2.83</td>
<td>14.86</td>
<td>-0.01</td>
<td>28.27</td>
<td>0.99993</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.47)</td>
<td>(14.33)</td>
<td>(0.41)</td>
<td>(14.73)</td>
<td>(6\times10^{-7})</td>
</tr>
<tr>
<td>95(^a)</td>
<td></td>
<td>8.88</td>
<td>70.76</td>
<td>-0.82</td>
<td>88.81</td>
<td>0.99999</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(2.51)</td>
<td>(22.15)</td>
<td>(1.97)</td>
<td>(25.08)</td>
<td>(2\times10^{-7})</td>
</tr>
<tr>
<td>107</td>
<td></td>
<td>0.54</td>
<td>4.27</td>
<td>-0.01</td>
<td>6.98</td>
<td>0.99998</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.18)</td>
<td>(1.98)</td>
<td>(0.21)</td>
<td>(2.32)</td>
<td>(2\times10^{-6})</td>
</tr>
<tr>
<td>108</td>
<td></td>
<td>4.13</td>
<td>32.62</td>
<td>-0.21</td>
<td>53.64</td>
<td>0.99995</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(14.22)</td>
<td>(181.4)</td>
<td>(0.02)</td>
<td>(184.83)</td>
<td>(2\times10^{-6})</td>
</tr>
<tr>
<td>Mean (Auctions 52-108)</td>
<td></td>
<td>5.07</td>
<td>43.24</td>
<td>-0.20</td>
<td>66.14</td>
<td>0.9999</td>
</tr>
</tbody>
</table>

\(^a\) Ex post revenue was higher than under truthful bidding

\(^b\) Using the upper envelope of marginal valuations

\(^c\) Defined as \( \frac{\text{Actual surplus}}{\text{Surplus from the efficient allocation}} \)

- employed mechanism performs quite well: loss versus the ideal mechanism is less than 8 basis points

- 223,000 T-bills sold to regular bidders during the sample period $\Rightarrow$ total loss worth less than 22 T-bills (0.01%)
Bidder Asymmetry

Affiliation of signals

- affiliation of signals - if signals affiliated, then the distribution of the noncompetitive bid would differ depending on the signal received

Table: Wilcoxon Rank Sum Test of Equality of Distributions $F_{s_1 \mid s_2}$

<table>
<thead>
<tr>
<th>Auctions — Sample split</th>
<th>1, 2</th>
<th>3, 4</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>{52, 55, 56, 60}</td>
<td>0.85</td>
<td>0.45</td>
<td></td>
<td></td>
</tr>
<tr>
<td>{61, 64, 65, 67}</td>
<td>0.78</td>
<td>0.88</td>
<td></td>
<td></td>
</tr>
<tr>
<td>{69, 72, 73, 75}</td>
<td>0.25</td>
<td>0.38</td>
<td></td>
<td></td>
</tr>
<tr>
<td>{76, 81, 82, 85}</td>
<td>0.12</td>
<td>0.05</td>
<td></td>
<td></td>
</tr>
<tr>
<td>{86, 87, 91, 92}</td>
<td>0.30</td>
<td>0.83</td>
<td></td>
<td></td>
</tr>
<tr>
<td>{94, 95, 99, 100}</td>
<td>0.40</td>
<td>0.85</td>
<td></td>
<td></td>
</tr>
<tr>
<td>{103, 104, 107, 108}</td>
<td>0.82</td>
<td>0.72</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* p-values of $H_0$: Samples are from the same continuous distribution.

Effect of quantity won on signals

- relationship between quantity won in auction $t$ and value for units in auction $t + 1$

Table: Testing dependence of signals and quantities won earlier

<table>
<thead>
<tr>
<th>Auditions</th>
<th>{52 – 60}</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>986,477.4</td>
<td>990,817.5</td>
</tr>
<tr>
<td></td>
<td>(49.93)</td>
<td>(1625.5)</td>
</tr>
<tr>
<td>$q_{t-1}$</td>
<td>885.6</td>
<td>-13576.3</td>
</tr>
<tr>
<td></td>
<td>(514.8)</td>
<td>(15638.4)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.09</td>
<td>0.003</td>
</tr>
<tr>
<td>N</td>
<td>33</td>
<td>250</td>
</tr>
</tbody>
</table>

* Std. errors in parentheses

Important questions

- What role does the possibility of noncompetitive bids play in policing bidding behavior?
  - in electricity markets - withholding part of the demand is impossible - what to do then?
  - maybe the auctioneer should sign private (option) contracts with some generators before the auction, so that the possibility of withholding is there

- Are 3 points "enough" to capture "almost all" of the surplus from a multiunit auction?
  - Is there an analogy between this simple bidding and simple linear pricing menus?
Conclusion

- Steps matter! (otherwise overestimating marginal valuations, and thus results biased against the uniform price auction)
- Bids can exceed marginal valuations in a uniform price auction!
- Empirical:
  - a method for evaluation of the employed mechanism
  - uniform price auctions of Czech T-bills perform well
  - unextracted total surplus of the bidders is less than 3 basis points, while efficiency loss is less than 5 basis points