Additional Material for "Equilibrium Fast Trading"

A. Derivation of Eq. (24) in the paper

By definition:

$$\frac{\partial \Delta}{\partial \alpha} = \frac{\partial \phi}{\partial \alpha} - \frac{\partial \psi}{\partial \alpha}. \tag{1}$$

Using the expression of ϕ and ψ in Eq. (19) and (21) in the text and the Leibniz rule, we obtain:

$$\frac{\partial \phi}{\partial \alpha} = -\frac{\partial S^*}{\partial \alpha}(2 - G(S^* - \sigma) - G(S^* + \sigma)), \tag{2}$$

and

$$\frac{\partial \psi}{\partial \alpha} = -2\mu(\lambda, \pi, r)\frac{\partial S^*}{\partial \alpha}(1 - G(S^*)). \tag{3}$$

To shorten notations, let: $\mu_0 \equiv \mu(\lambda, \pi, 0) = \lambda^s$ where the second equality comes from the definition of $\mu(\lambda, \pi, 0)$ and λ^s. Thus:

$$\begin{aligned}
\frac{\partial \Delta}{\partial \alpha} &= -\frac{\partial S^*}{\partial \alpha}[(2 - G(S^* - \sigma) - G(S^* + \sigma)) - 2\mu(\lambda, \pi, r)(1 - G(S^*))], \\
&= -\frac{\partial S^*}{\partial \alpha}[(2 - G(S^* - \sigma) - G(S^* + \sigma)) - 2(\mu(\lambda, \pi, r) - \mu_0)(1 - G(S^*)) - 2\mu_0(1 - G(S^*))], \\
&= -\frac{\partial S^*}{\partial \alpha} \times \left[\Delta\mathrm{Vol}(S^*(\alpha), \alpha) + (\mu_0 - \mu(\lambda, \pi, r))\,(1 - G(S^*)\right],
\end{aligned}$$

where the last equality comes from the expression for $\Delta\mathrm{Vol}(S^*(\alpha), \alpha)$ (see Eq. (13)) in the paper. When $r = 0$, the second term in the bracket is zero and:

$$\frac{\partial \Delta}{\partial \alpha} = -\frac{\partial S^*}{\partial \alpha} \times \Delta\mathrm{Vol}(a^*(\alpha), \alpha),$$

which is Eq. (24) in the paper.

B. Derivation of $\Delta(0)$ (Eq. (28) in the paper).

First, observe that:

$$E(|\delta| \, ||\delta| \leq x) = \frac{(2G(x)-1)}{2} \int_0^x \delta g(\delta) d\delta. \tag{4}$$

Using the definitions of $\phi(\alpha)$ and $\psi(\alpha)$ (Eq. (18) and (20) in the paper), the symmetry of $g(.)$, and the fact that $S^*(0) = 0$, we obtain:

$$\phi(0) = \epsilon(G(\sigma) - G(-\sigma)) + 2\int_0^{\overline{\delta}} \delta g(\delta) d\delta - 2\int_0^{\sigma} \delta g(\delta) d\delta,$$

and

$$\psi(0) = 2\mu(\lambda, \pi, r) \int_0^{\overline{\delta}} \delta g(\delta) d\delta.$$

Hence, using Eq. (4) in this appendix and the symmetry of $g(.)$, we deduce that

$$\Delta(0) = \phi(0) - \psi(0) = (1 - \mu(\lambda, \pi, r))E(|\delta|) + (2G(\sigma) - 1)(\sigma - E(|\delta| \, ||\delta| \leq \sigma)), \tag{5}$$

which, for $r = 0$, is Eq. (27) in the text.

C. Case $r > 0$

As shown in the paper, when $r > 0$,

$$\psi(\alpha) = 2\mu(\lambda, \pi, r) \int_{S^*(\alpha)}^{\overline{\delta}} (\delta - S^*(\alpha)) g(\delta) d\delta, \tag{6}$$

where $\mu(\lambda, \pi, r) = \lambda^s(\pi, \lambda) \frac{(1-(1-\lambda)\pi)}{(1-(1-\lambda)\pi(1+r)^{-1})}$. Thus, the discount rate just affects the results in which $\mu(\lambda, \pi, r)$ plays a role. Namely: Proposition 2, Corollary 1, and the expression for $\lambda(\sigma, C, \pi)$.

The proof of Proposition 2 holds for any value of μ. Thus, Proposition 2 is unchanged.

As shown in Part A of this appendix:

$$\frac{\partial \Delta(\alpha)}{\partial \alpha} = -\frac{\partial S^*}{\partial \alpha} \left[\Delta \mathrm{Vol}(S^*(\alpha), \alpha) + (\mu(\lambda, \pi, 0) - \mu(\lambda, \pi, r)) \, (1 - G(S^*) \right]. \tag{7}$$

When $r > 0$, the second term in bracket is strictly positive because $\mu(\lambda, \pi, r)$ decreases with r. Thus, Corollary 1 remains valid: Under the conditions of Lemma 3, $\Delta\text{Vol}(S^*(\alpha), \alpha) > 0$ and therefore $\frac{\partial\Delta(\alpha)}{\partial\alpha} > 0$. When the conditions in Lemma 3 are not satisfied, one can construct examples in which $\frac{\partial\Delta(\alpha)}{\partial\alpha} < 0$ for some values of α. This is true for $r = 0$ (see Example 2 in the paper) and therefore true at least for small values of r, by continuity.

Finally, proceeding as in the proof of Proposition 6 and using the expression for $\Delta(0)$ given in Eq. (5) in this appendix, one obtains that $\alpha^{SO} > 0$ (i.e., $\Delta(0) > C + C_{ext}(0)$) iff:

$$\frac{\text{E}(|\delta|) - C - (2G(\sigma) - 1)\text{E}(|\delta| \, ||\delta| \leq \sigma)}{\text{E}(|\delta|)} > \mu(\lambda, \pi, r). \tag{8}$$

The L.H.S of this inequality is less than 1. As μ increases with λ and is equal to 1 for $\lambda = 1$, we deduce that there is a threshold $\widehat{\lambda}(\sigma, C, \pi, r)$ (strictly less than 1 and possibly equal to zero if the L.H.S of (8) is negative), such that $\alpha^{SO} > 0$ iff $\lambda < \widehat{\lambda}(\sigma, C, \pi, r)$ (as obtained in Proposition 6 for $r = 0$). This threshold can be computed by replacing $\mu(\lambda, \pi, r)$ by its expression in (8), just as we do in the proof of Proposition 6. We obtain that:

$$\widehat{\lambda}(\sigma, C, \pi, r) = Max\{\frac{(1 + r - \pi)(\text{E}(|\delta|) - C - (2G(\sigma) - 1)\text{E}(|\delta| \, ||\delta| \leq \sigma))}{(1 + r)\text{E}(|\delta|) - \pi\,(\text{E}(|\delta|) - C - (2G(\sigma) - 1)\text{E}(|\delta| \, ||\delta| \leq \sigma))}, 0\} \tag{9}$$

First observe that if $r = 0$, $\widehat{\lambda}(\sigma, C, \pi, 0)$ is the same threshold as that derived in the paper in (41). Furthermore, $\widehat{\lambda}(\sigma, C, \pi, r)$ decreases with C, σ, and π, as obtained when $r = 0$ (see Proposition 6). Last, as claimed in Footnote 21 of the paper, $\widehat{\lambda}(\sigma, C, \pi)$ can be strictly positive even if $\pi = 1$ when $r > 0$.

D. Empirical Implication 2 (Example)

Effect of σ on the profitability of fast trading. In Implication 2 of Section 7, we claim that the effect of asset volatility (σ) on fast institutions' expected profit is positive when the level of fast trading is small and negative when the level of fast trading is high. Fig. A.1 below illustrates this claim when institutions' private valuations have a normal distribution with $\sigma_\delta = 4$.

Other parameters are: $\sigma = 5$ (dashed line), $\sigma = 4$ (plain line), and $\pi = 0.9$, $\lambda = 0.8$, $r = 0$. For $\alpha < 0.43$, fast institutions' expected profit is larger when volatility is high ($\sigma = 5$) whereas for $\alpha > 0.43$, fast institutions' expected profit is higher when volatility is low ($\sigma = 4$).