Common Agency and Public Good Provision under Asymmetric Information[1]

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This Version: 11th September 2009

Abstract

The provision of public goods under asymmetric information has most often been viewed as a mechanism design problem under the aegis of an uninformed mediator. This paper focuses on institutional contexts without such mediator. Contributors privately informed on their willingness to pay non-cooperatively offer contribution schedules to an agent who produces the public good on their behalf. In any separating and informative equilibrium of this common agency game under asymmetric information, instead of reducing marginal contributions to free-ride on others, principals do so to screen the agent's endogenous private information obtained from privately observing other principals' offers. Under weak conditions, existence of a differentiable equilibrium is shown. Equilibria are always ex post inefficient and interim efficient if and only if the type distribution has a linear inverse hazard rate. This points at the major inefficiency of contribution games under asymmetric information and stands in contrast with the more positive efficiency result that the common agency literature has unveiled when assuming complete information. Extensions of the model address direct contracting between principals, the existence of pooling uninformative equilibria and the robustness of our findings to the possibility that principals entertain more complex communication with their agent.

Keywords: Common agency, asymmetric information, public goods, ex post and interim efficiency.

JEL Codes: H41, D82.

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[1]We thank Vinicius Carrasco, Ivar Ekeland, Martin Hellwig, Paulo Monteiro, Mike Peters, Jerôme Pouyet, Jean-Charles Rochet, Aggey Semenov, and Jean Tirole for useful discussions, seminar participants in Toulouse, EPGE/FGV, PUC-Rio, IBMEC-Rio, WZB Berlin, Erasmus Rotterdam, ESEM Madrid, UCL, Marseille and UBC Vancouver for comments, the Editor Jeff Ely and two anonymous referees. Earlier versions of this paper were circulated under the title *"Common Agency with Informed Principals."* The second author thanks IDEI for its hospitality, CAPES and CNPG of Brazil for financial support.

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1 Introduction

Since Green and Laffont (1979) the provision of public goods under asymmetric information has most often been viewed as a mechanism design problem under the aegis of an uninformed mediator having a full commitment ability. This paper relaxes this assumption and focuses on cases without such mediator. Contributors privately informed on their willingness to pay non-cooperatively offer contribution schedules to an agent who produces the public good on their behalf.

Our first motivation for undertaking such analysis comes from observing that, in much real-world settings, centralized mechanisms and uninformed mediators with a strong ability to commit to those mechanisms are not available. Health, environment, multilateral foreign aid and other transnational public goods are all examples of public goods with voluntary provision by sovereign countries. There is no mediator to design the mechanisms that those countries should play to reveal their preferences. Politics and games of influence among interest groups offer other important examples. Key decision-makers might not have much commitment power to organize and ex ante design competition between interest groups. Instead, they only react ex post to the lobbying contributions they receive from those groups.[1] In those contexts, it is important to know whether a game of voluntary contributions fares well under asymmetric information, i.e., what are the positive and normative properties of the corresponding Bayesian-Nash equilibria.

Our second motivation is theoretical. Although earlier works on asymmetric information (Clarke 1971, Groves 1973) studied specific mechanisms for the provision of public goods, the bulk of the literature has departed from the analysis of real-world institutions to characterize instead properties of the whole set of incentive-feasible allocations.[2] In the standard framework (sometimes referred to as the centralized mechanism approach in what follows), an uninformed mediator moving first designs a mechanism for informed players. This mechanism induces an equilibrium allocation which is Bayesian incentive compatible, feasible (i.e., contributions cover the cost of the public good) and might respect the agents' veto constraints. No other institutional constraint on the kind of mechanisms or on the communication devices that can be used is considered. In this paper, we impose that such allocation is an equilibrium outcome of a game of voluntary contributions taking place under asymmetric information. In such a game, a privately informed contributor might want to offer a contribution schedule flexible enough to cope with different realizations of others' preferences. An agent collects contributions, endogenously learns something about the contributors' preferences from observing their mere offers and chooses the level of public good accordingly.

This institutional setting is thus viewed as a common agency game under asymmetric information with privately informed contributors non-cooperatively designing contribution schedules. We are interested in the general properties of such games both in terms of how information is aggregated, and in terms of ex post and interim efficiency.

[1] Grossman and Helpman (1994).

[2] This research strategy of the public good literature stands in sharp contrast with the way the literature on auctions has evolved. There, equal efforts have been devoted to the study of particular auction formats and to the characterization of the general properties of unrestricted auction mechanisms.

Our first important results are related to the process by which the equilibrium outcome aggregates contributors' private information. At a best response to what others offer, a given principal designs his own contribution not only to signal his preferences to the common agent but also to extract the endogenous private information that this agent may have learned from observing others' offers. Signaling turns out to be costless in our environment because of private values (the principals' private information does not enter directly into the agent's utility function) and risk-neutrality. Focusing on informative outcomes that aggregate information efficiently, we study separating Bayesian equilibria, i.e., contributors with different valuations offer different contribution schedules. In our private values environment, those contributions are the same as if the agent was perfectly informed on principals' valuations and out-of equilibrium beliefs following unexpected offers are irrelevant in characterizing the equilibrium. Instead, screening is costly. Each principal has to learn what the agent has endogenously learnt from observing others' contributions.[3] Standard mechanism design techniques can nevertheless be used to compute best responses. When choosing how much to contribute, each principal behaves actually as a monopsonist in front of an agent who is endogenously privately informed on the preferences of other contributors. By a standard screening argument,[4] this principal contributes less at the margin than his marginal valuation to decrease the agent's information rent. Intuitively, the agent can always ask for more from a given principal by pretending that others have not contributed enough. Each principal has then to reduce his own contribution to make that strategy less attractive to the agent.

As far as existence is concerned, we show that the marginal contribution in any equilibrium solves a complex functional equation with rather stringent boundary conditions. This equation links the equilibrium's marginal contribution, its inverse and its derivative. It is thus non-local by nature. Boundary conditions come from characterizing the bidding behavior of the two principals who have the highest and the lowest valuations, and who altogether implement a given output. We show that there always exists a differentiable equilibrium of the game under weak conditions on distributions. The idea is to analyze best responses in terms of the distribution of marginal contributions that a principal offers and to provide conditions under which that best-response mapping is monotonically decreasing: If principal 2's distribution of marginal contributions increases in the sense of first-order stochastic dominance, principal 1's own distribution decreases. This monotonicity helps defining a set of distributions which is stable by the best-response mapping and from which a fixed-point can be found using Schauder's Second Theorem. Finally, we also show uniqueness when the distribution of types is uniform.

Turning now to the normative properties of those equilibria, any equilibrium is necessarily ex post inefficient. For screening purposes, each principal always contributes less at the margin than what it is worth to him and "free-riding" arises. This is not to hide

[3]This is related to the notion of *"market information"* that Epstein and Peters (1999) stressed in multi-principals environments. Those authors derived general Revelation Principles for multi-principals games where each principal should try to learn from the agent whatever information he has on his own preferences but also on what he privately learns from observing others' offers. In a pure strategy Bayesian equilibrium as analyzed below, principals perfectly conjecture the strategy followed by others. They are a priori unaware of their exact types but may try to learn those types from asking the common agent about what he learns from observing offers made by other principals.

[4]Laffont and Martimort (2002, Chapter 3).

his type to the common agent as the centralized mechanism design approach predicts.[5] Instead, a principal induces less production to reduce the information rent that the agent gets from learning the preferences of others. Downward distortions below the first-best necessarily follow from this new source of distortion. The absence of a mediator forces privately informed contributors to communicate through a self-interested agent. Communication occurs via the offer of a contribution schedule which reveals the corresponding principal's type in any informative equilibrium.

Given that ex post efficiency fails and interim efficiency is a more relevant efficiency concept under asymmetric information, we ask whether equilibria are nevertheless interim efficient and under which circumstances if any.[6] The additional screening costs from having communication taking place through the agent explain why the Bayesian equilibria of the voluntary contribution game generally fail to be interim efficient. Interim efficiency is obtained if and only if the type distribution has a linear inverse hazard rate. We derive the symmetric equilibrium marginal contribution in that case. Beyond that non-generic case, public intervention under the aegis of an uniformed mediator is helpful in coordinating contributions. This points at the major inefficiency of contribution games under asymmetric information and stands in sharp contrast with the striking positive efficiency result that the common agency literature has unveiled when assuming complete information.[7]

Extensions of the model address direct contracting and communication between principals, the existence of pooling uninformative equilibria and the robustness of our findings to the possibility that principals entertain more complex communication with their agent.

Section 2 reviews the literature. Section 3 presents the model. Section 4 shows how to derive symmetric differentiable equilibria of the common agency game under asymmetric information. We present there also the Lindahl-Samuelson conditions satisfied at equilibrium and provide tractable examples. Section 5 discusses existence and uniqueness. Section 6 analyzes welfare properties of equilibria. Section 7 discusses several extensions of our model. Section 8 concludes. Proofs are relegated to an Appendix.

2 Review of the Literature

Following Wilson (1979) and Bernheim and Whinston (1986a), the common agency literature has developed an analytical framework to tackle a variety of important problems such as menu auctions, public goods provision through voluntary contributions,[8] or policy formation with competing lobbying groups in complete information environments.[9] Imposing that contributions are "truthful", i.e., reflect the relative preferences of the

[5] Laffont and Maskin (1982), Güth and Hellwig (1987), Rob (1989) and Mailath and Postlewaite (1990).

[6] Holmström and Myerson (1983) and Ledyard and Palfrey (1999). Because the common agent might get a positive rent in equilibrium from his endogenous private information, he might also receive a positive weight in the social welfare function maximized by the uninformed mediator offering the centralized mechanism designed to achieve a given interim efficient allocation.

[7] Bernheim and Whinston (1986).

[8] Laussel and Lebreton (1998).

[9] Grossman and Helpman (1994).

principals among alternatives, Bernheim and Whinston (1986a) reduced the equilibrium indeterminacy of those games and selected efficient equilibria.[10] With such truthful schedules, what a principal pays at the margin for inducing a change in the agent's decision is exactly what it is worth to him and the "free-riding" problem in public good provision cannot arise. Modulo truthfulness, common agency aggregates preferences efficiently under complete information.[11] Modelling private information on the principals' side justifies the use of nonlinear contributions for screening purposes in the first place. The "truthfulness" requirement is then replaced by incentive compatibility constraints. The cost of putting on firmer foundations the use of schedules is that ex post efficiency is lost and the conditions for interim efficiency become severe. In sharp contrast with complete information models, contribution games under asymmetric information are most often inefficient even in the interim sense. This gives a less optimistic view of decentralized bargaining.

Paralleling those complete information papers, Stole (1991), Martimort (1992, 1996a, 1996b), Mezzetti (1997), Biais, Martimort and Rochet (2000) and Martimort and Stole (2002, 2003, 2009) among others analyzed oligopolistic screening environments where different principals elicit information privately known by the common agent at the contracting stage. These papers stressed the impact of oligopolistic screening on the standard rent/efficiency trade-off. We focus instead on asymmetric information on the principals' side. Like in this earlier literature, the presence of competing principals introduces an additional distortion. In the standard common agency literature, a given principal needs to worry that the mechanism he offers affects the agent's choices of the contracting variables controlled by other principals. The distortion channel in our paper is different. Since principals cannot coordinate by communicating their types to a mediator, they do so through the agent and endow the latter with private information that the agent can exploit to obtain an information rent. The agent's private information vis-à-vis each principal is endogenous: it is what the agent may have learned from observing the other principals' offers.[12] The additional distortion due to the principals' non-cooperative behavior can thus be explained by their desire to extract the information rent associated to such endogenous information.

Contrasting with the use of schedules stressed by Bernheim and Whinston (1986), the complete information literature on voluntary provision of public goods has highlighted inefficiency and "free-riding" in models where contributors are restricted to offer fixed contributions (Bergstrom, Blume and Varian, 1986). Other solutions to this inefficiency problem include refunds (Bagnoli and Lipman, 1989) and multi-stage mechanisms in

[10] Multiplicity might still come from the flexibility in sharing the aggregate surplus among the contributing principals and their common agent (Bernheim and Whinston, 1986a).

[11] These results have been extended in many different directions. Dixit, Grossman and Helpman (1997) introduced redistributive concerns by relaxing the quasi-linearity assumption. Laussel and Lebreton (1998) studied incomplete information on the preferences of the common agent but focused on ex ante contracting when agency costs are null. Other extensions less directly relevant for the analysis of this paper include Prat and Rustichini (2003) who studied competition among principals trying to influence multiple agents and Bergemann and Välimäki (2003) who considered dynamic issues.

[12] Bond and Gresik (1997) studied the case where only one principal has private information and principals compete with piece-rate contracts. They showed that there exists an open set of inefficient equilibria. Bond and Gresik (1998) analyzed how tax authorities compete for a multinational firm's revenue when only one principal knows the firm's costs. In both papers, decisions are on private goods.

environments with partially verifiable information (Jackson and Moulin, 1992).

There exists a tiny literature on voluntary contributions for a 0-1 public good by privately informed agents. These works derive equilibrium strategy using techniques from the auction literature (Alboth, Lerner and Salev, 2001, Menezes, Monteiro and Temini, 2001). Menezes, Monteiro and Temini (2001) stressed the strong ex post inefficiency of equilibria whereas Laussel and Palfrey (2003) and Barbieri and Malueg (2008a, 2008b) found more positive results using interim efficiency. Our assumption that the level of public good is continuous invites the use of a differentiable approach. Interim efficiency is now much more stringent since it should apply not only on a line in the type space as in the 0-1 case (namely the set of types for which there is indifference between producing or not the public good) but on the whole type space. This is too demanding beyond the case of linear inverse hazard rates.

Finally, it is also useful to situate our contribution within the existing mechanism design literature on public goods. Since Clarke (1971) and Groves (1973), it is well-known that ex post efficiency is possible under dominant strategy implementation. D'Aspremont and Gerard-Varet (1979) showed that one can maintain budget balance and efficiency under Bayesian implementation. Laffont and Maskin (1982), Güth and Hellwig (1987), Rob (1989) and Mailath and Postlewaite (1990) stressed the role of participation constraints to generate inefficiency. A game of voluntary contributions ensures participation by principals, relies on Bayesian strategies, and finally generates a positive surplus for the agent. Hence, ex post inefficiency necessarily arises. When a centralized mechanism is offered by an uninformed mediator, inefficiencies are due to the contributors' incentives to hide their own types to this mediator: the so-called "free-riding" problem. Under common agency, as we will see below, contributors reveal instead their types by offering contracts to the agent but want to screen this agent according to what he has learned from others. This is no longer contributors who underestimate their valuations but their common agent who wants to claim to each principal that others have a lower willingness to pay: a different source of inefficiency in public good provision.

3 The Model

Consider two risk-neutral principals P_i $(i = 1, 2)$ who derive utility from consuming a public good which is produced in non-negative quantity q.[13,14] This public good may be an infrastructure of variable size, a charitable activity, or it may also have a more abstract interpretation as a policy variable in some lobbying games. The public good is excludable so that non-contributors do not enjoy the public good. P_i gets a utility $V_i(\theta_i, q, t_i) = \theta_i q - t_i$ from consuming q units of the good and paying an amount t_i.

Principals are privately informed on their respective valuations θ_i. Types are indepen-

[13]Extending our analysis to the case of more than two principals increases significantly complexity. Indeed, we will see below that each principal designs his contribution to screen others' types. Having more than two principals leads thus to a difficult multidimensional screening problem when computing each principal's best response. We leave those issues for further research.

[14]The public good can also be produced in quantity 0 or 1 and q is then viewed as its variable quality.

dently drawn from the same common knowledge and atomless distribution on $\Theta = [\underline{\theta}, \bar{\theta}]$ (we denote $\Delta\theta = \bar{\theta} - \underline{\theta} > 0$) with cumulative distribution function $F(\cdot)$ and everywhere positive and differentiable density $f = F'$. Unless specified otherwise, we assume that $\underline{\theta} > 0$ and $\bar{\theta} < \infty$ with $|f'(\theta)|$ being bounded.[15] The inverse hazard rate $R(\theta) = \frac{1-F(\theta)}{f(\theta)}$ is non-increasing. $E_\theta[\cdot]$ denotes the expectation operator with respect to θ.

Contributions are collected by a risk-neutral common agent A who produces at cost $C(q)$ the public good and whose utility function is $U(q, \sum_{i=1}^2 t_i) = \sum_{i=1}^2 t_i - C(q)$. $C(\cdot)$ is twice differentiable and convex with $C(0) = C'(0) = 0$ and $C'(\infty) = \infty$, where Inada conditions avoid corner solutions.

Benchmark. Let $q^{FB}(\theta_1, \theta_2)$ be the first-best level of public good. It is increasing in both arguments and satisfies the Lindahl-Samuelson conditions:

$$\sum_{i=1}^{2} \theta_i = C'(q^{FB}(\theta_1, \theta_2)).$$

Strategy space. Each principal P_i may offer any non-negative and continuous contribution schedule $t_i(\cdot)$ defined on a compact interval $\mathcal{Q} = [0, \bar{Q}]$ where $\bar{Q}$ is large enough (say larger than $q^{FB}(\bar{\theta}, \bar{\theta})$).

Timing. The sequence of events is as follows:

• Stage 0: Principals privately learn their types θ_i.

• Stage 1: Principals non-cooperatively and simultaneously offer the contributions $\{t_1(\cdot), t_2(\cdot)\}$.

• Stage 2: The agent accepts or refuses any of those contracts. If he refuses all contracts, the game ends with zero payoff for all players.

• Stage 3: The agent produces the level of public good q. Payments are made according to the agent's acceptance decisions and the chosen level of public good.

Together with the principals' preferences, the information structure, and strategy spaces, this timing defines our common agency game under incomplete information Γ. We consider pure-strategy Perfect Bayesian Equilibria (PBE) of Γ (in short equilibrium). Let $t_i(\cdot, \theta_i)$ denote an equilibrium strategy followed by principal P_i when his type is θ_i.

Definition 1 *A pair of strategy profiles $\{t_1(\cdot, \theta_1), t_2(\cdot, \theta_2)\}_{(\theta_1,\theta_2)\in\Theta^2}$ is an equilibrium of Γ if and only if:*

• *Principal P_i ($i = 1, 2$) with type θ_i finds it optimal to offer the contribution schedule $t_i(\cdot, \theta_i)$ given that he expects that principal P_{-i} follows the strategy profile $\{t_{-i}(\cdot, \theta_{-i})\}_{\theta_{-i}\in\Theta}$;*

• *The agent' updated beliefs on the principals' types follow Bayes' rule on the equilibrium path and are arbitrary elsewhere;*

• *The agent accepts contributions and chooses optimally the level of public good given those contributions and his beliefs on the principals' types.*

[15] Example 2 below provides an equilibrium characterization in the case of an exponential distribution. Theorem 6 applies to beta-density that may be zero at $\bar{\theta}$.

Because of symmetry between players, we will focus on symmetric equilibrium contributions and we may sometimes omit subscripts when obvious.

Remark 1 *Acceptance of all contributions is a weakly optimal strategy for the agent given that those contributions are non-negative. Note that the restriction to non-negative schedules is innocuous in this context. The agent would never choose an equilibrium output on the range of transfers offered by a given principal which are negative. He would prefer to refuse such schedule to increase his payoff.*[16]

Remark 2 *The strategy space that we consider allows principals to offer only contribution schedules. We postpone to Section 7.3 the analysis of the case where principals may offer more complex communication mechanisms in lines with the informed principal literature (say menus of such contribution schedules from which they may pick one).*[17,18]

Remark 3 *Existence of an optimal output at Stage 3 follows from compactness of $\mathcal{Q}$ and continuity of the schedules. In the sequel, we will impose further regularity assumptions on contributions to get sharper predictions.*

Remark 4 *In any separating equilibrium, the agent infers from each principal's contribution his type. In such equilibrium, the agent gets endogenous private information on both principals' types before making his own choice on the level of public good.*

Remark 5 *At Stages 2 and 3 of the game, the agent's decisions to accept and produce depend only on the contribution schedules he receives. In our private values context where the principals' types do not enter directly into the agent's utility function, these decisions do not depend on the agent's posterior beliefs following any offer made by one of the principals either on or off the equilibrium path. Hence, out-of equilibrium beliefs that sustain the equilibrium are arbitrary.*[19]

4 Characterizing Equilibria

4.1 Overview

We proceed as follows to compute P_i's best-response to a pure strategy profile $\{t_{-i}(\cdot, \theta_{-i})\}_{\theta_{-i}\in\Theta}$ followed by P_{-i}. First, we conjecture that P_{-i}'s strategy is separating, i.e., P_{-i} offers different contributions as his type changes. Our focus on separating equilibria is in the spirit

[16] In Section 7.3 we allow for negative transfers when principals offer inscrutable menus of mechanisms (i.e., menus of contributions schedules that do not reveal the principal's type) that are accepted or refused by the agent before he produces and the principals reveal their types. Ex post participation constraints are then replaced by interim ones and, in that case, it becomes quite natural to allow for transfers being possibly negative in some states of nature.

[17] Maskin and Tirole (1990, 1992) and Myerson (1983) among others.

[18] There is no cheap talk stage between players that could help them to replicate the existence of a mediator (Barany, 1992, Forges, 1990, Gerardi, 2002).

[19] See the Appendix for details.

of looking at equilibrium allocations that are informative as in Spence (1973) and Riley (1979).[20] Before choosing the level of public good, the agent gets thus endogenous private information on θ_{-i} by simply observing the mere offer $t_{-i}(\cdot, \theta_{-i})$ he receives. P_i must thus design his own contribution with an eye on the information rent that the agent gets from this endogenous information.[21] Second, we first do *as if* the agent was perfectly informed on P_i's type when the latter chooses his best response to the strategy profile $\{t_{-i}(q, \theta_{-i})\}_{\theta_{-i}\in\Theta}$ followed by P_{-i} in a pure strategy equilibrium. Third, we benefit of the private values environment (i.e., the principals' types do not enter directly into the agent's utility function) to show that the corresponding profile of contribution schedules is also a best response in the asymmetric information game Γ. Deviating towards another contribution schedule is suboptimal for any out-of equilibrium beliefs that the agent may hold following such unexpected offer. Finally, we notice that P_i's best response is itself separating and conveys information on P_i's type to the agent. Therefore, the agent gets endogenous private information on P_i's type also by simply observing his mere offer. This justifies that the same techniques can be used to compute also P_{-i}'s best response so that this approach holds the symmetric equilibrium we are looking for.

Running Example. To illustrate the above procedure, we will use throughout the quadratic-uniform example, i.e., $C(q) = \frac{q^2}{2}$ and types are uniformly distributed on $\Theta = [\underline{\theta}, \bar{\theta}]$ with $3\underline{\theta} > \bar{\theta}$. ■

4.2 Computing Best Responses

Following the procedure explained above, we assume that the agent is perfectly informed on the P_i 's type when the latter chooses his best response to the strategy profile $\{t_{-i}(q, \theta_{-i})\}_{\theta_{-i}\in\Theta}$ followed by P_{-i}. The Revelation Principle can be used to characterize any allocation that principal P_i may achieve by deviating towards any possible contribution schedule $t_i(q, \theta_i)$.[22] We thus focus on revelation mechanisms $\{t_i^D(\hat{\theta}_{-i}|\theta_i), q^D(\hat{\theta}_{-i}|\theta_i)\}_{\hat{\theta}_{-i}\in\Theta}$ inducing the agent to reveal to P_i what he has learned by observing P_{-i}'s offer.

Let $\hat{\theta}_{-i}$ be the agent's report on θ_{-i} (that he has learned from observing P_{-i}'s offer) to P_i in the truthful and direct revelation mechanism above. The agent's utility becomes:

$$\tilde{U}^D(\hat{\theta}_{-i}, \theta_{-i}|\theta_i) = t_i^D(\hat{\theta}_{-i}|\theta_i) + t_{-i}(q^D(\hat{\theta}_{-i}|\theta_i), \theta_{-i}) - C(q^D(\hat{\theta}_{-i}|\theta_i)).$$

Incentive compatibility yields the expression of the agent's information rent:

$$U^D(\theta_{-i}|\theta_i) = \tilde{U}^D(\theta_{-i}, \theta_{-i}|\theta_i) = \max_{\hat{\theta}_{-i}\in\Theta} \tilde{U}^D(\hat{\theta}_{-i}, \theta_{-i}|\theta_i). \quad (1)$$

For a fixed strategy profile $\{t_{-i}(q, \theta_{-i})\}_{\theta_{-i}\in\Theta}$ for P_{-i}, the mechanism $\{t_i^D(\hat{\theta}_{-i}|\theta_i), q^D(\hat{\theta}_{-i}|\theta_i)\}_{\hat{\theta}_{-i}\in\Theta}$ induces the allocation $\{U^D(\theta_{-i}|\theta_i), q^D(\theta_{-i}|\theta_i)\}_{\theta_{-i}\in\Theta}$.

[20] Such equilibria aggregates information efficiently which seems to be an interesting normative property, especially in view of assessing the ex post efficiency of equilibrium allocations. Section 7.2 analyzes instead the case of uninformative pooling equilibria.

[21] This points at the role that contributions play in a common agency environment: learning over what Epstein and Peters (1999) call "market information", i.e., over other principals' preferences that are reflected in their own offers to the agent.

[22] Martimort and Stole (2002).

Making P_i's endogenous screening problem about learning P_{-i}'s type from the agent tractable requires further conditions on P_{-i}'s contributions. The conditions below will thus be satisfied by the informative equilibrium under scrutiny. To simplify the analysis, we are now also considering contribution schedules which are piecewise three times differentiable so that the equilibrium output is differentiable.[23]

Definition 2 *A non-negative contribution $t_{-i}(q, \theta_{-i})$ is increasing in type (IT) when, at any differentiability point (q, θ_{-i}),*

$$\frac{\partial t_{-i}}{\partial \theta}(q, \theta_{-i}) \geq 0.$$

Under IT, principal P_{-i} contributes more if he has a greater valuation. Another natural requirement is that an upward shift in P_{-i}'s valuation increases also the equilibrium quantity, i.e., the same Spence-Mirrlees property as for the principals' preferences holds also for the contribution schedules.

Definition 3 *A non-negative contribution $t_{-i}(q, \theta_{-i})$ with margin $p_{-i}(q, \theta_{-i}) = \frac{\partial t_{-i}}{\partial q}(q, \theta_{-i})$ satisfies the Spence-Mirrlees Property (SMP) when, at any differentiability point (q, θ_{-i}),*

$$\frac{\partial p_{-i}}{\partial \theta}(q, \theta_{-i}) \geq 0.$$

Using standard techniques from the screening literature in monopolistic screening environments, the next lemma characterizes incentive compatible allocations that P_i may induce by choosing his own contribution schedule.

Lemma 1 *Assume that P_{-i} offers a non-negative contribution $t_{-i}(q, \theta_{-i})$. Any truthful and direct revelation mechanism $\{t_i^D(\hat{\theta}_{-i}|\theta_i), q^D(\hat{\theta}_{-i}|\theta_i)\}_{\hat{\theta}_{-i} \in \Theta}$ that P_i may offer to induce the allocation $\{U^D(\theta_{-i}|\theta_i), q^D(\theta_{-i}|\theta_i)\}_{\theta_{-i} \in \Theta}$ satisfies the following properties:*

- *$U^D(\theta_{-i}|\theta_i)$ is a.e. differentiable with respect to θ_{-i} with*

$$\frac{\partial U^D}{\partial \theta_{-i}}(\theta_{-i}|\theta_i) = \frac{\partial t_{-i}}{\partial \theta}(q^D(\theta_{-i}|\theta_i), \theta_{-i}) \geq 0 \tag{2}$$

when $t_{-i}(q, \theta_{-i})$ satisfies IT;

- *If $t_{-i}(q, \theta_{-i})$ satisfies SMP, $q^D(\theta_{-i}|\theta_i)$ is monotonically increasing and thus a.e. differentiable in θ_{-i} with*

$$\frac{\partial q^D}{\partial \theta_{-i}}(\theta_{-i}|\theta_i) \geq 0 \qquad a.e..; \tag{3}$$

- *If $t_{-i}(q, \theta_{-i})$ satisfies SMP, (3) is also sufficient for global optimality of the agent's problem (1).*

[23]Equilibria using forcing contributions can be constructed in this environment. See Section 7.2.

Running Example (cont.). Suppose that principal P_2 with type θ_2 offers the following revealing non-negative contribution:

$$t_2(q,\theta_2) = \max\left\{0, \left(\frac{\theta_2}{2} - \frac{\bar{\theta}}{6}\right) q + \frac{q^2}{6} + t_2^0(\theta_2)\right\} \tag{4}$$

where

$$t_2^0(\theta_2) = \frac{1}{12}(3\underline{\theta} - \bar{\theta})^2 - \frac{1}{6}\left(\frac{3}{2}(\theta_2 + \underline{\theta}) - \bar{\theta}\right)^2.$$

Notice that, on its positive range, $t_2(q,\theta_2)$ satisfies IT when $q \geq -2\frac{dt_2^0}{d\theta_2}(\theta_2) = \frac{3}{2}(\theta_2 + \underline{\theta}) - \bar{\theta}$ a property that holds for the equilibrium output given in (6) below. It also satisfies SMP. ■

Turning now to participation constraints, the agent accepts P_i's offer when:

$$U^D(\theta_{-i}|\theta_i) \geq \hat{U}_{-i}(\theta_{-i}), \quad \text{for all } \theta_{-i} \in \Theta \tag{5}$$

where $\hat{U}_{-i}(\theta_{-i}) = \max_{q\in\mathcal{Q}} \ t_{-i}(q,\theta_{-i}) - C(q)$ is the agent's rent when not taking P_i's contribution. Since $t_{-i}(q,\theta_{-i})$ is non-negative, the agent makes necessarily a non-negative profit at any profile (θ_i, θ_{-i}).

Running Example (cont.). By taking only the contribution schedule defined in (4), the agent gets a reservation payoff

$$\hat{U}_2(\theta_2) = \max_{q\geq 0} t_2(q,\theta_2) - \frac{q^2}{2} = \max\left\{0, \frac{1}{48}(3\theta_2 - \bar{\theta})^2 + t_2^0(\theta_2)\right\}$$

where the second-term in the right-hand side above is achieved by choosing the non-negative output $\hat{q}_2(\theta_2) = \frac{1}{4}(3\theta_2 - \bar{\theta})$ on the positive range of $t_2(q,\theta_2)$. ■

If the agent were informed on P_i's type θ_i, principal P_i would solve the following mechanism design problem at a best-response to any non-negative SMP profile $t_{-i}(q,\theta_{-i})$:

$$\mathcal{P}_i(\theta_i): \max_{\{U^D(\cdot|\theta_i);q^D(\cdot|\theta_i)\}} E_{\theta_{-i}}\left[\theta_i q^D(\theta_{-i}|\theta_i) + t_{-i}(q^D(\theta_{-i}|\theta_i),\theta_{-i}) - C(q^D(\theta_{-i}|\theta_i)) - U^D(\theta_{-i}|\theta_i)\right]$$

subject to (2), (3), and (5).

A solution to $\mathcal{P}_i(\theta_i)$ is an allocation $\{U^D(\theta_{-i}|\theta_i), q^D(\theta_{-i}|\theta_i)\}$ (or, equivalently, a direct revelation mechanism $\{t_i^D(\theta_{-i}|\theta_i), q^D(\theta_{-i}|\theta_i)\}$ that induces this allocation)[24] from which we can easily reconstruct the nonlinear contribution $t_i(q,\theta_i)$ offered by P_i with the simple formula $t_i(q,\theta_i) = t_i^D(\theta_{-i}|\theta_i)$ at $q = q^D(\theta_{-i}|\theta_i)$.[25]

The standard techniques for solving problems like $\mathcal{P}_i(\theta_i)$ in monopolistic screening environments consist in first neglecting the second-order condition (3), second assuming that the participation constraint (5) binds only at $\theta_i = \underline{\theta}$ to obtain an expression of the

[24] To simplify notations, the dependence on $t_{-i}(q,\theta_{-i})$ is implicit.

[25] This formula holds whether $q^D(\theta_{-i}|\theta_i)$ is strictly increasing in θ_{-i} or has flat parts on bunching areas if any.

agent's rent $U^D(\theta_{-i}|\theta_i)$, third integrating by parts the expected rent left to the agent to get an expression of the principal's virtual surplus function.

As shown in the Appendix, these first three steps of the analysis lead to the following reduced-form problem $\mathcal{P}_i'(\theta_i)$:

$$\mathcal{P}_i'(\theta_i): \max_{q^D(\cdot|\theta_i)} E_{\theta_{-i}}\left[\theta_i q^D(\theta_{-i}|\theta_i) + t_{-i}(q^D(\theta_{-i}|\theta_i), \theta_{-i}) - C(q^D(\theta_{-i}|\theta_i)) - R(\theta_{-i})\frac{\partial t_{-i}}{\partial \theta}(q^D(\theta_{-i}|\theta_i), \theta_{-i})\right].$$

A first difficulty is that the concavity of P_i's virtual surplus function in the maximand above depends on the other principal's offer $t_{-i}(q, \theta_{-i})$ which is an equilibrium construction. A second difficulty comes from checking that the second-order condition (3) holds. It turns out that both difficulties can be handled together when $t_{-i}(q, \theta_{-i})$ satisfies a couple of properties that are made explicit in condition (12) below.

Running Example (cont.). Let us find principal P_1's best response to $t_2(q, \theta_2)$ and assume that his type is revealed through the contract offer to the agent. As we saw above, such best response can be computed by means of a direct revelation mechanism $\{t_1^D(\hat{\theta}_2|\theta_1), q^D(\hat{\theta}_2|\theta_1)\}_{\hat{\theta}_2 \in \Theta}$.

Using standard techniques, let us thus write the agent's payoff when taking both schedules as:

$$U^D(\theta_2|\theta_1) = \max_{\hat{\theta}_2 \in \Theta} t_1^D(\hat{\theta}_2|\theta_1) + t_2(q^D(\hat{\theta}_2|\theta_1), \theta_2) - \frac{1}{2}(q^D(\hat{\theta}_2|\theta_1))^2.$$

The equilibrium output being chosen on the positive range of $t_2(q, \theta_2)$, we immediately get from the Envelope Theorem:

$$\frac{\partial U^D}{\partial \theta_2}(\theta_2|\theta_1) = \frac{\partial t_2}{\partial \theta_2}(q^D(\theta_2|\theta_1), \theta_2) = \frac{1}{2}q^D(\theta_2|\theta_1) \geq 0.$$

A key point for finding P_1's best response consists in determining where the participation constraint necessary to induce the agent's acceptance of principal P_1's contribution binds. Note that $\hat{U}_2(\underline{\theta}) = 0$ and that the slope of $\hat{U}_2(\theta_2)$ is lower than the slope of $U^D(\theta_2|\theta_1)$ provided principal P_1's marginal contribution is positive and induces more production than when the agent contracts only with principal P_2. This yields immediately:

$$U^D(\theta_2|\theta_1) = \int_{\underline{\theta}}^{\theta_2} \frac{1}{2}q^D(x|\theta_1)dx.$$

When using the direct revelation mechanism $\{t_1^D(\hat{\theta}_2|\theta_1), q^D(\hat{\theta}_2|\theta_1)\}_{\hat{\theta}_2 \in \Theta}$, principal P_1's expected payoff becomes:

$$\begin{aligned} & E_{\theta_2}\left[\theta_1 q^D(\theta_2|\theta_1) + t_2(q^D(\theta_2|\theta_1)) - \frac{1}{2}(q^D(\theta_2|\theta_1))^2 - U^D(\theta_2|\theta_1)\right] \\ = \; & E_{\theta_2}\left[\left(\theta_1 + \theta_2 - \frac{\bar{\theta}}{3}\right) q^D(\theta_2|\theta_1) - \frac{1}{3}(q^D(\theta_2|\theta_1))^2 - t_2^0(\theta_2)\right] \end{aligned}$$

where the equality follows from using $E_{\theta_2}\left[U^D(\theta_2|\theta_1)\right] = E_{\theta_2}\left[\frac{(\bar{\theta}-\theta_2)}{2}q^D(\theta_2|\theta_1)\right]$.

Pointwise optimization of P_1's virtual surplus yields the following expression of the output induced at a best response to $t_2(q,\theta_2)$:

$$q(\theta_1,\theta_2)=\frac{3}{2}(\theta_1+\theta_2)-\bar{\theta}. \tag{6}$$

Moreover, P_1's marginal contribution at a best response is such that:

$$\frac{\partial t_1}{\partial q}(q(\theta_1,\theta_2),\theta_1)=q(\theta_1,\theta_2)-\frac{\partial t_2}{\partial q}(q(\theta_1,\theta_2),\theta_1)=\frac{\theta_1}{2}-\frac{\bar{\theta}}{6}+\frac{q(\theta_1,\theta_2)}{3}\geq 0 \text{ when } 3\underline{\theta}>\bar{\theta}$$

and where the second equality follows from using the expression of $t_2(q,\theta_2)$ given in (4) on its positive range. Integrating yields the following expression of P_1's contribution for any output in its positive range:

$$t_1(q,\theta_1)=\left(\frac{\theta_1}{2}-\frac{\bar{\theta}}{6}\right)q+\frac{q^2}{6}+t_1^0(\theta_1) \tag{7}$$

where

$$t_1^0(\theta_1)=\frac{1}{12}(3\underline{\theta}-\bar{\theta})^2-\frac{1}{6}\left(\frac{3}{2}(\theta_1+\underline{\theta})-\bar{\theta}\right)^2$$

is chosen so that $U^D(\underline{\theta}|\theta_1)=0$ for all θ_1.[26] The contribution can finally be extended beyond the set of equilibrium outputs as in (4). A pair of such schedules forms thus a symmetric informative equilibrium in this quadratic-uniform example. ■

The next subsection offers a general analysis of such equilibria.

4.3 Symmetric Informative Equilibria

Consider symmetric equilibria which solve problems $\mathcal{P}_i(\theta_i)$ (or $\mathcal{P}_i'(\theta_i)$) for both principals. An important step of our analysis below will consist in showing that the contribution schedule solution to $\mathcal{P}_i(\theta_i)$ which was derived assuming that the agent has complete information on P_i's type is also a best-response in Γ, i.e., when P_i is privately informed. Indeed, note that the incentive and participation constraints (2), (3), and (5) do not depend on the agent's beliefs on P_i's type but only on the schedule that this principal offers. Hence, the agent's decisions to accept that contribution and to produce accordingly are also independent on his beliefs on P_i's type. Any deviation away from the contribution that P_i would optimally offer had the agent being informed on his type is thus dominated for any out-of equilibrium beliefs.

At a symmetric informative equilibrium with contribution $t(q,\theta)$ (resp. marginal contribution $p(q,\theta)$) satisfying SMP, we denote respectively the agent's output and rent as $q^D(\theta_{-i}|\theta_i)=q^D(\theta_i|\theta_{-i})=q(\theta_1,\theta_2)$ and $U^D(\theta_{-i}|\theta_i)=U^D(\theta_i|\theta_{-i})=U(\theta_1,\theta_2)$. The first-order condition for pointwise optimization of $\mathcal{P}_i'(\theta_i)$ is:

$$\theta_i+p(q(\theta_1,\theta_2),\theta_{-i})-C'(q(\theta_1,\theta_2))=R(\theta_{-i})\frac{\partial p}{\partial\theta}(q(\theta_1,\theta_2),\theta_{-i}) \text{ for } i=1,2. \tag{8}$$

[26] Notice that $\frac{\partial t_1}{\partial\theta_1}\geq 0$ if $q\geq -2\frac{dt_1^0}{d\theta_1}(\theta_1)=\frac{3}{2}(\theta_1+\underline{\theta})-\bar{\theta}$ as it will be the case for the output found in (6). This expression is symmetric of that obtained for $t_2^0(\theta_2)$.

This is the standard condition in screening model which says that the marginal surplus of the bilateral coalition between P_i and the agent (left-hand side of (8)) is equal to the marginal cost of the latter's information rent (right-hand side of (8)). The difficulty comes from the fact that the marginal contribution $p(q, \theta)$ is an equilibrium construction.

To complete the characterization of equilibrium marginal contributions, it is useful to rewrite the optimality condition for the agent's output given that he has accepted both contributions. This output must, on top of (8), also satisfy the following first-order condition of the agent's problem expressed in terms on nonlinear contributions:

$$\sum_{i=1}^{2} p(q(\theta_1, \theta_2), \theta_i) = C'(q(\theta_1, \theta_2)), \tag{9}$$

with the second-order condition

$$\sum_{i=1}^{2} \frac{\partial p}{\partial q}(q(\theta_1, \theta_2), \theta_i) - C''(q(\theta_1, \theta_2)) \leq 0. \tag{10}$$

Consider now an equilibrium output $q(\theta_1, \theta_2)$ increasing in each argument. For any given level of the public good $q = q(\theta_1, \theta_2)$, we can uniquely define the *conjugate* of type θ_i as the type $\psi(q, \theta_i)$ for principal P_{-i} such that $q(\theta_i, \psi(q, \theta_i)) = q$. Using conditions (8) and (9), $\psi(q, \theta)$ must be defined as:

$$\psi(q, \theta) = -p(q, \theta) + C'(q) + R(\theta)\frac{\partial p}{\partial \theta}(q, \theta), \quad \forall (q, \theta). \tag{11}$$

From now on, we will assume that

$$\frac{\partial \psi}{\partial q}(q, \theta) > 0 \text{ and } \frac{\partial \psi}{\partial \theta}(q, \theta) < 0. \tag{12}$$

These assumptions bear on an endogenous object, namely the equilibrium marginal contribution, but they allow a clear characterization of equilibrium properties.

Running Example (cont.). Going back on formula (6), we easily observe that $\psi(q, \theta) = \frac{2}{3}(\bar{\theta} + q) - \theta$ satisfies conditions (12). ■

Theorem 1 *A non-negative marginal contribution $p(q, \theta)$ arising at a symmetric informative equilibrium of Γ and satisfying*[27] *IT, SMP and (12) implements an output schedule $q(\theta_1, \theta_2)$ which is increasing in each argument and satisfies conditions (8), (9) and (10).*

Such equilibrium is separating and sustained by arbitrary out-of equilibrium beliefs.

Turning now to the output distortions and distributions of information rent in any such symmetric informative equilibrium, we obtain:

[27] From now on, an informative equilibrium satisfying IT and SMP will be called an equilibrium in short.

Theorem 2 *Any symmetric informative equilibrium with a marginal contribution $p(q,\theta)$ satisfying (12) implements an output schedule $q(\cdot)$ and a rent profile $U(\cdot)$ such that:*

- *Efficiency arises when both principals have the highest valuation ($q(\bar{\theta},\bar{\theta}) = q^{FB}(\bar{\theta},\bar{\theta})$) and output is downward distorted otherwise ($q(\theta_1,\theta_2) \leq q^{FB}(\theta_1,\theta_2)$ for all (θ_1,θ_2));*
- *The agent's information rent is such that*

$$U(\theta_1,\theta_2) \geq 0 \text{ with equality if } \theta_i = \underline{\theta} \text{ for at least one } i$$

where also

$$\frac{\partial U}{\partial \theta_i}(\theta_i,\underline{\theta}) = \frac{\partial t}{\partial \theta}(q(\theta_i,\underline{\theta}),\theta_i) = 0 \text{ for all } \theta_i. \tag{13}$$

At a best response, a principal induces less output from the agent than what is ex post efficient for their bilateral coalition. This downward distortion reduces the information rent that the agent gets from his endogenous private knowledge on the other principal's type. This distortion is captured by the right-hand side of (8) which is positive thanks to SMP. In this common agency game, inefficiency comes from the screening problem that each principal faces in contracting with an agent who is endogenously privately informed on the other principal's type.

This downward distortion should be contrasted with the usual "free-riding" problem for public good provision found in centralized Bayesian mechanisms (Laffont and Maskin, 1979, Rob, 1989, Mailath and Postlewaite, 1990). Free-riding comes there from the contributors' incentives to underestimate their valuations when reporting to a single mechanism designer. Under common agency instead, principals do not hide their own valuations to the common agent but each of these principals wants to screen the agent about the other principal's preferences. These are no longer contributors themselves who hide information but the agent who might pretend having received less contributions from each principal than what he really had.

Finally, the agent's rent is everywhere non-negative and zero when at least one of the principals has the lowest possible valuation. The equilibrium allocation is generally not budget balanced, and may generate some surplus that accrues to the agent.

4.4 Lindahl-Samuelson Conditions and Tractable Examples

From condition (11) we have:

$$\psi(q,\theta_i) + p(q,\theta_i) - C'(q) = R(\theta_i)\frac{\partial p}{\partial \theta}(q,\theta_i), \tag{14}$$

for all $q = q(\theta_i,\theta_{-i})$ and $(\theta_i,\theta_{-i}) \in \Theta^2$. This condition can be rewritten as:

$$\frac{\partial}{\partial \theta_i}[p(q,\theta_i)(1-F(\theta_i))] = (\psi(q,\theta_i) - C'(q))f(\theta_i).$$

This differential equation in θ_i can be integrated to get $p(q, \theta_i)$. Since $p(q, \theta_i)$ must remain bounded around $\theta_i = \bar{\theta}$ for all q, we obtain:

$$p(q, \theta_i) = C'(q) - \frac{1}{1 - F(\theta_i)} \int_{\theta_i}^{\bar{\theta}} \psi(q, x) f(x) dx. \tag{15}$$

Taking into account this expression of the marginal contributions and using (9) yield the following modified Lindahl-Samuelson conditions:

$$C'(q(\theta_1, \theta_2)) = \sum_{i=1}^{2} \frac{1}{1 - F(\theta_i)} \int_{\theta_i}^{\bar{\theta}} \psi(q(\theta_1, \theta_2), x) f(x) dx. \tag{16}$$

Conditions (15) and (16) might sometimes suffice to characterize the marginal contribution and output at an equilibrium.

Example 1. Let us extend our Running Example. Assume that the principals' types are still independently and uniformly distributed on $\Theta = [\underline{\theta}, \bar{\theta}]$. The following marginal contribution which is linear in type and satisfies SMP is part of a symmetric equilibrium:

$$p(q, \theta) = \frac{\theta}{2} - \frac{\bar{\theta}}{6} + \frac{C'(q)}{3}.$$

One can check that condition (8) holds so that this is a best-response for each principal to offer such a marginal contribution given that the other principal also does so.

The equilibrium output satisfies

$$C'(q(\theta_1, \theta_2)) = \frac{3}{2}(\theta_1 + \theta_2) - \bar{\theta}, \tag{17}$$

and thus $\psi(q, \theta) = \frac{2}{3}(\bar{\theta} + C'(q)) - \theta$ (with $\frac{\partial \psi}{\partial q}(q, \theta) > 0$ and $\frac{\partial \psi}{\partial \theta}(q, \theta) < 0$). Marginal contributions are always positive for any equilibrium output when $\Delta\theta$ is small enough, namely $3\underline{\theta} > \bar{\theta}$. ■

Example 2. Consider an exponential distribution on the unbounded support $\Theta = [\underline{\theta}, +\infty)$, with $1 - F(\theta) = exp\left(-r(\theta - \underline{\theta})\right)$, $r > 0$ and $\underline{\theta} > \frac{1}{r}$. Looking again for a symmetric equilibrium with a marginal contribution which is linear in type and satisfies SMP, we find:

$$p(q, \theta) = \theta - \frac{1}{r}.$$

Again (8) holds and the non-negative equilibrium output is such that

$$C'(q(\theta_1, \theta_2)) = \theta_1 + \theta_2 - \frac{2}{r}, \tag{18}$$

so that $\psi(q, \theta) = \frac{2}{r} + C'(q) - \theta$ (with again $\frac{\partial \psi}{\partial q}(q, \theta) > 0$ and $\frac{\partial \psi}{\partial \theta}(q, \theta) < 0$). Marginal contributions are always positive for any equilibrium output since $\underline{\theta} > \frac{1}{r}$. ■

5 Existence and Uniqueness

To get further insights on the structure of equilibria, it is useful to describe an equilibrium in terms of its isoquant lines $\theta_2 = \psi(q, \theta_1)$. Rewriting conditions (8) and (9) along such isoquant yields:

$$\begin{aligned} p(q,\theta) + p(q,\psi(q,\theta)) &= C'(q), &(19)\\ \psi(q,\theta) - p(q,\psi(q,\theta)) &= R(\theta)\frac{\partial p}{\partial \theta}(q,\theta), &(20)\end{aligned}$$

for all (q, θ), where q is in the range of the equilibrium schedule of outputs $q(\cdot)$.[28]

For a distribution with finite support, positive density and having $|f'(\theta)|$ bounded, those two equations are already quite informative on the shape of the marginal contribution at its boundaries on any isoquant. For any q such that $2\bar{\theta} \geq C'(q) \geq \bar{\theta} + \underline{\theta} - \frac{1}{2f(\underline{\theta})}$, the highest type on the q-isoquant is $\bar{\theta}$ whereas the lowest type $\underline{\theta}(q) \geq \underline{\theta}$ is increasing in q. We show in the Appendix (Lemma 6) that marginal contributions at those boundaries satisfy:

$$p(q,\underline{\theta}(q)) = \underline{\theta}(q),\ \frac{\partial p}{\partial \theta}(q,\underline{\theta}(q)) = \frac{1}{2},\ \ p(q,\bar{\theta}) = C'(q) - \underline{\theta}(q) < \bar{\theta} \text{ and } \ \frac{\partial p}{\partial \theta}(q,\bar{\theta}) > 0.$$

Solving for (19) and (20) at a fixed q means looking for a function $x(\theta) = p(q, \theta)$ increasing in θ (and thus invertible) on a domain $[\underline{\theta}(q), \bar{\theta}]$ which satisfies the following non-standard functional equation:

$$R(\theta)\dot{x}(\theta) - x(\theta) + C'(q) = x^{-1}(C'(q) - x(\theta)) \quad (21)$$

with the boundary conditions[29]

$$x(\underline{\theta}(q)) = \underline{\theta}(q) \text{ and } x(\bar{\theta}) = C'(q) - \underline{\theta}(q). \quad (22)$$

(21) is not a standard differential equation since it depends not only on the function and its (non-negative) derivative but also on its inverse. Standard results do not apply to guarantee existence and uniqueness of such a solution. Moreover, the boundary conditions (22) are such that (21) has a singularity at $\bar{\theta}$. Hence, the analysis needed to prove existence has to rely on a global approach. In this respect, a more tractable way to prove existence is to work with the equilibrium distribution of marginal prices.[30] Doing so turns out also to provide new intuition on how each principal computes his best response.

Fix q and denote by $G(p, q)$ the cumulative distribution of marginal price $p(q, \theta)$ on that isoquant, i.e., $G(p, q) = \Pr[p(q, \theta) \leq p]$. Since we are interested in deriving equilibria

[28]Notice that, by definition of a conjugate type, it must also be that $\psi(q, \psi(q, \theta)) = \theta$, for all $\theta \in [\underline{\theta}, \bar{\theta}]$.

[29]We focus on the case $2\bar{\theta} \geq C'(q) \geq \bar{\theta} + \underline{\theta} - \frac{1}{2f(\underline{\theta})}$, i.e., outputs close enough to the first-best when both principals have the highest valuation since it appears to be the most interesting case. Lower output levels correspond to less stringent boundary conditions which are thus less constraining for the equilibrium characterization.

[30]A similar trick is used by Leininger, Linhart and Radner (1989) for double auctions and Wilson (1993) for nonlinear pricing.

with strictly increasing marginal contribution, $G(p,q)$ has no atom. Denote then by $g(p,q) = \frac{\partial G}{\partial p}(p,q)$ the corresponding density. A priory, only agents with type $\theta \geq \underline{\theta}(q)$ may lie on that isoquant q and the boundary conditions (22) tell us that the range of prices $p(q,\theta)$ must be $[\underline{\theta}(q), C'(q) - \underline{\theta}(q)]$. By the monotonicity of $p(\theta,q)$, we have $G(p(q,\theta),q) = F(\theta)$ and $g(p(q,\theta))\frac{\partial p}{\partial \theta}(q,\theta) = f(\theta)$ and we may extend $G(p,q)$ for $p \in [\underline{\theta}, \underline{\theta}(q)]$ with the convention that types $\theta \leq \underline{\theta}(q)$ contribute at the margin their valuation, i.e., $p(q,\theta) = \theta$.

The equilibrium condition (21) can be rewritten using the definition of $G(\cdot,q)$ as

$$F\left(C'(q) - p + \frac{1 - G(p,q)}{g(p,q)}\right) = G(C'(q) - p, q).$$

From this, we obtain the following functional equation:

$$\frac{\frac{\partial G}{\partial p}(p,q)}{1 - G(p,q)} = \frac{1}{F^{-1}(G(C'(q) - p, q)) - C'(q) + p}. \tag{23}$$

The boundary conditions (22) yield

$$G(\underline{\theta}(q), q) = F(\underline{\theta}(q)) \text{ and } G(C'(q) - \underline{\theta}(q), q) = 1. \tag{24}$$

The next theorem provides our existence result. For this we need the following technical assumption on the hazard function:[31]

$$\min_{\theta \in \Theta} \quad \theta + R(\theta) = \bar{\theta}. \tag{25}$$

Theorem 3 *Assume that (25) holds. A solution $G(p,q)$ to the system (23)-(24) (or alternatively a solution $p(q,\theta)$ to (21)-(22)) exists. This solution $G(p,q)$ (resp. $p(q,\theta)$) is increasing in p (resp. θ).*

It is instructive to sketch the proof of Theorem 3. The first step is to consider the sequence of distributions of marginal contributions that each principal plays in turn at a best response to what the other offers starting from the simple case where one principal, say P_1, would myopically offer a marginal contribution always equal to his own valuation. Under the weak condition (25), principal P_2 reacts by himself offering a distribution which, at each iteration, dominates in the sense of first-order stochastic dominance that offered at the round before. For P_1, this is the reverse; each iterate is dominated by the previous one. Intuitively, a principal finds it worth to offer higher marginal contributions if the other offers lower contributions and vice-versa; the best-response mapping is monotonically

[31] Since $R(\overline{\theta}) = 0$ and $R'(\overline{\theta}) = -1$, it is straightforward that the convexity of the hazard function guarantees (25). Another sufficient condition is the log-concavity of the density. Notice that $R'(\theta) = -1 - R(\theta)\frac{d}{d\theta}(\ln(f(\theta)))$ and $R''(\theta) = -R'(\theta)\frac{d}{d\theta}(\ln(f(\theta))) - R(\theta)\frac{d^2}{d\theta^2}(\ln(f(\theta)))$ which imply that every critical point $\theta^* < \overline{\theta}$ of the minimization problem (25) is such that $\frac{d}{d\theta}(\ln(f(\theta)))\,|_{\theta=\theta^*} = 0$. Thus, $R''(\theta^*) = -R(\theta^*)\frac{d^2}{d\theta^2}(\ln(f(\theta)))\,|_{\theta=\theta^*} \geq 0$ whenever $\ln(f(\theta))$ is a concave function. Therefore, every critical point is a local minimum which implies that $\theta = \overline{\theta}$ is the unique minimum. It is straightforward to check that uniform, and normal or exponential distributions restricted to finite supports satisfy log-concavity of the density.

decreasing. This iterative process converges towards a set of distributions which is stable in the following sense: if any principal offers a distribution of marginal prices from this set, the other principal's best response lies also in it. Schauder's Second Theorem[32] guarantees then existence of a distribution in that stable set which is a fixed-point.

Theorem 3 gives us the existence of a solution $p(q,\theta)$ to (21) for a given isoquant q. We must also check that, as q increases, the corresponding $\psi(q,\theta)$ derived from the knowledge of $p(q,\theta)$ increases in q to ensure concavity of the principals' problems as requested by Theorem 1. Using that $\psi(q,\underline{\theta}(q)) = \bar{\theta}$ in any equilibrium and differentiating with respect to q yields $\frac{\partial \psi}{\partial q}(q,\theta) > 0$ in the neighborhood of $\underline{\theta}(q)$. Hence, concavity holds when $\Delta\theta$ is small enough. The monotonicity of output follows from Theorem 2.

We have been silent so far about uniqueness. In this respect, we have:

Theorem 4 *Assume that types are uniformly distributed, then the solution to the system (23)-(24) is unique.*

This result is of some importance for what follows. In the case of a uniform distribution, the unique separating equilibrium is given by (17) and, anticipating on Theorem 6 below, it is interim efficient.

With an unbounded support however, $\underline{\theta}(q)$ is not properly defined, and there is no boundary condition that must be satisfied by the price schedule at $\bar{\theta} = +\infty$. This indeterminacy opens the door to a multiplicity of equilibria as shown by the example below.

Example 2 (cont.). Assume that types are distributed according to an exponential distribution $F(\theta) = 1 - \exp(-r(\theta - \underline{\theta}))$ on $[\underline{\theta}, \infty)$ with $\underline{\theta} > 1/r$. There exists a whole continuum of equilibria $p(q,\theta)$ which solve (21). Those equilibria are such that $p(q,\theta) < \theta - 1/r$. Inefficiencies in any of those equilibria are stronger than in the equilibrium where $p(q,\theta) = \theta - 1/r$ that we already exhibited above.[33] ■

6 Welfare Properties

6.1 Ex Post Inefficiency

The following necessary implementability condition makes it easy to check whether a given output schedule cannot be implemented as a common agency equilibrium.

Lemma 2 *Any equilibrium output $q(\cdot)$ must satisfy:*

$$E_{(\theta_1,\theta_2)}\left[\left(\sum_{i=1}^{2}\theta_i - R(\theta_i)\right)q(\theta_1,\theta_2) - C(q(\theta_1,\theta_2))\right] \geq 2\underline{\theta}q(\underline{\theta},\underline{\theta}) - C(q(\underline{\theta},\underline{\theta})) > 0. \quad (26)$$

[32]Burton (2005, Chapter 3).

[33]See the proof in the Appendix.

Condition (26) says that the expected virtual surplus (where marginal valuations θ_i are replaced by their virtual values $\theta_i - R(\theta_i)$) is worth at least the whole surplus generated in the worst scenario where both principals have the lowest type. This is similar to the standard feasibility condition that arises in asymmetric information models with independent types once Bayesian incentive compatibility, ex post budget balanced and individual rationality constraints are aggregated altogether.[34] In the contexts used so far in this literature, there is no restriction in the centralized mechanisms that an uninformed mediator can use to implement an allocation and this condition turns out to be also sufficient: Given any output schedule satisfying the implementability condition, one can find transfers which are ex post budget-balanced, Bayesian incentive compatible and individual rational for the informed players. Here, the added requirement is that the allocation should arise as the equilibrium of a common agency game and budget balance is replaced by the weaker requirement that the agent's information rent is non-negative. Condition (26) is here no longer sufficient for implementation as a common agency equilibrium. Indeed, such an allocation must also solve the functional equation (16).

Nevertheless, the necessary condition (26) is enough to get sharp results. Indeed, Examples 1 and 2 above already showed existence of ex post inefficient equilibria. Equipped with condition (26), it is straightforward to check that ex post inefficiency always arises.

Theorem 5 *The first-best output $q^{FB}(\theta_1, \theta_2)$ never satisfies condition (26) and thus cannot be achieved at any common agency equilibrium under asymmetric information.*

This result echoes the discussion after Theorem 2 but it sharpens it. Equipped with Theorems 2 and 5, we can conclude that there is always some downward distortion of the equilibrium output below the first-best for at least a set of types with non-zero measure. This contrasts sharply with the case of complete information where common agency games have efficient equilibria sustained with "truthful" schedules.

6.2 Interim Inefficiency

Under asymmetric information, one can still be interested in the normative properties of common agency equilibria provided that *interim efficiency* is used as the welfare criterion. We now investigate under which circumstances an equilibrium might be interim efficient.

Interim efficient allocations are obtained as solutions of a centralized mechanism design problem.[35] An uninformed mediator offers a centralized mechanism to both principals, who then report their types to this mediator. This mediator maximizes a weighted sum of the principals' and the agent's utilities with the weights given to different types of principals being possibly different. Because we want to replicate with such centralized mechanism a symmetric common agency equilibrium, we consider symmetric weights which do not depend on the principal's identity.

[34]Myerson and Satterthwaite (1983) developed such condition for the case of bargaining, whereas Laffont and Maskin (1982), Güth and Hellwig (1987), Mailath and Postlewaite (1990), Ledyard and Palfrey (1999), and Hellwig (2003) did so for the case of public goods.

[35]Holmström and Myerson (1983).

Lemma 3 *An interim efficient profile $q(\theta_1, \theta_2)$ non-decreasing in each argument (resp. increasing) is such that there exist positive social weights $\alpha(\theta) > 0$*[36] *such that $\int_{\underline{\theta}}^{\bar{\theta}} \alpha(\theta) f(\theta) d\theta \leq 1$*[37] *and*

$$\sum_{i=1}^{2} b(\theta_i) = C'(q(\theta_1, \theta_2)) \tag{27}$$

where $b(\theta_i) = \theta_i - R(\theta_i)(1 - \tilde{\alpha}(\theta_i))$ is non-decreasing (resp. increasing) in θ_i and $\tilde{\alpha}(\theta_i) = \frac{1}{1-F(\theta_i)} \int_{\theta_i}^{\bar{\theta}} \alpha(x) f(x) dx$.

Equation (27) is again a Lindhal-Samuelson condition under asymmetric information where valuations are replaced by virtual valuations reflecting the weights that different types have in the social welfare function that is maximized by the uninformed mediator in charge of finding such interim efficient allocation.

Examples 1 and 2 (cont.). For a uniform distribution having support $\Theta = [\underline{\theta}, \bar{\theta}]$, the solution found in (17) remains interim efficient with the uniform weight $\alpha(\theta) = \frac{1}{2}$ for all θ. Even though the type distribution has unbounded support, positive results can also be found for Example 2 with the uniform weight $\alpha(\theta) = 0$, i.e., principals have no weight in the social welfare objective maximized by the uninformed mediator. ■

Altogether (16) and (27) show that any increasing candidate function $b(\cdot)$ must solve the following functional equation:

$$\sum_{i=1}^{2} b(\theta_i) = \sum_{i=1}^{2} \frac{1}{1 - F(\theta_i)} \int_{\theta_i}^{\bar{\theta}} b^{-1}(b(\theta_1) + b(\theta_2) - b(x)) f(x) dx \quad \forall (\theta_1, \theta_2) \in \Theta^2. \tag{28}$$

This condition is rather stringent. As a result, it is not surprising that there are few candidates for such $b(\cdot)$ and $F(\cdot)$ functions that altogether ensure interim efficiency.

Theorem 6 *A symmetric equilibrium of a common agency game is interim efficient if and only if the inverse hazard rate $R(\theta)$ is linear.*

Theorem 6 implies that the only possibility for interim efficiency in the case of distributions having finite support arises with the β-density function $f(\theta) = \frac{1+\beta}{\Delta\theta^{1+\beta}}(\bar{\theta} - \theta)^{\beta}$ (for $\beta \geq 0$).[38] The function $b(\cdot)$ is then linear ($b(\theta) = \frac{(\beta+3)\theta - \bar{\theta}}{\beta+2}$), isoquants have slope -1 in the

[36] We focus on the case where all types receive a positive social weight in the social welfare criterion. Without this assumption, we would get the unpalatable conclusion that giving only a Dirac mass to types $\bar{\theta}$ trivially achieves efficiency since the equilibrium output has no distortion at the top. Also, given that we focused above in separating equilibria with strictly monotonically increasing allocations as described in Theorem 1, we restrict to social weights that induce monotonically increasing allocations as well.

[37] This inequality captures the possibility that the common agent receives a positive weight in the social welfare function maximized by the uninformed mediator. Remember that Theorem 2 shows that, in any common agency symmetric equilibrium, the agent gets a non-negative ex post rent $U(\theta_1, \theta_2)$ which should be accounted for when evaluating welfare. This distinguishes our notion of interim efficiency from that used when it is assumed that budget is always balanced ex post (as in Ledyard and Palfrey, 1999).

[38] Note that $f(\bar{\theta}) = 0$ for that density so that Lemma 6 does not apply. In particular, $\frac{\partial^2 p}{\partial q \partial \theta}(q, \theta) \neq \frac{1}{2}$ for $\beta > 0$.

(θ_1, θ_2) space, and social weights are uniform ($\alpha(\theta) = \frac{1}{\beta+2}$). Marginal contributions are linear in type and positive for $\Delta\theta$ small enough:

$$p(q,\theta) = \frac{C'(q)}{\beta+3} + \bar{\theta}\left(\frac{\beta+1}{\beta+3}\right) - (\bar{\theta}-\theta)\left(\frac{\beta+1}{\beta+2}\right).$$

The derivative $\frac{\partial p}{\partial \theta}(q,\theta) = \frac{\beta+1}{\beta+2} > 0$ is independent of type and output. This is the SMP term that determines the distortions induced by each principal at a best response. When it is constant, each principal induces a distortion which does not depend on the other's type. This reduces the scope for manipulations by the agent and ensures interim efficiency.

An immediate corollary of Theorem 6 follows.

Theorem 7 *Public intervention through an uninformed mediator improves on the equilibrium outcome unless the inverse hazard rate $R(\theta)$ is linear.*

Although the common agency institution implements an interim efficient allocation for linear inverse hazard rates, beyond that case, players strictly gain from appealing to an uninformed mediator to collect contributions and move the outcome towards the interim efficiency frontier with a centralized mechanism.

This is an important insight. Contribution games under asymmetric information are unlikely to be efficient even in the weaker sense of interim efficiency. Beyond the linear inverse hazard rate case, those games entail too much screening with each principal trying to learn the other's type through the agent compared to a more centralized design with an uninformed mediator collecting direct messages from the privately informed principals.

7 Discussion

This section investigates a few extensions of our basic framework and discusses some modeling assumptions.

7.1 Delegation

The output distortion in our common agency game comes from the fact that each principal tries to screen the agent's endogenous information. This indirect communication seems overly costly. An alternative to the game of voluntary contributions could be for one principal, say P_{-i}, to provide the public good himself and to have direct communication between principals.[39] Assuming that the agent has no particular advantage in producing the public good himself, this would amount to consider that the principals' objective functions are now respectively:

[39]Matthews and Postlewaite (1989) analyzed the gains of allowing unmediated communication between bidders in a double-auction. Contrary to us, they did not give any productive role to one of those bidders.

$$V_i(\theta_i, q, t) = \theta_i q - t \text{ and } V_{-i}(\theta_{-i}, q, t) = \theta_{-i} q - C(q) + t.$$

Consider now the case where a mechanism is designed by an uninformed mediator who gives all bargaining power to principal P_i. It is straightforward to check that the optimal output obtained this way solves:[40,41]

$$\theta_i + \theta_{-i} - R(\theta_{-i}) = C'(q_i(\theta_i, \theta_{-i})).$$

The valuation θ_{-i} of the principal with no bargaining power is replaced by the lower virtual valuation $\theta_{-i} - R(\theta_{-i})$.

Furthermore, assuming now that types are uniformly distributed on $[\underline{\theta}, \bar{\theta}]$, and that each principal might have all bargaining power with probability one half, we find:

$$C'(q(\theta_1, \theta_2)) = \frac{1}{2}(C'(q_1(\theta_1, \theta_2)) + C'(q_2(\theta_2, \theta_1)))$$

where $q(\theta_1, \theta_2)$ is the (unique from Theorem 4) equilibrium output obtained in the common agency game which is defined by (17). Indeed, under common agency, both principals have the same bargaining power and their valuations θ_i are replaced by virtual valuations $\theta_i - \frac{1}{2}R(\theta_i)$ with only a weight one half on the inverse hazard rate distortion term. From this, we obtain immediately:

Proposition 1 *Assume that types are uniformly distributed on $[\underline{\theta}, \bar{\theta}]$, then the average output implemented with asymmetric bargaining situations $\frac{1}{2}(q_1(\theta_1, \theta_2) + q_2(\theta_2, \theta_1))$ is greater (resp. lower, equal) than the equilibrium output under common agency if $C'(\cdot)$ is concave (resp. convex, linear).*

When $C(q) = \frac{q^2}{2}$ as in our Running Example, the equilibrium output under common agency is the exact mean between those implemented by delegating with probability one half the contracting power to either principal. Because virtual valuations under common agency only entail half of the inverse hazard rate distortions, common agency reduces output fluctuations around that mean. Strict concavity of the surplus function implies that an ex ante efficiency criterion would select common agency rather than an institution that delegates with probability one half all contracting power to either principal. This result justifies our focus on a game with voluntary contributions in the first place.

7.2 Pooling and Bunching

Pooling Equilibria. Our focus on separating informative equilibria where principals reveal their types through their offers made clearer what is the agent's endogenous information vis-à-vis each of them. This made also the analysis of information aggregation more relevant by stressing the most favorable case for it.

[40] Assuming that $\Delta\theta$ is small enough to get positive output and marginal contributions.

[41] This result arises also when principal P_1 offers himself the mechanism (Mylovanov, 2005).

In contrast, it is possible to construct uninformative equilibria. In such equilibria, all types of a given principal pool and offer the same contribution that specifies a payment for a given output target q^*, the agent learns nothing form observing that contribution, the other principal has nothing to screen about and is forced to agree on this output target if any production takes place. Consider thus the forcing contribution:

$$t^*(q) = \begin{cases} \frac{C(q^*)}{2} > 0 & \text{for } q = q^* > 0 \\ 0 & \text{for } q \neq q^*. \end{cases} \tag{29}$$

When both principals offer this contract, they share equally the cost of producing q^*.

Denote also respectively by $\hat{W}(\theta) = \max_q \theta q - C(q)$ and $\hat{q}(\theta) = \arg\max_q \theta q - C(q)$ the aggregate payoff of a bilateral coalition between a principal with type θ and the agent and the corresponding optimal (increasing) output.

Proposition 2 *Assume that* $2\underline{\theta}q^* - C(q^*) \geq 2\hat{W}(\underline{\theta})$ *and* $q^* \geq \hat{q}(\bar{\theta})$.[42] *There exists a pooling equilibrium in which both principals offer* $t^*(q)$ *whatever their types. This equilibrium is sustained with arbitrary out-off equilibrium beliefs.*

Running Example (cont.). The forcing contributions above give us an example of a non-differentiable equilibrium. Assuming that the cost function is quadratic, we immediately observe that the best such symmetric forcing contracts[43] implement an output equal to $q^* = \underline{\theta} + \bar{\theta}$ giving an ex ante welfare worth $W^P = \frac{1}{2}(\underline{\theta} + \bar{\theta})^2$. Instead, tedious computations show that the linear equilibrium yields a lower ex ante welfare worth $W^S = W^P - \frac{\Delta\theta^2}{16}$. In other words, principals are somewhat able to weaken competition with those rather inflexible contracts. ■

Bunching. The pooling equilibria above are such all types of principals offer the same contribution and the agent chooses a fixed output. Starting from the separating equilibria stressed above, one can construct other equilibria which still induce the agent to choose a fixed output if his type belongs to some interval with a positive measure although the principals' types are revealed through contract offers. Coming back to our Running Example, consider indeed the following schedules:

$$t(q, \theta_i) = \begin{cases} 0 & \text{if } q < q^* \\ \left(\frac{\theta_i}{2} - \frac{\bar{\theta}}{6}\right) q + \frac{q^2}{6} + t_i^0(\theta_i), & \text{otherwise,} \end{cases} \tag{30}$$

where $q^* \in (3\underline{\theta} - \bar{\theta}, 2\bar{\theta})$ and $t_i^0(\theta_i) = \frac{(q^*)^2}{12} - \left(\frac{\theta_i}{2} - \frac{\bar{\theta}}{6}\right) q^*$. Those schedules are such that all types $(\theta_1, \underline{\theta})$ or $(\underline{\theta}, \theta_2)$ get zero rent. They are discontinuous at some q^* that lies in the range of equilibrium outputs defined in (6) for the informative equilibrium. It can be checked that for all pairs (θ_1, θ_2) such that $\frac{3}{2}(\theta_1 + \theta_2) - \bar{\theta} \leq q^*$, the agent chooses q^* when offered those contributions. Bunching arises due to the discontinuity at q^*.[44]

[42] It can be easily seen that the set of such q^* is non-empty when $q^{FB}(\underline{\theta}, \underline{\theta}) \geq \hat{q}(\bar{\theta})$, i.e., $2\underline{\theta} \geq \bar{\theta}$.

[43] For this to be an equilibrium, we need to check the condition $2\underline{\theta}q^* - C(q^*) \geq 2\hat{W}(\underline{\theta})$ from Proposition 2. For our Running Example, this amounts to $\underline{\theta} \geq (\sqrt{2} - 1)\bar{\theta}$, which is slightly stronger than the assumption $3\underline{\theta} \geq \bar{\theta}$.

[44] To check that those schedules are best responses to each other, one has only to check that the forcing contract region (i.e., $q < q^*$) does not induce deviation for each principal. The condition is similar to that in Proposition 2, $2\theta_i q^* - C(q^*) \geq 2\hat{W}(\theta_i)$ for $\hat{q}(\theta_i) < q^*$ which becomes $q^* \in ((2\sqrt{2} - 1)\underline{\theta}, (2\sqrt{2} + 1)\underline{\theta})$.

7.3 Communication

Our previous analysis has focused on a particular strategy space for competing principals: the space of nonlinear contributions. Although it is quite natural, it might restrict communication since all information revelation takes place through the choice of a particular schedule and happens thus prior to the agent's choices on acceptance and production. One may wonder whether there would be any gain for principals to send messages to their common agent after the offer stage, or equivalently, to offer a menu of such contributions, from which they will later pick one after the agent's acceptance.

Suppose that principal P_i can offer any more general mechanism consisting of a collection of contribution schedules $\tilde{t}_i(q, \cdot) = \{\tilde{t}_i(q, \hat{\theta}_i)\}_{\hat{\theta}_i \in \Theta}$. The output q is the agent's choice and $\hat{\theta}_i$ a message sent by that principal at a communication stage that takes place following the agent's acceptance. From the Revelation Principle, there is no loss of generality focusing on such direct communication when computing best-responses in any pure strategy equilibrium. For technical reasons, we assume that $\tilde{t}_i(q, \hat{\theta}_i)$ is continuous in q and $\hat{\theta}_i$. We consider also the following sequence of events where the agent chooses an output after principals have picked schedules within the menu of contributions they respectively proposed. Finally, communication opens new possibilities for contracting and, in particular, principals may find it attractive to offer "inscrutable" menus of contribution schedules that do not reveal their types, letting the agent only break even between accepting or not such offer in expectations. Accordingly, the strategy space of contributions is enlarged by allowing also for negative transfers if needed. Payments follow according to the principals' and the agent's choices. Denote by Γ^* the game thereby modified by appending these communication possibilities.

We now show the robustness of any separating equilibrium in Γ as defined through Theorem 1 to such extension of the strategy space. An equilibrium in Γ yields payoffs to principals that remain equilibrium payoffs in Γ^*. Given $t(q, \theta)$ a separating (equilibrium) strategy in Γ, we define a degenerate extension of this strategy in Γ^* as a collection of contribution schedules $t^*(q, \cdot|\theta)$ such that $t^*(q, \hat{\theta}_i|\theta) = t(q, \theta)$ for all $\hat{\theta}_i \in \Theta$. With such degenerate extension, principal P_i's contribution does not depend on his message $\hat{\theta}_i$.

Proposition 3 *Take any symmetric equilibrium in Γ corresponding to the contribution schedule $t(q, \theta_i)$. There exists a perfect Bayesian equilibrium of Γ^* such that principal P_i with type θ_i offers a degenerate menu $t^*(q, \hat{\theta}_i|\theta_i) = t(q, \theta_i)$ for all $(\hat{\theta}_i, \theta_i)$.*[45]

[45]The literature on informed principal problems (Myerson, 1983, Maskin and Tirole, 1990, 1992) in monopolistic environments has stressed the value of pooling offers where different types of principals offer the same mechanism (a menu of contribution schedules) delaying communication to a later stage. Such delayed communication is attractive when the agent is risk-averse (because it allows to pool incentive constraints), or under common values (because it avoids signaling distortions). With private values and risk-neutrality, no such benefit arises as shown by Mylovanov (2005) in a model as ours with a continuum of types. Allowing communication does not break equilibrium. Proposition 3 confirms that result. (The proof in the Appendix constructs the out-of equilibrium beliefs explicitly and uses the compactness of the menu to prove that each principal finds it optimal to offer the informative mechanism $t^*(q, \cdot|\theta_i)$ at a best response.) More precisely, each principal has also in his best-response correspondence in Γ^* a degenerate menu of contributions which are all equal to his equilibrium strategy in Γ and all information revelation takes place at the offer stage. In a related vein, Peters (2003) found conditions under which

When communication is allowed, we may also ask whether there is the possibility of sustaining "inscrutable" equilibria in which all principals pool and offer the same menu of contributions, so that nothing is learned by the agent and his acceptance decision takes place in expectations. Such *ex ante* acceptance could relax participation constraints and reduce screening distortions.

Two remarks are in order. First, Section 7.2 shows that there exist pooling equilibria with forcing contributions which proves existence of such "inscrutable" equilibria. However, pooling is by and large induced by the nature of those non-differentiable contributions that force all types of principals to agree on equal-sharing of the cost of implementing a given output target. Second, moving back to more flexible differentiable contributions, next proposition shows an impossibility result.

Proposition 4 *There does not exist any perfect Bayesian equilibrium of* Γ^* *such that both principals* P_i *pool whatever their types* θ_i *and offer the same "inscrutable" menu of differentiable contribution schedules* $\{t(q, \hat{\theta}_i)\}_{\hat{\theta}_i \in \Theta}$.

The intuition behind this proposition can be grasped in two steps. First, observe that, with an inscrutable offer by principal P_2, principal P_1 and the agent have symmetric but incomplete information on θ_2 at the time of contracting. Under such ex ante contracting, it is well known that the differentiable "sell-out" contribution schedule $t_1^S(q, \theta_1) = \theta_1 q - V^S(\theta_1)$ (where $V^S(\theta_1)$ is principal P_1's payoff) maximizes the bilateral payoff of the coalition between that principal and the agent. Provided P_2's offer is itself differentiable in q, this is the unique way to maximize this bilateral payoff. Offering the menu $\{t^S(q, \hat{\theta}_1)\}_{\hat{\theta}_1 \in \Theta}$ is thus part of an inscrutable best response for principal P_1. By the same token, principal P_2 also offers a menu of sell-out contracts. Given those offers, the agent chooses an efficient output. But such ex post efficient allocation cannot be implemented from Theorem 5 which yields a contradiction.

8 Conclusion

Let us summarize the main findings of our analysis.

First, modeling private information on the principals' preferences in a common agency game justifies the use of nonlinear contributions for screening purposes, whereas such strategy space is given a priori in previous complete information models. Doing so introduces incentive compatibility conditions which replace the "truthfulness" requirement used earlier on. Under asymmetric information, principals reveal their types to the common agent through their mere offer of contributions and try to learn about the types of others which have been endogenously learned by the common agent from observing these offers.

Second, ex post inefficiency always arises at equilibrium contrary to complete information models. The reason is not the standard "free-riding" phenomenon stressed by the

principals do not gain from offering more than a take-it-or-leave-it offer in common agency environments with complete information on their preferences.

centralized mechanism design approach but it comes now from the desire of each principal to screen the agent about the endogenous information he has learned from observing others' offers. The common agent at the nexus of all information sets may indeed pretend that each principal contributes less than what he really does.

Third, the weaker criterion of interim efficiency may be satisfied by some separating equilibria only when the inverse hazard rate of the types distribution is linear. This suggests that principals might generally find it worth agreeing on more centralized mechanisms to improve on the equilibrium outcome achieved with voluntary contributions.

Fourth, and from a more technical viewpoint, we developed techniques to prove existence of at least one differentiable equilibrium which solves a complex functional equation linking the marginal contribution, its type derivative, and its inverse. The difficulty in solving that equation comes from having boundary conditions at both ends of the types interval. Existence has to follow from a global approach. The techniques we developed are likely to be valuable beyond the specific examples analyzed here to tackle existence in other settings where principals offer contribution schedules in an effort to control a common agent's choice. Uniqueness is proved for the uniform distribution.

Finally, although we restricted principals to make single take-it-or-leave-it offers, we show that the separating equilibria we focus on are robust when principals may entertain more complex communication with their agent. Other extensions that were investigated dealt with the existence of pooling and uninformative equilibria and the possibility of direct communication between principals. The latter provided a justification of the common agency institution as a means of maximizing ex ante welfare compared with more random and asymmetric allocations of the bargaining power between principals.

A few other extensions of our framework would be worth to pursue. Indeed, we have so far focused on the case where principals have no means of communicating one with the other. The motivation for doing so is twofold. First, and from a practical viewpoint, this may be viewed as describing equilibrium behavior when principals do not know each other before contributing or when opening communicating channels between principals is prohibitively costly or even forbidden. Coming back on our earlier motivating examples, the first situation may capture what happens when different sovereign countries contribute for a transnational public good whereas the second case is more likely when different governmental bodies, separated by "Chineese walls", contribute to the financing of a public good. Second, and from a theoretical viewpoint, this focus on non-communication between principals gives us a reference point to analyze in future research the benefits of adding either direct or mediated communication. Following Agastya, Menezes and Sengupta (2006) who studied equilibria in a game of voluntary contributions for a 0-1 project appended with a cheap-talk stage, we might conjecture that more equilibria might arise when such communication is possible.

A particular way by which communication takes place is when principals contribute sequentially.[46] Distortions might then depend on whether offers are publicly observable by subsequent principals or not. In the latter case, we would be back to an analysis of

[46]See the related work of Marx and Matthews (2000).

the Stackelberg timing whereas our previous analysis focused on simultaneous offers.[47] In the former case, we would have also to take into account how the first contributors may manipulate beliefs of subsequent contributors to reduce its own contribution.[48]

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Appendix

Proof of Lemma 1: The proof is standard (see for instance Laffont and Martimort 2002, Chapter 3) and is thus omitted. ■

Proof of Theorem 1: First, we assume that the agent is informed on P_i's type and we transform problem $\mathcal{P}_i(\theta_i)$ to get $\mathcal{P}'_i(\theta_i)$. Then, we compute P_i's best response $\{t_i(q,\theta_i)\}_{\theta_i\in\Theta}$ to a strategy profile $\{t_{-i}(q,\theta_{-i})\}_{\theta_{-i}\in\Theta}$ satisfying IT and SMP. From this, we derive the optimality conditions (8). To do so, we also assume quasi-concavity of the agent's problem and the fact that the participation constraint (5) binds only at $\theta_{-i}=\underline{\theta}$. Second, we show that these conditions are indeed satisfied.

• **Pointwise optimization:** Consider P_i's best response to a strategy profile $\{t_{-i}(q,\theta_{-i})\}_{\theta_{-i}\in\Theta}$ used by P_{-i} and satisfying IT. $U^D(\theta_{-i}|\theta_i)$ is thus weakly increasing in θ_{-i} and (5) is binding at $\theta_{-i}=\underline{\theta}$ only provided that the marginal contribution $p_i(q,\theta_i)$ is positive (we show this last claim below). Integrating by parts, we then obtain:

$$E_{\theta_{-i}}\left[U^D(\theta_{-i}|\theta_i)\right]=E_{\theta_{-i}}\left[R(\theta_{-i})\frac{\partial t_{-i}}{\partial\theta}(q(\theta_{-i}|\theta_i),\theta_{-i})\right]+\hat{U}_{-i}(\underline{\theta}).$$

Inserting this latter expression into P_i's objective function, and neglecting the second-order condition (3) (that will be checked below) we obtain the reduced-form problem

$$\mathcal{P}_i'(\theta_i): \quad \max_{q^D(\cdot|\theta_i)} E_{\theta_{-i}}\left[S_i(q(\theta_{-i}|\theta_i),\theta_i,\theta_{-i})\right] \tag{A1}$$

where $S_i(q,\theta_i,\theta_{-i})$ denotes principal P_i's virtual surplus defined as

$$S_i(q,\theta_i,\theta_{-i}) = \theta_i q + t_{-i}(q,\theta_{-i}) - C(q) - R(\theta_{-i})\frac{\partial t_{-i}}{\partial \theta}(q,\theta_{-i}).$$

Define

$$\psi_{-i}(q,\theta_{-i}) = -p_{-i}(q,\theta_{-i}) + C'(q) + R(\theta_{-i})\frac{\partial p_{-i}}{\partial \theta}(q,\theta_{-i}). \tag{A2}$$

$S_i(q,\theta_i,\theta_{-i})$ is concave (resp. strictly concave) in q when

$$\frac{\partial^2 S_i}{\partial q^2}(q,\theta_i,\theta_{-i}) = \frac{\partial p_{-i}}{\partial q}(q,\theta_{-i}) - C''(q) - R(\theta_{-i})\frac{\partial^2 p_{-i}}{\partial\theta\partial q}(q,\theta_{-i}) \leq 0 \text{ (resp. } < 0)$$

which is true when

$$\frac{\partial \psi_{-i}}{\partial q}(q,\theta_{-i}) \geq 0 \text{ (resp. } > 0). \tag{A3}$$

(A3) yields the first condition in (12) for a symmetric equilibrium (where the index $-i$ has been suppressed). Under strict concavity, optimizing pointwise the virtual surplus in (A1) gives thus a unique output $q^D(\theta_{-i}|\theta_i)$ (which is interior since $\bar{Q}$ is large enough) implemented at a best response which satisfies

$$\frac{\partial S_i}{\partial q}(q^D(\theta_{-i}|\theta_i),\theta_i,\theta_{-i}) = 0 \Leftrightarrow \theta_i = \psi_{-i}(q^D(\theta_{-i}|\theta_i),\theta_{-i}). \tag{A4}$$

Hence condition (8) holds at a symmetric equilibrium satisfying IT and SMP.

Differentiating (A4) with respect to θ_i, we obtain:

$$\frac{\partial^2 S_i}{\partial q^2}\frac{\partial q^D}{\partial \theta_i} = -1 \tag{A5}$$

which yields the monotonicity property, under strict concavity

$$\frac{\partial q^D}{\partial \theta_i} > 0.$$

Therefore, principal P_i offers different output schedules as his type changes so that the family $t_i(q,\theta_i)$ is separating in θ_i.

Differentiating (A4) with respect to θ_{-i}, we obtain:

$$\frac{\partial^2 S_i}{\partial q^2}\frac{\partial q^D}{\partial \theta_{-i}} = R(\theta_{-i})\frac{\partial^2 p_{-i}}{\partial \theta^2} - \left(1 - \dot{R}(\theta_{-i})\right)\frac{\partial p_{-i}}{\partial \theta}. \tag{A6}$$

Differentiating (A2) with respect to θ_{-i} allows us to simplify (A6) to get

$$\frac{\partial^2 S_i}{\partial q^2}\frac{\partial q^D}{\partial \theta_{-i}} = \frac{\partial \psi_{-i}}{\partial \theta}. \tag{A7}$$

Hence, the other monotonicity property

$$\frac{\partial q^D}{\partial \theta_{-i}}(\theta_{-i}|\theta_i) \geq 0 \text{ (resp. } > 0)$$

and SMP ensures that the second-order condition for the agent's problem (3) holds when

$$\frac{\partial \psi_{-i}}{\partial \theta}(q, \theta_{-i}) \leq 0 \text{ (resp. } < 0). \tag{A8}$$

Again (A8) yields the second condition in (12) at a symmetric equilibrium with an output schedule increasing in both arguments.

• **Implementation of the best response through a nonlinear contribution** $t_i(q, \theta_i)$**:** At a best response to P_{-i}'s offer $t_{-i}(q, \theta_{-i})$ (with margin $p_{-i}(q, \theta_{-i})$), P_i cannot do better than offering himself a direct revelation mechanism which implements the increasing output $q^D(\cdot|\theta_i)$ satisfying

$$\theta_i + p_{-i}(q^D(\theta_{-i}|\theta_i), \theta_{-i}) - C'(q^D(\theta_{-i}|\theta_i)) = \theta_i - p_i(q^D(\theta_{-i}|\theta_i), \theta_i) = R(\theta_{-i})\frac{\partial p_i}{\partial \theta}(q^D(\theta_{-i}|\theta_i), \theta_{-i}) \tag{A9}$$

(which gives (8) at a symmetric equilibrium).

Denote the inverse function for $q^D(\theta_{-i}|\theta_i)$ by $\theta^D_{-i}(\cdot|\theta_i)$. We can reconstruct the nonlinear schedule $t_i(q, \theta_i)$ that P_i could as well offer as $t_i(q, \theta_i) = t_i^D(\theta^D_{-i}(q|\theta_i)|\theta_i)$ for q in the range of $q^D(\cdot|\theta_i)$. For $q \geq q^D(\bar{\theta}|\theta_i)$, we extend that schedule in a smooth-pasting way with a constant slope $p_i(q, \theta_i) = \theta_i$, i.e., $t_i(q, \theta_i) = t_i(q^D(\bar{\theta}|\theta_i), \theta_i) + \theta_i(q - q^D(\bar{\theta}|\theta_i))$ where $t_i(q^D(\bar{\theta}|\theta_i), \theta_i) = t_i(q^D(\underline{\theta}|\theta_i), \theta_i) + \int_{q^D(\underline{\theta}|\theta_i)}^{q^D(\bar{\theta}|\theta_i)} p_i(q, \theta_i)dq$ and $t_i(q^D(\underline{\theta}|\theta_i), \theta_i)$ is determined through the binding participation constraint $U^D(\underline{\theta}|\theta_i) = \hat{U}_{-i}(\underline{\theta})$. Note that this upward extension satisfies IT and SMP. For $q \leq q^D(\underline{\theta}|\theta_i)$, $t_i(q, \theta_i)$ is also extended in a smooth-pasting way below $q^D(\underline{\theta}|\theta_i)$ as a non-negative schedule by the formula $t_i(q, \theta_i) = \max\left\{0, t_i(q^D(\underline{\theta}|\theta_i), \theta_i) + \int_{q^D(\underline{\theta}|\theta_i)}^{q} p(x, \theta_i)dx\right\}$ where we take the following extension $p(x, \theta_i) = \int_{\underline{\theta}}^{\theta_i} \frac{\partial p}{\partial \theta_i}(q^D(\underline{\theta}|y), y)dy$ for all $x \leq q^D(\underline{\theta}|\theta_i)$. This downward extension satisfies IT and SMP by construction.

When written in terms of contribution schedules, the first- and second-order conditions for the agent's problem can be expressed as:

$$p_i(q^D(\theta_{-i}|\theta_i), \theta_i) + p_{-i}(q^D(\theta_{-i}|\theta_i), \theta_{-i}) = C'(q^D(\theta_{-i}|\theta_i)), \tag{A10}$$

and

$$\frac{\partial p_i}{\partial q}(q^D(\theta_{-i}|\theta_i), \theta_i) + \frac{\partial p_{-i}}{\partial q}(q^D(\theta_{-i}|\theta_i), \theta_{-i}) - C''(q^D(\theta_{-i}|\theta_i)) \leq 0$$

which give (9) and (10) at a symmetric equilibrium.

• **The agent's participation constraint (5) binds at** $\underline{\theta}$: We proceed with several lemmata. Define first the output level $\hat{q}_{-i}(\theta_{-i})$ when the agent does not take P_i's contribution as an arbitrary selection in the correspondence $\arg\max_{q \in \mathcal{Q}} \; t_{-i}(q, \theta_{-i}) - C(q)$ with $\hat{q}_{-i}(\theta_{-i}) = 0$ (resp. > 0) if $\max_{q \in \mathcal{Q}} \; t_{-i}(q, \theta_{-i}) - C(q) = 0$ (resp. $\max_{q \in \mathcal{Q}} \; t_{-i}(q, \theta_{-i}) - C(q) > 0$).

Lemma 4 *Assume that* $p_i(q, \theta_i) \geq 0$ *for all* (q, θ_i) *and that* $t_i(q, \theta_i)$ *satisfies SMP. Then, for any* (θ_i, θ_{-i})*, we have:*

$$q^D(\theta_{-i}|\theta_i) \geq q^D(\theta_{-i}|\underline{\theta}) \geq \hat{q}_{-i}(\theta_{-i}). \tag{A11}$$

Proof: By the definitions of $\hat{q}(\theta_{-i})$ and $q^D(\theta_{-i}|\underline{\theta})$ respectively, we have:

$$t_{-i}(\hat{q}(\theta_{-i}), \theta_{-i}) - C(\hat{q}(\theta_{-i})) \geq t_{-i}(q^D(\theta_{-i}|\underline{\theta}), \theta_{-i}) - C(q^D(\theta_{-i}|\underline{\theta})),$$

$$t_i(q^D(\theta_{-i}|\underline{\theta}), \underline{\theta}) + t_{-i}(q^D(\theta_{-i}|\underline{\theta}), \theta_{-i}) - C(q^D(\theta_{-i}|\underline{\theta})) \geq t_i(\hat{q}(\theta_{-i}), \underline{\theta}) + t_{-i}(\hat{q}(\theta_{-i}), \theta_{-i}) - C(\hat{q}(\theta_{-i})).$$

Adding up these inequalities, we get

$$t_i(q^D(\theta_{-i}|\underline{\theta}), \underline{\theta}) - t_i(\hat{q}(\theta_{-i}), \underline{\theta}) = \int_{\hat{q}(\theta_{-i})}^{q^D(\theta_{-i}|\underline{\theta})} p_i(x, \underline{\theta}) dx \geq 0.$$

Since marginal transfers are positive, the last inequality is true only if $q^D(\theta_{-i}|\underline{\theta}) \geq \hat{q}(\theta_{-i})$.

Moreover, if $p_i(q, \theta_i)$ satisfies SMP, $p_i(q^D(\theta_{-i}|\theta_i), \theta_i) \geq p_i(q^D(\theta_{-i}|\theta_i), \underline{\theta})$ and, from (A10), we obtain:

$$p_i(q^D(\theta_{-i}|\theta_i), \underline{\theta}) + p_{-i}(q^D(\theta_{-i}|\theta_i), \theta_{-i}) - C'(q^D(\theta_{-i}|\theta_i)) \leq 0.$$

Quasi-concavity of the agent's problem at $(\underline{\theta}, \theta_{-i})$ yields finally $q^D(\theta_{-i}|\theta_i) \geq q^D(\theta_{-i}|\underline{\theta})$.

Thus, we necessarily have $\hat{q}_{-i}(\theta_{-i}) \leq q^D(\theta_i|\theta_{-i}) = q^D(\theta_{-i}|\theta_i)$ for all θ_i. ■

Lemma 5 *Assume that* $p_i(q, \theta_i) \geq 0$ *for all* (q, θ_i)*. Then,* $U^D(\theta_{-i}|\theta_i) \geq \hat{U}_{-i}(\theta_{-i})$ *for any* (θ_i, θ_{-i}) *if* $U^D(\underline{\theta}|\theta_i) \geq \hat{U}_{-i}(\underline{\theta})$ *holds.*

Proof: Using the Envelope Theorem, we get $\frac{\partial U^D}{\partial \theta_{-i}}(\theta_{-i}|\theta_i) = \frac{\partial t_{-i}}{\partial \theta}(q^D(\theta_{-i}|\theta_i), \theta_{-i})$ and $\frac{\partial \hat{U}_{-i}}{\partial \theta_{-i}}(\theta_{-i}) = \frac{\partial t_{-i}}{\partial \theta}(\hat{q}_{-i}(\theta_{-i}), \theta_{-i})$. Hence, we always get:

$$\frac{\partial \hat{U}_{-i}}{\partial \theta_{-i}}(\theta_{-i}) = \frac{\partial t_{-i}}{\partial \theta}(\hat{q}_{-i}(\theta_{-i}), \theta_{-i}) \leq \frac{\partial t_{-i}}{\partial \theta}(q^D(\theta_{-i}|\theta_i), \theta_{-i}) = \frac{\partial U^D}{\partial \theta_{-i}}(\theta_{-i}|\theta_i)$$

where the last inequality follows from (A11) and the fact that $t_{-i}(q, \theta_{-i})$ satisfies SMP. Therefore, $U^D(\theta_{-i}|\theta_i) \geq \hat{U}_{-i}(\theta_{-i})$ for any (θ_i, θ_{-i}) if $U^D(\underline{\theta}|\theta_i) \geq \hat{U}_{-i}(\underline{\theta})$ holds. ■

Lemma 5 shows that the agent's participation constraint (5) binds necessarily at $\underline{\theta}$, $U^D(\underline{\theta}|\theta_i) = \hat{U}_{-i}(\underline{\theta})$ at any P_i's best response to a strategy profile $\{t_{-i}(q, \theta_{-i})\}_{\theta_{-i} \in \Theta}$ satisfying IT and SMP used by P_{-i} when this best response also satisfies SMP and has a positive marginal contribution. ■

- **Non-negative transfers:** Observe also that $U^D(\underline{\theta}|\theta_i) = \hat{U}_{-i}(\underline{\theta})$ implies

$$t_i(q^D(\underline{\theta}|\theta_i)) = t_i(\hat{q}(\underline{\theta})) - C(\hat{q}(\underline{\theta})) - \left(t_i(q^D(\underline{\theta}|\theta_i)) - C(q^D(\underline{\theta}|\theta_i))\right) \geq 0$$

where the last inequality follows from the definition of $\hat{q}(\underline{\theta})$. This gives, for any θ_{-i},

$$t_i(q^D(\theta_{-i}|\theta_i)) - t_i(q^D(\underline{\theta}|\theta_i)) = \int_{q^D(\theta_{-i}|\theta_i)}^{q^D(\underline{\theta}|\theta_i)} p(x|\theta_i)dx \geq 0$$

when marginal contributions are positive. This in turn implies that $t_i(q^D(\theta_{-i}|\theta_i)) \geq 0$ for any equilibrium output. Finally, the extension defined above respects non-negativity. ■

• **Out-of equilibrium beliefs and best responses in Γ:** The analysis above has assumed that the agent was informed on P_i's type when the latter computes his best response to the strategy profile $\{t_{-i}(q, \theta_{-i})\}_{\theta_{-i}\in\Theta}$ satisfying IT and SMP used by P_{-i}. We show first that the strategy profile $\{t_i(q, \theta_i)\}_{\theta_i\in\Theta}$ is also a best response to $\{t_{-i}(q, \theta_{-i})\}_{\theta_{-i}\in\Theta}$ in the game Γ where the agent is a priori uninformed on P_i's type. Second, we show that any off-equilibrium beliefs sustain the strategy profile $\{t_i(q, \theta_i)\}_{\theta_i\in\Theta}$ as a best response in the game Γ where principals are privately informed.

Consider the collection of strategies $\{t_i(q, \hat{\theta}_i)\}_{\hat{\theta}_i\in\Theta}$. These strategies are all distinct so that, if played in a separating equilibrium, P_i reveals his type θ_i to the agent when he chooses $t_i(q, \theta_i)$. We want to prove that this menu of contributions is incentive compatible for P_i. Denote by $\tilde{V}_i(\theta_i, \hat{\theta}_i)$ principal P_i's payoff when his type is θ_i and he picks the strategy $t_i(q, \hat{\theta}_i)$ for some $\hat{\theta}_i \in \Theta$. Denote also $V_i(\theta_i) = \tilde{V}_i(\theta_i, \theta_i)$ the equilibrium payoff.

Facing the contributions $t_i(q, \hat{\theta}_i)$ and $t_{-i}(q, \theta_{-i})$, the agent chooses the quantity $q^D(\theta_{-i}|\hat{\theta}_i)$. $\tilde{V}_i(\theta_i, \hat{\theta}_i)$ can be written as:

$$\begin{aligned}\tilde{V}_i(\theta_i, \hat{\theta}_i) &= E_{\theta_{-i}}\left[\theta_i q^D(\theta_{-i}|\hat{\theta}_i) - t_i(q^D(\theta_{-i}|\hat{\theta}_i), \hat{\theta}_i)\right] \\ &= E_{\theta_{-i}}\left[S_i(q^D(\theta_{-i}|\hat{\theta}_i), \hat{\theta}_i, \theta_{-i}) + (\theta_i - \hat{\theta}_i)q^D(\theta_{-i}|\hat{\theta}_i)\right] - \hat{U}_{-i}(\underline{\theta}).\end{aligned}$$

We can now compute:

$$\frac{\partial \tilde{V}_i}{\partial \hat{\theta}_i}(\theta_i, \hat{\theta}_i) = E_{\theta_{-i}}\left[\frac{\partial S_i}{\partial q}(q^D(\theta_{-i}|\hat{\theta}_i), \theta_i, \theta_{-i})\frac{\partial q^D}{\partial \hat{\theta}_i}(\theta_{-i}|\hat{\theta}_i)\right].$$

Since $S_i(\cdot, \theta_i, \theta_{-i})$ is a strictly concave function with critical point at $q = q^D(\theta_{-i}|\theta_i)$ and $\frac{\partial q^D}{\partial \hat{\theta}_i}(\theta_{-i}|\hat{\theta}_i) \geq 0$, we have:

$$\frac{\partial \tilde{V}_i}{\partial \hat{\theta}_i}(\theta_i, \hat{\theta}_i) \geq 0 \text{ (resp. =) if and only if } \theta_i \geq \hat{\theta}_i. \text{ (resp. =)}. \tag{A12}$$

Condition (A12) shows then that the collection of strategies $\{t_i(q, \hat{\theta}_i)\}_{\hat{\theta}_i\in\Theta}$ is incentive compatible from principal P_i's viewpoint.

Consider now a deviation by principal P_i with type θ_i to a contribution schedule $z_i(q)$ such that $z_i(q) \notin \{t_i(q, \hat{\theta}_i)\}_{\hat{\theta}_i\in\Theta}$. Facing the contributions $z_i(q)$ and $t_{-i}(q, \theta_{-i})$, the agent chooses a quantity $\tilde{q}(\theta_{-i}|\theta_i)$ in the correspondence $\arg\max_q z_i(q) + t_{-i}(q, \theta_{-i}) - C(q)$ whatever his beliefs on P_i's type. Now observe that an upper bound on the payoff for such deviation is obtained when we replace the agent's participation constraint (5) by the weaker requirement

$$U^D(\theta_{-i}|\theta_i) \geq 0.$$

In this relaxed problem, the agent's participation constraint binds necessarily at $\underline{\theta}$ only. The payoff in any such deviation is thus no greater than

$$E_{\theta_{-i}}\left[S_i(\tilde{q}(\theta_{-i}|\theta_i),\theta_i,\theta_{-i})\right]-\hat{U}_{-i}(\underline{\theta})\leq V_i(\theta_i)=E_{\theta_{-i}}\left[S_i(q^D(\theta_{-i}|\theta_i),\theta_i,\theta_{-i})\right]-\hat{U}_{-i}(\underline{\theta}),$$

where the right-hand side is principal P_i's payoff when he offers $t_i(q,\theta_i)$. This proves that $t_i(q,\theta_i)$ is a best response in Γ when principal P_i's type is θ_i. ■

Proof of Theorem 2:

• **First-best at the top**: Using (8) for $\theta_1=\theta_2=\bar{\theta}$ and (9) yields the result.

• **Downward distortions**: Observe that SMP and (8) altogether imply $\theta_i\geq p(q(\theta_1,\theta_2),\theta_i)$. Summing over i and taking into account (9) yield $\theta_1+\theta_2=C'(q^{FB}(\theta_1,\theta_2))\geq C'(q(\theta_1,\theta_2))$ with equality only when $\theta_1=\theta_2=\bar{\theta}$.

• **Non-negative rent for the agent and equilibrium contributions**: From Lemma 5, we know that in any symmetric equilibrium

$$U(\theta_i,\underline{\theta})=t(q(\theta_i,\underline{\theta}),\underline{\theta})+t(q(\theta_i,\underline{\theta}),\theta_i)-C(q(\theta_i,\underline{\theta}))=\hat{U}_{-i}(\underline{\theta})\text{ for all }\theta_i$$

where, using above notations, $\hat{U}_{-i}(\underline{\theta})=t(\hat{q}(\underline{\theta}),\underline{\theta})-C(\hat{q}(\underline{\theta}))\geq 0$. For $\theta_i=\underline{\theta}$, we get:

$$U(\underline{\theta},\underline{\theta})=2t(q(\underline{\theta},\underline{\theta}),\underline{\theta})-C(q(\underline{\theta},\underline{\theta}))=t(\hat{q}(\underline{\theta}),\underline{\theta})-C(\hat{q}(\underline{\theta})).$$

Suppose $\hat{U}_{-i}(\underline{\theta})>0$, then observe that $t(\hat{q}(\underline{\theta}),\underline{\theta})>C(\hat{q}(\underline{\theta}))>0$ and thus $U(\underline{\theta},\underline{\theta})<2t(\hat{q}(\underline{\theta}),\underline{\theta})-C(\hat{q}(\underline{\theta}))$, a contradiction with the definition of $q(\underline{\theta},\underline{\theta})$. Hence, we have necessarily $U(\underline{\theta},\underline{\theta})=0$ which means $t(q(\underline{\theta},\underline{\theta}),\underline{\theta})=\frac{C(q(\underline{\theta},\underline{\theta}))}{2}>0$. Therefore, we get

$$U(\theta_i,\underline{\theta})=\hat{U}_{-i}(\underline{\theta})=0\text{ for all }\theta_i. \tag{A13}$$

For $\theta_i\geq\underline{\theta}$, observe that

$$t(q(\theta_i,\underline{\theta}),\underline{\theta})=t(q(\underline{\theta},\underline{\theta}),\underline{\theta})+\int_{q(\underline{\theta},\underline{\theta})}^{q(\theta_i,\underline{\theta})}p(x,\underline{\theta})dx$$

so that (A13) yields

$$t(q(\theta_i,\underline{\theta}),\theta_i)=C(q(\theta_i,\underline{\theta}))-\frac{C(q(\underline{\theta},\underline{\theta}))}{2}-\int_{q(\underline{\theta},\underline{\theta})}^{q(\theta_i,\underline{\theta})}p(x,\underline{\theta})dx. \tag{A14}$$

Differentiating with respect to θ_i yields:

$$\frac{\partial t}{\partial\theta_i}(q(\theta_i,\underline{\theta}),\theta_i)+(p(q(\theta_i,\underline{\theta}),\theta_i)+p(q(\theta_i,\underline{\theta}),\underline{\theta})-C'(q(\theta_i,\underline{\theta}))\frac{\partial q}{\partial\theta_i}(\theta_i,\underline{\theta})=0.$$

Taking thus into account the agent's first-order condition (9), we obtain $\frac{\partial t}{\partial\theta_i}(q(\theta_i,\underline{\theta}),\theta_i)=0$ and finally (13).

Moreover, we have:

$$t(q(\theta_i,\theta_j),\theta_i) - t(q(\theta_i,\underline{\theta}),\theta_i) = \int_{q(\theta_i,\underline{\theta})}^{q(\theta_i,\theta_j)} p(x,\theta_i)dx > 0$$

which, taken in tandem with (A14), defines the transfer $t(q(\theta_i,\theta_j),\theta_i)$ on the range of equilibrium outputs $q(\theta_i,\theta_j)$.

Note also that

$$\frac{\partial t}{\partial \theta_i}(q(\theta_i,\theta_j),\theta_i) = \frac{\partial t}{\partial \theta_i}(q(\theta_i,\underline{\theta}),\theta_i) + \int_{q(\theta_i,\underline{\theta})}^{q(\theta_i,\theta_j)} \frac{\partial p}{\partial \theta}(x,\theta_i)dx = \int_{q(\theta_i,\underline{\theta})}^{q(\theta_i,\theta_j)} \frac{\partial p}{\partial \theta}(x,\theta_i)dx \geq 0$$

and thus

$$\frac{\partial t}{\partial \theta_i}(q(\theta_i,\theta_j),\theta_i) \geq 0 \text{ for } \theta_j \geq \underline{\theta}$$

when SMP holds. Using (2), we deduce from that the agent's rent is everywhere non-negative and zero only when $\theta_i = \underline{\theta}$ for at least one i. ■

Boundaries conditions for the system (20):

Lemma 6 *The following properties hold:*

• *For q such that $2\bar{\theta} \geq C'(q) \geq \bar{\theta} + \underline{\theta} - \frac{1}{2f(\underline{\theta})}$, the highest type on the q-isoquant is $\bar{\theta}$ whereas the lowest type $\underline{\theta}(q) \geq \underline{\theta}$ is increasing in q and defined by the condition:*

$$C'(q) = \bar{\theta} + \underline{\theta}(q) - \frac{1}{2}R(\underline{\theta}(q)); \tag{A15}$$

Marginal contributions at these boundaries satisfy:

$$p(q,\underline{\theta}(q)) = \underline{\theta}(q), \quad \frac{\partial p}{\partial \theta}(q,\underline{\theta}(q)) = \frac{1}{2}; \tag{A16}$$

$$p(q,\bar{\theta}) = C'(q) - \underline{\theta}(q) < \bar{\theta}, \quad \frac{\partial p}{\partial \theta}(q,\bar{\theta}) > 0; \tag{A17}$$

• *For q such that $C'(q) \leq \bar{\theta} + \underline{\theta} - \frac{1}{2f(\underline{\theta})}$, the lowest type on the $q-$isoquant is $\underline{\theta}$ and the highest one is $\bar{\theta}(q)$ with:*

$$p(q,\bar{\theta}(q)) = \bar{\theta}(q) - R(\underline{\theta})\frac{\partial p}{\partial \theta}(q,\underline{\theta}), \quad \frac{\partial p}{\partial \theta}(q,\bar{\theta}(q)) = \frac{\underline{\theta} - p(q,\underline{\theta})}{R(\bar{\theta}(q))} > 0; \tag{A18}$$

$$p(q,\underline{\theta}) < \underline{\theta}, \quad \frac{\partial p}{\partial \theta}(q,\underline{\theta}) > 0. \tag{A19}$$

Proof: First consider a q-isoquant which crosses the vertical axis at $\bar{\theta}$. Define $\underline{\theta}(q)$ such that $\underline{\theta}(q) = \psi(q,\bar{\theta})$ (and thus $\bar{\theta} = \psi(q,\underline{\theta}(q))$). From the equilibrium conditions (20) taken respectively at $\underline{\theta}(q)$ and $\bar{\theta}$, we get:

$$p(q,\underline{\theta}(q)) = \underline{\theta}(q) \text{ and } \bar{\theta} + p(q,\underline{\theta}(q)) - C'(q) = R(\underline{\theta}(q))\frac{\partial p}{\partial \theta}(q,\underline{\theta}(q)), \tag{A20}$$

which is the first part of (A16).

From SMP and $\underline{\theta}(q) < \bar{\theta}$, we get $\bar{\theta} > p(q, \bar{\theta})$, which gives the first part of (A17).

Using (20), we get:

$$\frac{\partial \psi}{\partial \theta}(q, \theta) = -\frac{\frac{\partial p}{\partial \theta}(q, \theta)}{\frac{\partial p}{\partial \theta}(q, \psi(q, \theta))} = -\frac{R(\psi(q, \theta))(\psi(q, \theta) + p(q, \theta) - C'(q))}{R(\theta)(\theta - p(q, \theta))}. \tag{A21}$$

Using (A21) to evaluate $\frac{\partial \psi}{\partial \theta}(q, \theta)$ at $\underline{\theta}(q)$ and using Lhospital's rule yield

$$\frac{\partial \psi}{\partial \theta}(q, \underline{\theta}(q)) = -\frac{\partial \psi}{\partial \theta}(q, \underline{\theta}(q)) \frac{\dot{R}(\bar{\theta})(\bar{\theta} + p(q, \underline{\theta}(q)) - C'(q))}{R(\underline{\theta}(q))\left(1 - \frac{\partial p}{\partial \theta}(q, \underline{\theta}(q))\right)} = \frac{\partial \psi}{\partial \theta}(q, \underline{\theta}(q)) \frac{\frac{\partial p}{\partial \theta}(q, \underline{\theta}(q))}{1 - \frac{\partial p}{\partial \theta}(q, \underline{\theta}(q))}$$

where we have used $\dot{R}(\bar{\theta}) = -1$ and (A20) to get the last equality. The only possibility for having $\frac{\partial \psi}{\partial \theta}(q, \underline{\theta}(q)) < 0$ is $\frac{\partial p}{\partial \theta}(q, \underline{\theta}(q)) = \frac{1}{2}$ which is the second part of (A16) and gives also (A15). This and (A21) yields the second part of (A17). Therefore, $\underline{\theta}(q)$ is defined by (A15) and, given that $R(\cdot)$ is decreasing, this can only be possible when $C'(q) \geq \bar{\theta} + \underline{\theta} - \frac{1}{2f(\underline{\theta})}$.

For $C'(q) < \bar{\theta} + \underline{\theta} - \frac{1}{2f(\underline{\theta})}$, the conditions coming from the equilibrium behavior of types $\underline{\theta}$ and $\bar{\theta}(q)$ are given by (A18) and (A19). ■

Proof of Theorem 3: Fix q such that $2\overline{\theta} \geq C'(q) \geq \bar{\theta} + \underline{\theta} - \frac{1}{2f(\underline{\theta})}$. The boundary condition (A16) can be used to integrate (23) and get $G(\cdot, q)$ as a solution to:

$$1 - G(p, q) = (1 - F(\underline{\theta}(q)))exp\left(-\int_{\underline{\theta}(q)}^{p} \frac{dx}{F^{-1}(G(C'(q) - x, q)) - C'(q) + x}\right). \tag{A22}$$

Consider now the mapping $\Phi(\cdot)$ such that:

$$1 - \Phi(G)(p) = (1 - F(\underline{\theta}(q))) \exp\left(-\int_{\underline{\theta}(q)}^{p} \frac{dx}{F^{-1}(G(C'(q) - x)) - C'(q) + x}\right). \tag{A23}$$

An equilibrium distribution $G(\cdot, q)$ (defined on $[\underline{\theta}(q), C'(q) - \underline{\theta}(q)]$ and extended on $[\underline{\theta}, C'(q) - \underline{\theta}(q)]$ as explained in the text) is thus a fixed-point of the mapping $\Phi(\cdot)$.

Several facts immediately follow from the definition (A23).

- Boundary conditions: $\Phi(G)(\underline{\theta}(q)) = F(\underline{\theta}(q))$, and $\Phi(G)(C'(q) - \underline{\theta}(q)) = 1$[49] when $G(\underline{\theta}(q), q) = F(\underline{\theta}(q))$;
- $\Phi(\cdot)$ is monotonically decreasing and thus $\Phi^2(\cdot)$ is monotonically increasing: $G_1 \leq G_2$ implies $\Phi(G_1) \geq \Phi(G_2)$.

[49]Notice that from (A23), $\lim_{p \to C'(q) - \underline{\theta}(q)} \Phi(G)(p) \leq 1$. Hence, $\Phi(G)$ is then a distribution function well defined at $C'(q) - \underline{\theta}(q)$ as 1.

Consider the function

$$\mathcal{I}(p) = \begin{cases} 1 & \text{if } p \in (\underline{\theta}(q), C'(q) - \underline{\theta}(q)], \\ F(p) & \text{if } p \in [\underline{\theta}, \underline{\theta}(q)]. \end{cases}$$

This is not a distribution admitting a density function as required by our formalism. However, we may still apply twice the mapping $\Phi(\cdot)$ above to it to generate such distribution. For $p \in [\underline{\theta}(q), C'(q) - \underline{\theta}(q))$, we have:

$$1-\Phi(\mathcal{I})(p) = (1-F(\underline{\theta}(q)))\exp\left(-\int_{\underline{\theta}(q)}^{p} \frac{dx}{\bar{\theta} - C'(q) + x}\right) = (1-F(\underline{\theta}(q)))\left(\frac{\bar{\theta} + \underline{\theta}(q) - C'(q)}{\bar{\theta} - C'(q) + p}\right)$$

with

$$\Phi(\mathcal{I})(C'(q) - \underline{\theta}(q)) = 1 \text{ and } \lim_{p \to C'(q) - \underline{\theta}(q)} \Phi(\mathcal{I})(p) < 1.$$

One can check that

$$\Phi(F)(p) = \begin{cases} 1 & \text{if } p > \underline{\theta}(q) \\ \underline{\theta}(q) & \text{if } p = \underline{\theta}(q) \end{cases} \text{ and } \Phi^2(F) = \Phi(\mathcal{I}) \tag{A24}$$

Moreover, we want to find a condition ensuring that the mapping $\Phi(\cdot)$ will be onto and that the distribution of price at any iteration starting from $\Phi(\mathcal{I})(\cdot)$ never crosses $F(\cdot)$ to avoid infinite terms in the denominator on the right-hand side of (A23). A sufficient condition is that $\Phi(\mathcal{I})(p) \geq F(p)$ for all $p \in [\underline{\theta}(q), C'(q) - \underline{\theta}(q)]$. This amounts to:

$$\chi(p) = (1 - F(p))(\bar{\theta} + p - C'(q)) - (1 - F(\underline{\theta}(q)))(\bar{\theta} + \underline{\theta}(q) - C'(q)) \geq 0. \tag{A25}$$

Note that $\chi(\underline{\theta}(q)) = 0$ and that $\chi(\cdot)$, which is quasi-concave under the assumption $\dot{R}(p) \leq 0$, achieves its maximum at $p^* < C'(q) - \underline{\theta}(q)$ such that $\bar{\theta} + p^* - C'(q) = \frac{1-F(p^*)}{f(p^*)}$. Hence, (A25) holds when $\chi(C'(q) - \underline{\theta}(q)) > 0$. This last condition holds when $\frac{1-F(x)}{\bar{\theta}-x}$ increases with x, a sufficient condition is $\min_{\theta\in\Theta} \quad \theta + R(\theta) = \bar{\theta}$.

Consider now the following sequence $\phi_n = \Phi^n(\phi_0)$ with $\phi_0 = F$. One can easily show that ϕ_{2k} is increasing whereas ϕ_{2k+1} is decreasing in k. Moreover, $\phi_2 < 1 = \phi_1$ and thus, by iterating, we get $\phi_{2k} \leq \phi_{2k+1}$. Moreover, as soon as $n \geq 2$, $\phi_n(\underline{\theta}(q)) = F(\underline{\theta}(q))$ and $\phi_n(C'(q) - \underline{\theta}(q)) = 1$. Now, denote by $\underline{\phi}$ and $\bar{\phi}$ the respective limits of ϕ_{2k} and ϕ_{2k+1}. We have: $\underline{\phi} \leq \bar{\phi}$, $\quad \underline{\phi} = \Phi(\bar{\phi})$ and $\bar{\phi} = \Phi(\underline{\phi})$. Note that $\underline{\phi}(\underline{\theta}(q)) = \bar{\phi}(\underline{\theta}(q)) = F(\underline{\theta}(q))$ and $\underline{\phi}(C'(q) - \underline{\theta}(q)) = \bar{\phi}(C'(q) - \underline{\theta}(q)) = 1$. $\underline{\phi}(\cdot)$ and $\bar{\phi}(\cdot)$ are by definition both differentiable at $C'(q) - \underline{\theta}(q)$. Moreover, $\dot{\phi}_{2k}(C'(q) - \underline{\theta}(q))$ is decreasing in k and $\dot{\phi}_{2k+1}(C'(q) - \underline{\theta}(q))$ increasing in k so that, in the limit, $+\infty > \dot{\underline{\phi}}(C'(q) - \underline{\theta}(q)) \geq \dot{\bar{\phi}}(C'(q) - \underline{\theta}(q)) > 0 = \dot{\phi}_1(C'(q) - \underline{\theta}(q))$.

Define first $N = \{G(\cdot)|\ G(\cdot)$ is increasing and $\underline{\phi}(p) \leq G(p) \leq \bar{\phi}(p)$ for all $p \in [\underline{\theta}(q), C'(q) - \underline{\theta}(q)]\}$. Clearly, N is convex and non-empty. Let us also define:

$$N^* = \{G(\cdot)|\ G(\cdot) \text{ is increasing and } \underline{\phi}(p) \leq G(p) \leq \bar{\phi}(p) \text{ for all } p \in [\underline{\theta}(q), C'(q) - \underline{\theta}(q)]$$

$$\text{and } |G(p) - G(p')| \leq K|p - p'|\}$$

where $K < +\infty$ is chosen below. $\Phi(\cdot)$ maps N into N^*. Indeed, from the Theorem of Intermediate Values, we have:

$$|\Phi(G)(p) - \Phi(G)(p')| = |\dot{\Phi}(G)(\zeta)||p - p'|$$

for some $\zeta \in [p, p']$ where

$$|\dot{\Phi}(G)(\zeta)| = \frac{1 - F(\underline{\theta}(q))}{F^{-1}(G(C'(q) - \zeta)) - C'(q) + \zeta} \exp\left(-\int_{\underline{\theta}(q)}^{\zeta} \frac{dx}{F^{-1}(G(C'(q) - x)) - C'(q) + x}\right).$$

Using that $\underline{\phi} \leq G \leq \bar{\phi}$, we get

$$|\Phi(\dot{G})(\zeta)| \leq \frac{1 - F(\underline{\theta}(q))}{F^{-1}(\underline{\phi}(C'(q) - \zeta)) - C'(q) + \zeta} \exp\left(-\int_{\underline{\theta}(q)}^{\zeta} \frac{dx}{F^{-1}(\bar{\phi}(C'(q) - x)) - C'(q) + x}\right)$$

$$= \frac{1 - \underline{\phi}(\zeta)}{F^{-1}(\underline{\phi}(C'(q) - \zeta)) - C'(q) + \zeta}.$$

The right-hand side above is in fact a bounded function of ζ over $[\underline{\theta}(q), C'(q) - \underline{\theta}(q)]$. Indeed, using Lhospital rule, we have:

$$\lim_{\zeta \to C'(q) - \underline{\theta}(q)} \frac{1 - \underline{\phi}(\zeta)}{F^{-1}(\underline{\phi}(C'(q) - \zeta)) - C'(q) + \zeta} = -\frac{\dot{\underline{\phi}}(C'(q) - \underline{\theta}(q))}{1 - \frac{\dot{\underline{\phi}}(\underline{\theta}(q))}{f(\underline{\theta}(q))}}.$$

Using $\underline{\phi} = \Phi(\bar{\phi})$ and thus

$$\frac{\dot{\underline{\phi}}(p)}{1 - \underline{\phi}(p)} = \frac{1}{F^{-1}(\bar{\phi}(C'(q) - p)) - C'(q) + p}$$

taken at $p = \underline{\theta}(q)$ yields

$$\dot{\underline{\phi}}(\underline{\theta}(q)) = \frac{1 - F(\underline{\theta}(q))}{\bar{\theta} + \underline{\theta}(q) - C'(q)} = 2f(\underline{\theta}(q)).$$

Hence, we get

$$\lim_{\zeta \to C'(q) - \underline{\theta}(q)} \frac{1 - \underline{\phi}(\zeta)}{F^{-1}(\underline{\phi}(C'(q) - \zeta)) - C'(q) + \zeta} = \dot{\underline{\phi}}(C'(q) - \underline{\theta}(q)).$$

Finally, denote $K' = \sup_{\zeta \in [\underline{\theta}(q), C'(q) - \underline{\theta}(q)]} \frac{1 - \underline{\phi}(\zeta)}{F^{-1}(\underline{\phi}(C'(q) - \zeta)) - C'(q) + \zeta} < +\infty$. Take now $K = \sup\{K', \sup_\zeta \dot{\underline{\phi}}(\zeta), \sup_\zeta \dot{\bar{\phi}}(\zeta)\}$. Such value of K ensures that N^* is non-empty because at least $\underline{\phi}$ and $\bar{\phi}$ are in it. Moreover, by Ascoli Theorem, N^* is compact.

Finally, $\Phi(\cdot)$ is continuous on N. To show that consider two distributions G and H in N. We have:

$$\Phi(G)(p) - \Phi(H)(p) = (1 - F(\underline{\theta}(q))$$

$$\times\left(\exp\left(-\int_{\underline{\theta}(q)}^{p}\frac{dx}{F^{-1}(H(C'(q)-x))-C'(q)+x}\right)-\exp\left(-\int_{\underline{\theta}(q)}^{p}\frac{dx}{F^{-1}(G(C'(q)-x))-C'(q)+x}\right)\right). \tag{A26}$$

First, note that $H\leq\bar{\phi}$ implies

$$\exp\left(-\int_{\underline{\theta}(q)}^{p}\frac{dx}{F^{-1}(H(C'(q)-x))-C'(q)+x}\right)\leq\exp\left(-\int_{\underline{\theta}(q)}^{p}\frac{dx}{F^{-1}(\bar{\phi}(C'(q)-x))-C'(q)+x}\right)$$

and similarly, $G\leq\bar{\phi}$ implies

$$\exp\left(-\int_{\underline{\theta}(q)}^{p}\frac{dx}{F^{-1}(G(C'(q)-x))-C'(q)+x}\right)\leq\exp\left(-\int_{\underline{\theta}(q)}^{p}\frac{dx}{F^{-1}(\bar{\phi}(C'(q)-x))-C'(q)+x}\right).$$

Now fix ϵ arbitrarily small. There exists η such that for $p\geq C'(q)-\underline{\theta}(q)-\eta$, both right-hand sides above are less than ϵ and thus $|\Phi(G)(p)-\Phi(H)(p)|\leq 2\epsilon$. For $p\in[\underline{\theta}(q),C'(q)-\underline{\theta}(q)-\eta]$, the right-hand side of (A26) can be made arbitrarily small, say less than 2ϵ, by taking H close enough to G with respect to $||\cdot||_\infty$. Gathering everything $||\Phi(G)-\Phi(H)||_\infty=\sup_p|\Phi(G)(p)-\Phi(H)(p)|\leq 2\epsilon$ which ensures continuity.

Therefore $\Phi(\cdot)$ is a compact mapping from N onto $N^*\subseteq N$. Existence of $G(\cdot,q)$ follows then Schauder's Second Theorem (Burton 2005, p.184) which states that a compact mapping on a convex non-empty subset of a Banach space N has a fixed point. ■

Proof of Theorem 4: If $G(\cdot)$ (we omit the dependence on q for simplicity) corresponds to the marginal price distribution in a symmetric equilibrium, then it must be a solution of the following system of ordinary differential equations (where $H(p)=G(C'(q)-p)$):

$$\frac{\dot{G}(p)}{1-G(p)}=\frac{1}{F^{-1}(H(p))-C'(q)+p}, \tag{A27}$$

$$\frac{\dot{H}(p)}{1-H(p)}=-\frac{1}{F^{-1}(G(p))-p}, \tag{A28}$$

for all $p\in[\underline{\theta}(q),C'(q)-\underline{\theta}(q)]$ with the boundary conditions

$$G(\underline{\theta}(q))=H(C'(q)-\underline{\theta}(q))=F(\underline{\theta}(q)),G(C'(q)-\underline{\theta}(q))=H(\underline{\theta}(q))=1. \tag{A29}$$

Let $F(\theta)=\frac{\theta-\underline{\theta}}{\Delta\theta}$ be the uniform cumulative distribution on the interval $[\underline{\theta},\overline{\theta}]$ (where $\Delta\theta=\bar{\theta}-\underline{\theta}$). Then, $F^{-1}(x)=\Delta\theta x+\underline{\theta}$ and $\underline{\theta}(q)=\frac{2}{3}(C'(q)-\bar{\theta})$ ($\underline{\theta}(q)\geq\underline{\theta}$ requires thus $3\underline{\theta}+\bar{\theta}\geq C'(q)$). The system becomes

$$\dot{G}(p)(\Delta\theta H(p)+\underline{\theta}-C'(q)+p)=1-G(p),\quad \dot{H}(p)(\Delta\theta G(p)+\underline{\theta}-t)=-1+H(p).$$

Adding up these equations we get

$$\Delta\theta[\dot{G}(p)H(p)+G(p)\dot{H}(p)]+(\underline{\theta}-C'(q))\dot{G}(p)+\underline{\theta}\dot{H}(p)+p[\dot{G}(p)-\dot{H}(p)]=G(p)-H(p).$$

Integrating, there exists a constant of integration K such that:

$$\Delta\theta G(p)H(p) + p(G(p) - H(p)) + (\underline{\theta} - C'(q))G(p) + \underline{\theta}H(p) = K.$$

At any equilibrium, this constant is uniquely determined. Indeed, at $t = \underline{\theta}(q)$ we have that $G(p) = F(\underline{\theta}(q))$ and $H(\underline{\theta}(q)) = 1$ and therefore $K = (\underline{\theta}(q) + \underline{\theta} - C'(q))\frac{(\underline{\theta}(q)-\underline{\theta})}{\Delta\theta} > 0$. Inserting into (A28) yields:

$$H(p) = \frac{K + (C'(q) - \beta - p)G(p)}{\Delta\theta G(p) + \underline{\theta} - p}.$$

Substituting into (A27), we get

$$\frac{\dot{G}(p)}{1 - G(p)} = \frac{\Delta\theta G(p) + \underline{\theta} - p}{\Delta\theta K + (C'(q) - \underline{\theta} - p)(p - \underline{\theta})}. \tag{A30}$$

Notice that, given that $G(p)$ is an equilibrium, $\Delta\theta K + (C'(q) - \underline{\theta} - p)(p - \underline{\theta}) > 0$ for all $p \in (\underline{\theta}(q), C'(q) - \underline{\theta}(q))$. This implies that (A30) is an ordinary differential equation which is regular on $\Delta\theta K + (C'(q) - \underline{\theta} - p)(p - \underline{\theta}) > 0$ and the local uniqueness of a solution holds at any such p.

Suppose then that there are two symmetric equilibria distributions in two putative distinct equilibria with the same boundary conditions, i.e., two fixed-points G_1 and G_2 for $\Phi(\cdot)$ such that $G_1(\underline{\theta}(q)) = G_2(\underline{\theta}(q)) = F(\underline{\theta}(q))$, $G_1(C'(q) - \underline{\theta}(q)) = G_2(C'(q) - \underline{\theta}(q)) = 1$. Then, one of these distributions cannot dominate the other in the sense of first-order stochastic dominance; they necessarily cross each other at least once on $(\underline{\theta}(q), C'(q) - \underline{\theta}(q))$. Suppose otherwise, i.e., $G_1(p) \leq G_2(p)$ for $p \in (\underline{\theta}(q), C'(q) - \underline{\theta}(q))$. Using that $\Phi(\cdot)$ is monotonic, we get $G_1 = \Phi(G_1) \geq G_2 = \Phi(G_2)$ and, finally, $G_1 = G_2$. But then, G_1 and G_2 must cross at some $p_0 \in (\underline{\theta}(q), C'(q) - \underline{\theta}(q))$ and both satisfy (A30) for the same K. However, this is a contradiction with the local uniqueness for a solution to (A30). Hence, global uniqueness of a solution follows. ■

Proof of Example 2 (cont.): Equations (19) and (20) can be first transformed into a system of first-order differential equations to get both the marginal contribution of a given type and the identity of his conjugate. Using (19), we get:

$$\frac{\partial p}{\partial \theta}(q, \theta) = r(\psi(q, \theta) + p(q, \theta) - C'(q)). \tag{A31}$$

From equation (A21), we obtain:

$$\frac{\partial p}{\partial \theta}(q, \theta) = -\frac{\partial p}{\partial \theta}(q, \psi(q, \theta))\frac{\partial \psi}{\partial \theta}(q, \theta).$$

Differentiating (A31) with respect to θ, using the last expression, replacing θ by $\psi(q, \theta)$ in (A31) and (19) yield:

$$\frac{\partial p}{\partial \theta}(q, \theta)\,(1 - r(\theta - p(q, \theta))) + (\theta - p(q, \theta))\frac{\partial^2 p}{\partial \theta^2}(q, \theta) = 0. \tag{A32}$$

The solutions to this differential equation do not depend on q and we denote $u(\theta) = \theta - \frac{1}{r} - p(q,\theta)$. We look for such non-negative solutions $u(\cdot)$ with $0 < \dot{u}(\theta) \leq 1$ where the last inequality is needed to satisfy SMP. (A32) can also be written as

$$\ddot{u}(\theta)(ru(\theta)+1) + ru(\theta)(1-\dot{u}(\theta)) = 0.$$

Defining $\phi(\cdot)$ as $\dot{u}(\theta) = \phi(u(\theta))$, we get:

$$\phi'(u)\frac{\phi(u)}{1-\phi(u)} = -\frac{ru}{1+ru}.$$

A first quadrature yields:

$$\phi(u) + \ln(1-\phi(u)) = -\lambda + u - \frac{1}{r}\ln(1+ru)$$

where λ is some constant. Since the function $\phi + \ln(1-\phi)$ is monotonically decreasing on $[0,1)$, it is invertible. Denote $G(\cdot)$ its inverse defined over $\mathbb{R}_-$. We obtain:

$$\dot{u}(\theta) = G\left(-\lambda + u(\theta) - \frac{1}{r}\ln(1+ru(\theta))\right). \tag{A33}$$

Take now any initial value $u(\underline{\theta}) \in (0, \underline{\theta} - \frac{1}{r})$ and consider the solution $u(\cdot)$ to (A33) with this initial condition when $\lambda > u(\underline{\theta}) - \frac{1}{r}\ln(1+ru(\underline{\theta}))$. $u(\cdot)$ is, non-negative, strictly increasing and has a slope less than 1, so that it never reaches the boundary $v(\theta) = \theta - \frac{1}{r}$. Using the Theorem of Uniqueness for the solution to such differential equation (Hirsh and Smale 1974, p.164), it can also be shown that such solution converges without reaching it towards a limit u_∞ defined as $\lambda = u_\infty - \frac{1}{r}\ln(1+ru_\infty)$. ■

Proof of Lemma 2: Denote P_i's ex post payoff for a given pair (θ_i, θ_{-i}) as:

$$V_i(\theta_i,\theta_{-i}) = \theta_i q(\theta_i,\theta_{-i}) - t(q(\theta_i,\theta_{-i}),\theta_i).$$

Simple algebra gives:

$$U(\theta_1,\theta_2) + \sum_{i=1}^{2} V_i(\theta_i,\theta_{-i}) = \left(\sum_{i=1}^{2}\theta_i\right) q(\theta_1,\theta_2) - C(q(\theta_1,\theta_2)). \tag{A34}$$

From the fact that $U(\underline{\theta},\underline{\theta}) = 0$ in any symmetric equilibrium, we must have:

$$2V(\underline{\theta},\underline{\theta}) = 2\underline{\theta}q(\underline{\theta},\underline{\theta}) - C(q(\underline{\theta},\underline{\theta})) > 0. \tag{A35}$$

Indeed, we have $\underline{\theta} - p(q(\underline{\theta},\underline{\theta}),\underline{\theta}) > 0$ from SMP and (8). Using (9), we get:

$$2\underline{\theta} > C'(q(\underline{\theta},\underline{\theta})) > \frac{C(q(\underline{\theta},\underline{\theta}))}{q(\underline{\theta},\underline{\theta})}$$

where the last inequality follows from the strict convexity of $C(\cdot)$, $C(0) = 0$ and the fact that $q(\underline{\theta},\underline{\theta}) > 0$ when $p(q,\underline{\theta}) > 0$ and $C'(0) = 0$.

We also obtain the following expressions of the partial derivatives of $V(\cdot)$:

$$\frac{\partial V_i}{\partial \theta_{-i}}(\theta_i,\theta_{-i}) = (\theta_i - p(q(\theta_i,\theta_{-i}),\theta_i))\,\frac{\partial q}{\partial \theta_{-i}}(\theta_i,\theta_{-i}) = R(\theta_{-i})\frac{\partial p}{\partial \theta}(q(\theta_i,\theta_{-i}),\theta_{-i})\frac{\partial q}{\partial \theta_{-i}}(\theta_i,\theta_{-i}). \tag{A36}$$

and

$$\frac{\partial V_i}{\partial \theta_i}(\theta_i,\theta_{-i}) = q(\theta_i,\theta_{-i}) + (\theta_i - p(q(\theta_i,\theta_{-i}),\theta_i))\,\frac{\partial q}{\partial \theta_i}(\theta_i,\theta_{-i}) - \frac{\partial t}{\partial \theta}(q(\theta_i,\theta_{-i}),\theta_i)$$

$$= q(\theta_i,\theta_{-i}) + R(\theta_{-i})\frac{\partial^2 U}{\partial \theta_1 \partial \theta_2}(\theta_1,\theta_2) - \frac{\partial U}{\partial \theta_i}(\theta_1,\theta_2). \tag{A37}$$

Integrating (A37) yields

$$V_i(\theta_i,\theta_{-i}) = \phi(\theta_{-i}) + \int_{\underline{\theta}}^{\theta_i} q(x,\theta_{-i})dx + R(\theta_{-i})\frac{\partial U}{\partial \theta_{-i}}(\theta_i,\theta_{-i}) - U(\theta_i,\theta_{-i}) \tag{A38}$$

for some function $\phi(\cdot)$. Because $U(\underline{\theta},\theta_{-i}) = 0$ for all θ_{-i}, one gets

$$V_i(\underline{\theta},\theta_{-i}) = \phi(\theta_{-i}). \tag{A39}$$

Inserting the expressions obtained from (A38) and (A39) into (A34) yields:

$$-U(\theta_i,\theta_{-i}) + \sum_{i=1}^{2} R(\theta_i)\frac{\partial U}{\partial \theta_i}(\theta_i,\theta_{-i})$$

$$= \left(\sum_{i=1}^{2}\theta_i\right) q(\theta_1,\theta_2) - C(q(\theta_1,\theta_2)) - \sum_{i=1}^{2}\left(\phi(\theta_i) + \int_{\underline{\theta}}^{\theta_i} q(x,\theta_{-i})dx\right). \tag{A40}$$

Simple integrations by parts show that:

$$E_{(\theta_1,\theta_2)}\left[-U(\theta_1,\theta_2) + \sum_{i=1}^{2} R(\theta_i)\frac{\partial U}{\partial \theta_i}(\theta_1,\theta_2)\right] = E_{(\theta_1,\theta_2)}\left[U(\theta_1,\theta_2)\right].$$

Because in any equilibrium $U(\theta_1,\theta_2) \geq 0$, we must have from (A40):

$$E_{(\theta_1,\theta_2)}\left[\left(\sum_{i=1}^{2}\theta_i\right) q(\theta_1,\theta_2) - C(q(\theta_1,\theta_2)) - \sum_{i=1}^{2}\int_{\underline{\theta}}^{\theta_i} q(x,\theta_{-i})dx\right] \geq \sum_{i=1}^{2} E_{\theta_i}\left[\phi(\theta_i)\right].$$

Integrating by parts the left-hand side above yields the following inequality:

$$E_{(\theta_1,\theta_2)}\left[\left(\sum_{i=1}^{2}\theta_i - R(\theta_i)\right) q(\theta_1,\theta_2) - C(q(\theta_1,\theta_2))\right] \geq \sum_{i=1}^{2} E_{\theta_i}\left[\phi(\theta_i)\right].$$

To get (26), note that $\phi'(\theta_{-i}) \geq 0$ from (A39) and that $\phi(\underline{\theta}) > 0$ is given by (A35).

In passing, using (A38), integrating by parts and taking into account that $U(\theta_i,\underline{\theta}) = 0$ show also that

$$E_{\theta_{-i}}\left[V(\theta_i,\theta_{-i})\right] = E_{\theta_{-i}}\left[\phi(\theta_{-i})\right] + \int_{\underline{\theta}}^{\theta_i} E_{\theta_{-i}}\left[q(x,\theta_{-i})\right]dx \geq \phi(\underline{\theta}) > 0.$$

Hence, the principals' interim participation constraints are satisfied. ■

Proof of Theorem 5: Define first

$$J(\theta_2) = E_{\theta_1}\left[\left(\sum_{i=1}^{2}\theta_i - R(\theta_i)\right)q^{FB}(\theta_1,\theta_2) - C(q^{FB}(\theta_1,\theta_2))\right] \text{ and } I = E_{\theta_2}\left[J(\theta_2)\right].$$

Integrating by parts and using $\frac{d}{dx}(x(F(x)-1)) = xf(x) - 1 + F(x)$, we have:

$$J(\theta_2) = (\underline{\theta} + \theta_2 - R(\theta_2))\,q^{FB}(\underline{\theta},\theta_2) - C(q^{FB}(\underline{\theta},\theta_2))$$
$$+\int_{\underline{\theta}}^{\bar{\theta}} \frac{\partial q^{FB}}{\partial \theta_1}(\theta_1,\theta_2)\left(\theta_1 + \theta_2 - R(\theta_2) - C'(q^{FB}(\theta_1,\theta_2))\right)(1-F(\theta_1))d\theta_1.$$

Using the definition of $q^{FB}(\cdot)$ to simplify the last integral yields:

$$J(\theta_2) = (\underline{\theta} + \theta_2 - R(\theta_2))\,q^{FB}(\underline{\theta},\theta_2) - C(q^{FB}(\underline{\theta},\theta_2)) - R(\theta_2)\int_{\underline{\theta}}^{\bar{\theta}} \frac{\partial q^{FB}}{\partial \theta_1}(\theta_1,\theta_2)(1-F(\theta_1))d\theta_1.$$

Therefore, taking expectations with respect to θ_2 yields:

$$I = E_{\theta_2}\left[(\underline{\theta} + \theta_2 - R(\theta_2))\,q^{FB}(\underline{\theta},\theta_2) - C(q^{FB}(\underline{\theta},\theta_2))\right]$$
$$-\int_{\underline{\theta}}^{\bar{\theta}}\int_{\underline{\theta}}^{\bar{\theta}} \frac{\partial q^{FB}}{\partial \theta_1}(\theta_1,\theta_2)(1-F(\theta_2))(1-F(\theta_1))d\theta_1 d\theta_2.$$

The first term can again be integrated by parts to get:

$$E_{\theta_2}\left[(\underline{\theta} + \theta_2 - R(\theta_2))\,q^{FB}(\underline{\theta},\theta_2) - C(q^{FB}(\underline{\theta},\theta_2))\right] = 2\underline{\theta}q^{FB}(\underline{\theta},\underline{\theta}) - C(q^{FB}(\underline{\theta},\underline{\theta}))$$
$$-\int_{\underline{\theta}}^{\bar{\theta}} \frac{\partial q^{FB}}{\partial \theta_2}(\underline{\theta},\theta_2)(\underline{\theta} + \theta_2 - C'(q^{FB}(\underline{\theta},\theta_2)))(1-F(\theta_2))d\theta_2$$

where the last integral is zero by the definition of $q^{FB}(\cdot)$. Gathering everything, we get:

$$I = 2\underline{\theta}q^{FB}(\underline{\theta},\underline{\theta}) - C(q^{FB}(\underline{\theta},\underline{\theta})) - \int_{\underline{\theta}}^{\bar{\theta}}\int_{\underline{\theta}}^{\bar{\theta}} \frac{\partial q^{FB}}{\partial \theta_1}(\theta_1,\theta_2)(1-F(\theta_2))(1-F(\theta_1))d\theta_1 d\theta_2$$
$$< 2\underline{\theta}q^{FB}(\underline{\theta},\underline{\theta}) - C(q^{FB}(\underline{\theta},\underline{\theta})).$$

Hence, (26) does not hold for the first-best. ■

Proof of Lemma 3: The uninformed mediator offers a centralized mechanism $\{T_1(\theta_i,\theta_{-i}), T_2(\theta_i,\theta_{-i}), q(\theta_i,\theta_{-i})\}$. Denote P_i's expected payoff when his type is θ_i as:

$$V_i(\theta_i) = \theta_i E_{\theta_{-i}}\left[q(\theta_i,\theta_{-i}) - T_i(\theta_i,\theta_{-i})\right].$$

Denote also the agent's payoff as

$$U(\theta_1,\theta_2) = \sum_{i=1}^{2} T_i(\theta_i,\theta_{-i}) - C(q(\theta_i,\theta_{-i})).$$

Incentive compatibility implies

$$\dot{V}_i(\theta_i) = E_{\theta_{-i}}\left[q(\theta_i, \theta_{-i})\right] \tag{A41}$$

and

$$E_{\theta_{-i}}[q(\theta_i, \theta_{-i})] \text{ non-decreasing in } \theta_i. \tag{A42}$$

Voluntary participation by the principals and the agent requires respectively:

$$V_i(\theta_i) \geq 0 \quad \forall \theta_i \tag{A43}$$

$$U(\theta_1, \theta_2) \geq 0 \quad \forall (\theta_1, \theta_2). \tag{A44}$$

The uninformed mediator maximizes now the following objective function:[50]

$$E_{(\theta_1,\theta_2)}\left[\sum_{i=1}^{2} \alpha'(\theta_i) f(\theta_i) V(\theta_i) + \beta U(\theta_1, \theta_2)\right] \text{ subject to (A41), (A43) and (A44)}$$

for some weights $\alpha'(\cdot)$ to be made precise below. The characterization of those interim efficient allocations follows then closely Ledyard and Palfrey (1999). First, (A41) implies

$$V_i(\theta_i) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} E_{\theta_{-i}}\left[q(x, \theta_{-i})\right] dx$$

where we use symmetry to set $V_1(\underline{\theta}) = V_2(\underline{\theta}) = V(\underline{\theta}) \geq 0$. Then, observe that

$$E_{(\theta_1,\theta_2)}\left[\left(\sum_{i=1}^{2} \theta_i\right) q(\theta_1, \theta_2) - C(q(\theta_1, \theta_2)) - \sum_{i=1}^{2} V_i(\theta_i)\right] = E_{(\theta_1,\theta_2)}\left[U(\theta_1, \theta_2)\right] \geq 0$$

where the last inequality follows from (A44). Integrating by parts the left-hand side above, one gets

$$E_{(\theta_1,\theta_2)}\left[\left(\sum_{i=1}^{2} \theta_i - R(\theta_i)\right) q(\theta_1, \theta_2) - C(q(\theta_1, \theta_2))\right] \geq 2V(\underline{\theta}). \tag{A45}$$

Integrating by parts the mediator's objective function, we get:

$$\beta\left(E_{(\theta_1,\theta_2)}\left[\left(\sum_{i=1}^{2} \theta_i - R(\theta_i)\right) q(\theta_1, \theta_2) - C(q(\theta_1, \theta_2))\right]\right)$$

$$+\sum_{i=1}^{2} \int_{\underline{\theta}}^{\bar{\theta}} (1 - F(\theta_i))\tilde{\alpha}'(\theta_i) E_{\theta_{-i}}[q(\theta_i, \theta_{-i})]\theta_i + 2V(\underline{\theta})\left(\int_{\underline{\theta}}^{\bar{\theta}} \alpha'(\theta) f(\theta) d\theta - \beta\right) \tag{A46}$$

where $\tilde{\alpha}'(\theta_i) = \frac{1}{1-F(\theta_i)} \int_{\theta_i}^{\bar{\theta}} \alpha'(\theta) f(\theta) d\theta$. Hence, any interim efficient allocation must maximize (A46) subject to (A45). Denote λ the multiplier of this last constraint. Optimizing the corresponding Lagrangian pointwise yields:

$$C'(q(\theta_1, \theta_2)) = \sum_{i=1}^{2} \theta_i - R(\theta_i)\left(1 - \frac{\tilde{\alpha}'(\theta_i)}{\lambda + \beta}\right)$$

[50] We neglect (A42) which is checked ex post.

which is the solution when the monotonicity condition (A42) holds; and $V(\underline{\theta})$ is not infinite when $\frac{\tilde{\alpha}'(\bar{\theta})}{\beta+\lambda} \leq 1$. Denoting $\alpha(\theta) = \frac{\tilde{\alpha}'(\theta)}{\beta+\lambda}$ yields (27).

Reciprocally, the fact that a common agency equilibrium satisfies (27) implies that one can find transfers which implement the corresponding output. Take $T_i(\theta_i, \theta_{-i}) = t(q(\theta_i, \theta_{-i}), \theta_i)$ where $t(\cdot)$ is the symmetric contribution schedule. ■

Proof of Theorem 6: Interim efficient equilibrium links necessarily the equilibrium output $Q(\theta) = q(\theta, \theta)$ along the diagonal and the function $b(\theta)$ because (27) also implies

$$C'(Q(\theta)) = 2b(\theta)$$

with the extra condition that $Q(\bar{\theta}) = q^{FB}(\bar{\theta}, \bar{\theta})$ since $b(\bar{\theta}) = \bar{\theta}$.

We now prove a lemma which significantly restricts the kind of equilibrium schedules which may be looked for.

Lemma 7 *Any informative equilibrium of a common agency game which is interim efficient satisfies:*

$$\frac{\partial^2 p}{\partial \theta \partial q}(Q(\theta), \theta) = 0 \quad \forall \theta \in \Theta. \tag{A47}$$

Proof: Along the diagonal where both principals have the same type θ, we must have:

$$b(\theta) = p(Q(\theta), \theta) \text{ and } \theta - b(\theta) = R(\theta)\frac{\partial p}{\partial \theta}(Q(\theta), \theta). \tag{A48}$$

Let fix an isoquant defined as $\theta_2 = \psi(Q(\tilde{\theta}), \theta_1)$ for some $\tilde{\theta} \in \Theta$. From (A48), we have:

$$\sum_{i=1}^{2} \theta_i - C'(Q(\tilde{\theta})) = \sum_{i=1}^{2} R(\theta_i)\frac{\partial p}{\partial \theta}(Q(\theta_i), \theta_i). \tag{A49}$$

Along such isoquant, we have also

$$\theta_i - p(Q(\tilde{\theta}), \theta_i) = R(\theta_{-i})\frac{\partial p}{\partial \theta}(Q(\tilde{\theta}), \theta_{-i}) \text{ for } i = 1, 2.$$

Summing over i, we get:

$$\sum_{i=1}^{2} \theta_i - C'(Q(\tilde{\theta})) = \sum_{i=1}^{2} R(\theta_i)\frac{\partial p}{\partial \theta}(Q(\tilde{\theta}), \theta_i). \tag{A50}$$

Gathering (A49) and (A50) yields along the isoquant:

$$R(\theta_1)\frac{\partial p}{\partial \theta}(Q(\tilde{\theta}), \theta_1) + R(\psi(Q(\tilde{\theta}), \theta_1))\frac{\partial p}{\partial \theta}(Q(\tilde{\theta}), \psi(Q(\tilde{\theta}), \theta_1))$$

$$= R(\theta_1)\frac{\partial p}{\partial \theta}(Q(\theta_1), \theta_1) + R(\psi(Q(\tilde{\theta}), \theta_1))\frac{\partial p}{\partial \theta}(Q(\psi(Q(\tilde{\theta}), \theta_1)), \psi(Q(\tilde{\theta}), \theta_1)). \tag{A51}$$

This identity should hold for all θ_1. We now look at the Taylor expansions of both the right- and left-hand sides of (A51) around $\tilde{\theta}$.

Using (A21) and the fact that $\tilde{\theta} = \psi(Q(\tilde{\theta}), \theta)$ yields first

$$\frac{\partial \psi}{\partial \theta}(Q(\tilde{\theta}), \tilde{\theta}) = -1.$$

Differentiating once more (A21) with respect to θ and evaluating at $\tilde{\theta}$ yields also:

$$\frac{\partial^2 \psi}{\partial \theta^2}(Q(\tilde{\theta}), \tilde{\theta}) = -2\left(\frac{\dot{R}(\tilde{\theta})}{R(\tilde{\theta})} - \frac{1 - \frac{\partial p}{\partial \theta}(Q(\tilde{\theta}), \tilde{\theta})}{\tilde{\theta} - \frac{C'(Q(\tilde{\theta}))}{2}}\right).$$

For an interim efficient equilibrium (if any), it must be that $0 \leq 2\tilde{\theta} - C'(Q(\tilde{\theta})) = 2R(\tilde{\theta})(1 - \tilde{\alpha}(\tilde{\theta})) \leq 2R(\tilde{\theta})$ and $2\tilde{\theta} - C'(Q(\tilde{\theta})) = 2R(\tilde{\theta})\frac{\partial p}{\partial \theta}(Q(\tilde{\theta}), \tilde{\theta})$ so that $\frac{\partial p}{\partial \theta}(Q(\tilde{\theta}), \tilde{\theta}) \leq 1$. Since $\dot{R}(\tilde{\theta}) < 0$, we have $\frac{\partial^2 \psi}{\partial \theta^2}(Q(\tilde{\theta}), \tilde{\theta}) > 0$. The right- and left-hand sides of (A51) are equal at $\theta_1 = \tilde{\theta}$ and have both zero first-order derivative at this point. The second-order derivative for the left-hand side evaluated at $\theta_1 = \tilde{\theta}$ is

$$\frac{\partial^2 \psi}{\partial \theta^2}(Q(\tilde{\theta}), \tilde{\theta})\left(\dot{R}(\tilde{\theta})\frac{\partial p}{\partial \theta}(Q(\tilde{\theta}), \tilde{\theta}) + R(\tilde{\theta})\frac{\partial^2 p}{\partial \theta^2}(Q(\tilde{\theta}), \tilde{\theta})\right).$$

The second-order derivative for the right-hand side at $\theta_1 = \tilde{\theta}$ is instead

$$\frac{\partial^2 \psi}{\partial \theta^2}(Q(\tilde{\theta}), \tilde{\theta})\left(\dot{R}(\tilde{\theta})\frac{\partial p}{\partial \theta}(Q(\tilde{\theta}), \tilde{\theta}) + R(\tilde{\theta})\left(\frac{\partial^2 p}{\partial \theta^2}(Q(\tilde{\theta}), \tilde{\theta}) + \frac{\partial^2 p}{\partial \theta \partial q}(Q(\tilde{\theta}), \tilde{\theta})\dot{Q}(\tilde{\theta})\right)\right).$$

Since $\dot{Q}(\tilde{\theta}) > 0$ holds, these second-order derivatives can only be equal when (A47) holds. ■

Condition (A47) is of course very demanding since, taken with the equilibrium conditions, it fully characterizes the equilibrium $Q(\cdot)$ along the diagonal.

From (14) that we differentiate with respect to q, we have indeed:

$$\frac{\partial \psi}{\partial q}(Q(\theta), \theta) - C''(Q(\theta)) + \frac{\partial p}{\partial q}(Q(\theta), \theta) = R(\theta)\frac{\partial^2 p}{\partial \theta \partial q}(Q(\theta), \theta) = 0.$$

Using also the identity $\psi(Q(\theta), \theta) = \theta$ and differentiating with respect to θ yield:

$$\frac{\partial \psi}{\partial q}(Q(\theta), \theta)\dot{Q}(\theta) + \frac{\partial \psi}{\partial \theta}(Q(\theta), \theta) = 1.$$

Using $\frac{\partial \psi}{\partial \theta}(Q(\theta), \theta) = -1$, we finally find

$$\frac{\partial p}{\partial q}(Q(\theta), \theta) = C''(Q(\theta)) + \frac{2}{\dot{Q}(\theta)}.$$

Moreover, using $2p(Q(\theta), \theta) = C'(Q(\theta))$ and differentiating with respect to θ yields

$$\left(2\frac{\partial p}{\partial q}(Q(\theta), \theta) - C''(Q(\theta))\right)\dot{Q}(\theta) + \frac{\partial p}{\partial \theta}(Q(\theta), \theta) = 0.$$

Finally, we have:

$$2\theta - C'(Q(\theta)) = R(\theta)\frac{\partial p}{\partial \theta}(Q(\theta),\theta) = (4 - C''(Q(\theta))\dot{Q}(\theta))R(\theta). \quad \text{(A52)}$$

Integrating the differential equation in $Q(\cdot)$ (A52) with the boundary condition requested by interim efficiency (i.e., $C'(Q(\bar{\theta})) = 2\bar{\theta}$) shows that the only candidate for an interim efficient equilibrium has an increasing output along the diagonal given by:

$$C'(Q(\theta)) = 2b(\theta) = 2\left(\theta - \frac{1}{1-F(\theta)}\int_\theta^{\bar{\theta}}(1-F(x))dx\right). \quad \text{(A53)}$$

Putting equations (28) (for $\theta = \theta_1 = \theta_2$) and (A53) together we get

$$b(\theta) = \theta - \frac{1}{1-F(\theta)}\int_\theta^{\bar{\theta}}(1-F(x))dx \text{ and } b(\theta) = \frac{1}{1-F(\theta)}\int_\theta^{\bar{\theta}} b^{-1}(2b(\theta)-b(x))f(x)dx. \quad \text{(A54)}$$

Simple differentiation of those two equalities with respect to θ shows that necessarily:

$$\dot{b}(\theta) = 2 - \frac{f(\theta)}{(1-F(\theta))^2}\int_\theta^{\bar{\theta}}(1-F(x))dx \text{ and } \frac{\dot{b}(\theta)}{1-F(\theta)}\int_\theta^{\bar{\theta}}\frac{f(x)}{\dot{b}(b^{-1}(2b(\theta)-b(x)))}dx = 1.$$

Suppose now that $\dot{b}(\theta)$ which must be positive (by assumption) is not everywhere constant. Then, because Θ is compact, $\dot{b}(\theta)$ achieves its maximum (resp. its minimum) at some $\tilde{\theta}$ (resp. $\tilde{\theta}'$). Either $\tilde{\theta}$ or $\tilde{\theta}'$ is necessarily different from $\bar{\theta}$ if $\dot{b}(\theta)$ is not constant. Assume thus $\tilde{\theta} < \bar{\theta}$. Then for any $x > \tilde{\theta}$, $b(\cdot)$ increasing implies $b^{-1}(2b(\widetilde{\theta}) - b(x)) < \tilde{\theta}$, and thus $\dot{b}(b^{-1}(2b(\widetilde{\theta}) - b(x))) < \dot{b}(\tilde{\theta})$, and finally $\frac{\dot{b}(\tilde{\theta})}{1-F(\tilde{\theta})}\int_{\tilde{\theta}}^{\bar{\theta}}\frac{f(x)}{\dot{b}(b^{-1}(2b(\widetilde{\theta})-b(x)))}dx > 1$. A contradiction. If $\tilde{\theta}' < \bar{\theta}$, one shows similarly that $\frac{\dot{b}(\tilde{\theta}')}{1-F(\tilde{\theta}')}\int_{\tilde{\theta}'}^{\bar{\theta}}\frac{f(x)}{\dot{b}(b^{-1}(2b(\tilde{\theta}')-b(x)))}dx < 1$.

Since $\dot{b}(\theta) = \beta$ for some $\beta \geq 0$ and $b(\bar{\theta}) = \bar{\beta}$, we immediately obtain, $b(\theta) = \bar{\beta} + \beta(\theta - \bar{\beta})$. Inserting into (A54) yields that $R(\theta) = \frac{2-\beta}{1-\beta}(\bar{\theta} - \theta)$. This gives a beta-density function $f(\theta) = \frac{1+\eta}{(\bar{\theta}-\underline{\theta})^{1+\eta}}(\bar{\theta} - \theta)^\eta$ where $\beta = 2 + \frac{1}{\eta}$ which ensures that $\dot{R}(\theta) < 0$. ■

Proof of Proposition 1: Immediate from the text. ■

Proof of Proposition 2: Suppose that principal P_2 offers $t^*(q)$ whatever his own type. The agent learns nothing from this offer and has no endogenous private information. Consider principal P_1's best-response. Two possibilities arises. First, he may "agree" with principal P_2 and induce the agent to produce q^*. This is done by offering also $t^*(q)$ whatever P_1's type. This yields payoff

$$W^*(\theta_1) = \theta_1 q^* - \frac{C(q^*)}{2}.$$

The second possibility is that principal P_1 deviates and induces another output. The best of such deviation should solve:

$$\max_{\{q,t_1(\cdot,\theta_1)\}} \theta_1 q - t_1(q,\theta_1) \text{ subject to } t_1(q,\theta_1) - C(q) \geq \max\left\{0, -\frac{C(q^*)}{2}\right\} = 0$$

where the latter condition is the agent's participation constraint.[51] This best deviation implements the output $\hat{q}(\theta)$ with a forcing contract

$$t(q,\theta_1)=\begin{cases}C(\hat{q}(\theta_1))>0 & \text{for } q=\hat{q}(\theta_1)\\ 0 & \text{for } q\neq\hat{q}(\theta_1)\end{cases} \tag{A55}$$

and gives payoff $\hat{W}(\theta_1)$ to the deviating principal. This deviation is unprofitable for all θ_1 when:

$$W^*(\theta_1)=\theta_1 q^*-\frac{C(q^*)}{2}\geq\hat{W}(\theta_1),\quad \forall\theta_1\in\Theta. \tag{A56}$$

Since $W^{*\prime}(\theta_1)=q^*\geq\hat{q}(\bar{\theta})\geq\hat{q}(\theta_1)=\hat{W}'(\theta_1)$, (A56) holds everywhere if it holds also at $\underline{\theta}$. Hence, offering $t^*(q)$ is a best-response for all θ_1 under the assumptions of the proposition.

■

Proof of Proposition 3: Suppose that P_{-i} with type θ_{-i} offers $t^*(q,\cdot|\theta_{-i})$ such that $t^*(q,\hat{\theta}_i|\theta_{-i})=t(q,\theta_{-i})$ for all $\hat{\theta}_{-i}\in\Theta$ on the equilibrium path when playing in Γ^*. This choice reveals of course all information on P_{-i}'s type to the agent who gets endogenous private information on P_{-i}'s type against P_i from that exactly as when playing Γ.

Consider P_i's best response. First, notice that P_i can achieve the same payoff as in Γ by offering also the degenerate menu $t^*(q,\cdot|\theta_i)$ such that $t^*(q,\hat{\theta}_i|\theta_i)=t(q,\theta_i)$ for all $\hat{\theta}_i\in\Theta$. Indeed, the agent's decision to accept that degenerate menu and to produce accordingly are the same as in Γ.

Suppose now that P_i makes any other offer, say a menu $\tilde{t}_i(q,\cdot)\neq t^*(q,\cdot|\theta_i)$, we want to find out-of equilibrium beliefs for the agent that makes offering this menu a suboptimal strategy for the deviating principal. Consider first the lower envelope of the offered menu defined as $z_i(q)=\min_{\hat{\theta}_i\in\Theta}\tilde{t}_i(q,\hat{\theta}_i)$ for all $q\in\mathcal{Q}$. By continuity of $\tilde{t}_i(q,\cdot)$ in $\hat{\theta}_i$ and compactness of Θ, the Theorem of the Maximum ensures that such lower envelope $z_i(q)$ is well-defined and continuous in q. Define also accordingly any arbitrary selection within the non-empty compact values and upper semi-continuous correspondence $\arg\min_{\hat{\theta}_i\in\Theta}\tilde{t}_i(q,\hat{\theta}_i)$ as $\hat{\theta}_i^0(q)$. For any θ_{-i}, define also $q(\theta_{-i})$ a measurable selector from the non-empty compact values correspondence $\arg\max_{q\in\mathcal{Q}} z_i(q)+t(q,\theta_{-i})-C(q)$. Such selector exists from the Measurable Maximum Theorem (Aliprantis and Border, 1999, p. 570) since the above maximand is a Carathéodory function. Such measurable selector allows us to compute the deviating principal's expected payoff in a meaningful way. Choose now out-of equilibrium beliefs that put mass one on $\hat{\theta}_i^0(q(\theta_{-i}))$ following any deviation by principal P_i. These beliefs minimize the agent's rent from his endogenous private information. Using the definition of $z_i(q)$, observe that, following the deviating menu offer $\tilde{t}_i(q,\cdot)$, P_i gets thus at most the expected payoff $E_{\theta_{-i}}[\theta_i q(\theta_{-i})-z_i(q(\theta_{-i}))]$.

[51] This participation constraint takes into account first the possibility to produce q^* at a loss and second the possibility of refusing all contracts. Note again that this participation constraint is the same for any beliefs that the agent may have following principal P_1's unexpected deviation.

Note then that the contribution $z_i(q)$ could also have been offered when playing Γ and accepted by any type of the agent if $\max_{q \in \mathcal{Q}} \quad z_i(q) + t(q, \theta_{-i}) - C(q) \geq 0$ for any type θ_{-i}. Such contribution implements the output schedule $q(\theta_{-i})$. Then, by definition of the equilibrium strategy $t(q, \theta_i)$ in Γ, we necessarily have

$$E_{\theta_{-i}}[\theta_i q(\theta_{-i}) - z_i(q(\theta_{-i}))] \leq E_{\theta_{-i}}[\theta_i q(\theta_i, \theta_{-i}) - t(q(\theta_i, \theta_{-i}), \theta_{-i})]$$

where $q(\theta_i, \theta_{-i})$ is the equilibrium output in Γ. This ends the proof that the deviating offer $\tilde{t}_i(q, \cdot)$ is dominated. ■

Proof of Proposition 4: Take any menu of differentiable contribution schedules $\{t_2^*(q, \hat{\theta}_2)\}_{\hat{\theta}_2 \in \Theta}$ which is incentive compatible for principal P_2 and inscrutable, i.e., all types of that principal offer this menu and the agent's prior beliefs on principal P_2's types are unchanged following such offer. When either accepting this menu or refusing, the agent gets:

$$\hat{U}_2 = \max \left\{ 0, E_{\theta_2} \left[\max_q t_2^*(q, \theta_2) - C(q) \right] \right\}.$$

Take a menu of contribution schedules $\{t_1(q, \hat{\theta}_1)\}_{\hat{\theta}_1 \in \Theta}$ which is incentive compatible for principal P_1 and also inscrutable. For the agent to accept both menus of contributions, the following participation constraint must hold:

$$E_{(\theta_1, \theta_2)} \left[\max_q \{ t_1(q, \theta_1) + t_2^*(q, \theta_2) - C(q) \} \right] \geq \hat{U}_2. \tag{A57}$$

For differentiable schedules, incentive compatibility for the agent implies the following first-order condition at any equilibrium output $q(\theta_1, \theta_2)$:

$$\frac{\partial t_1}{\partial q}(q(\theta_1, \theta_2), \theta_1) + \frac{\partial t_2^*}{\partial q}(q(\theta_1, \theta_2), \theta_2) = C'(q(\theta_1, \theta_2)). \tag{A58}$$

Lemma 8 *In any best response to the inscrutable menu $\{t_2^*(q, \hat{\theta}_2)\}_{\hat{\theta}_2 \in \Theta}$, principal P_1 with type θ_1 gets:*

$$V_1^S(\theta_1) = E_{\theta_2} \left[\max_q \{ \theta_1 q + t_2^*(q, \theta_2) - C(q) \} \right] - \hat{U}_2. \tag{A59}$$

Proof: Consider principal P_1 with type θ_1. He can always deviate by offering a degenerate menu $\{t_1(q, \hat{\theta}_1)\}_{\hat{\theta}_1 \in \Theta}$ such that $t_1(q, \hat{\theta}_1) = t_1^S(q, \theta_1)$ for all $\hat{\theta}_1$ where $t_1^S(q, \theta_1)$ is the sell-out contract

$$t_1^S(q, \theta_1) = \theta_1 q - V_1^S(\theta_1) \tag{A60}$$

with $V_1^S(\theta_1)$ satisfying (A59) and being the principal's deviation payoff.

Such sell-out contract aligns the objective of principal P_1 with that of the agent. It induces an output $q(\theta_1, \theta_2)$ which is efficient for their bilateral coalition (given the contributions received from principal P_2) and it maximizes their expected bilateral payoff when expectations are taken over principal P_2's type which is unknown at the time of acceptance in any inscrutable equilibrium. This output is thus such that:

$$\theta_1 + \frac{\partial t_2^*}{\partial q}(q(\theta_1, \theta_2), \theta_2) = C'(q(\theta_1, \theta_2)). \tag{A61}$$

Finally, $V_1^S(\theta_1)$ is adjusted to leave the agent indifferent between taking this degenerate menu, in which case his beliefs on the principal's deviating types are irrelevant, or not.

Lastly, at any best response in the game Γ^*, principal P_1 gets precisely $V_1^S(\theta_1)$ whatever his type. Indeed, such best response would give a set of incentive compatible payoffs $(V_1(\theta_1))_{\theta_1 \in \Theta}$ for principal P_1 which, by definition, must weakly Pareto dominate the payoff vector $(V_1^S(\theta_1))_{\theta_1 \in \Theta}$. However, the payoff vector $(V_1^S(\theta_1))_{\theta_1 \in \Theta}$ is undominated within the set of payoffs achievable with incentive compatible allocations and thus there cannot be other equilibrium payoffs.

To see that the payoff vector $(V_1^S(\theta_1))_{\theta_1 \in \Theta}$ is undominated, observe first that this payoff vector also maximizes the ex ante payoff of principal P_1, namely $E_{(\theta_1, \theta_2)}\left[\theta_1 q - t_1(q, \theta_1)\right]$, over the set of all incentive feasible allocations that induce the agent to accept principal P_1's contract. Indeed, because of risk-neutrality and ex ante contracting, the best ex ante incentive compatible mechanisms obviously implements the bilateral efficient output that solves (A61). It does so with menus of contributions $\{t_1(q, \theta_1)\}_{\theta_1 \in \Theta}$ of the form $t_1(q, \theta_1) = \theta_1 q - \alpha(\theta_1)$ which leave the agent residual claimant for his output decision. Note that $E_{\theta_1}\left[\alpha(\theta_1)\right]$ is then the principal's ex ante payoff which is set so that the agent's ex ante participation constraint holds as an equality, namely

$$E_{\theta_1}\left[\alpha(\theta_1)\right] = E_{(\theta_1, \theta_2)}\left[\max_q \{\theta_1 q + t_2^*(q, \theta_2) - C(q)\}\right] - \hat{U}_2 = E_{\theta_1}\left[V^S(\theta_1)\right].$$

This last equality shows that the payoff vector $(V_1^S(\theta_1))_{\theta_1 \in \Theta}$ is indeed undominated.[52] ■

Lemma 9 *In any best response to the inscrutable menu $\{t_2^*(q, \hat{\theta}_2)\}_{\hat{\theta}_2 \in \Theta}$ such that $t_1(q, \theta_1)$ is differentiable in q, principal P_1 with type θ_1 offers $t_1^S(q, \theta_1)$ as part of his menu $\{t_1(q, \hat{\theta}_1)\}_{\hat{\theta}_1 \in \Theta}$.*

Proof: Any menu of differentiable schedules $\{t_1(q, \hat{\theta}_1)\}_{\hat{\theta}_1 \in \Theta}$ in principal P_1's best-response correspondence must actually satisfy both (A58) and (A61) when his type is θ_1. Hence, we necessarily have $\frac{\partial t_1}{\partial q}(q, \theta_1) = \theta_1$. This implies, after integration, that $t_1(q, \theta_1) = \theta_1 q - h(\theta_1)$ for some $h(\cdot)$ but we know that $h(\theta_1) = V^S(\theta_1)$ from Lemma 8. ■

Altogether, Lemmata 8 and 9 imply also that principal P_1 offering the inscrutable incentive compatible menu $\{t_1^S(q, \hat{\theta}_1)\}_{\hat{\theta}_1 \in \Theta}$ of sell-out contracts is a best response whatever his type θ_1. This is the unique such menu with differentiable schedules. By the same token, if there exists any inscrutable equilibrium of Γ^*, principal 2 also does the same and offers the inscrutable menu $\{t_2^S(q, \hat{\theta}_2)\}_{\hat{\theta}_1 \in \Theta}$ where

$$t_2^S(q, \theta_2) = \theta_2 q - V_2^S(\theta_2). \tag{A62}$$

[52]As this proof shows, there may be many different ways of distributing payoffs between the different types θ_1 of principal P_1 from an ex ante viewpoint, but only one such allocation corresponds to a best response in the game where the principal already knows his type when making his offer to the agent.

Finally, inserting into (A61) yields the first-best output $q(\theta_1,\theta_2) = q^{FB}(\theta_1,\theta_2)$. From (A59), and denoting first-best welfare as $W^{FB}(\theta_1,\theta_2) = (\theta_1+\theta_2)q^{FB}(\theta_1,\theta_2) - C(q^{FB}(\theta_1,\theta_2))$, equilibrium payoffs for the principals satisfy the system of equations:

$$V_i^S(\theta_i) = E_{\theta_{-i}}\left[W^{FB}(\theta_i,\theta_{-i}) - V_{-i}^S(\theta_{-i})\right] - \hat{U}_{-i} \text{ for } i = 1,2 \tag{A63}$$

with

$$\hat{U}_{-i} = \max\left\{0, E_{\theta_{-i}}\left[\hat{W}(\theta_{-i}) - V_{-i}^S(\theta_{-i})\right]\right\}. \tag{A64}$$

It is immediate to derive from (A63):

$$\dot{V}_i^S(\theta_i) = E_{\theta_{-i}}\left[q^{FB}(\theta_i,\theta_{-i})\right] \tag{A65}$$

and thus

$$V_i^S(\theta_i) = V_i^S(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} E_{\theta_{-i}}\left[q^{FB}(x,\theta_{-i})\right] dx. \tag{A66}$$

From (A63) and taking expectations over θ_i, we get also

$$\sum_{i=1}^{2} E_{\theta_i}\left[V_i^S(\theta_i)\right] = E_{(\theta_1,\theta_2)}\left[W^{FB}(\theta_1,\theta_2)\right] - \hat{U}_j \text{ for } j = 1,2 \tag{A67}$$

and thus

$$\hat{U}_1 = \hat{U}_2 = \hat{U}.$$

Using (A66) and integrating by parts in the left-hand side of (A67) yields:

$$\sum_{i=1}^{2} V_i^S(\underline{\theta}) = E_{(\theta_1,\theta_2)}\left[\sum_{i=1}^{2}(\theta_i - R(\theta_i))q^{FB}(\theta_1,\theta_2) - C(q^{FB}(\theta_1,\theta_2))\right] - \hat{U}. \tag{A68}$$

But using (A63) to express $V_1^S(\underline{\theta})$, (A66) to express $V_2^S(\theta_2)$ and integrating by parts, we get also:

$$\sum_{i=1}^{2} V_i^S(\underline{\theta}) = E_{\theta_2}\left[(\underline{\theta} + \theta_2 - R(\theta_2))q^{FB}(\underline{\theta},\theta_2) - C(q^{FB}(\underline{\theta},\theta_2))\right] - \hat{U}.$$

We already know from the proof of Theorem 5 that

$$E_{\theta_2}\left[(\underline{\theta} + \theta_2 - R(\theta_2))q^{FB}(\underline{\theta},\theta_2) - C(q^{FB}(\underline{\theta},\theta_2))\right] = 2\underline{\theta}q^{FB}(\underline{\theta},\underline{\theta}) - C(q^{FB}(\underline{\theta},\underline{\theta})).$$

Hence, we get:

$$\sum_{i=1}^{2} V_i^S(\underline{\theta}) = 2\underline{\theta}q^{FB}(\underline{\theta},\underline{\theta}) - C(q^{FB}(\underline{\theta},\underline{\theta})) - \hat{U}.$$

Inserting into (A68) implies

$$2\underline{\theta}q^{FB}(\underline{\theta},\underline{\theta}) - C(q^{FB}(\underline{\theta},\underline{\theta})) = E_{(\theta_1,\theta_2)}\left[\sum_{i=1}^{2}(\theta_i - R(\theta_i))q^{FB}(\theta_1,\theta_2) - C(q^{FB}(\theta_1,\theta_2))\right].$$

But, we know from Theorem 5 that this equality never holds. Hence, there does not exist any equilibrium where both principals offer inscrutable mechanisms with differentiable schedules. ■