Optimal Investment under liquidity constraints *

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1 Introduction

Mathematical finance applied to investment theory gave rise about thirty years ago to a new field known as real options. In traditional real option models, the underlying state variable is the *observable* value of an investment project that could be undertaken at a fixed cost. The real option literature has emphasized the ability of firms to delay their irreversible investment decisions. In the presence of sunk costs, this flexibility in the timing of investment is valuable because it gives firms the option to wait for new information. As a result, optimal investment policies are mathematically determined as the solution to optimal stopping problems and prescribe to invest above the point at which expected discounted cash-flows cover sunk costs, in contrast with the usual net present value rule. The pioneered model is due to McDonald and Siegel [13] and has been extended in various ways by many authors (see for instance Dixit and Pyndick [8] for an overview of this literature). An important common feature of this literature is to assume that the investment decision can be made independently of the financing of the sunk cost. This amounts to consider that capital markets are perfect so that any project with positive net present value will find a funding (Modigliani and Miller [14]). However, capital markets are not perfect, external financing is costly and firms accumulate cash to cover investment needs without resorting to the market.[1] Despite strong empirical evidences, the real option literature has somewhat neglected market imperfections and, typically, the role of cash holdings in the firms' investment decision. Very few papers focus on the level of self-financing that a firm should optimally decide in a dynamic setting. A first attempt in that direction is Boyle and Guthrie [3]. More recently, Asvanunt, Broadie and Sundaresan [2] develop a corporate model with interactions between cash reserves and investment opportunity when the firm has some outstanding debt. Hugonnier, Malamud and Morellec [10]

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[1] Among many other papers, see for instance Bates, Kahle and Stulz [4].

considers the interactions between cash holdings, dividend distribution and capacity expansion when firms face uncertainty regarding their ability to raise external funds and have to pay a search cost to meet outside investors.

In this note, we try to merge the real option and the corporate finance literature by focusing on the optimal investment policy for a cash-constrained firm. More precisely, we make the strong assumption that the firm has no access to capital markets. As a consequence, the cash reserves of the firm must always remains non-negative to meet operating costs and the firm value is computed as the expected value of dividends payment. In this framework, when facing an investment opportunity, shareholders have both a profitability concern (the optimal time to undertake a growth opportunity) and a liquidity concern (the risk to be forced to liquidate a profitable project). The model presented above takes the result of Décamps and Villeneuve [6] and studies the consequences of liquidity constraints on the decision to invest in a new project.

2 Optimal Investment in perfect capital markets

2.1 The benchmark model

As a benchmark, we begin with the seminal model of McDonald and Siegel [13]. We start with a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, a filtration $(\mathscr{F}_t)_{t\geq 0}$ and a Brownian Motion $W = (W_t)_{t\geq 0}$ with respect to $\mathscr{F}_t$. We assume that a decision maker continuously observes the instantaneous cash-flow X of a project where $X = (X_t)_{t\geq 0}$ is a Geometric Brownian Motion with drift μ and volatility σ,

$$dX_t = \mu X_t dt + \sigma X_t dW_t.$$

We denote by r the constant risk-free rate and we assume that $\mu < r$. The decision maker's problem is to decide when to invest in this project at a fixed cost I. After the investment is made, the firm generates cash-flow forever. As a result, the sum of the discounted expected future cash-flows if investment is made at time t is

$$\mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} X_s \, ds \right] = \frac{X_t}{r-\mu}.$$

Thus framed, the decision maker's problem takes the form of an optimal stopping problem. Because X is the only state variable, the set of admissible strategies is the set of stopping times adapted to $\mathscr{F}_t$ denoted by $\mathscr{T}$. That is, the value function associated to the investment opportunity is defined as

$$V(x) = \sup_{\tau \in \mathscr{T}} \mathbb{E}_x \left[\frac{X_t}{r-\mu} - I \right].$$

Under this formulation, it is easy to prove that the optimal investment strategy belongs to the set of threshold strategies T_y where T_y is the hitting time of y by the process X. Specifically, the investment option should be exercised the first time that the value of the investment project exceeds a critical threshold, the so-called optimal exercise boundary. The exercise boundary can be explicitly computed using a standard verification theorem based on the smooth-fit principle (see for instance Dixit and Pyndick [8], Part III). This leads to an explicit expression for V,

$$V(x) = L(x,x^*)\left(\frac{x}{r-\mu} - I\right),$$

where $L(x,y) = \mathbb{E}_x(e^{-rT_y})$ and where x^* represents the optimal exercise boundary or the level of cash-flow above which it is optimal to invest. We have,

$$x^* = \frac{\xi}{\xi-1} I(r-\mu)$$

with

$$\xi = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right) + \frac{2r}{\sigma^2}},$$

and, due to the properties of the Geometric Brownian motion, $L(x,x^*) = \left(\frac{x}{x^*}\right)^{\xi}$.

2.2 Discussion

It is worth to mention that the seminal model of McDonald and Siegel is based on several implicit assumptions, two of which are particulary strong: perfect information on the future cash-flows and perfect capital markets. In particular, it is not necessary to assume that the decision maker has the possibility to self-finance the sunk cost I. In a perfect capital market, she has the possibility to access to outside financing by issuing shares. Let us describe a possible financing contract between the decision maker and an outside investor. Because X is perfectly observable, the expected profit Π if the investment is made at time t is denoted by $\Pi(X_t)$ with $\Pi(x) = \frac{x}{r-\mu}$. Therefore, the outside financier may propose the following contract: if the decision maker invests at a level y, the investor will ask for a proportion δ of shares that satisfies $\delta\Pi(y) = I$. Along this contract, the expected payoff for the decision maker will be $(1-\delta)\Pi(y) = \Pi(y) - I$. Therefore, the decision maker has to choose the optimal level y that maximizes her profit, that is,

$$\max_y L(x,y)(\Pi(y) - I),$$

which is equivalent to the decision problem of the benchmark model. Therefore, a decision maker that cannot afford to self-finance I invests optimally at the same level of investment x^* if she signs the contract described above. As a consequence, under the assumption of perfect capital markets, the investment decision is made independently of the financing decision which is in the spirit of the Modigliani-Miller theorem. The objective of this note is to relax the assumption of perfect capital markets and to illustrate the consequences of liquidity constraints on the investment decisions. Before developing our model, we emphasize that taking into account costly external financing may lead to challenging stopping problems. Let us consider a decision maker who needs to finance the investment cost I. Assume that banks are in perfect competition and offer consol bonds with the following covenant: if the borrower is unable to pay the coupon, the firm is forced to default. As a consequence, the market price at time t of the bond is

$$D(X_t,c) = \mathbb{E}_t\left[\int_t^{t+T_c\circ\theta_t} e^{-r(s-t)} c\, ds\right] = \frac{c}{r}\left(1-\left(\frac{X_t}{c}\right)^{\alpha}\right) 1\!\!1_{\{X_t\geq c\}}$$

where

$$\alpha = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)+\frac{2r}{\sigma^2}}.$$

Consequently, the expected project's payoff for the decision-maker investing at time t by entering in the debt contract with covenant to finance the sunk cost I is

$$\Pi_t = \mathbb{E}_t\left[\int_t^{t+T_c\circ\theta_t} e^{-r(s-t)}(X_s - c)\, ds\right],$$

where θ_t is the shift operator[2] and where c has to be computed so that $D(x,c)\geq I$, otherwise the debtholders refuse to lend I. Because the competition between banks is assumed to be perfect, we must have $D(X_t,c) = I$ if the investment is made at time t . The participation constraint $D(x,c) = I$ has several important consequences. We observe that for a fixed x, the function $D(x,.)$ defined on $[0,x]$ is convex with $D(x,0) = D(x,x) = 0$. thus, $D(x,.)$ reaches a maximum at $c^*(x) = (1-\alpha)^{-\frac{1}{\alpha}}x$ and therefore the participation constraint is satisfied if and only if

$$D(x,c^*(x)) = -\alpha(1-\alpha)^{-(\frac{1}{\alpha}+1)}\frac{x}{r} \geq I. \tag{2.1}$$

For any x satisfying the participation constraint (2.1), there are two levels c_1, c_2 with $c_1 \leq c^* \leq c_2$ for which $D(x,c_i) = I$. It is obvious that the decision-maker will choose the smallest coupon c_1 and thus $\Pi(X_t)$ can be expressed as

$$\Pi(x) = \mathbb{E}\left[\int_0^{T_{c_1}} e^{-rs}X_s\, ds\right] - I = \frac{x}{r-\mu} - \frac{c_1}{r-\mu}\left(\frac{x}{c_1}\right)^{\alpha}.$$

Using $\frac{c_1}{r}\left(1-\left(\frac{x}{c_1}\right)^{\alpha}\right) = I$, we may rewrite Π as

$$\Pi(x) = \frac{x-c_1(x)}{r-\mu} - \frac{rI}{r-\mu}$$

and thus, the decision-maker has to solve the optimal stopping problem

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}_x\left[e^{-r\tau}\Pi(X_\tau)\right]. \tag{2.2}$$

To the best of our knowledge, solving (2.2) remains an open question. In particular, there is no guarantee that the optimal stopping time is a threshold strategy. To circumvent the difficulty, we will assume that the decision-maker can store her cash-flow X and thus control the payment process. This is the topic of the next section.

[2] See for instance Revuz and Yor [19] page 36 for the definition of the shift operator.

3 Optimal stopping for a cash-constrained firm

3.1 The model

We consider a firm with an activity in place that generates a cash-flow process. The firm faces liquidity constraints because it has no access to capital markets. Consequently, the firm defaults as soon as the cash process hits the threshold 0. The manager of the firm acts in the best interest of its shareholders and maximizes the expected present value of dividends up to default. At any time the firm has the option to invest in a real option that increases the drift of the cash generating process from μ_0 to $\mu_1 > \mu_0$ without affecting its volatility σ. This growth opportunity requires a fixed investment cost I that must be financed only by using the cash reserve.

The mathematical formulation of our problem is as follows. We start with a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, a filtration $(\mathscr{F}_t)_{t\geq 0}$ and a Brownian Motion $W = (W_t)_{t\geq 0}$ with respect to $\mathscr{F}_t$. In the sequel, $\mathscr{Z}$ denotes the set of positive non-decreasing right continuous and $\mathscr{F}_t$-adapted processes and $\mathscr{T}$, the set of $\mathscr{F}_t$-adapted stopping times. A control policy $\pi = (Z_t^\pi, \tau^\pi; t \geq 0)$ models a dividend/investment policy and is said to be admissible if Z_t^π belongs to $\mathscr{Z}$ and if τ^π belongs to $\mathscr{T}$. We denote the set of all admissible controls by Π. The control component Z_t^π therefore corresponds to the total amount of dividends paid out by the firm up to time t and the control component τ^π represents the investment time in the growth opportunity. A given control policy $(Z_t^\pi, \tau^\pi; t \geq 0)$ fully characterizes the associated investment process $(I_t^\pi)_{t\geq 0}$ which belongs to $\mathscr{Z}$ and is defined by relation $I_t = I\,1\!\!1_{t\geq \tau^\pi}$. We denote by X_t^π the cash reserve of the firm at time t under a control policy $\pi = (Z_t^\pi, \tau^\pi; t \geq 0)$. The dynamic of the cash process X_t^π satisfies

$$dX_t^\pi = (\mu_0 1\!\!1_{t<\tau^\pi} + \mu_1 1\!\!1_{t\geq\tau^\pi})dt + \sigma dW_t - dZ_t^\pi - dI_t^\pi, \quad X_{0^-}^\pi = x.$$

For a given admissible control π, we define the time of bankruptcy by

$$\tau_0^\pi = \inf\{t \geq 0 \,:\, X_t^\pi \leq 0\},$$

and the firm value V_π by

$$V_\pi(x) = \mathbb{E}_x\left[\int_0^{\tau_0^\pi} e^{-rs} dZ_s^\pi\right].$$

The objective is to find the optimal return function which is defined as

$$V(x) = \sup_{\pi\in\Pi} V_\pi(x), \tag{3.3}$$

and the optimal policy $\pi^\star$ such that

$$V_{\pi^\star}(x) = V(x).$$

Thus, we model the interaction between dividends and investment as a mixed singular control/optimal stopping problem. We show that problem (3.3) can be reduced to a stopping problem that we solve quasi explicitly.

3.2 Value of the firm with no growth option.

Assume for the moment that the firm has only access to one of the two technologies (say, technology $i = 0$ for drift μ_0 and technology $i = 1$ for drift μ_1). The cash process $X_i = (X_{i,t})_{t \geq 0}$ therefore satisfies

$$dX_{i,t} = \mu_i dt + \sigma dW_t - dZ_{i,t}.$$

We are back in the classical distribution problem studied in Jeanblanc and Shiryaev [11], Radner and Shepp [16] or Asmussen and Taksar [1], the firm value is $V_i(X_{i,t \wedge \tau_{i,0}})$ where

$$V_i(x) = \sup_{Z_i \in \mathscr{Z}} \mathbb{E}_x \left[\int_0^{\tau_{i,0}} e^{-rs} dZ_{i,s} \right]. \tag{3.4}$$

Computations are explicit and we have:

Proposition 3.1 *(Jeanblanc and Shiryaev (1995))*

(i) (Firm value)

– *The firm value V_i is given by:*

$$\begin{cases} V_i(x) = \dfrac{f_i(x)}{f_i'(x_i)} & 0 \leq x \leq x_i, \\ V_i(x) = x - x_i + V_i(x_i), & x \geq x_i, \end{cases} \tag{3.5}$$

where

$$f_i(x) = e^{\alpha_i^+ x} - e^{\alpha_i^- x}, \; x_i = \frac{1}{\alpha_i^+ - \alpha_i^-} \ln \left(\frac{\alpha_i^-}{\alpha_i^+} \right)^2, \tag{3.6}$$

and where $\alpha_i^- < 0 < \alpha_i^+$ are the roots of the equation $\frac{1}{2}\sigma^2 x + \mu_i x - r = 0$.

(ii) (Optimal policy)

– *The process $L^* = \{L_t^*; t \geq 0\}$ defined by*

$$L_t^* = (x - x_i)^+ \, 1\!\!1_{t=0} + L_t^{x_i} \, 1\!\!1_{t>0} \tag{3.7}$$

is an optimal policy for problem (3.4). In Equation (3.7), the process $L^{x_i^*}$ denotes the solution to the Skohorod problem at x_i for the drifted Brownian motion $\mu_i t + B_t$, that is $L_t^{x_i} = \max \left[0, \max_{0 \leq s \leq t} (\mu_i s + \sigma W_s - x_i) \right]$.

It is worth noting that the function f_i defined on $[0, \infty)$ is non negative, increasing, concave on $[0, x_i]$, convex on $[x_i, \infty)$ and satisfies $f_i^{'} \geq 1$ on $[0, \infty)$ together with $\mathscr{L}_i f_i - r f_i = 0$ on $[0, x_i]$ where $\mathscr{L}_i$ is the infinitesimal generator of the drifted Brownian motion

$\mu_i t + \sigma W_t$. Remark also that V_i is concave on $[0, x_i]$ and linear above x_i. Finally, it is also important to note that there is no obvious comparison between x_0 and x_1 (see for instance Rochet and Villeneuve [18] Proposition 2). Coming back to our problem (3.3), we deduce from these standard results that the strategies

$$\pi^0 = (Z_t^0, 0) = \left((x - x_0)_+ 1\!\!1_{t=0} + L_t^{x_0}(\mu_0, W) 1\!\!1_{t>0}\,,\, \infty\right), \tag{3.8}$$

and

$$\pi^1 = (Z_t^1, 0) = \left((x - I) - x_1)_+ 1\!\!1_{t=0} + L_t^{x_1}(\mu_1, W) 1\!\!1_{t>0}\,,\, 0\right) \tag{3.9}$$

lead to the inequalities $V(x) \geq V_0(x)$ and $V(x) \geq V_1(x - I)$. Strategy π^0 corresponds to the investment policy *"never invest in the growth option (and follow the associated optimal dividend policy)"*, while strategy π^1 corresponds to the investment policy *"invest immediately in the growth option (and follow the associated optimal dividend policy)"*. Finally, note that, because the inequality $x - I \leq 0$ leads to immediate bankruptcy, the firm value $V_1(x - I)$ is defined by:

$$\begin{cases} V_1(x-I) = \max\left(0, \dfrac{f_1(x-I)}{f_1'(x_1)}\right), & 0 \leq x \leq x_1 + I, \\ V_1(x-I) = x - I - x_1 + \frac{\mu_1}{r}, & x \geq x_1 + I. \end{cases} \tag{3.10}$$

3.3 Value of the firm with a growth option

The dynamic programming principle[3] gives the following representation for the value function

$$V(x) = \sup_{\pi \in \Pi} \mathbb{E}_x \left[\int_0^{(\tau^\pi \wedge \tau_0^\pi)^-} e^{-rs} dZ_s^\pi + e^{-r(\tau^\pi \wedge \tau_0^\pi)} V_1(X^\pi_{(\tau^\pi \wedge \tau_0^\pi)^-} - I) \right], \tag{3.11}$$

Because $V(X_t) = V_1(X_t)$ on the set $\{t > \tau\}$, the strategy $Z_s^\pi = 0$ for $0 \leq s \leq t$ and $\tau^\pi = t$ leads to

$$V(x) \geq \mathbb{E}\left[e^{-r(t \wedge \tau_0^\pi)} V(R_{t \wedge \tau_0^\pi}) \right],$$

where $R = (R_t)_{t \geq 0}$ denotes the cash reserve process generated by the activity in place in absence of dividend distribution, that is $dR_t = \mu_0 dt + \sigma dW_t$. It results from the Markov property that the process $(e^{-r(t \wedge \tau_0^\pi)} V(R_{t \wedge \tau_0^\pi}))_{t \geq 0}$ is a supermartingale which dominates the function $\max(V_0(.), V_1(. - I))$. Thus, according to optimal stopping theory, V dominates the Snell envelope of the process $(\max(V_0(R_t), V_1(R_t - I)))_{t \geq 0}$. Let us consider the stopping time problem with value function

$$\phi(x) = \sup_{\tau \in \mathscr{T}} \mathbb{E}_x \left[e^{-r(\tau \wedge \tau_0)} \max(V_0(R_{\tau \wedge \tau_0}), V_1(R_{\tau \wedge \tau_0} - I)) \right], \tag{3.12}$$

where $\tau_0 = \inf\{t \geq 0 \,:\, R_t \leq 0\}$. In accordance with the foregoing, we have $V \geq \phi$. The following result characterizes the value function in terms of ϕ.

Theorem 3.1 *For all $x \in [0, \infty)$, $V(x) = \phi(x)$.*

The rest of the note is devoted to the proof of Theorem 3.1.

[3] We refer to Décamps and Villeneuve [6] Proposition 3.1 for a proof.

3.3.1 A verification Theorem.

Proving Theorem 3.1 amounts to show the reverse inequality $V(x) \le \phi(x)$. This requires a verification result for the Hamilton-jacobi-Bellman (HJB) equation associated to problem (3.11). One indeed expects from the dynamic programming principle, the value function to satisfy the HJB equation

$$\max(1-v', \mathscr{L}_0 v - rv, V_1(.-I)-v) = 0. \tag{3.13}$$

The next proposition shows that any piecewise function C^2 which is a supersolution to the HJB equation (3.13) is a majorant of the value function V.

Proposition 3.2 *(verification result for the HJB equation) Suppose we can find a positive function $\tilde{V}$ piecewise C^2 on $(0,+\infty)$ with bounded first derivatives*[4] *and such that for all $x>0$,*

(i) $\mathscr{L}_0\tilde{V} - r\tilde{V} \le 0$ *in the sense of distributions,*

(ii) $\tilde{V}(x) \ge V_1(x-I)$*,*

(iii) $\tilde{V}'(x) \ge 1$*,*

with the initial condition $\tilde{V}(0)=0$ then, $\tilde{V}(x) \ge V(x)$ for all $x \in [0,\infty)$.

Proof of Proposition 3.2 We have to prove that for any control policy $\pi = (Z_t^\pi, \tau^\pi; t \ge 0)$, $\tilde{V}(x) \ge V_\pi(x)$ for all $x>0$. Let us write the process $Z_t^\pi = Z_t^{\pi,c} + Z_t^{\pi,d}$ where $Z_t^{\pi,c}$ is the continuous part of Z_t^π and $Z_t^{\pi,d}$ is the pure discontinuous part of Z_t^π. Using a generalized Itô's formula (see Dellacherie and Meyer [7], Theorem VIII-25 and Remark c) page 349), we can write

$$\begin{aligned} e^{-r(\tau^\pi\wedge\tau_0^\pi)}\tilde{V}(X^\pi_{(\tau^\pi\wedge\tau_0^\pi)^-}) \;&=\; \tilde{V}(x) + \int_0^{(\tau^\pi\wedge\tau_0^\pi)^-} e^{-rs}(\mathscr{L}_0\tilde{V}(X_s^\pi) - r\tilde{V}(X_s^\pi))\,ds \\ &+\; \int_0^{(\tau^\pi\wedge\tau_0^\pi)^-} e^{-rs}\tilde{V}'(X_s^\pi)\,\sigma dW_t - \int_0^{(\tau^\pi\wedge\tau_0^\pi)^-} e^{-rs}\tilde{V}'(X_s^\pi)\,dZ_s^c \\ &+\; \sum_{s<\tau^\pi\wedge\tau_0^\pi} e^{-rs}(\tilde{V}(X_s^\pi) - \tilde{V}(X_{s-}^\pi)). \end{aligned}$$

Since $\tilde{V}$ satisfies (i), the second term of the right hand side is negative. On the other hand, the first derivative of $\tilde{V}$ being bounded, the third term is a square integrable martingale. Taking expectations, we get

$$\begin{aligned} \mathbb{E}_x\left[e^{-r(\tau^\pi\wedge\tau_0^\pi)}\tilde{V}(X^\pi_{(\tau^\pi\wedge\tau_0^\pi)^-})\right] \;&\le\; \tilde{V}(x) - \mathbb{E}_x\left[\int_0^{(\tau^\pi\wedge\tau_0^\pi)^-} e^{-rs}\tilde{V}'(X_s^\pi)\,dZ_s^{\pi,c}\right] \\ &+\; \mathbb{E}_x\left[\sum_{s<\tau\wedge\tau_0} e^{-rs}(\tilde{V}(X_s^\pi) - \tilde{V}(X_{s-}^\pi))\right]. \end{aligned}$$

[4] in the sense of Definition 4.8 page 271 in Karatzas and Shreve [12].

Since $\tilde{V}'(x) \geq 1$ for all $x > 0$, we have $\tilde{V}(X_s^\pi) - \tilde{V}(X_{s-}^\pi) \leq X_s^\pi - X_{s-}^\pi$. Therefore, using the equality $X_s^\pi - X_{s-}^\pi = -(Z_s^\pi - Z_{s-}^\pi)$ for $s < \tau^\pi \wedge \tau_0^\pi$, we finally get

$$\begin{aligned}
\tilde{V}(x) &\geq \mathbb{E}_x\left[e^{-r(\tau^\pi \wedge \tau_0^\pi)}\tilde{V}(X^\pi_{(\tau^\pi \wedge \tau_0^\pi)-})\right] + \mathbb{E}_x\left[\int_0^{(\tau^\pi \wedge \tau_0^\pi)^-} e^{-rs}\tilde{V}'(X_s^\pi)\, dZ_s^{\pi,c}\right] \\
&\quad + \mathbb{E}_x\left[\sum_{s<\tau\wedge\tau_0} e^{-rs}(Z_s^\pi - Z_{s-}^\pi)\right] \\
&\geq \mathbb{E}_x\left[e^{-r(\tau^\pi \wedge \tau_0^\pi)}V_1(X^\pi_{(\tau^\pi \wedge \tau_0^\pi)^-} - I)\right] + \mathbb{E}_x\left[\int_0^{(\tau^\pi \wedge \tau_0^\pi)^-} e^{-rs}\, dZ_s^\pi\right] \\
&= V_\pi(x),
\end{aligned}$$

where assumptions (ii) and (iii) have been used for the second inequality. ◇

We know already that $V \geq \phi$. Thus, to complete the proof of Theorem 3.1, it remains simply to verify that ϕ satisfies the assumption of Proposition 3.2. This will clearly imply the reverse inequality $V(x) \leq \phi(x)$. To achieve this goal we start by solving explicitly optimal stopping problem (3.12).

3.3.2 Solution to optimal stopping problem ϕ.

First, we have to know when $V_1(. - I)$ dominates V_0. According to Décamps and Villeneuve [6] Proposition 2.2, we have

Proposition 3.3 *The following holds.*

$$V(x) = V_0(x) \text{ for all } x \geq 0 \text{ if and only if } \left(\frac{\mu_1 - \mu_0}{r}\right) \leq (x_1 + I) - x_0.$$

Hereafter, we will denote by (H1) the strict inequality

$$\left(\frac{\mu_1 - \mu_0}{r}\right) > (x_1 + I) - x_0$$

ensuring that the growth opportunity is worthwhile. Note that for all positive x, $V(x) \geq \phi(x) \geq \theta(x)$ where θ is the value function of optimal stopping problem

$$\theta(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x\left[e^{-r(\tau\wedge\tau_0)}V_1(R_{\tau\wedge\tau_0} - I)\right], \tag{3.14}$$

where $\tau_0 = \inf\{t \geq 0 \,:\, R_t \leq 0\}$. The value function θ represents the option to invest in the growth opportunity when the manager decides to postpone dividend payments until investment. In line with the intuition underlying Theorem 3.1, one anticipates that, if, for all positive x, the option value $\theta(x)$ is larger than $V_0(x)$ then, we have the equalities $V(x) = \phi(x) = \theta(x)$. A crucial point will be to show that the inequality $\theta(x) > V_0(x)$ holds for all positive x, if and only if it is satisfied at the threshold x_0 that triggers distribution of dividend when the firm is run under the technology in place (see Lemma 3.8 hereinafter). In such a situation, the optimal dividend/investment policy will be to postpone dividend distribution, to invest at a certain threshold b in the growth opportunity and to pay out any surplus above x_1 as dividend. Next proposition specifies all these points and derives the solution to optimal stopping problem ϕ.

Proposition 3.4 *The following holds.*

(A) If condition (H1) is satisfied then,

(i) If $\theta(x_0) > V_0(x_0)$ then, the value function ϕ satisfies for all positive x, $\phi(x) = \theta(x)$.

(ii) If $\theta(x_0) \leq V_0(x_0)$ then, the value function ϕ has the following structure.

$$\phi(x) = \begin{cases} V_0(x) \quad 0 \leq x \leq a, \\ V_0(a)\mathbb{E}_x[e^{-r\tau_a} 1\!\!1_{\tau_a < \tau_c}] + V_1(c-I)\mathbb{E}_x[e^{-r\tau_c} 1\!\!1_{\tau_a > \tau_c}] = Ae^{\alpha_0^+ x} + Be^{\alpha_0^- x} \quad a \leq x \leq c, \\ V_1(x-I) \quad x \geq c, \end{cases}$$

where $\tau_a = \inf\{t \geq 0 : R_t \leq a\}$ and $\tau_c = \inf\{t \geq 0 : R_t \geq c\}$ and where A, B, a, c are determined by the continuity and smooth-fit C^1 conditions at a and c:

$$\begin{aligned} &\phi(a) = V_0(a), \\ &\phi(c) = V_1(c-I), \\ &\phi'(a) = V_0'(a), \\ &\phi'(c) = V_1'(c-I). \end{aligned}$$

(B) If condition (H1) is not satisfied then, for all positive x, $\phi(x) = V_0(x)$.

Figures 1 and 2 illustrate cases (i) and (ii) of Proposition 3.4. We establish Proposition 3.4 through a series of lemmas. The first one derives quasi explicitly the value function θ.

Lemma 3.6 *The value function θ is defined by*

$$\begin{cases} \theta(x) = \dfrac{f_0(x)}{f_0(b)} V_1(b-I) \quad x \leq b, \\ \theta(x) = V_1(x-I), \quad x \geq b, \end{cases} \tag{3.15}$$

where f_0 is defined in (3.6) and where $b > I$ is defined by the smooth-fit principle

$$\frac{V_1'(b-I)}{f_0'(b)} = \frac{V_1(b-I)}{f_0(b)}. \tag{3.16}$$

Proof of Lemma 3.6 It follows from Dayanik and Karatzas [5] (Corollary 7.1) that the optimal value function θ is C^1 on $[0,\infty)$ furthermore, from Villeneuve [20] (Theorem 4.2. and Proposition

4.6) a threshold strategy is optimal. This allows us to use a standard verification procedure and to write the value function θ in terms of the free boundary problem:

$$\begin{cases} \mathscr{L}_0\theta(x) - r\theta(x) = 0, \ 0 \le x \le b, \ \text{and} \ \mathscr{L}_0\theta(x) - r\theta(x) \le 0, \ x \ge b, \\ \theta(b) = V_1(b-I), \ \theta'(b) = V_1^{'}(b-I). \end{cases} \tag{3.17}$$

Standard computations lead to the desired result. ◇

The next Lemma characterizes the stopping region of optimal stopping problem ϕ.

Lemma 3.7 *The stopping region S of problem ϕ satisfies $S = S_0 \cup S_1$ with*

$$S_0 = \{0 < x < \tilde{x} \,|\, \phi(x) = V_0(x)\}$$

and

$$S_1 = \{x > \tilde{x} \,|\, \phi(x) = V_1(x-I)\},$$

where $\tilde{x}$ is the unique crossing point of the value functions $V_0(.)$ and $V_1(x-.)$.

Proof of Lemma 3.7 According to Optimal Stopping Theory (see El Karoui [9], Theorems 10.1.9 and 10.1.12 in Øksendal [15]), the stopping region S of problem ϕ satisfies

$$S = \{x > 0 \,|\, \phi(x) = \max(V_0(x), V_1(x-I))\}.$$

Now, from Proposition 5.13 and Corollary 7.1 by Dayanik-Karatzas [5], the hitting time $\tau_S = \inf\{t : R_t \in \mathscr{S}\}$ is optimal and the optimal value function is C^1 on $[0,\infty)$. Moreover, it follows from Lemma 4.3 from Villeneuve [20] that $\tilde{x}$, defined as the unique crossing point of the value functions $V_0(.)$ and $V_1(x-.)$, does not belong to S. Hence, the stopping region can be decomposed into two subregions $S = S_0 \cup S_1$ with

$$S_0 = \{0 < x < \tilde{x} \,|\, \phi(x) = V_0(x)\},$$

and

$$S_1 = \{x > \tilde{x} \,|\, \phi(x) = V_1(x-I)\}.$$

◇

We now obtain Assertion (i) of Proposition 3.4 as a byproduct of the next Lemma.

Lemma 3.8 *The following assertions are equivalent:*

(i) $\theta(x_0) > V_0(x_0)$.

(ii) $\theta(x) > V_0(x)$ *for all* $x > 0$.

(iii) $S_0 = \emptyset$.

Proof of Lemma 3.8.
$(i) \Longrightarrow (ii)$. We start with $x \in (0,x_0)$. Let us define $\tau_{x_0} = \inf\{t : R_t < x_0\} \in \mathscr{T}$. The inequality $\theta(x_0) > V_0(x_0)$ together with the initial condition $\theta(0) = V_0(0) = 0$ implies

$$\mathbb{E}_x\left[e^{-r(\tau_{x_0}\wedge\tau_0)}\left(\theta(R_{\tau_{x_0}\wedge\tau_0}) - V_0(R_{\tau_{x_0}\wedge\tau_0})\right)\right] > 0.$$

Itô's formula gives

$$\begin{aligned} 0 &< \mathbb{E}_x\left[e^{-r(\tau_{x_0}\wedge\tau_0)}\left(\theta(R_{\tau_{x_0}\wedge\tau_0}) - V_0(R_{\tau_{x_0}\wedge\tau_0})\right)\right] \\ &= \theta(x) - V_0(x) + \mathbb{E}_x\left[\int_0^{\tau_{x_0}\wedge\tau_0} e^{-rt}\left(\mathscr{L}_0\theta(R_t) - r\theta(R_t)\right)dt\right] \\ &\leq \theta(x) - V_0(x), \end{aligned}$$

where the last inequality follows from (3.17). Thus, $\theta(x) > V_0(x)$ for all $0 < x \leq x_0$. Assume now that $x > x_0$. We distinguish two cases. If $b > x_0$, it follows from (3.5) and (3.15) that, $\theta(x) > V_0(x)$ for $x \leq x_0$ is equivalent to $\theta^{'}(x_0) > 1$. Then, the convexity properties of f_0 yields to $\theta^{'}(x) > 1$, for all $x > 0$. If, on the contrary, $b \leq x_0$ then, $\theta(x) = V_1(x-I)$ for all $x \geq x_0$. Since $V_1^{'}(x-I) \geq 1$ for all $x \in [I,\infty)$, the smooth fit principle implies $\theta^{'}(x) \geq 1$ for all $x \geq x_0$. Therefore, the function $\theta - V_0$ is increasing for $x \geq x_0$ which ends the proof.

$(ii) \Longrightarrow (iii)$. Simply remark that equations (3.14) and (3.12) give $\phi \geq \theta$. Therefore, we have, $\phi(x) \geq \theta(x) > V_0(x)$ for all $x > 0$ which implies the emptyness of S_0.

$(iii) \Longrightarrow (i)$. Suppose $S_0 = \emptyset$ and let us show that $\theta = \phi$. This will clearly imply $\theta(x_0) = \phi(x_0) > V_0(x_0)$ and thus (i). From Optimal Stopping theory, the process $(e^{-r(t\wedge\tau_0\wedge\tau_S)}\phi(X_{t\wedge\tau_0\wedge\tau_S}))_{t\geq 0}$ is a martingale. Moreover, on the event $\{\tau_S < t\}$, we have $\phi(R_{\tau_S}) = V_1(R_{\tau_S} - I)$ *a.s.* It results that

$$\begin{aligned} \phi(x) &= \mathbb{E}_x\left[e^{-r(t\wedge\tau_S)}\phi(R_{t\wedge\tau_S})\right] \\ &= \mathbb{E}_x\left[e^{-r\tau_S}V_1(R_{\tau_S} - I)1\!\!1_{\tau_S<t}\right] + \mathbb{E}_x\left[e^{-rt}\phi(R_t)1\!\!1_{t<\tau_S}\right] \\ &\leq \theta(x) + \mathbb{E}_x\left[e^{-rt}\phi(R_t)\right]. \end{aligned}$$

Now, it follows from (3.5), (3.10) that $\phi(x) \leq Cx$ for some positive constant C. This implies $\mathbb{E}_x[e^{-rt}\phi(R_t)]$ converges to 0 as t goes to infinity. We therefore deduce that $\phi \leq \theta$ and thus that $\phi = \theta$. ◇

Figure 1 represents the value function of the optimal stopping problem ϕ under the assumptions of Lemma 3.8.

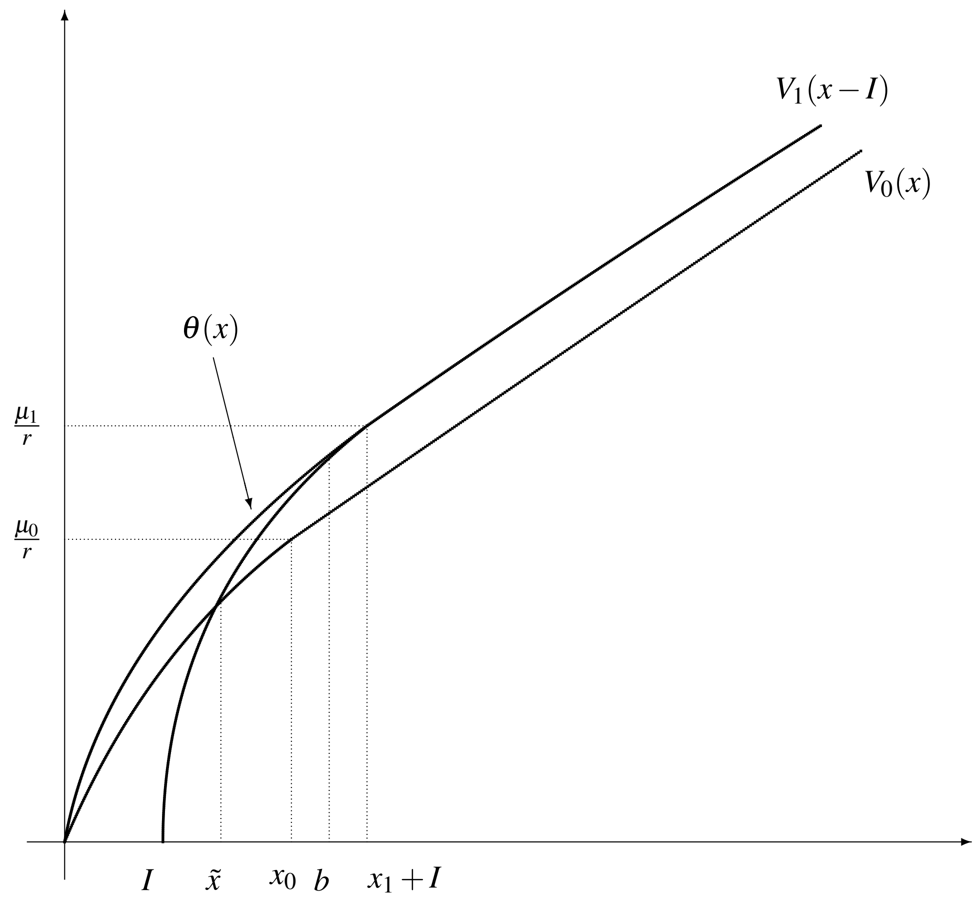

Figure 1: $\theta(x_0) > V_0(x_0)$

Assertion (ii) of Proposition 3.4 relies on the following lemma.

Lemma 3.9 *Assume $\theta(x_0) \leq V_0(x_0)$ then, there are two positive real numbers $a \geq x_0$ and $c \leq x_1 + I$ such that*

$$S_0 =]0, a] \text{ and } S_1 = [c, +\infty[.$$

Proof of Lemma 3.9 From the previous Lemma we know that the inequality $\theta(x_0) \leq V_0(x_0)$ implies $S_0 \neq \emptyset$. We start the proof with the shape of the subregion S_0. Take $x \in S_0$, we have to prove that any $y \leq x$ belongs to S_0. As a result, we will define $a = \sup\{x < \tilde{x} | x \in S_0\}$. Now, according to Proposition 5.13 by Dayanik and Karatzas [5], we have

$$\phi(y) = \mathbb{E}_y \left[e^{-r(\tau_S \wedge \tau_0)} \max(V_0(R_{\tau_S \wedge \tau_0}), V_1(R_{\tau_S \wedge \tau_0} - I)) \right].$$

Since $x \in S_0$, $x < \tilde{x}$ and thus $\tau_S = \tau_{S_0}$ $\mathbb{P}^y$ a.s. for all $y \leq x$. Hence,

$$\begin{aligned} \phi(y) &= \mathbb{E}_y \left[e^{-r(\tau_{S_0} \wedge \tau_0)} V_0(R_{\tau_{S_0} \wedge \tau_0}) \right] \\ &\leq V_0(y), \end{aligned}$$

where the last inequality follows from the supermartingale property of the process $(e^{-r(t\wedge\tau_0)}V_0(R_{t\wedge\tau_0}))_{t\geq 0}$. Now, assuming that $a<x_0$, (i.e. $\phi(x_0)>V_0(x_0)$) yields the contradiction:

$$\begin{aligned}
\phi(a) &= V_0(a)\\
&= \mathbb{E}_a\left[e^{-r\tau_{x_0}}1\!\!1_{\tau_{x_0}<\tau_0}V_0(R_{\tau_{x_0}})\right]\\
&\leq \mathbb{E}_a\left[e^{-r\tau_{x_0}}V_0(R_{\tau_{x_0}})\right]\\
&< \mathbb{E}_a\left[e^{-r\tau_{x_0}}\phi(R_{\tau_{x_0}})\right]\\
&\leq \phi(a),
\end{aligned}$$

where the second equality follows from the martingale property of the process $(e^{-r(t\wedge\tau_{x_0}\wedge\tau_0)}V_0(R_{t\wedge\tau_{x_0}\wedge\tau_0}))_{t\geq 0}$ under $\mathbb{P}^a$ and the last inequality follows from the supermartingale property of the process $(e^{-r(t\wedge\tau_0)}\phi(R_{t\wedge\tau_0}))_{t\geq 0}$.

The shape of the subregion S_1 is a direct consequence of Lemma 4.4 by Villeneuve [20]. The only difficulty is to prove that $c\leq x_1+I$. Let us consider $x\in(a,c)$, and let us introduce the stopping times $\tau_a=\inf\{t:R_t=a\}$, and $\tau_c=\inf\{t:R_t=c\}$, we have:

$$\begin{aligned}
\phi(x) &= \mathbb{E}_x\left[e^{-r(\tau_a\wedge\tau_c)}\max(V_0(R_{\tau_a\wedge\tau_c}),V_1(R_{\tau_a\wedge\tau_c}-I))\right]\\
&\leq \mathbb{E}_x\left[e^{-r(\tau_a\wedge\tau_c)}(R_{\tau_a\wedge\tau_c}-(x_1+I)+\frac{\mu_1}{r})\right]\\
&= x-(x_1+I)+\frac{\mu_1}{r}+\mathbb{E}_x\left[\int_0^{\tau_a\wedge\tau_c}e^{-rs}(\mu_0-r(R_s-(x_1+I))-\mu_1)\,ds\right].
\end{aligned}$$

Remark that, on the stochastic interval $[0,\tau_a\wedge\tau_c]$, $R_s\geq a\geq x_0 \quad \mathbb{P}^x$ a.s. and thus

$$\mu_0-r(R_s-(x_1+I))-\mu_1\leq\mu_0-r(x_0-(x_1+I))-\mu_1<0,$$

by condition (H1). Therefore, $\phi(x)\leq x-(x_1+I)+\frac{\mu_1}{r}$ for $x\in(a,c)$. We conclude remarking that, assuming the inequality $c>x_1+I$ would yield to the contradiction

$$\frac{\mu_1}{r}=V_1(x_1)<\phi(x_1+I)\leq\frac{\mu_1}{r}.$$

$\diamond$

Figure 2 represents the value function of the optimal stopping problem ϕ under the assumptions of Lemma 3.9.

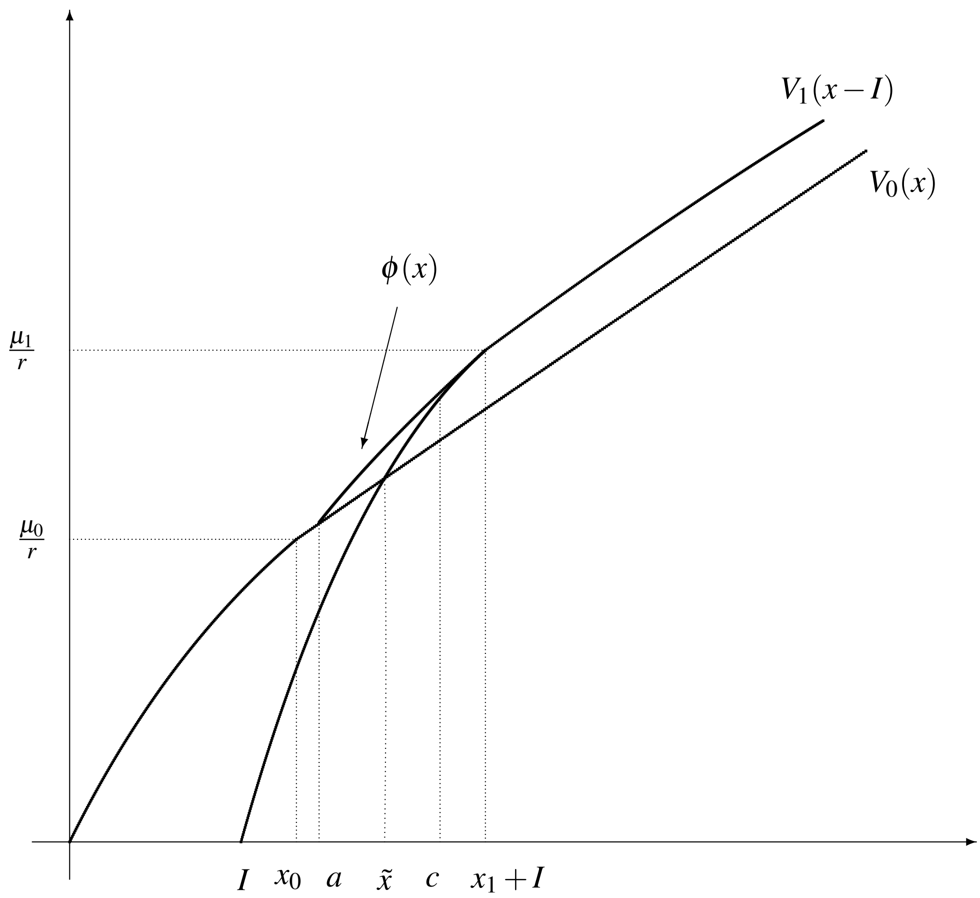

Figure 2: $\theta(x_0) < V_0(x_0)$

We now finish the proof of Proposition 3.4. It follows from Lemma 3.9 that the structure of the value function ϕ in assertion (ii) of Proposition 3.4 is a direct consequence of continuity and smooth-fit C^1 properties. Finally, consider case (B) of Proposition 3.4 and therefore assume that condition $(H1)$ is not satisfied. Similar arguments to those used for studying optimal stopping problem θ easily yield to the relation

$$V_0(x) = \sup_{\tau \in \mathscr{T}} \mathbb{E}_x \left[e^{-r(\tau \wedge \tau_0)} V_0(R_{\tau \wedge \tau_0} - I) \right].$$

The equality $V(x) = \phi(x)$ follows then from Proposition 3.3. $\diamond$

As a final remark note that, if $\theta(x_0) = V_0(x_0)$ then, we have that $a = x_0$, $c = b$ and the value functions ϕ and θ coincide. Indeed, using same argument than in the first part of the proof of Lemma 3.8, we easily deduce from $\theta(x_0) = V_0(x_0)$ that $\theta(x) = V_0(x) = \phi(x)$ for $x \leq x_0$. Furthermore, (3.5) and (3.15) imply that, $\theta(x_0) = V_0(x_0)$ is equivalent to $\theta'(x_0) = V'(x_0) = 1$, which implies that $a = x_0$. The equality $c = b$ follows then from relations (3.15) and (3.16). To summarize, if $\theta(x_0) = V_0(x_0)$ then, θ is the lowest supermartingale that majorizes $e^{-r(\tau\wedge\tau_0)} \max(V_0(R_{\tau\wedge\tau_0}), V_1(R_{\tau\wedge\tau_0} - I))$ from which it results that $\theta = \phi$.

3.3.3 ϕ as a super solution to HJB equation (3.13).

We are now ready to prove that ϕ satisfies the assumptions of Proposition 3.2. Formally,

Proposition 3.5 *ϕ is a supersolution to HJB equation (3.13).*

Proof of Proposition 3.5 The result clearly holds if, for all positive x, $\phi(x) = V_0(x)$ (that is, if condition (H1) is not satisfied). Assume now that condition (H1) is satisfied. Two cases have to be considered.

i) $\theta(x_0) > V_0(x_0)$.
In this case, $\phi = \theta$ according to part (i) of Proposition 3.4. It remains to check that the function θ satisfies the assumptions of Proposition 3.2. But, according to optimal stopping theory, $\theta \in C^2[(0,\infty)\setminus b)]$, $\mathscr{L}_0\theta - r\theta \leq 0$ and clearly $\theta \geq V_1(.-I)$. Moreover, it is shown in the first part of the proof of Lemma 3.8 that $\theta^{'}(x) \geq 1$ for all $x > 0$. Finally, let us check that $\theta^{'}$ is bounded above in the neighbourhood of zero. Clearly we have that

$$\theta(x) \leq \sup_{\tau\in\mathscr{T}} \mathbb{E}_x\left[e^{-r(\tau\wedge\tau_0)}V_1(R_{\tau\wedge\tau_0})\right],$$

furthermore, the process $(e^{-r(t\wedge\tau_0)}V_1(R_{t\wedge\tau_0}))_{t\geq 0}$ is a supermartingale since $\mu_1 > \mu_0$. Therefore $\theta \leq V_1$, the boundedness of the first derivative of θ follows then from Equation (3.10).

ii) $\theta(x_0) \leq V_0(x_0)$.
In this case, the function ϕ is characterized by part (ii) of Proposition 3.4. Thus, $\phi = V_0$ on $(0,a)$, $\phi = V(.-I)$ on $(c,+\infty)$ and $\phi(x) = Ae^{\alpha_0^+ x} + Be^{\alpha_0^- x}$ on (a,c). Hence, ϕ will be a supersolution if we prove that $\phi^{'}(x) \geq 1$ for all $x > 0$. In fact, it is enough to prove that $\phi^{'}(x) \geq 1$ for $x \in (a,c)$ because $V_0^{'} \geq 1$ and $V_1^{'}(.-I) \geq 1$. The smooth fit principle gives $\phi^{'}(a) = V_0^{'}(a) \geq 1$ and $\phi^{'}(c) = V_1^{'}(c-I) \geq 1$. Clearly, ϕ is convex in a right neighbourhood of a. Therefore, if ϕ is convex on (a,c), the proof is over. If not, the second derivative of ϕ given by $A(\alpha_0^+)^2e^{\alpha_0^+ x} + B(\alpha_0^-)^2e^{\alpha_0^- x}$ vanishes at most one time on (a,c), say in d. Therefore,

$$1 \leq \phi^{'}(a) \leq \phi^{'}(x) \leq \phi^{'}(d) \text{ for } x \in (a,d),$$

and

$$1 \leq \phi^{'}(c) \leq \phi^{'}(x) \leq \phi^{'}(d) \text{ for } x \in (d,c),$$

which completes the proof of Proposition 3.5 and thus concludes the proof of Theorem 3.1. ◇

4 Future works.

While the real option literature has emerged and developed within the framework of perfect capital markets, few papers have been interested in the financing of investment costs. However, when the

assumption of perfect capital markets is released, new issues are emerging that have an interest both in Mathematics and Finance. In particular, the interactions between liquidity management and investment policies lead to the study of mixed stochastic control problems that are relatively scarce in the applied probability literature. In the particular case where the firm have no access to external financing, the real option problem associated to the optimal investment for a cash-constrained firm is tackled by solving a stopping problem with a non linear payoff that exhibits interesting properties in terms of investment decisions that are not predicted by the standard real option theory. In the standard real option literature as well as in the optimal dividend policy literature, increasing the volatility of the cash process has an unambiguous effect: Greater uncertainty increases both the option value to invest (see McDonald and Siegel [13]), and the threshold that triggers distribution of dividend (see Rochet and Villeneuve [18]). An interesting feature of our model is that an increase of the volatility can kill the growth option. Because the difference $x_1 - x_0$ considered as a function of the volatility σ tends to $\frac{\mu_1-\mu_0}{r}$ when σ tends to infinity. This implies that for large volatility, condition (H1) is never satisfied and thus that the growth opportunity is worthless which is in contradiction with the positive effect of uncertainty on the option value to invest in the standard model of real option.
The study can be extended in two directions. From a mathematical viewpoint, it would be interesting to know if the main result (Theorem 3.1) remains valid if one models the dynamics of cash reserves with a more general class of regular diffusion. From a financial viewpoint, it would be natural and more realistic to release the liquidity constraints by assuming that firms have access to outside financing. In the state of our knowledge, this extension, if we focus on debt financing, leads to the same type of problems that the ones described in the discussion of Section 2.

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