Dominance and Competitive Bundling*

Sjaak Hurkens[†] Doh-Shin Jeon[‡] Domenico Menicucci[§]

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Abstract

We study bundling by a dominant multi-product firm facing competition from a rival multi-product firm. Compared to competition under independent pricing, competition under pure bundling reduces (increases) each firm's profit for low (high) levels of dominance, while for intermediate levels of dominance, it increases the dominant firm's profit but reduces the rival's profit. The latter result provides a justification for the use of contractual bundling to build entry barrier. When we allow for mixed bundling, we find a threshold level of dominance above which the unique outcome is the one under pure bundling.

Key words: Competitive Bundling, Tying, Leverage, Entry Barrier

JEL Codes: D43, L13, L41.

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[†]Institute for Economic Analysis (CSIC) and Barcelona GSE, sjaak.hurkens@iae.csic.es.

[‡]Toulouse School of Economics, dohshin.jeon@gmail.com.

[§]Universita degli Studi di Firenze, domenico.menicucci@unifi.it.

1 Introduction

Does bundling (or tying) soften or intensify competition? Does bundling build a barrier to entry? These are two classic questions related to bundling: the first is analyzed in the literature on competitive bundling and the second in the literature on leverage theory of tying. The leverage theory is defined as one about a monopolist leveraging its market power in the tying good to the market of the tied product in order to monopolize it.[1] In reality, however, there are many cases in which the firm attempting to use bundling is not a pure monopolist in any single market but rather a dominant firm in at least one market. Then, bundling may be used to transfer market power from one market to the other or, simply, to increase the firm's overall market power. In this paper we study bundling by a dominant multi-product firm and offer novel insights with regards to the theory of competitive bundling and the leverage theory of tying.

Even if a firm has a legal monopoly over a product because of patents or copyrights, it often does face competition. For instance, consider two of the most publicized cases of bundling: bundling of aircraft engine and avionics in the GE/Honeywell merger case and Microsoft's bundling of Windows and Internet Explorer. In the proposed merger of GE and Honeywell, GE was supposed to have 52.5% market share in the engine market and Honeywell 50-60% in avionics (Nalebuff, 2002) and the European Commission opposed it for the concern that the merged entity would drive out rivals through bundling. In the Microsoft case, Windows was supposed to have 90% market share in the OS market for Intel-compatible PCs.[2] Furthermore, if we consider the two Supreme Court cases which first adopted the doctrine of leverage theory to make block booking (i.e. bundling of movies) per se illegal, these cases are about licensing movies to movie theaters (U.S. v. Paramount Pictures, 1948) or to TV stations (U.S. v Loew's, 1962). It would be impossible to argue that any of the movie studios or distribution companies involved in the two cases was a monopoly. More recently (May 2013), US senator John McCain proposed a bill to force cable service providers (such as Cablevision) and content providers (such as Viacom, owner of MTV and Nickelodeon) to use "à la carte" pricing, that is, to unbundle. The main concern

[1] For instance, according to Whinston (1990), leverage theory of tying is that "tying provides a mechanism whereby a firm with monopoly power in one market can use the leverage provided by this power to foreclose sales in, and thereby monopolize, a second market (p.837)."

[2] See Evans *et al.* (2000).

of the senator is the high price paid by customers for bundles of TV channels, many of which they do not watch. At the same time, Cablevision has filed a lawsuit against Viacom as it considers that Viacom's obligation to acquire the bundle of core and suite networks forecloses Cablevision from distributing competing networks that consumers would likely prefer.

By contrast, the existing formalized theory of leverage considers a pure monopoly in at least one market leveraging its monopoly power to another market. For instance, in his baseline model, Whinston (1990) finds that pre-commitment to tying induces the incumbent to be aggressive and thereby reduces the profit of a rival firm in the market of the tied product. Hence, tying may induce a potential competitor to stay out of the market if there is a fixed cost of entry. Notice, however, that tying also reduces the profit of the incumbent when entry does occur. In this sense, (contractual) tying is not credible because the incumbent would undo the bundling decision after observing that entry deterrence did not succeed. Only if the incumbent uses technical tying (which requires it to incur a high cost to undo the tying), entry deterrence may be successful. Nalebuff (2004) finds that bundling two products reduces the profit of an entrant in a single market in a credible way, but this result crucially relies on the assumption that the incumbent is a Stackelberg leader in setting prices.[3]

In this paper, we study bundling by a dominant multi-product firm (called A) facing competition from a rival multi-product firm (called B).[4] Motivated by the previous literature on bundling, we ask the following questions and answer them in terms of the level of A's dominance:

- When is pure bundling profitable for A? (i.e., when is A's profit higher under pure bundling than under independent pricing?)
- When does pure bundling build an entry barrier against B? (i.e., when is B's profit lower under pure bundling than under independent pricing?)

[3] Choi and Stefanadis (2001) and Carlton and Waldman (2002) consider a system monopoly and allow for entry in each complementary component, but their mechanism of leverage requires pre-commitment to bundling as in Whinston (1990). A notable exception is Peitz (2008) who builds an *example* in which the incumbent's tying is credible and reduces the rival's profit.

[4] Alternatively, we can assume that A competes against specialized firms as in Denicolo' (2000) and Nalebuff (2000). In this case pure bundling generates the additional effect that each specialized firm becomes less aggressive because she will not internalize the externality of her price on the other specialized firm's profit. To isolate the main effects, we consider the case in which B is a multi-product firm.

- How does mixed bundling affect these results? Is A's pure bundling credible when mixed bundling is allowed?

To answer these questions, we consider three different *simultaneous* pricing games: the game of independent pricing, the game of pure bundling, and the game of mixed bundling. In our model, each of the two firms produces two products (called 1 and 2). In each product market, there is both horizontal and vertical differentiation. To isolate the main effects, we assume that the markets are symmetric. Firm A is assumed to be dominant and we formalize this by supposing that product Aj gives a higher value than product Bj by $\alpha \geq 0$, for $j = 1, 2$. Thus α is a measure of A's dominance and is our key parameter.

We have two sets of novel results. First, when we compare competition under independent pricing with competition between two pure bundles, we find (i) for low levels of dominance, bundling reduces each firm's profit; (ii) for intermediate levels of dominance, bundling increases the dominant firm's profit but reduces the rival's profit; (iii) for high levels of dominance, bundling increases each firm's profit. Second, when we allow for mixed bundling, we find a threshold level of dominance above which the unique equilibrium outcome is the outcome obtained under pure bundling.

These results imply that for an intermediate level of dominance, pure bundling of firm A builds a credible barrier to entry against B. This result provides a justification for the use of contractual bundling to build an entry barrier. In contrast, for very high levels of dominance, bundling does not build a barrier to entry against B (but is still profitable for A). In this case, if A had the power to dictate the terms of competition, the most effective way to deter entry would be to enforce competition in independent pricing, which is completely opposite to Whinston's insight. Therefore, as far as entry barriers are concerned, "the mutual leverage of dominance" in our model[5] is quite different from "the leverage of monopoly" in Whinston (1990).

We generalize a two-dimensional Hotelling model of Matutes and Régibeau (1988) in two respects. First, we add the vertical differentiation parameter α introduced above: $\alpha = 0$ corresponds to their model. Second, we suppose that consumers' locations on the Hotelling line are i.i.d., each with a log-concave symmetric density function; this family of densities includes the uniform density considered by Matutes and Régibeau (1988). Our main simpli-

[5]Given that we consider symmetric markets, leverage is mutual in that if bundling allows A to leverage dominance from market 1 to market 2, then leverage from market 2 to market 1 occurs at the same time.

fying assumption is full coverage, which means that all consumers are served under either regime. Hence, the difference between complements and independent products does not appear.

The intuition for our first main result regarding pure bundling can be provided in terms of the following two effects: *demand size effect* and *demand elasticity effect*. To explain the demand size effect, suppose that for both products firm A's product is located at 0 and B's product at 1. Let p^*_{ij} for $i = A, B$ and $j = 1, 2$ represent the equilibrium price under independent pricing. A positive α implies that the marginal consumer is located at $x(\alpha) \in (1/2, 1]$ under independent pricing. Under bundling, what matters is the distribution of the average location, which is *more peaked* around the mean than that of the individual location, in the sense that the probability of a given size deviation from the mean is smaller for the average location than for the individual one for any symmetric log-concave density function (Fang and Norman, 2006). Consider now bundling such that firm A charges $P_A = p^*_{A1} + p^*_{A2}$ for the bundle and, likewise, firm B charges $P_B = p^*_{B1} + p^*_{B2}$ for the bundle. Then, the average location of the marginal consumer is still equal to $x(\alpha)$, but the fact that the distribution of the average location is more peaked implies that bundling increases the demand of A (and hence increases A's profit) and reduces the demand of B.[6]

We now turn to the demand elasticity effect. The fact that the distribution of the average location is more peaked than that of the individual location implies that the demand under bundling is more elastic (resp. less elastic) if the average location of the marginal consumer is close to the mean (resp. close to 1). The average location of the marginal consumer is closer to 1 as firm A's dominance increases. Therefore, bundling makes A more aggressive (less aggressive) if α is smaller than (larger than) some threshold. Since firm A always benefits from the demand size effect, when α is equal to the threshold, bundling increases A's profit but reduces B' profit. In general, for low levels of dominance the competition-intensifying demand elasticity effect dominates the demand size effect and hence bundling reduces every firm's profit; this generalizes the finding Matutes and Régibeau (1988) obtained for $\alpha = 0$. Likewise, for high levels of dominance the competition-softening demand elasticity effect dominates the demand size effect and bundling increases every firm's profit. For intermediate levels of dominance, bundling is profitable for A but reduces B's profit.

[6]Indeed, Fang and Norman (2006) find this demand size effect of bundling in a monopoly setting in which valuations are i.i.d., each with a symmetric and log-concave density.

In Section 4, we consider a simultaneous pricing game in which each firm is allowed to use mixed bundling, and establish that if A is dominant enough then the unique equilibrium outcome is the outcome obtained under pure bundling. To provide an intuition for this result, we first note that pure bundling is always an equilibrium outcome since for each firm pure bundling is a best response to pure bundling. In order to see why it is the unique equilibrium outcome, we prove that for given prices of B, the problem of firm A to find optimal prices boils down to the problem of a two-product monopolist facing a single consumer with suitably distributed valuations. If the minimum valuation for each good is large enough, then the monopolist finds it optimal to use a pure bundling strategy instead of a mixed bundling strategy in order to avoid that some types of consumer buy only one object, which would make the monopolist lose a (substantial) profit from selling the other object.[7] A large minimum valuation in the monopoly setting is equivalent to strong dominance of A with respect to B in the duopoly, which induces firm A to play a pure bundling strategy.

The bundling (or tying) literature can be divided into three categories.[8] The first one includes papers that study bundling as a price discrimination device for a monopolist (Schmalensee, 1984, McAfee et al. 1989, Salinger 1995, Armstrong 1996, Bakos and Brynjolfsson, 1999, Fang and Norman, 2006, Chen and Riordan 2012). The second is about competitive bundling where entry and exit is not an issue (Matutes and Régibeau 1988, Economides, 1989, Carbajo, De Meza and Seidmann, 1990, Chen 1997, Denicolo 2000, Nalebuff, 2000, Armstrong and Vickers, 2010, Carlton, Gans, and Waldman, 2010, Thanassoulis 2011, Hahn and Kim 2012).[9] The last is about leverage theory of bundling in which the

[7]In fact, this result holds if the virtual valuation for each object is always positive – a property satisfied if the support of valuations is shifted far enough to the right on the real line. Moreover, the monopolist chooses a price for the bundle which is higher than the minimum valuation for the bundle, implying that some types of consumers buy nothing. This occurs because very few types have a valuation for the bundle which is close to the minimum valuation (Armstrong, 1996).

[8]See Choi (2011) for a recent survey of bundling literature.

[9]Economides (1989) generalizes Matutes and Régibeau (1988) by showing that profits are higher under compatibility when n symmetric firms compete. Hahn and Kim (2012) extend Matutes and Régibeau (1988) by introducing cost asymmetry, which generates results similar to those in our Proposition 3. However, there are a number of important differences between their paper and our paper. First (and the most important), we are mainly motivated by solving a puzzle in the bundling literature, namely, to identify conditions under which bundling is credible and builds an entry barrier. In contrast, they focus only on the compatibility literature. Second, as a consequence, we study mixed bundling, which they don't. Third, we consider the

main motive of bundling is to deter entry or induce the exit of rival firms in the competitive segment of the market (Whinston, 1990, Choi and Stefanadis 2001, Carlton and Waldman, 2002, Nalebuff, 2004, Peitz 2008, Jeon and Menicucci, 2006, 2012).[10]

We contribute to each of these categories. We contribute to the theory of competitive bundling by building a general framework that includes as a special case the model of Matutes and Régibeau (1988) and showing that the level of dominance of a firm is a crucial parameter such that their finding is completely reversed for strong dominance. We also provide a general intuition in terms of the demand size and the demand elasticity effect. Studying bundling by a dominant firm and its implications on entry barriers, we bridge the gap between the real world and the leverage theory which studies a monopolist's bundling. Most importantly, we show that for intermediate levels of dominance, pure bundling builds an entry barrier and is credible. This result provides a justification for the use of contractual bundling to deter entry.[11] We also contribute to the monopoly's price discrimination theory of bundling by identifying sufficient conditions on the valuations' distribution that make mixed bundling dominated by pure bundling. This complements McAfee, McMillan and Whinston (1989), who identify sufficient conditions for mixed bundling to strictly dominate independent pricing. However, they do not study conditions under which pure bundling generates the highest profit. Furthermore, we use some results in Pavlov (2011) on bi-dimensional mechanism design to prove that in suitable environments, a pure bundling mechanism is superior to any

class of log-concave and symmetric densities, which include the uniform density while they consider only uniform density. Hence, our proofs are based on general properties of log-concave and symmetric densities while their proofs are based on direct computations. Last (related to the third point), we give a unified intuition based on the demand size effect and the demand elasticity effect while their intuition is mainly based on the effects identified by Matutes and Régibeau; in particular, they do not mention the demand size effect (or its equivalent).

[10]Jeon and Menicucci (2006, 2012) study competitive bundling and derive implications on entry barriers. They consider a common agency setting under complete information in which firms sell portfolios of digital products and find that in the absence of the buyer's budget constraint, all equilibria under bundling are efficient and each firm obtains a profit equal to the social marginal contribution of its portfolio (and hence bundling does not build any entry barrier). However, in the presence of the budget constraint, bundling builds an entry barrier since it allows firms with large portfolios to capture all the budget.

[11]When we study a benchmark in which A is monopoly in product 1 and faces competition only in product 2, we rediscover the findings of Whinston (1990): pure bundling reduces both firms' profits and hence building an entry barrier requires pre-commitment to pure bundling. The analysis can be obtained from the authors upon request.

other incentive compatible and individually rational selling mechanism.

The paper is organized as follows. We present our model in Section 2, compare independent pricing and pure bundling in Section 3 and study mixed bundling in Section 4. Concluding remarks are gathered in Section 5.

2 Model

We consider competition between two multi-product firms A and B, each producing two different products, 1 and 2, to address the question of whether bundling helps a firm to increase its static profit and to foreclose the other firm. Let ij denote product j produced by firm i, for $i = A, B$ and $j = 1, 2$. Each consumer has a unit demand for each product j.

We consider a model of both vertical and horizontal differentiation. Regarding the latter, let $x_j \in [0, 1]$ denote a generic consumer's location in terms of product j. For each product, firm A is located at $y_A = 0$ and firm B is located at $y_B = 1$. The gross utility that a consumer with location x_j obtains from consuming product ij is given by $v_{ij} - t|x_j - y_i|$, where $v_{ij} > 0$ is the same for all consumers, $t > 0$ is the usual transportation cost parameter and $|x_j - y_i|$ denotes the distance between the consumer's and the firm's location. The gross utility that a consumer with location (x_1, x_2) obtains from consuming products $i1$ and $i'2$ is simply the sum $v_{i1} - t|x_1 - y_i| + v_{i'2} - t|x_2 - y_{i'}|$. We assume that v_{ij} is sufficiently high so that every consumer consumes one unit of each product. Hence, the two products can be interpreted as products that can be independently consumed or complements.[12]

We assume that x_1 and x_2 are identically and independently distributed with support $[0, 1]$, c.d.f. F and density f such that $f(x) > 0$ for all $x \in (0, 1)$. Moreover, we assume that f is differentiable, symmetric around 1/2, and log-concave, i.e., $\log(f)$ is a concave function. This implies that f is weakly increasing on $[0, 1/2]$ and weakly decreasing on $[1/2, 1]$. It also implies that $\log(F)$ and $\log(1 - F)$ are both concave.[13]

Given our assumption of large v_{ij}, a crucial role is played by the difference between v_{Aj} and v_{Bj}, representing vertical differentiation. In order to isolate the effect of dominance, we assume that $v_{A1} - v_{B1} = v_{A2} - v_{B2} \geq 0$ and let $\alpha \equiv v_{A1} - v_{B1}$ represent the dominance of firm

[12] In the case of complements, one firm's decision to bundle its products has the same effect as the firm's choice to make its products incompatible with the products of the other firm. This interpretation is followed in Matutes and Régibeau (1988).

[13] See for example Bagnoli and Bergstrom (2005).

A. For each firm i, let c_i denote the marginal cost of production for product ij (for $j = 1, 2$). We assume that c_A and c_B are large enough that each consumer buys only one unit of each of the two products. However, allowing $c_A \neq c_B$ generates the same results obtained when the dominance of firm A is $\alpha + c_B - c_A$ and marginal costs are equal; thus we assume without loss of generality that $c_A = c_B$. But in this case marginal costs have only an additive effect on prices, hence we simplify notation by setting $c_A = c_B = 0$ and interpreting equilibrium prices as profit margins.

Let p_{ij} be the price charged by firm i for product ij under independent pricing. Under bundling, P_i denotes the price charged by firm i for the bundle of products $i1$ and $i2$. We study three different games of simultaneous pricing played by the two firms.

- Game of independent pricing [**IP**]: firm A chooses p_{A1} and p_{A2}, and firm B chooses p_{B1} and p_{B2}.
- Game of pure bundling [**PB**]: firm A chooses P_A for the bundle of $A1$ and $A2$, and firm B chooses P_B for the bundle of $B1$ and $B2$.
- Game of mixed bundling [**MB**]: firm A chooses P_A, p_{A1} and p_{A2}, and firm B chooses P_B, p_{B1} and p_{B2}.

In Section 3, we study the first two games and compare them. In Section 4, we study the third game. In addition, the analysis of Section 3 allows us to study the following three-stage game:

- Stage one: firm B chooses between entering or not.
- Stage two: if B has entered, each firm chooses between IP and PB.
- Stage three: firms compete in prices in the game determined by their choices at stage two.

Notice that if in stage two at least one firm has chosen PB, then competition in stage three occurs between the two pure bundles.[14] Therefore competition in independent prices occurs only if both firms have chosen IP. Both firms' choosing PB is always an equilibrium

[14]Indeed, suppose that firm A, for instance, has chosen PB and firm B has chosen IP. Then each consumer either buys the pure bundle of A or the two products of firm B, which are therefore viewed as a bundle.

in stage two but it may involve playing a weakly dominated strategy. We will impose that firms do not play weakly dominated strategies. In particular, (IP, IP) is the outcome only if this is the preferred outcome for both firms. In Section 4 we assume that firms are unable to precommit (before prices are chosen) to a certain class of strategies such as pure bundling or independent pricing, and therefore each firm can choose any mixed bundling strategy.

3 Independent pricing vs. pure bundling

3.1 Preliminaries

If x_1 and x_2 are two random variables that are i.i.d. with log-concave density function f with support $[0,1]$, then their average, $(x_1+x_2)/2$, is distributed with density function $\tilde{f}$ which is also log-concave (see An, 1998, Cor. 1). If, moreover, f is symmetric around $1/2$ (that is, $f(x) = f(1-x)$ for each $x \in [0,1]$), then the distribution of the average is symmetric and strictly more-peaked around $1/2$ (the mean) than that of each original variable (Proschan, 1965).[15] That is, for any $t \in (0, 1/2)$,

$$\int_t^{1-t} f(s)ds < \int_t^{1-t} \tilde{f}(s)ds.$$

The density function of the average is given by

$$\tilde{f}(x) = \int_0^{2x} 2f(2x-s)f(s)ds \text{ for } x \leq 1/2$$

and

$$\tilde{f}(x) = \tilde{f}(1-x) = \int_0^{2-2x} 2f(2-2x-s)f(s)ds \text{ for } x \geq 1/2.$$

The c.d.f. of the average is $\tilde{F}(x) = \int_0^x \tilde{f}(s)ds$. Note that the more-peakedness of the distribution of the average can be equivalently expressed as $\tilde{F}(x) > F(x)$ for all $x \in (1/2, 1)$.

In the special case of uniform distribution, we have $f(x) = 1$, $F(x) = x$ and, for $x \geq 1/2$, $\tilde{f}(x) = 4(1-x)$, $\tilde{F}(x) = 1 - 2(1-x)^2$. These functions are depicted in Figure 1.

For later reference we point out two properties of the density functions of the individual and the average locations. We will use them later when comparing the outcomes of independent pricing with pure bundling.

[15] Observe that for density functions that are not log-concave, the average is not necessarily more-peaked than the original distribution (see for instance the Cauchy distribution). This explains our restriction to log-concave densities.

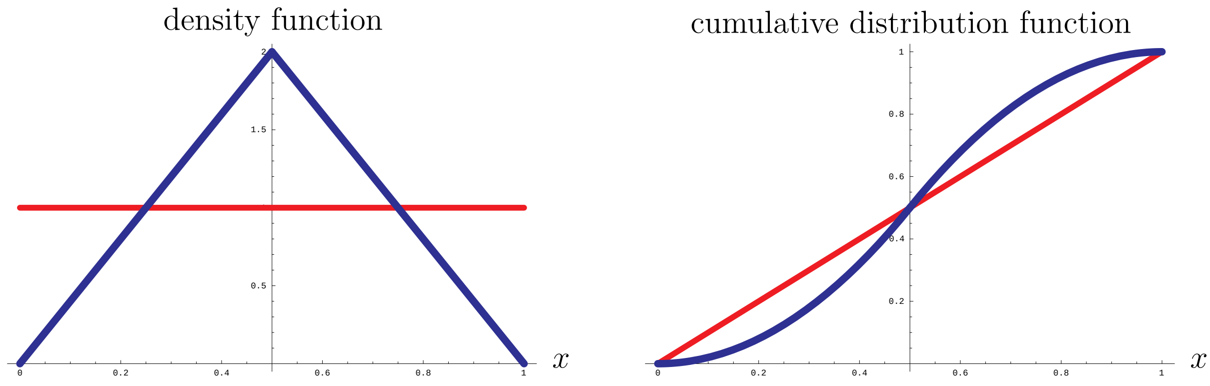

Figure 1: Density and distribution functions for the uniform case. The thicker (blue) graphs correspond to average valuations (bundling).

Lemma 1. *Let f be a log-concave density function which is symmetric around $1/2$ with support $[0,1]$. Let $\tilde{f}$ be the density function of the average of two independently and identically distributed random variables that are distributed according to the density function f. Then*

(i) $\tilde{f}(1/2) > f(1/2)$.

(ii) $\tilde{f}(x) \leq 4(1-x)f(x)^2$ for all $x \geq 1/2$, and thus

$$\lim_{x \to 1} \frac{\tilde{f}(x)}{f(x)} = 0.$$

3.2 Independent Pricing

When firms compete in independent prices, we can consider each market in isolation. Moreover, since the markets for the two products are symmetric, the equilibrium prices in the two markets will be the same, and we here only need to solve for the equilibrium in one of the markets. Thus we restrict attention to price competition on the Hotelling line where consumers are distributed with density f. Recall that firm A (B) is located at 0 (1) and firm A offers a product that is valued α higher than the product of firm B. Hence, given prices p_A and p_B, the indifferent consumer is located at

$$x(\alpha, p_A, p_B) = \frac{1}{2} + \sigma(\alpha - p_A + p_B), \tag{1}$$

where $\sigma = 1/(2t)$ (unless, of course, p_A and p_B are such that every type of consumer prefers A, or every type prefers B). For simplicity, we will often suppress the arguments and simply write x for the location of the indifferent consumer.

We suppose for now that the distribution and the parameters are such that independent pricing leads to an interior equilibrium, that is such that both firms obtain positive market share.[16] Then the first-order conditions must be satisfied at the equilibrium prices.

Since marginal costs are assumed to be zero, the profit functions are

$$\pi_A = p_A F(x), \qquad \pi_B = p_B(1 - F(x)),$$

and the first-order conditions are[17]

$$0 = F(x) - \sigma p_A f(x), \qquad 0 = 1 - F(x) - \sigma p_B f(x).$$

If p_A^*, p_B^* are the equilibrium prices and x^* denotes the equilibrium location of the indifferent consumer, then we have

$$x^* = \frac{1}{2} + \sigma\alpha - \sigma(p_A^* - p_B^*) = \frac{1}{2} + \sigma\alpha + \frac{1 - 2F(x^*)}{f(x^*)}.$$

Hence, the equilibrium location of the indifferent consumer is a fixed point of the mapping:

$$X^\alpha : x \mapsto \frac{1}{2} + \sigma\alpha + \frac{1 - 2F(x)}{f(x)}. \tag{2}$$

Notice that $\frac{1-2F(x)}{f(x)} = -\frac{F(x)}{f(x)} + \frac{1-F(x)}{f(x)}$. As we mentioned in Section 2, log-concavity of f implies that (i) $-\frac{F(x)}{f(x)}$ is decreasing; and (ii) $\frac{1-F(x)}{f(x)}$ is decreasing. Hence, X^α is weakly decreasing and a unique fixed point $x^* < 1$ exists, provided that $\lim_{x\to 1} X^\alpha(x) < 1$. The equilibrium prices are then also unique. Clearly, at $\alpha = 0$ we have $x^* = 1/2$ and Proposition 1(i) establishes that x^* is increasing in α, hence $x^* > 1/2$ for $\alpha > 0$. If α is sufficiently large and $f(1) > 0$, then $x^* = 1$.

Proposition 1 (Independent Pricing)**.** *(i) Suppose that $(\sigma\alpha - 1/2)f(1) < 1$. Then the independent pricing game has a unique and interior equilibrium, characterized by the unique fixed point $x^*(\alpha)$ of X^α, and $x^*(\alpha) \in [\frac{1}{2}, 1)$. The function $x^*(\alpha)$ is increasing and concave for $\alpha \geq 0$. The equilibrium prices are*

$$p_A^*(\alpha) = \frac{F(x^*(\alpha))}{\sigma f(x^*(\alpha))}, \qquad p_B^*(\alpha) = \frac{1 - F(x^*(\alpha))}{\sigma f(x^*(\alpha))}.$$

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[16] Proposition 1 characterizes when an interior equilibrium exists.

[17] Notice that $\frac{d\pi_A}{dp_A} = 0$ suffices to maximize π_A because $\frac{d\pi_A}{dp_A} = F(x)[1 - \sigma p_A \frac{f(x)}{F(x)}]$, and because log-concavity of F implies that $\frac{f(x)}{F(x)}$ is decreasing in x, and thus increasing in p_A. Hence, if p_A^* solves the first-order condition, then $\frac{d\pi_A}{dp_A} < 0$ for $p_A > p_A^*$ and $\frac{d\pi_A}{dp_A} > 0$ for $p_A < p_A^*$. A similar argument reveals that $\frac{d\pi_B}{dp_B} = 0$ suffices to maximize π_B with respect to p_B.

The equilibrium profits are

$$\pi_A^*(\alpha) = \frac{F(x^*(\alpha))^2}{\sigma f(x^*(\alpha))}, \qquad \pi_B^*(\alpha) = \frac{(1 - F(x^*(\alpha)))^2}{\sigma f(x^*(\alpha))}.$$

p_A^* *and* π_A^* *are increasing in* α*, while* p_B^* *and* π_B^* *are decreasing in* α*.*

(ii) Suppose that $(\sigma\alpha - 1/2)f(1) \geq 1$*. Then the independent pricing game has a unique equilibrium, and it is such that firm A's market share is 1. The equilibrium prices and profits are*

$$p_A^*(\alpha) = \pi_A^*(\alpha) = \alpha - 1/(2\sigma), \qquad p_B^*(\alpha) = \pi_B^*(\alpha) = 0.$$

3.3 Pure Bundling

The analysis performed for independent pricing straightforwardly extends to the case in which firms A and B compete under bundling. Given P_A, P_B chosen by the firms, let $p_A = P_A/2$ and $p_B = P_B/2$. Then consumers with average location $x < \tilde{x}$ will buy the bundle from firm A, while consumers with average location $x > \tilde{x}$ will buy the bundle from firm B, where $\tilde{x}$ is given by:

$$\tilde{x} = \frac{1}{2} + \sigma\left(\alpha - p_A + p_B\right).$$

The equilibrium bundle prices are found in a way very similar to the analysis of independent pricing, since the game under bundling can be considered as a competition between two firms each offering one product – in fact, a bundle. The only difference is that the underlying density function is the density $\tilde{f}$ of the average location, and not the density f of the individual location. Let us define

$$\tilde{X}^\alpha : x \mapsto \frac{1}{2} + \sigma\alpha + \frac{1 - 2\tilde{F}(x)}{\tilde{f}(x)}. \tag{3}$$

Since $\tilde{f}$ is log-concave, we obtain (as above) that $\tilde{X}^\alpha$ is decreasing in x. Since $\lim_{x\to 1}\tilde{f}(x) = 0$, $\tilde{X}^\alpha$ always admits a unique fixed point $x^{**}(\alpha) < 1$, and equilibrium prices and profits can be expressed in terms of this fixed point. Hence, under pure bundling we always obtain a unique and interior equilibrium in which both firms have positive market share.[18]

[18] We find at work here the same principle which makes it optimal to exclude some consumers for a multiproduct monopolist: see Armstrong (1996).

Proposition 2 (Pure Bundling)**.** *The pure bundling pricing game has a unique equilibrium, characterized by the unique fixed point* $x^{**}(\alpha)$ *of* $\tilde{X}^{\alpha}$*, and* $x^{**}(\alpha) \in [\frac{1}{2}, 1)$*. The function* $x^{**}(\alpha)$ *is increasing and concave for* $\alpha \geq 0$*. The equilibrium prices are*

$$P_A^{**}(\alpha) = \frac{2\tilde{F}(x^{**}(\alpha))}{\sigma \tilde{f}(x^{**}(\alpha))}, \qquad P_B^{**}(\alpha) = \frac{2(1-\tilde{F}(x^{**}(\alpha)))}{\sigma \tilde{f}(x^{**}(\alpha))}.$$

The equilibrium profits are

$$\Pi_A^{**}(\alpha) = \frac{2\tilde{F}(x^{**}(\alpha))^2}{\sigma \tilde{f}(x^{**}(\alpha))}, \qquad \Pi_B^{**}(\alpha) = \frac{2(1-\tilde{F}(x^{**}(\alpha)))^2}{\sigma \tilde{f}(x^{**}(\alpha))}.$$

P_A^{**} *and* Π_A^{**} *are increasing in* α *while* P_B^{**} *and* Π_B^{**} *are decreasing in* α*.*

3.4 Illustration through Uniformly distributed locations

For the special case of uniformly distributed locations we can get explicit expressions for equilibrium prices and profits.

Proposition 3. *In the case of uniformly distributed locations and* $\sigma = 1/2$ *(that is,* $t = 1$*) and* $\alpha < 3$ *(to guarantee we are in case (i) of Proposition 1) we have*

independent pricing $x^*(\alpha) = (3+\alpha)/6$, $p_A^*(\alpha) = 2x^*(\alpha)$, $p_B^*(\alpha) = 2(1-x^*(\alpha))$, $\pi_A^*(\alpha) = 2(x^*(\alpha))^2$, $\pi_B^*(\alpha) = 2(1-x^*(\alpha))^2$.

pure bundling $x^{**}(\alpha) = \left(7+\alpha-\sqrt{9-2\alpha+\alpha^2}\right)/8$, $P_A^{**}(\alpha) = (1-2(1-x^{**}(\alpha))^2)/(1-x^{**}(\alpha))$, $P_B^{**}(\alpha) = 2(1-x^{**}(\alpha))$, $\Pi_A^{**}(\alpha) = (1-2(1-x^{**}(\alpha))^2)^2/(1-x^{**}(\alpha))$, $\Pi_B^{**}(\alpha) = 4(1-x^{**}(\alpha))^3$.

It is easily verified that the inequality $\Pi_A^{**}(\alpha) > 2\pi_A^*(\alpha)$ holds if and only if $\alpha > 1.415$. The inequality $\Pi_B^{**}(\alpha) > 2\pi_B^*(\alpha)$ holds if and only if $\alpha > 2.376$.

3.5 Comparing independent pricing and pure bundling

We now study how bundling affects each firm's profit in comparison to independent pricing. We find that the impact of bundling on profits depends crucially on the level of dominance that firm A has over firm B. This impact can be decomposed into two effects, the demand size effect and the demand elasticity effect. We first provide an intuition, based on the two effects, of our main result and then formally establish the result.

3.5.1 Heuristic Intuition

Demand size effect. Suppose that firms A and B sell the two products independently and set prices (for each of their two products) p_A and p_B. Suppose furthermore that at these prices firm A has a market share larger than one half, that is the indifferent consumer in each market is located at $x > 1/2$. The demand for each product of firm A equals $F(x)$. Now assume that both firms bundle their two products at a price equal to the sum of the previous prices, that is $P_A = 2p_A$ and $P_B = 2p_B$. Then the indifferent consumer is the consumer whose average location is equal to x. The demand for A's bundle is $\tilde{F}(x)$. Since the distribution of the average location is more-peaked around 1/2 than the distribution of the individual location for any symmetric log-concave density f, we have $\tilde{F}(x) > F(x)$ for each $x \in (1/2, 1)$. Hence, the demand after bundling, for given prices, increases for the dominant firm (firm A) and decreases for the dominated firm (firm B), unless firm A covers the whole market to start with. For instance, in Figure 1, fix the location of the marginal consumer at $x \in (1/2, 1)$. Then, it is clear that bundling increases the demand for the dominant firm. These arguments apply, in particular, when p_A and p_B are equal, respectively, to p_A^* and p_B^*, their equilibrium values under independent pricing, since they imply $x^* > 1/2$ for $\alpha > 0$.

Demand elasticity effect. After bundling, firms will have incentives to change their prices away from $P_A = 2p_A^*$ and $P_B = 2p_B^*$. Whether they want to charge higher or lower prices depends on how bundling affects demand elasticity, which in turn depends on the location of the marginal consumer and hence on the level of A's dominance. From Figure 1, for low levels of dominance (that is, when x^* is not much larger than 1/2), bundling makes the demand more elastic: a given decrease in the average price of a bundle generates a higher boost in demand than the same decrease in the prices of single products because the distribution of the average location is more-peaked around 1/2 than the distribution of individual locations. On the other hand, for high levels of dominance (that is, when x^* is close to 1), bundling makes the demand less elastic: because $\tilde{f}(x)/f(x)$ converges to zero as x goes to one, for x close enough to one, a given decrease in the average price of a bundle generates a smaller boost in demand than the same decrease in the prices of single products. In summary, bundling changes the elasticity of demand such that firms compete more aggressively for low levels of dominance, but less aggressively for high levels of dominance. Figure 2 illustrates this for the case of the uniform distribution.

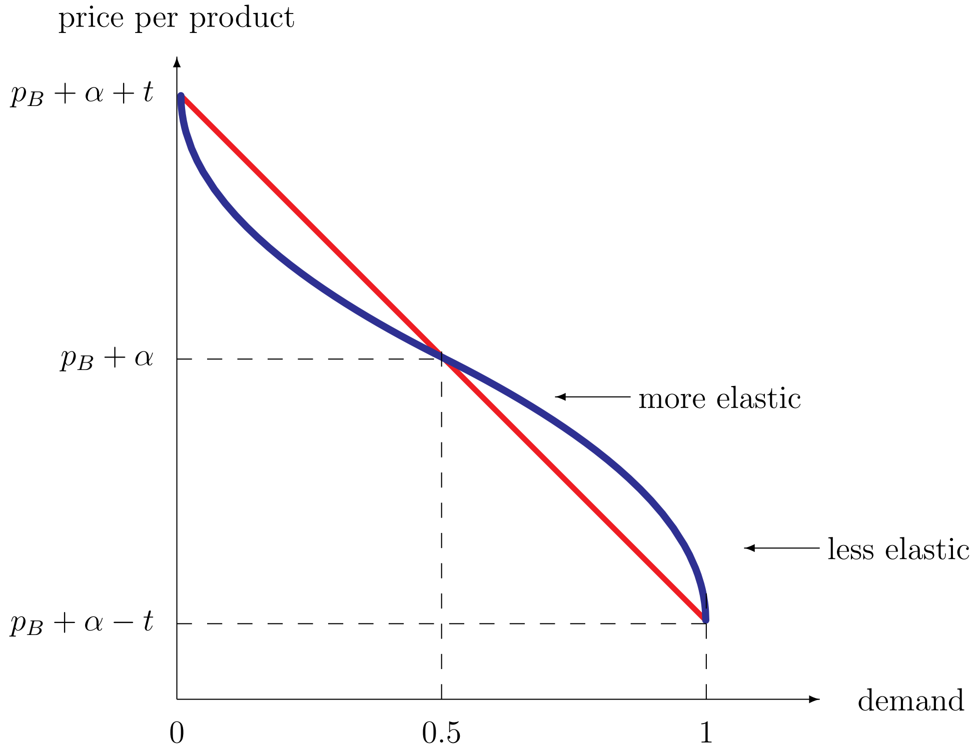

Figure 2: Demand for each of A's products when sold independently (thinner red line) and sold as bundle (thicker blue line), for given price per product p_B of firm B.

More precisely, at equilibrium prices for each firm the elasticity of demand of the own product with respect to the own price must be equal to one. For instance, at the independent pricing equilibrium, the elasticity of demand for firm A equals

$$\varepsilon_A = -\frac{D'(p)p}{D(p)} = \frac{\sigma f(x^*)p_A^*}{F(x^*)} = 1.$$

When firms A and B bundle their products without changing prices, elasticity equals

$$\tilde{\varepsilon}_A = \frac{\sigma \tilde{f}(x^*)p_A^*}{\tilde{F}(x^*)} = \frac{\tilde{f}(x^*)F(x^*)}{f(x^*)\tilde{F}(x^*)}.$$

For relatively low levels of dominance (that is, x^* slightly larger than $1/2$), this elasticity is strictly larger than 1 because $F(1/2) = \tilde{F}(1/2)$ and $\tilde{f}(1/2) > f(1/2)$. This implies that firm A has strong incentives to lower its price (per product). On the other hand, if firm A has large dominance, then $F(x^*)$ and $\tilde{F}(x^*)$ are both close to 1 but $\tilde{f}(x^*)/f(x^*)$ is close to zero, implying that $\tilde{\varepsilon}_A$ is (much) smaller than 1. Hence firm A will want to increase its price. Similar arguments show that firm B has an incentive to lower (increase) its price for low (high) levels of dominance.

Combining the two effects. For very small dominance, the demand elasticity effect dominates the demand size effect because the latter is negligible, implying that bundling reduces each firm's profit. As the level of A's dominance increases, the competition-strengthening demand elasticity effect becomes weaker such that there is a first cut-off level of dominance above which bundling increases A's profit due to the demand size effect. At this cut-off level, bundling still reduces B's profit because B suffers from both the demand size effect and the demand elasticity effect. As the level of A's dominance further increases, the demand elasticity effects starts to soften competition. It turns out that there is a second cut-off level of dominance (which is larger than the first cut-off) above which bundling increases each firm's profit.

3.5.2 Analysis

Let us first consider the extreme case where firm A has no advantage over firm B, that is, $\alpha = 0$. In this case, there is no demand size effect and each obtains half of the market both with independent pricing and with bundling. However, because of the competition-strengthening demand elasticity effect, the equilibrium bundle prices are strictly lower than the sum of the equilibrium prices under independent pricing. Precisely, because $f(1/2) < \tilde{f}(1/2)$ and $F(1/2) = \tilde{F}(1/2)$ hold, we have

$$2p_A^* = \frac{2F(1/2)}{\sigma f(1/2)} > \frac{2\tilde{F}(1/2)}{\sigma \tilde{f}(1/2)} = P_A^{**}.$$

Likewise $2p_B^* > P_B^{**}$. Bundling thus hurts both firms when firms are symmetric.

By continuity, bundling still hurts both firms for low but positive values of α, despite the fact that firm A benefits from the demand size effect. In order to see that firm A indeed obtains strictly higher market share under bundling, observe that

$$\frac{d}{d\alpha}F(x^*(\alpha)) = f(x^*(\alpha))\frac{dx^*}{d\alpha},$$

while

$$\frac{d}{d\alpha}\tilde{F}(x^{**}(\alpha)) = \tilde{f}(x^{**}(\alpha))\frac{dx^{**}}{d\alpha}.$$

Evaluating these expressions at $\alpha = 0$ yields $f(1/2)\sigma/3$ and $\tilde{f}(1/2)\sigma/3$, respectively (see the proof of Proposition 1 in the appendix). The latter expression is strictly higher than the former, so that we indeed conclude that $\tilde{F}(x^{**}(\alpha)) > F(x^*(\alpha))$ for small $\alpha > 0$.

Now consider the opposite extreme case where firm A has such a huge advantage that firm B obtains higher market share under bundling than under independent pricing. It is immediate to see that such levels of dominance exist when $f(1) > 0$ (but in fact, we can prove their existence also if $f(1) = 0$) because then firm B has zero market share under independent pricing for high levels of dominance (Proposition 1(ii)), while it obtains always a positive market share under bundling (Proposition 2). We show in Lemma 2 below that when firm B obtains higher market share under bundling, both firms benefit from bundling as it increases the profits of both of them.

We now analyze the case of intermediate levels of α. To that end we introduce some notation for the sets of dominance levels where firm A's and B's market share and total profit are higher under bundling than under independent pricing. After defining these sets, we establish some inclusion relations between them.

Definition 1.

(i) $\mathcal{A}^+_{MS} = \{\alpha \geq 0 \ : \ \tilde{F}(x^{**}(\alpha)) \geq F(x^*(\alpha))\}$

(ii) $\mathcal{A}^+_{\pi A} = \{\alpha \geq 0 \ : \ \Pi^{**}_A(\alpha) \geq 2\pi^*_A(\alpha)\}$

(iii) $\mathcal{A}^+_{\pi B} = \{\alpha \geq 0 \ : \ \Pi^{**}_B(\alpha) \geq 2\pi^*_B(\alpha)\}$

Let furthermore $\mathcal{A}^-_K = [0, \infty) \setminus \mathcal{A}^+_K$ *for* $K \in \{MS, \pi A, \pi B\}$

Lemma 2. *We have the following strict superset relations between the various dominance level sets:*

$$\mathcal{A}^+_{\pi A} \supset \mathcal{A}^+_{\pi B} \supset \mathcal{A}^-_{MS}.$$

In words, the Lemma says that if firm B obtains a strictly higher market share under bundling, then it benefits from bundling and that whenever firm B benefits from bundling, so does firm A. Moreover, there are levels of dominance for which firm B benefits from bundling despite obtaining lower market share and there are levels of dominance for which only firm A benefits from bundling. This complements our earlier observation that bundling hurts both firms for low levels of dominance.

An immediate consequence of this lemma is that there are three regions of dominance levels with distinct effects of bundling on the firms' profits. Namely, for $\alpha \in \mathcal{A}^-_{\pi A}$, both firms are hurt by bundling, and $\mathcal{A}^-_{\pi A}$ includes values of α close to zero. For $\alpha \in \mathcal{A}^+_{\pi B}$, both firms benefit from bundling and $\mathcal{A}^+_{\pi B}$ includes large values of α. Finally, for α in the intermediate region $\mathcal{A}^+_{\pi A} \cap \mathcal{A}^-_{\pi B}$, only firm A benefits from bundling ($\mathcal{A}^+_{\pi A} \cap \mathcal{A}^-_{\pi B} \neq \varnothing$ by Lemma 2). Under the assumption that these sets are convex[19], there exist cutoff levels $\alpha_{\pi A} < \alpha_{\pi B}$ such that bundling hurts both firms when $\alpha < \alpha_{\pi A}$, helps both firms when $\alpha > \alpha_{\pi B}$, and only helps the dominant firm when $\alpha \in (\alpha_{\pi A}, \alpha_{\pi B})$. We record this result in the following proposition.

Proposition 4. *There exist levels of dominance $\alpha_{\pi B} > \alpha_{\pi A} > 0$ such that*

(i) profits of firm A are strictly higher under bundling if and only if $\alpha > \alpha_{\pi A}$ and

(ii) profits of firm B are strictly higher under bundling if and only if $\alpha > \alpha_{\pi B}$.

We are now in a position to solve the three-stage game outlined in section 2 by backward induction. In the last stage firms choose the equilibrium prices corresponding to the bundling game whenever at least one firm has chosen PB. This is so because if, for example, firm A chooses PB and firm B chooses IP, then effectively competition will be in bundles, and equilibrium prices will be P^{**}_A for the price of A's bundle, and $P^{**}_B/2$ for each of B's individual products. Substituting equilibrium payoffs from stage 3 yields the following game to be played in the second stage.

	PB	IP
PB	Π^{**}_A, Π^{**}_B	Π^{**}_A, Π^{**}_B
IP	Π^{**}_A, Π^{**}_B	$2\pi^*_A, 2\pi^*_B$

Of course, both firms have a weakly dominant strategy. It is a weakly dominant strategy for firm A to choose PB when $\alpha > \alpha_{\pi A}$ and to choose IP when $\alpha < \alpha_{\pi A}$. Similarly, it is a weakly dominant strategy for firm B to choose PB when $\alpha > \alpha_{\pi B}$ and to choose IP when $\alpha < \alpha_{\pi B}$.

The bundling outcome will prevail when $\alpha > \alpha_{\pi A}$ and it will then be enforced by firm A. For very low levels of dominance ($\alpha < \alpha_{\pi A}$), firm A may be tempted to threaten to use

[19] We conjecture that this must be true, but have not been able to provide a proof. We therefore just assume this to be the case.

bundling in order to deter entry, as it would lower firm B's profit. However, such a threat is not credible because firm A would choose independent pricing once B has entered. For very high levels of dominance ($\alpha > \alpha_{\pi B}$) bundling is profitable for firm B. Hence, firm A cannot use bundling to deter entry. In fact, in order to deter entry, firm A would need to threaten to use IP. Not only is this not credible, firm B can in fact force bundling by choosing PB unilaterally. Only for intermediate levels of dominance ($\alpha_{\pi A} < \alpha < \alpha_{\pi B}$), bundling is profitable for A (and thus credible) and hurts firm B. Hence, bundling could be used as a foreclosure strategy if firm B's entry cost is sufficiently high.

Assuming that firms will use their weakly dominant strategies in stage two, firm B will enter (by incurring the fixed cost of entry $K > 0$) when $\alpha < \alpha_{\pi A}$ and $2\pi_B^* - K > 0$, and also when $\alpha > \alpha_{\pi A}$ and $\Pi_B^{**} - K > 0$. Otherwise firm B will stay out.

3.6 Extensions

We briefly discuss below how our results extend to the case of positive correlation in tastes and the case of asymmetric dominance.

To introduce positive correlation in tastes,[20] suppose that a fraction $\rho \in (0, 1]$ of consumers have perfectly correlated locations, while the rest have locations independently and identically distributed as above. Given (ρ, ρ') satisfying $1 \geq \rho > \rho' > 0$, the distribution of the average location for ρ is less peaked than that of the average location for ρ'. Therefore, a greater positive correlation weakens both the demand size effect and the demand elasticity effect but the two effects still exist except for the limit case of perfect correlation. In the latter case, the distribution of the average location is identical to that of the individual location, which implies that bundling has no effect.

Consider now the case of asymmetric dominance in which α_i is the degree of dominance of firm A in market i, for $i = 1, 2$. Suppose $\alpha_1 > \alpha_2$ and that A maintains the aggregate dominance, i.e. $\alpha_1 + \alpha_2 > 0$. Since the analysis of bundling depends on $\alpha_1 + \alpha_2$ only, the effect of bundling on the profits can be analyzed in two steps: the effect of the change from (α_1, α_2) to $\alpha_1' = \alpha_2' = (\alpha_1 + \alpha_2)/2$ under independent pricing, and the effect of the change from independent pricing to bundling when $\alpha_1' = \alpha_2' = (\alpha_1 + \alpha_2)/2$ (which has been analyzed in Section 3.5). It turns out that at least for some log-concave and symmetric densities

[20]For instance, Gandal *et al.* (2013) find strong positive correlation of consumer values for spreadsheets and word-processors when they empirically study Microsoft's bundling of the two products.

including the uniform density, each firm's profit from product j is convex with respect to α_j under independent pricing. This implies that when the asymmetry $\alpha_1 - \alpha_2$ increases while keeping the aggregate dominance $\alpha_1 + \alpha_2$ constant, each firm's total profit increases under independent pricing, and this makes bundling less profitable (or more unprofitable) for each firm with respect to independent pricing. The intuition can be given in terms of the margin effect. For instance, A's margin from product 1 is higher than its margin from product 2 under independent pricing when $\alpha_1 > \alpha_2$. However, bundling reduces the sale of the high-margin product and increases the sale of the low-margin product. Therefore, an increase in asymmetry $\alpha_1 - \alpha_2$ reduces the range of A's aggregate dominance under which bundling increases A's profit but expands the range under which bundling reduces B's profit.[21]

In summary, our results imply that a firm which is dominant in one market can leverage its dominance to another market (in which it is not dominant) in a credible way for foreclosure purposes. To give a sharp illustration of such leverage, consider the case in which A and B compete in three different markets (instead of two), with uniformly distributed locations, $t = 1$, and dominance levels $\alpha_1 = \alpha_2 = 3$, $\alpha_3 = -0.1$. This implies that under independent pricing, firm A is monopolist in markets 1 and 2, but B slightly dominates A in market 3. Straightforward calculations show that firm A benefits from bundling (profit increases from 4.467 to 4.631) and B suffers considerably from bundling (profit goes down from 0.534 to 0.055). For some range of fixed costs of entry firm A could thus foreclose firm B. This extension to more than two products with asymmetric dominance is particularly relevant to the on-going lawsuit between Cablevision and Viacom mentioned in the introduction.

3.7 Social welfare

In this subsection we compare static social welfare under independent pricing and bundling. We find that the effects of bundling are non-monotonic in the dominance level. Recall from Propositions 1 and 2 that the market share is always interior under bundling while, under independent pricing, firm A's share becomes one when $(\sigma\alpha - 1/2)f(1) \geq 1$. We have:

[21] In the case in which no firm has aggregate dominance ($\alpha_1 + \alpha_2 = 0$) but each firm is dominant in a different market ($\alpha_1 > 0 > \alpha_2$), bundling reduces each firm's profit. This is because there is no demand size effect, but there is a negative margin effect for each firm, and the demand elasticity effect induces each firm to compete more aggressively under bundling (as the marginal consumer is located at the mean).

Proposition 5. *(i) Both under independent pricing and under bundling, the market share of firm A is too low from the point of view of social welfare for any $\alpha > 0$ as long as it is interior.*

(ii) Bundling reduces social welfare either for $\alpha (\geq 0)$ small enough or for $\alpha (\geq 0)$ large enough (in particular, when $(\sigma\alpha - 1/2)f(1) \geq 1$). For intermediate values of α, bundling may increase or reduce social welfare.

In order to understand how Proposition 5(i) is obtained, notice that in the market for a given object and for a given location of the marginal consumer x, social welfare under independent pricing is given by $W(x) = \alpha F(x) - T(x)$ where $T(x) = t\int_0^x zf(z)dz + t\int_x^1 (1-z)f(z)dz$ is the average transportation cost in that market, which is increasing for $x > 1/2$. Likewise, given the average location of the marginal consumer x, social welfare under bundling is given by $\tilde{W}(x) = \alpha\tilde{F}(x) - \tilde{T}(x)$ where $\tilde{T}(x) = t\int_0^x z\tilde{f}(z)dz + t\int_x^1 (1-z)\tilde{f}(z)dz$. Both W and $\tilde{W}$ are maximized by the same location $x_w = \frac{1}{2} + \sigma\alpha$ if $\frac{1}{2} + \sigma\alpha < 1$ and $x_w = 1$ if $\frac{1}{2} + \sigma\alpha \geq 1$. However, in equilibrium the dominant firm is not aggressive enough, because $x^*(\alpha) < x_w$ and $x^{**}(\alpha) < x_w$ for any $\alpha > 0$,[22] except when $(\sigma\alpha - 1/2)f(1) \geq 1$ (in which case $x^*(\alpha) = x_w = 1$ under independent pricing).

From the formulations of W and $\tilde{W}$, we infer that bundling increases the first term in welfare if and only if it increases A's market share, i.e. if and only if $\tilde{F}(x^{**}(\alpha)) > F(x^*(\alpha))$. This inequality holds for α not too large (see the proof of Lemma 2 in the appendix). For the second term in W and $\tilde{W}$, note that $T(x) < \tilde{T}(x)$ holds for any $x \in [\frac{1}{2}, 1)$ since consumers cannot mix and match under bundling. For small values of α, the difference $\alpha\tilde{F}(x^{**}(\alpha)) - \alpha F(x^*(\alpha))$ is positive but small. Hence, the negative effect of bundling on transportation costs dominates and we have $W(x^*(\alpha)) > \tilde{W}(x^{**}(\alpha))$. For large values of α, we have already noticed that $x^*(\alpha) = x_w = 1$ and hence again $W(x^*(\alpha)) > \tilde{W}(x^{**}(\alpha))$. However, for some intermediate values of α bundling may significantly increase the market share of firm A, such that $\tilde{W}(x^{**}(\alpha)) > W(x^*(\alpha))$ even though $\tilde{T}(x^{**}(\alpha)) > T(x^*(\alpha))$. For instance, for the uniform distribution, $\tilde{W}(x^{**}(\alpha)) > W(x^*(\alpha))$ holds if and only if $\alpha \in (1.071t, 2.306t)$.

Our comparison above has been static in the sense that we have considered a given duopolistic market structure. However, we know from Subsection 3.5 that bundling may help

[22]This occurs because the function X^α introduced in (2) is such that $X^\alpha(\frac{1}{2} + \sigma\alpha) < \frac{1}{2} + \sigma\alpha$. A similar remark applies to $\tilde{X}^\alpha$ introduced in (3).

firm A to erect an entry barrier against firm B. For instance, for the uniform distribution, bundling is credible and reduces B's profit for $\alpha \in (1.415t, 2.376t)$, which largely overlaps with the interval for which bundling increases static welfare. Therefore, one should be very cautious in generating policy implications on bundling from static welfare analysis, in particular for those competition authorities whose objective is to maximize consumer surplus.

4 Mixed Bundling

In this section we consider the case in which, if firm B enters, each firm is unrestricted in the choice of the pricing strategy, and thus can practice mixed bundling. This means that firm i $(= A, B)$ chooses a price P_i for the bundle of its own products and a price $p_i = p_{ij}$ for each single product $j = 1, 2$. Thus each consumer buys the bundle of a firm i and pays P_i, or buy one object from each firm and pays $p_A + p_B$. The main result in this section is that when α is sufficiently large, we find the same equilibrium outcome described by Proposition 2 in Section 3.3 on pure bundling because for firm A a pure bundling strategy is superior to any alternative strategy when it has a large advantage over firm B.

Without loss of generality, we assume that $P_i \le 2p_i$ holds for $i = A, B$ and that each consumer willing to buy both products of i buys i's bundle. As a consequence, each consumer chooses one alternative among AA, AB, BA, BB, where for instance AB means buying products $A1$ and $B2$.[23] In order to describe the preferred alternative of each type of consumer, we introduce $x' \equiv \frac{1}{2} + \frac{\alpha + P_B - p_A - p_B}{2t}$ and $x'' \equiv \frac{1}{2} + \frac{\alpha + p_A + p_B - P_A}{2t}$, with $x' \le x''$ since $P_A \le 2p_A$, $P_B \le 2p_B$. We find:

- Type (x_1, x_2) buys AA if and only if $x_1 \le x''$, $x_2 \le x''$, $x_1 + x_2 \le x' + x''$.[24]
- Type (x_1, x_2) buys AB if and only if $x_1 \le x'$, $x_2 > x''$.
- Type (x_1, x_2) buys BA if and only if $x_1 > x''$, $x_2 \le x'$.
- Type (x_1, x_2) buys BB if and only if $x_1 > x'$, $x_2 > x'$, $x_1 + x_2 > x' + x''$.

Let $S_{ii'}$ and $\mu_{ii'}$ denote, respectively, the set of types who choose ii' and the measure of $S_{ii'}$ for $ii' = AA$, AB, BA, BB; note that $\mu_{AB} = \mu_{BA}$. If $0 < x'$ and $x'' < 1$, then each of

[23]For a consumer located at (x_1, x_2), her payoff from AB is $2v - \alpha - tx_1 - t(1 - x_2) - p_A - p_B$.

[24]As the distribution of locations is atomless, how indifferences are broken does not affect the results.

the sets S_{AA}, S_{AB}, S_{BB} has a positive measure, and precisely the measures are given by:

$$\mu_{AA} = F(x')F(x'') + \int_{x'}^{x''} F(x' + x'' - x_1) f(x_1) dx_1; \qquad \mu_{AB} = F(x')[1 - F(x'')]; \quad (4)$$
$$\mu_{BB} = [1 - F(x')][1 - F(x'')] + \int_{x'}^{x''} [1 - F(x' + x'' - x_1)] f(x_1) dx_1.$$

If instead $x' \leq 0$ and/or $x'' \geq 1$, then $\mu_{AB} = 0$ as in Section 3.3[25] and we have:

$$S_{AA} = \{(x_1, x_2) \in [0,1]^2 : x_1 + x_2 \leq x' + x''\}, S_{BB} = \{(x_1, x_2) \in [0,1]^2 : x_1 + x_2 > x' + x''\}.$$

In either case the firms' profits are given by

$$\pi_A = P_A \mu_{AA} + 2p_A \mu_{AB}; \qquad \pi_B = P_B \mu_{BB} + 2p_B \mu_{AB}.$$

Consider a NE $(p_A^*, P_A^*, p_B^*, P_B^*)$ with the corresponding measures, $\mu_{AA}^*, \mu_{AB}^*, \mu_{BB}^*$ for S_{AA}, S_{AB}, S_{BB}. We define a mixed bundling equilibrium and a pure bundling equilibrium as follows:

Definition 2. *We say that $(p_A^*, P_A^*, p_B^*, P_B^*)$ is a mixed bundling NE if $\mu_{AB}^* > 0$; it is a pure bundling NE if instead $\mu_{AB}^* = 0$.*

It is almost immediate to see that a pure bundling NE exists for any values of parameters as, for each firm, pure bundling is a best response to pure bundling.[26] In subsection 4.2 we show that no mixed bundling NE exists when the dominance of firm A is sufficiently strong.

In order to establish this result, we first prove a result in the theory of monopoly bundling. Precisely, we consider a two-product monopolist facing a single consumer with valuations (v_1, v_2) which are independently distributed with support $[\underline{v}_1, \bar{v}_1] \times [\underline{v}_2, \bar{v}_2]$. We prove that if the virtual valuation for each object is positive at any point in $[\underline{v}_1, \bar{v}_1] \times [\underline{v}_2, \bar{v}_2]$, then the monopolist does not want to sell any object alone to any type of consumer, but rather uses a pure bundling strategy (weaker conditions suffice if (v_1, v_2) are identically distributed).

This result is relevant to our duopoly setting because, given (p_B, P_B) chosen by firm B, the problem of maximizing A's profit with respect to (p_A, P_A) is equivalent to that of

[25] Precisely, if $x' < 0$ then each type of consumer prefers BB to AB and to BA. If $x'' > 1$, then each type of consumer prefers AA to AB and to BA.

[26] Let P_A^{**}, P_B^{**} be the equilibrium prices from Proposition 2. Under mixed bundling, $(p_A^{**}, P_A^{**}, p_B^{**}, P_B^{**})$ is a NE if p_A^{**} and p_B^{**} are large enough, as for firm $A\,(B)$ it is impossible to induce any type of consumer to choose AB or BA since $P_B = P_B^{**}$ and a large p_B imply $x' < 0$ for any $p_A \geq 0$, thus $S_{AB} = S_{BA} = \emptyset$ ($P_A = P_A^{**}$ and a large p_A imply $x' > 1$ for any $p_B \geq 0$, thus $S_{AB} = S_{BA} = \emptyset$).

maximizing the profit of a two-product monopolist facing a consumer with valuations (v_1, v_2) suitably distributed, each with support $[P_B - p_B + \alpha - t, P_B - p_B + \alpha + t]$. Our results from the monopoly setting imply that for a large α the monopolist plays a pure bundling strategy since the virtual valuation for each object increases with α. This reveals that a large α makes it optimal for firm A to use a pure bundling strategy in the duopoly setting.

4.1 Pure bundling as the optimal pricing strategy for a monopolist

A monopolist, denoted by M, produces two products at zero marginal cost.[27] Each type of consumer wants to consume at most one unit of each product and is characterized by her valuations (v_1, v_2) for the two products. Her payoff is given by her gross utility minus the payment to M. Her gross utility is $v_1 + v_2$ if she consumes both products, is v_j if she consumes only object j (for $j = 1, 2$), is zero if she consumes nothing. The valuation v_j has a support $[\underline{v}_j, \bar{v}_j]$ with $\bar{v}_j > \underline{v}_j \geq 0$, a c.d.f. G_j and a density g_j which is continuous and strictly positive in $[\underline{v}_j, \bar{v}_j]$, for $j = 1, 2$. Let $V \equiv [\underline{v}_1, \bar{v}_1] \times [\underline{v}_2, \bar{v}_2]$. Finally, v_1 and v_2 are independently distributed.

A pricing strategy of M consists of (p_1, p_2, P) where $P \geq 0$ is the price of the bundle and $p_j \geq 0$ the price of object j for $j = 1, 2$. Without loss of generality, we consider (p_1, p_2, P) satisfying $P \leq p_1 + p_2$ (an independent pricing strategy is such that $p_1 + p_2 = P$).

Let S_1, S_2, S_{12} denote the set of types who, respectively, buy object 1 only, object 2 only, the bundle. Let μ_1, μ_2, μ_{12} denote the measure of S_1, S_2, S_{12}, respectively. Then, M's profit is given by:

$$\pi = \mu_1 p_1 + \mu_2 p_2 + \mu_{12} P.$$

A type (v_1, v_2) belongs to S_1 if and only if[28] $v_1 \geq p_1$ (i.e., buying only object 1 is better than buying nothing) and $v_2 < P - p_1$ (i.e., buying only object 1 is better than buying the bundle).[29] Hence

$$\mu_1(p_1, p_2, P) = [1 - G_1(p_1)]G_2(P - p_1).$$

[27] See footnote 32 for the case of positive marginal costs.

[28] As a tie-breaking rule we assume that each consumer who is indifferent between two or more alternatives chooses the alternative which maximizes her gross utility. However, since the distribution of types is atomless, how indifferences are broken does not affect the results.

[29] These two inequalities, jointly with $P \leq p_1 + p_2$, imply $v_1 - p_1 > v_2 - p_2$. Hence buying only object 1 is better than buying only object 2.

Notice that if $p_1 > \bar{v}_1$ and/or $p_1 > P - \underline{v}_2$, then $S_1 = \emptyset$ and $\mu_1 = 0$ since for each type, $v_1 < p_1$ and/or $v_2 > P - p_1$. However, π remains unchanged if M lowers p_1 to satisfy $p_1 = \min\{\bar{v}_1, P - \underline{v}_2\}$, since then still $\mu_1 = 0$.[30] Therefore, without loss of generality, we assume that M chooses p_1 satisfying $p_1 \leq \min\{\bar{v}_1, P - \underline{v}_2\}$.

Likewise, $(v_1, v_2) \in S_2$ if and only if $v_2 \geq p_2$ and $v_1 < P - p_2$. Hence

$$\mu_2(p_1, p_2, P) = [1 - G_2(p_2)]G_1(P - p_2),$$

and we assume without loss of generality that M chooses p_2 such that $p_2 \leq \min\{\bar{v}_2, P - \underline{v}_1\}$.[31]

Finally, $(v_1, v_2) \in S_{12}$ if and only if $v_1 + v_2 - P \geq \max\{0, v_1 - p_1, v_2 - p_2\}$, which is equivalent to $v_1 + v_2 \geq P$, $v_1 \geq P - p_2$, $v_2 \geq P - p_1$. Hence

$$\mu_{12}(p_1, p_2, P) = \int_{P-p_1}^{p_2} [1 - G_1(P - v_2)]g_2(v_2)dv_2 + [1 - G_1(P - p_2)][1 - G_2(p_2)].$$

We define a mixed bundling strategy and a pure bundling strategy as follows.

Definition 3. *We say that (p_1, p_2, P) is a mixed bundling strategy if $\mu_1(p_1, p_2, P) > 0$ and/or $\mu_2(p_1, p_2, P) > 0$; it is a pure bundling strategy if $\mu_1(p_1, p_2, P) = \mu_2(p_1, p_2, P) = 0$.*

Therefore, (p_1, p_2, P) is a pure bundling strategy if $P = \min\{p_1 + \underline{v}_2, p_2 + \underline{v}_1\}$.

As a benchmark, consider a single-product monopolist such that the consumer's valuation for the object has support $[\underline{v}, \bar{v}]$, c.d.f. G and density g which is continuous and positive in $[\underline{v}, \bar{v}]$. Then it is well known that the profit-maximizing price is either $\underline{v}$ or solves the equation $J(x) = 0$ with $J(x) \equiv x - \frac{1-G(x)}{g(x)}$ for $x \in [\underline{v}, \bar{v}]$. In particular, the optimal price is equal to $\underline{v}$ if $J(x) > 0$ for each $x \in [\underline{v}, \bar{v}]$. $J(x)$ is often called the "virtual valuation" of type x (Myerson, 1981) and represents the marginal contribution to M's profit made by the sale of the object to a consumer with valuation x.

In our setting of two-product monopolist, the virtual valuation for object i of a type (v_1, v_2) is $J_j(v_j) \equiv v_j - \frac{1-G_j(v_j)}{g_j(v_j)}$ for $v_j \in [\underline{v}_j, \bar{v}_j]$ and $j = 1, 2$. Let $J_j^m \equiv \min_{x \in [\underline{v}_j, \bar{v}_j]} J_j(x)$. Hence, $J_j^m > 0$ is equivalent to $J_j(v_j) > 0$ for any $v_j \in [\underline{v}_j, \bar{v}_j]$. Next proposition establishes that the optimal strategy for M is a pure bundling strategy when the virtual valuation for each object is positive for all types.

[30]This reduction of p_1 does not affect μ_2 nor μ_{12} since, given $p_1 \geq \min\{\bar{v}_1, P - \underline{v}_2\}$, for no type p_1 affects the type's preferred alternative among buying only object 2, buying the bundle, and buying nothing.

[31]Notice that $p_1 \leq \bar{v}_1$, $p_2 \leq \bar{v}_2$ and $P \leq p_1 + p_2$ imply $P - p_1 \leq \bar{v}_2$ and $P - p_2 \leq \bar{v}_1$.

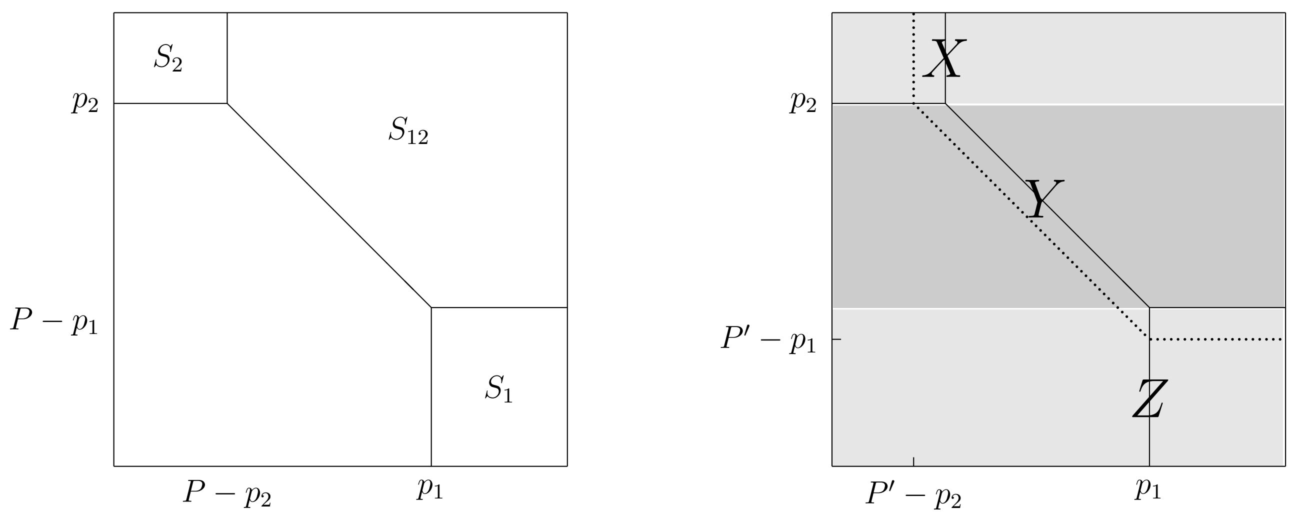

Fig. 3a Fig. 3b

Figure 3: Illustration of the proof of Proposition 6.

Proposition 6. *Suppose that v_1 and v_2 are independently distributed. If $J_1^m > 0$ and $J_2^m > 0$, for any given mixed bundling strategy, there is a pure bundling strategy satisfying $P = \min\{p_1 + \underline{v}_2, p_2 + \underline{v}_1\}$ which gives M a higher profit. Therefore, M's profit is maximized by a pure bundling strategy.*[32]

Let P^* denote the optimal pure bundling price, i.e. the solution to the problem of $\max_P P \Pr\{v_1 + v_2 \geq P\}$. Proposition 6 establishes that if $J_1^m > 0$ and $J_2^m > 0$, then P^*, jointly with $p_1 = P^* - \underline{v}_2$ and $p_2 = P^* - \underline{v}_1$, is the optimal strategy since each mixed bundling strategy is suboptimal.[33] This result holds since, by definition, under a mixed

[32] If the marginal cost for producing object j is constant and equal to $c_j > 0$, for $j = 1, 2$, then the result in Proposition 6 holds if $J_1^m > c_1$ and $J_2^m > c_2$.

[33] We specify $p_1 = P^* - \underline{v}_2$ and $p_2 = P^* - \underline{v}_1$ because of the restriction on (p_1, p_2) that we previously

bundling strategy, some types of consumer buy only one object, for instance only object 2 ($\mu_2 > 0$). Then, it is profitable to sell these consumers also object 1, as the virtual valuation of object 1 is positive for any type of consumer.

To explain how Proposition 6 is obtained, we consider a mixed bundling strategy (p_1, p_2, P) satisfying $p_2 + \underline{v}_1 < p_1 + \underline{v}_2 < P$ and show that a small reduction in the price of the bundle from P to $P' = P - \varepsilon$ (with $\varepsilon > 0$ and small) is profitable with the help of Figure 3(a)-(b). Figure 3(a) represents the sets S_1, S_2, S_{12} given the initial mixed bundling strategy. In Figure 3(b), we consider the reduction in the price of the bundle and partition V into three subsets X, Y, Z such that $X \equiv \{(v_1, v_2) \in V : v_2 \geq p_2\}$, $Y \equiv \{(v_1, v_2) \in V : v_2 \in [P - p_1, p_2)\}$, $Z \equiv \{(v_1, v_2) \in V : v_2 < P - p_1\}$. We show below that the reduction in the price of the bundle is profitable in each of the three regions X, Y, Z.

First, regarding the region Z, it is straightforward to see that the reduction in the price of the bundle is profitable because it induces some types in Z to buy the bundle rather than buying nothing, or buying only object 1. Regarding the region X, notice that every type in this set buys at least object 2 under (p_1, p_2, P). For any type buying object 2, the implicit price of object 1 is $P - p_2$; therefore a type in X buys also object 1 (i.e., buys the bundle) if and only if $v_1 \geq P - p_2$. Hence, for the types in X, the reduction in the price of the bundle has the effect of reducing the price of object 1 and $J_1(P - p_2) \geq J_1^m > 0$ implies that the profit of M from these types increases. Regarding Y, for each $v_2 \in [P - p_1, p_2)$, let $Y(v_2) \subset \{(v_1, v_2)$ such that $v_1 \in [\underline{v}_1, \bar{v}_1]\}$ be the horizontal segment in V such that the valuation for object 2 is equal to v_2; thus $Y = \cup_{v_2 \in [P-p_1,p_2)} Y(v_2)$. Each type in $Y(v_2)$ buys the bundle if $v_1 \geq P - v_2$, buys nothing if $v_1 < P - v_2$. Reducing the price of the bundle from P to P' has an effect on the types in $Y(v_2)$ similar to the effect on the types in region X, but the profit increase from the types in $Y(v_2)$ who buy the bundle under (p_1, p_2, P') but buy nothing under (p_1, p_2, P) is P', which is larger than $P' - p_2$, the profit increase from the types in X who buy the bundle under (p_1, p_2, P') but buy only object 2 under (p_1, p_2, P).

Formally, we find that

$$\begin{aligned} \frac{\partial \pi}{\partial P} &= [1 - G_2(p_2)][1 - G_1(P - p_2) - (P - p_2)g_1(P - p_2)] \\ &\quad + \int_{P-p_1}^{p_2} [1 - G_1(P - v_2) - Pg_1(P - v_2)]g_2(v_2)dv_2 - (P - p_1)[1 - G_1(p_1)]g_2(P - p_1), \end{aligned} \tag{5}$$

imposed without loss of generality, i.e., $p_1 \leq \min\{\bar{v}_1, P - \underline{v}_2\}$ and $p_2 \leq \min\{\bar{v}_2, P - \underline{v}_1\}$. More generally, P^* together with $p_1 \geq P^* - \underline{v}_2$ and $p_2 \geq P^* - \underline{v}_1$ is optimal.

where each of the first, the second, the third terms refers, respectively, to region X, Y, Z. Notice that $J_1^m > 0$ implies that $1 - G_1(P - p_2) - (P - p_2)g_1(P - p_2) < 0$ and $\int_{P-p_1}^{p_2}[1 - G_1(P - v_2) - Pg_1(P - v_2)]g_2(v_2)dv_2 \leq 0$. This implies $\frac{\partial \pi}{\partial P} < 0$ since the last term is negative.

If we consider the case of $p_2 + \underline{v}_1 < P = p_1 + \underline{v}_2$ instead of $p_2 + \underline{v}_1 < p_1 + \underline{v}_2 < P$, we find again that a reduction in the price of the bundle is profitable because (i) region Z is empty in this case; (ii) in regions X and Y the previous arguments still apply.[34]

We note that the optimal pure bundling price P^* is larger than $\underline{v}_1 + \underline{v}_2$, as is shown by Armstrong (1996), even when $J_1^m > 0$ and $J_2^m > 0$ hold. In fact, if we let H and h denote the c.d.f. and the density of $v_1 + v_2$, then $h(\underline{v}_1 + \underline{v}_2) = 0$. Therefore, the virtual valuation for the bundle of a type with $v_1 + v_2$ close to $\underline{v}_1 + \underline{v}_2$ is negative and it is optimal to have $P^* > \underline{v}_1 + \underline{v}_2$. This implies that there always exists a positive measure of types who buy nothing in the optimal pure bundling strategy.[35]

The case of i.i.d. valuations Here we consider the case in which v_1 and v_2 are i.i.d., each with support $[\underline{v}, \bar{v}]$ and c.d.f. G; hence M may restrict to consider only (p_1, p_2, P) satisfying $p_1 = p_2 \equiv p$. Moreover, for the application to the duopoly setting in Subsection 4.2, we also assume that the objects are weak complements in the sense that the buyer enjoys a synergy $s \geq 0$, and thus obtains a gross surplus of $v_1 + v_2 + s$, if she consumes both objects. The synergy has the same effect on μ_1, μ_2, μ_{12} as a reduction of the bundle price from P to $P - s$. Finally, we define $J(x) \equiv x - \frac{1-G(x)}{g(x)}$ and $J^m \equiv \min_{x \in [\underline{v}, \bar{v}]} J(x)$.

Proposition 7. *Suppose that (i) v_1 and v_2 are independently distributed with $\underline{v}_1 = \underline{v}_2 \equiv \underline{v}$, $\bar{v}_1 = \bar{v}_2 \equiv \bar{v}$ and $G_1 = G_2 \equiv G$; (ii) the surplus of type v_1, v_2 is $v_1 + v_2 + s$ if she consumes both objects, for a given $s \geq 0$. Then, if $\underline{v} + s + J^m > 0$, for any given mixed bundling strategy there is a pure bundling strategy satisfying $P = p + \underline{v} + s$ which gives M a higher*

[34] In the previous arguments we used only the assumption $J_1^m > 0$, and not $J_2^m > 0$, because the initial mixed bundling strategy is such that $p_2 + \underline{v}_1 \leq p_1 + \underline{v}_2$. If conversely $p_1 + \underline{v}_2 < p_2 + \underline{v}_1$, then $J_1^m > 0$ implies that P can be profitably reduced to $p_2 + \underline{v}_1$ (hence $\mu_2 = 0$), but not that P should be reduced to $p_1 + \underline{v}_2$. For instance, if (v_1, v_2) is uniformly distributed over $[6, 7] \times [0, 10]$, then $J_1^m = 5 > 0 > -10 = J_2^m$ and the maximal profit under pure bundling is 6.806 (with $P = 8.25$), but $p_1 = 6$, $p_2 = 5$, $P = 11$ imply $\mu_1 = \frac{1}{2}$, $\mu_2 = 0$, $\mu_{12} = \frac{1}{2}$ and generate a profit of 8.5 (> 6.806).

[35] If v_1, v_2 are not independently distributed, then Proposition 6 extends in a natural way. Precisely, letting $g_{j|k}$ and $G_{j|k}$ denote the conditional density and the conditional c.d.f. of v_j given v_k (for $j, k = 1, 2$, $j \neq k$), mixed bundling is suboptimal if $v_j - \frac{1-G_{j|k}(v_j|v_k)}{g_{j|k}(v_j|v_k)} > 0$ for any $(v_j, v_k) \in V$, for $j, k = 1, 2$, $j \neq k$.

profit. Therefore, M's profit is maximized by a pure bundling strategy.

Consider first the case of $s = 0$, which brings us back to additive preferences for the consumer. Then Proposition 7 strengthens the result in Proposition 6 for the specific case of i.i.d. valuations, as it establishes that mixed bundling is suboptimal even though $J(x) < 0$ for some x, provided that $\underline{v} + J^m > 0$. The result follows because $G_1 = G_2$ allows to combine the first and the third term of $\frac{\partial \pi}{\partial P}$ in (5) to prove that $\frac{\partial \pi}{\partial P}$ is always negative even though sometimes the first term is positive.[36]

In the case of $s > 0$, as it is intuitive, the positive synergy makes pure bundling more attractive for M, and thus a less restrictive condition suffices to make pure bundling optimal.

Relationship with McAfee, McMillan and Whinston (1989) McAfee *et al.* (1989) consider the same model we have studied (allowing for correlated valuations). Their main result is that any independent pricing strategy (i.e. any strategy satisfying $P = p_1 + p_2$) is suboptimal under a suitable restriction on the distribution of v_1, v_2, which is always satisfied by independent distributions. But they do not study conditions under which pure bundling generates the highest profit. Precisely, let p_1^*, p_2^* denote the optimal prices under independent pricing. Then, they show that the strategy $(p_1^*, p_2^*, p_1^* + p_2^* - \varepsilon)$ is superior to $(p_1^*, p_2^*, p_1^* + p_2^*)$. Hence, any independent pricing strategy is inferior to a suitable mixed bundling strategy. An implicit assumption in their analysis is that p_1^* and p_2^* are such that, in our notation, $p_1^* > \underline{v}_1$ and $p_2^* > \underline{v}_2$. Conversely, our assumptions $J_1^m > 0$ and $J_2^m > 0$ imply $p_1^* = \underline{v}_1$ and $p_2^* = \underline{v}_2$, and hence reducing P below $p_1^* + p_2^*$ definitely reduces M's profit. Rather, Proposition 6 proves that M should choose the optimal pure bundling price and combine it with p_1, p_2 sufficiently high to make $\mu_1 = \mu_2 = 0$.

Stochastic mechanisms We have focused above on the class of selling mechanisms in which M chooses a price schedule, which specifies a price for each object and a price for the bundle. However, if $\mathcal{M}$ denotes the set of all incentive compatible and individually

[36] When $s = 0$ the inequality $\underline{v} + J^m > 0$ is not necessary in order for pure bundling to be optimal. For instance, Pavlov (2011) shows that if (v_1, v_2) is uniformly distributed over $V = [\omega, \omega+1]^2$, then pure bundling is M's optimal strategy as long as $\omega > 0.077$, even though $J^m = \omega - 1$. In fact, Pavlov (2011) shows that when $\omega > 0.077$ pure bundling is superior also to any other (incentive compatible and individually rational) selling mechanism, including stochastic mechanisms. Propositions 6-7 do not allow for stochastic mechanisms, but apply to any distributions. Proposition 8 below provides a result about stochastic mechanisms.

rational mechanisms, then there exist many mechanisms in $\mathcal{M}$ which are not characterized by a price schedule, for instance stochastic mechanisms specifying that a certain type of buyer receives an object with a probability in $(0,1)$. Our results above do not establish that pure bundling is optimal when M can choose an arbitrary mechanism in $\mathcal{M}$.[37] Next result identifies conditions under which pure bundling is superior to any other mechanism in $\mathcal{M}$.

Proposition 8. *Suppose that v_1, v_2 are i.i.d., each with support $[\underline{v}, \bar{v}]$ and with an increasing density g such that $g(\underline{v})\underline{v} \geq \frac{3}{2}$. Then a suitable pure bundling mechanism is the optimal selling mechanism among all mechanisms in $\mathcal{M}$.*

Notice that, given i.i.d. valuations and an increasing density, the condition $J^m > 0$ is equivalent to $g(\underline{v})\underline{v} > 1$, and thus the assumption $g(\underline{v})\underline{v} \geq \frac{3}{2}$ is not much more restrictive.[38]

Mechanism design in a bidimensional setting is typically complicated, but Pavlov (2011) shows that in a symmetric environment with increasing g, it is possible to reduce the problem to a one-dimensional screening problem in which, conditional on participation, each type receives for sure her most preferred object. In this problem the screening variable is the probability that the buyer receives also her less preferred object, as a function of her reported valuation for that object; an unusual feature is that the set of participating types depends on the mechanism itself. Pavlov (2011) uses this formulation to find the optimal selling mechanism when v_1, v_2 are uniformly distributed with $V = [\omega, \omega + 1]^2$. We use it to show that pure bundling is the optimal mechanism under suitable assumptions.

4.2 Duopoly and the pure bundling NE

In this subsection, we first provide a result, Lemma 3, which allows to apply the result of Proposition 7 to our duopoly setting, in order to prove that firm A plays a pure bundling strategy if α is sufficiently large. Given (p_B, P_B) chosen by firm B, let $b_1 \equiv P_B - p_B + \alpha$, $b_2 \equiv p_B + \alpha$, and then consider a monopoly setting in which v_1, v_2 are i.i.d., each with support $[b_1 - t, b_1 + t]$, and c.d.f. $G(x) = F(\frac{x - b_1 + t}{2t})$ for $x \in [b_1 - t, b_1 + t]$;[39] moreover, assume that the consumer enjoys a synergy $s = b_2 - b_1 \geq 0$ if she consumes both objects.

[37] Manelli and Vincent (2006) and Pavlov (2011) give examples in which the optimal mechanism is stochastic. Conversely, for the case of a single good, Riley and Zeckhauser (1983) prove that the optimal mechanism consists of posting a suitable take-it-or-leave-it price; thus no stochastic mechanism is optimal.

[38] In fact, in the proof we provide less restrictive (but less immediate) sufficient conditions.

[39] Recall that F is the c.d.f. of the individual locations in the duopoly setting introduced in Section 2.

Lemma 3. *In the duopoly setting given (p_B, P_B), the profit of firm A from charging (p_A, P_A) is equal to the monopolist's profit from charging (p, P) in the above described setting, with $p = p_A, P = P_A$.*

This result holds because μ_{12} and μ_1 (μ_2 is equal to μ_1 given identical distributions) are equal to μ_{AA} and to μ_{AB} in (4), respectively, if $p = p_A$ and $P = P_A$.

We can combine Lemma 3 and Proposition 7 to find conditions under which firm A wants to play a pure bundling strategy, which ultimately implies that no mixed bundling NE exists.

Proposition 9. *Consider competition between two multi-product firms in which each firm can use a mixed bundling strategy.*
(i) When each consumer's location is i.i.d. with a log-concave density f such that $f(1) > 0$, there exists no mixed bundling NE if $\alpha \geq t + \frac{t}{f(1)}$.
(ii) When each consumer's location is i.i.d. with a uniform density, there exists no mixed bundling NE if $\alpha \geq t$.

Proposition 9(i) is a corollary of Proposition 7, and relies on verifying that $\alpha \geq t + \frac{t}{f(1)}$ makes the condition $\underline{v} + s + J^m > 0$ satisfied when $\underline{v} = b_1 - t$, $s = b_2 - b_1$, and J^m is obtained from the c.d.f. $F(\frac{x - b_1 + t}{2t})$.

In the case of the uniform distribution, we find $t + \frac{t}{f(1)} = 2t$, but Proposition 9(ii) relies on the particular features of this distribution to establish that no mixed bundling NE exists if $\alpha \geq t$.[40] In order to see how this stronger result is obtained, fix p_B, P_B arbitrarily and let M_A denote the set of (p_A, P_A) such that $\mu_{AB} > 0$. Whereas Proposition 9(i) is proved by showing that $\frac{\partial \pi_A}{\partial P_A}$ is negative at each $(p_A, P_A) \in M_A$ if $\alpha \geq t + \frac{t}{f(1)} = 2t$, for the uniform distribution we can show that if $\alpha \in [\frac{9}{8}t, 2t)$, there exists no $(p_A, P_A) \in M_A$ such that $\frac{\partial \pi_A}{\partial P_A} = 0$ and $\frac{\partial \pi_A}{\partial p_A} = 0$ are both satisfied; therefore no mixed bundling strategy is optimal for firm A when $\alpha \in [\frac{9}{8}t, 2t)$. Conversely, if $\alpha \in [t, \frac{9}{8}t)$ then it is optimal for A to play (p_A, P_A) in M_A if p_B is small, but we prove that no optimal mixed bundling strategy of A induces B to play a small p_B.

It is interesting to notice that a well-established result in the literature is that mixed bundling reduces profits with respect to independent pricing, at least for symmetric firms: see

[40] Numeric analysis suggests that (i) no mixed bundling NE exists as long as $\alpha \geq 0.72t$; (ii) when a mixed bundling NE exists, the firms' equilibrium profits are lower than under independent pricing.

Armstrong and Vickers (2010) and references therein.[41] Propositions 4 and 9(i), conversely, prove that if one firm's dominance over the other is strong enough, that is if $\alpha \geq t + \frac{t}{f(1)}$ and $\alpha > \alpha_{\pi B}$, then mixed bundling boils down to pure bundling, and each firm's profit is larger under mixed bundling than under independent pricing.

5 Conclusion

By studying a dominant firm's bundling, we have provided new insights about when bundling intensifies or softens competition and when bundling reduces the dominated firm's profit. In particular, we find that for intermediate levels of dominance, pure bundling is profitable for the dominant firm but hurts the dominated firm. Entry barriers can thus be created in a credible way, without having to rely on an assumption of commitment power. This finding provides a justification for the use of contractual bundling for foreclosure purposes. Whinston (1990) suggests technical tying as a commitment device to (pure) bundling, but his baseline model does not explain the use of contractual bundling, which is also often used.

We have briefly discussed extensions of our model to asymmetric dominance levels and correlation of valuations. Further extensions to allowing more than two products and competition against specialized firms are left for future research.

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6 Appendix

6.1 Proof of Lemma 1.

(i) Note first that

$$\tilde{f}(1/2) = 2\int_0^1 f(s)^2 ds = 4\int_0^{1/2} f(s)^2 ds > 4\int_{1/4}^{1/2} f(s)^2 ds.$$

Next, observe that for a log-concave function f we have for any a and b in $[0,1]$

$$f(a)f(b) \leq f(\frac{a+b}{2})^2, \tag{6}$$

because concavity of $\log(f)$ implies the inequality in

$$\log[f(a)f(b)] = \log[f(a)] + \log[f(b)] \leq 2\log[f(\frac{a+b}{2})] = \log[f(\frac{a+b}{2})^2].$$

In particular, taking $b = 1/2$ and $a = 2s - 1/2$ yields $f(s)^2 \geq f(2s-1/2)f(1/2)$ for $s > 1/4$. And thus

$$\int_{1/4}^{1/2} f(s)^2 ds \geq f(1/2)\int_{1/4}^{1/2} f(2s-1/2)ds = f(1/2)\int_0^{1/2} \frac{1}{2}f(y)dy = f(1/2)/4.$$

Combining these results we obtain $\tilde{f}(1/2) > f(1/2)$.

(ii) Observe that (6) also holds for $a = 2-2x-s$ and $b = s$. Hence, $f(2-2x-s)f(s) \leq f(x)^2$ for any $x \in [\frac{1}{2}, 1]$, $s \in [0, 2-2x]$. Therefore, we have:

$$\tilde{f}(x) = \int_0^{2-2x} 2f(2-2x-s)f(s)ds \leq 2\int_0^{2-2x} f(x)^2 ds = 4(1-x)f(x)^2.$$

The limiting result follows immediately. ■

6.2 Proof of Proposition 1.

(i) The condition of the proposition implies that $\lim_{x\to 1} X^{\alpha}(x) < 1$, so that the fixed point $x^*(\alpha) < 1$.

Now we show that $x^*(\alpha)$ is increasing and concave for $\alpha \geq 0$. By taking the derivative w.r.t. α on both sides of the equation $X^{\alpha}(x^*(\alpha)) = x^*(\alpha)$, one obtains immediately

$$\frac{dx^*(\alpha)}{d\alpha} = \frac{\sigma}{3 + \frac{1-2F(x^*(\alpha))}{f(x^*(\alpha))}\frac{f'(x^*(\alpha))}{f(x^*(\alpha))}}.$$

Moreover, it follows that $\frac{dx^*(\alpha)}{d\alpha}$ is a strictly positive and weakly decreasing function of α: First note that both $(1-2F(x))/f(x)$ and $f'(x)/f(x)$ are non-positive for $x \geq 1/2$. Next observe that both functions are decreasing because of log-concavity of f. The product of these two functions is thus positive and increasing. Since $x^*(\alpha)$ is increasing in α, it follows that $\frac{dx^*(\alpha)}{d\alpha}$ is decreasing.

Next, the equilibrium price of firm A is increasing in α because (i) $x^*(\alpha)$ is increasing in α and (ii) $F(x)/f(x)$ is increasing in x by log-concavity. The equilibrium profit of firm A is then also increasing because both equilibrium price (as seen above) and market share $(F(x^*(\alpha)))$ are increasing. Similarly, the equilibrium price and profit for firm B are decreasing.

(ii) In this case, no interior equilibrium exists. Necessarily, $p_B^* = 0$ and firm A corners the market. The lowest price to corner the market, given $p_B^* = 0$, is $p_A^* = \alpha - 1/(2\sigma)$. Clearly firm A has no incentive to set a lower price (as demand cannot be increased). Firm A has also no incentive to increase its price because the marginal profit, evaluated at p_A^*, equals

$$\frac{d\pi_A}{dp_A} = F(1) - \sigma p_A^* f(1) \leq 0,$$

where the inequality follows directly from $(\sigma\alpha - 1/2)f(1) \geq 1$. By virtue of the remark in footnote 17, it follows that $\frac{d\pi_A}{dp_A} < 0$ for any $p_A > p_A^*$. ■

6.3 Proof of Lemma 2.

It will be convenient to define some auxiliary dominance level sets. Let

$$\mathcal{A}^+_{DENS} = \{\alpha \geq 0 : \ \tilde{f}(x^{**}(\alpha)) \geq f(x^*(\alpha))\}, \qquad \mathcal{A}^-_{DENS} = \{\alpha \geq 0 : \ \tilde{f}(x^{**}(\alpha)) < f(x^*(\alpha))\},$$

$$\mathcal{A}^+_{PA} = \{\alpha \geq 0 : \ P_A^{**}(x^{**}(\alpha)) \geq 2p_A^*(x^*(\alpha))\}, \qquad \mathcal{A}^+_{PB} = \{\alpha \geq 0 : \ P_B^{**}(x^{**}(\alpha)) \geq 2p_B^*(x^*(\alpha))\}.$$

Step 1. We first establish that if the dominance level belongs to $\mathcal{A}^-_{MS}$, then both firms will set higher total prices and obtain higher profits under bundling. Let $\bar{\alpha} \in \mathcal{A}^-_{MS}$ be such a dominance level, that is $\tilde{F}(x^{**}(\bar{\alpha})) < F(x^*(\bar{\alpha}))$. As the distribution of the average location is more peaked, this implies that $x^*(\bar{\alpha}) > x^{**}(\bar{\alpha})$. From (2) and (3) we know that for any α such that $(\sigma\alpha - 1/2)f(1) < 1$ we have[42]

$$x^{**}(\alpha) - x^*(\alpha) = \frac{1 - 2\tilde{F}(x^{**}(\alpha))}{\tilde{f}(x^{**}(\alpha))} - \frac{1 - 2F(x^*(\alpha))}{f(x^*(\alpha))}. \tag{7}$$

In particular, for $\bar{\alpha}$ the left-hand side of (7) is negative. Eq. (7) can then only hold if $\tilde{f}(x^{**}(\bar{\alpha})) < f(x^*(\bar{\alpha}))$. That is, $\bar{\alpha} \in \mathcal{A}^-_{DENS}$. Using the expressions for equilibrium prices of firm B from Propositions 1 and 2 we conclude that $P^{**}_B(\bar{\alpha}) > 2p^*_B(\bar{\alpha})$. As firm B also obtains higher market share under bundling, firm B's profit is higher under bundling as well. Now firm A could set bundle price $P_A = 2p^*_A$ and obtain higher market share, and thus higher profits than what he obtains in the independent pricing equilibrium. The optimal bundle price for firm A yields at least as much profit. As we know that in equilibrium firm A obtains less market share than under independent pricing, the optimal bundle price must be such that $P^{**}_A(\bar{\alpha}) > 2p^*_A(\bar{\alpha})$.

Step 2. Next we focus on dominance levels for which firm A obtains higher market share under bundling, that is $\alpha \in \mathcal{A}^+_{MS}$. We will show that

$$(\mathcal{A}^+_{\pi A} \cap \mathcal{A}^+_{MS}) \supseteq (\mathcal{A}^+_{PA} \cap \mathcal{A}^+_{MS}) \supseteq (\mathcal{A}^-_{DENS} \cap \mathcal{A}^+_{MS}) \supseteq (\mathcal{A}^+_{PB} \cap \mathcal{A}^+_{MS}) \supseteq (\mathcal{A}^+_{\pi B} \cap \mathcal{A}^+_{MS}).$$

It is straightforward that $\mathcal{A}^+_{\pi A} \cap \mathcal{A}^+_{MS} \supseteq \mathcal{A}^+_{PA} \cap \mathcal{A}^+_{MS}$. Namely, for dominance levels for which firm A sets higher total price and obtains higher market share under bundling, profits are automatically higher under bundling.

Note that for $\alpha \in \mathcal{A}^-_{DENS} \cap \mathcal{A}^+_{MS}$, $P^{**}_A = 2\tilde{F}(x^{**})/(\sigma\tilde{f}(x^{**})) > 2F(x^*)/(\sigma f(x^*)) = 2p^*_A$, because the numerator is larger (and positive) and the denominator is strictly smaller (but positive) on the left-hand side. This shows that $\mathcal{A}^+_{PA} \cap \mathcal{A}^+_{MS} \supseteq \mathcal{A}^-_{DENS} \cap \mathcal{A}^+_{MS}$.

Note that for $\alpha \in \mathcal{A}^+_{DENS} \cap \mathcal{A}^+_{MS}$, $P^{**}_B = 2(1-\tilde{F}(x^{**}))/(\sigma\tilde{f}(x^{**})) \leq 2(1-F(x^*))/(\sigma f(x^*)) = 2p^*_B$, because the numerator is smaller (and positive) and the denominator is larger (and positive) on the left-hand side. Moreover, the inequality must be strict. Namely, the inequality could only be binding when both $f(x^*) = \tilde{f}(x^{**})$ and $F(x^*) = \tilde{F}(x^{**})$. But this would imply

[42] If $(\sigma\bar{\alpha} - 1/2)f(1) \geq 1$, then Proposition 1(ii) applies and thus $\Pi^{**}_B(\bar{\alpha}) > 2\pi^*_B(\bar{\alpha}) = 0$, $P^{**}_B(\bar{\alpha}) > 2p^*_B(\bar{\alpha}) = 0$. Minor changes to the arguments below establish that $\Pi^{**}_A(\bar{\alpha}) > 2\pi^*_A(\bar{\alpha})$, $P^{**}_A(\bar{\alpha}) > 2p^*_A(\bar{\alpha})$.

that $x^* = x^{**}$ because of (7). However, this is incompatible with $F(x^*) = \tilde{F}(x^{**})$ because $\tilde{F}(x) > F(x)$ for any $x \in (1/2, 1)$. This proves that $\mathcal{A}^-_{DENS} \cap \mathcal{A}^+_{MS} \supseteq \mathcal{A}^+_{PB} \cap \mathcal{A}^+_{MS}$.

It is straightforward that $\mathcal{A}^+_{PB} \cap \mathcal{A}^+_{MS} \supseteq \mathcal{A}^+_{\pi B} \cap \mathcal{A}^+_{MS}$. Namely, for dominance levels for which firm B obtains higher total profit under bundling, despite having smaller market share under bundling, it must be that the total price is higher under bundling.

Step 3. We show that the set relations are strict. We know that $\tilde{f}(x^{**}(0)) > f(x^*(0))$, but if $f(1) > 0$ and $\alpha \geq 1/(\sigma f(1)) + 1/(2\sigma)$ then necessarily $\tilde{f}(x^{**}(\alpha)) < f(1) = f(x^*(\alpha))$.[43] There must then exist level $\alpha_{DENS} > 0$ for which $\tilde{f}(x^{**}(\alpha_{DENS})) = f(x^*(\alpha_{DENS}))$. In the hypothetical case that there exist multiple such levels, we choose the maximal one. We claim that $\alpha_{DENS} \in \mathcal{A}^+_{MS} \cap \mathcal{A}^+_{\pi A} \cap \mathcal{A}^-_{\pi B}$.

It is clear that $\alpha_{DENS} \in \mathcal{A}^+_{MS}$. Namely, suppose it is not true. Then firm B has strictly higher market share under bundling, and thus both firms would obtain higher profits under bundling (from Step 1). However, $f(x^*(\alpha_{DENS})) = \tilde{f}(x^{**}(\alpha_{DENS}))$ and $F(x^*(\alpha_{DENS})) > \tilde{F}(x^{**}(\alpha_{DENS}))$ contradict $\Pi^{**}_A(\alpha_{DENS}) > 2\pi^*_A(\alpha_{DENS})$ (from Propositions 1 and 2.).

Using again Propositions 1 and 2, it easily follows that $\alpha_{DENS} \in \mathcal{A}^+_{\pi A}$. It must also be true that firm B has strictly lower profits under bundling. Namely, profits for firm B could at best be equal under bundling, but this would require that firm B's market share is exactly the same under both pricing regimes. We have already seen before that this is impossible as it would imply that $x^{**}(\alpha_{DENS}) = x^*(\alpha_{DENS})$ by Eq. (7), and thus $\tilde{F}(x) = F(x)$ for $x = x^{**}(\alpha_{DENS})$. This thus proves the strictness of the first superset relation.

In order to prove the second, let $\alpha_{MS} > 0$ be such that market shares of the two firms are equal under both regimes, that is, $F(x^*(\alpha_{MS})) = \tilde{F}(x^{**}(\alpha_{MS}))$. Such a level exists if $f(1) > 0$ as firm B has a strictly lower market share under bundling for small positive dominance levels, while for $\alpha \geq 1/(\sigma f(1)) + 1/(2\sigma)$ his market share is zero under independent pricing (Prop. 1) but positive under bundling (Prop. 2).[44] In the hypothetical case that there exist multiple levels of dominance with this property, we choose the maximal one. We claim that $\alpha_{MS} > \alpha_{DENS}$, and therefore $\tilde{f}(x^{**}(\alpha_{MS})) < f(x^*(\alpha_{MS}))$, which implies $\alpha_{MS} \in \mathcal{A}^+_{\pi B}$ even though $\alpha \notin \mathcal{A}^-_{MS}$

This follows easily from, on the one hand, observing that $\alpha_{MS} = \alpha_{DENS}$ leads to a contradiction, as again by Eq. (7) we would deduce that $x^{**}(\alpha_{MS}) = x^*(\alpha_{MS})$, which is

[43]We show below in Step 3.2 that if $f(1) = 0$, then $\tilde{f}(x^{**}(\alpha)) < f(x^*(\alpha))$ still holds for a large α.

[44]We show below in Step 3.1 that if $f(1) = 0$, then $F(x^*(\alpha)) > \tilde{F}(x^{**}(\alpha))$ still holds for a large α.

impossible. On the other hand, $\alpha_{MS} < \alpha_{DENS}$ is impossible because of the assumption that α_{MS} has been chosen as the maximal level of dominance for which market shares are equal under the two regimes. It implies that for higher levels, in particular for α_{DENS}, market share is strictly lower for firm A under bundling. But we have already established before that $\alpha_{DENS} \in \mathcal{A}^+_{MS}$. ■

Step 3.1 Proof that $F(x^*(\alpha)) > \tilde{F}(x^{**}(\alpha))$ if $f(1) = 0$ and α is large.
Let $W(x) = x + \frac{2F(x)-1}{f(x)}$ and $\tilde{W}(x) = x + \frac{2\tilde{F}(x)-1}{\tilde{f}(x)}$, two strictly increasing functions. Given $\alpha > 0$, we know that $x^*(\alpha), x^{**}(\alpha)$ are such that $W(x^*(\alpha)) = \frac{1}{2} + \sigma\alpha$, $\tilde{W}(x^{**}(\alpha)) = \frac{1}{2} + \sigma\alpha$. For a large α, $x^*(\alpha)$ and $x^{**}(\alpha)$ are close to 1. Thus, given x close to 1, we select $y(x)$ as the unique y such that $\tilde{F}(y) = F(x)$. We prove that $\tilde{W}(y(x)) > W(x)$ for x close to 1, hence $\tilde{W}(y(x^*(\alpha))) > W(x^*(\alpha)) = \frac{1}{2} + \sigma\alpha$ for a large α, which implies $x^{**}(\alpha) < y(x^*(\alpha))$ and thus $\tilde{F}(x^{**}(\alpha)) < \tilde{F}(y(x^*(\alpha))) = F(x^*(\alpha))$ for a large α.

In order to prove $\tilde{W}(y(x)) > W(x)$, we notice that $\tilde{F}(y(x)) = F(x)$ makes the inequality equivalent to $(y(x)-x)\tilde{f}(y(x))+(2F(x)-1)[1-\frac{\tilde{f}(y(x))}{f(x)}] > 0$. We prove that $\lim_{x\uparrow 1} \frac{\tilde{f}(y(x))}{f(x)} = 0$, hence $\lim_{x\uparrow 1}\left((y(x) - x)\tilde{f}(y(x)) + (2F(x) - 1)[1 - \frac{\tilde{f}(y(x))}{f(x)}]\right) = 1$.

First notice that $\tilde{f}(x)$ can be written as $2\int_{2x-1}^{1} f(2x-z)f(z)dz$ for $x \in [\frac{1}{2}, 1]$, after using the substitution $z = 1 - s$ in the equality $\tilde{f}(x) = 2\int_0^{2-2x} f(2-2x-s)f(s)ds$ given at the beginning of Section 3. In the following we denote with $f^{(i)}$ the i-th derivative of f ($f^{(0)}$ denotes f), and with $\tilde{f}^{(i)}$ the i-th derivative of $\tilde{f}$ ($\tilde{f}^{(0)}$ denotes $\tilde{f}$), for $i = 1, 2, ...$ Arguing by induction from $\tilde{f}(x) = 2\int_{2x-1}^{1} f(2x-z)f(z)dz$ we obtain that for $x \in [\frac{1}{2}, 1]$ and for $n = 1, 2, ...$, the n-th derivative of $\tilde{f}$ is

$$\tilde{f}^{(n)}(x) = -2^{n+1}\sum_{i=0}^{n-1} f^{(i)}(1)f^{(n-1-i)}(2x-1) + 2^{n+1}\int_{2x-1}^{1} f^{(n)}(2x-z)f^{(0)}(z)dz \qquad (8)$$

Now we pick $n \geq 1$ such that $f^{(0)}(1) = f^{(1)}(1) = ... = f^{(n-1)}(1) = 0$, but $f^{(n)}(1) \neq 0$, and let $\gamma \equiv f^{(n)}(1)$. Then we approximate $f, F, \tilde{f}, \tilde{F}$ in a left neighbourhood of $x = 1$ using Taylor's formula. Precisely, a straightforward application of Taylor's formula to f and F yields (9) ($f^{(n)}$ is the $(n+1)$-th derivative of F), and notice that from (9) we infer that $\gamma < 0$ if n is odd, $\gamma > 0$ if n is even. From (8) we obtain $\tilde{f}^{(1)}(1) = ... = \tilde{f}^{(2n)}(1) = 0$ and $\tilde{f}^{(2n+1)}(1) = -2^{2n+2}\gamma^2 \neq 0$, which yields (10).

$$f(x) = \frac{\gamma}{n!}(x-1)^n + \eta_f(x), \qquad F(x) = 1 + \frac{\gamma}{(n+1)!}(x-1)^{n+1} + \eta_F(x) \qquad (9)$$

$$\tilde{f}(y) = -\frac{2^{2n+2}\gamma^2}{(2n+1)!}(y-1)^{2n+1} + \eta_{\tilde{f}}(y), \quad \tilde{F}(y) = 1 - \frac{2^{2n+2}\gamma^2}{(2n+2)!}(y-1)^{2n+2} + \eta_{\tilde{F}}(y) \qquad (10)$$

with

$$\lim_{x\uparrow 1}\frac{\eta_f(x)}{(x-1)^n}=\lim_{x\uparrow 1}\frac{\eta_F(x)}{(x-1)^{n+1}}=\lim_{y\uparrow 1}\frac{\eta_{\tilde f}(y)}{(y-1)^{2n+1}}=\lim_{y\uparrow 1}\frac{\eta_{\tilde F}(y)}{(y-1)^{2n+2}}=0 \qquad (11)$$

Let $\varepsilon>0$ be close enough to zero to satisfy $\varepsilon<\frac{2^{2n+2}\gamma^2}{(2n+1)!}$ and $\varepsilon<\frac{|\gamma|}{(n+1)!}$, and notice that (11) implies the existence of a $\delta>0$ such that

$$\begin{cases} -\varepsilon(1-x)^n<\eta_f(x)<\varepsilon(1-x)^n \\ -\varepsilon(1-x)^{n+1}<\eta_F(x)<\varepsilon(1-x)^{n+1}\end{cases} \quad \text{for each } x \ \in \ (1-\delta,1) \qquad (12)$$

$$\begin{cases} -\varepsilon(1-y)^{2n+1}<\eta_{\tilde f}(y)<\varepsilon(1-y)^{2n+1} \\ -\varepsilon(1-y)^{2n+2}<\eta_{\tilde F}(y)<\varepsilon(1-y)^{2n+2}\end{cases} \quad \text{for each } y \ \in \ (1-\delta,1) \qquad (13)$$

Combining (10) and (13) we obtain

$$1-(\frac{2^{2n+2}\gamma^2}{(2n+2)!}+\varepsilon)(1-y)^{2n+2}<\tilde F(y)<1-(\frac{2^{2n+2}\gamma^2}{(2n+2)!}-\varepsilon)(1-y)^{2n+2} \qquad \text{for each } y\in(1-\delta,1)$$

Therefore $y(x)$, the solution to $\tilde F(y)=F(x)$, is such that

$$1-\left(\frac{1-F(x)}{\frac{2^{2n+2}\gamma^2}{(2n+2)!}-\varepsilon}\right)^{\frac{1}{2n+2}}<y(x)<1-\left(\frac{1-F(x)}{\frac{2^{2n+2}\gamma^2}{(2n+2)!}+\varepsilon}\right)^{\frac{1}{2n+2}} \qquad (14)$$

From (9) and (12) we obtain

$$1-(\frac{|\gamma|}{(n+1)!}+\varepsilon)(1-x)^{n+1}<F(x)<1-(\frac{|\gamma|}{(n+1)!}-\varepsilon)(1-x)^{n+1} \qquad \text{for each } x\in(1-\delta,1)$$

and, jointly with (14), this leads to

$$1-\left(\frac{(\frac{|\gamma|}{(n+1)!}+\varepsilon)(1-x)^{n+1}}{\frac{2^{2n+2}\gamma^2}{(2n+2)!}-\varepsilon}\right)^{\frac{1}{2n+2}}<y(x)<1-\left(\frac{(\frac{|\gamma|}{(n+1)!}-\varepsilon)(1-x)^{n+1}}{\frac{2^{2n+2}\gamma^2}{(2n+2)!}+\varepsilon}\right)^{\frac{1}{2n+2}} \qquad (15)$$

Moreover, (9)-(10) and (12)-(13) yield

$$(\frac{|\gamma|}{n!}-\varepsilon)(1-x)^n \ < \ f(x)<(\frac{|\gamma|}{n!}+\varepsilon)(1-x)^n \qquad \text{for each } x\in(1-\delta,1)$$

$$(\frac{2^{2n+2}\gamma^2}{(2n+1)!}-\varepsilon)(1-y)^{2n+1} \ < \ \tilde f(y)<(\frac{2^{2n+2}\gamma^2}{(2n+1)!}+\varepsilon)(1-y)^{2n+1} \qquad \text{for each } y\in(1-\delta,1)$$

The latter two inequalities, combined with (15), yield

$$\begin{aligned}\frac{\tilde f(y(x))}{f(x)} \ &< \ \frac{(\frac{2^{2n+2}\gamma^2}{(2n+1)!}+\varepsilon)(1-y(x))^{2n+1}}{(\frac{|\gamma|}{n!}-\varepsilon)(1-x)^n}<\frac{(\frac{2^{2n+2}\gamma^2}{(2n+1)!}+\varepsilon)\left(\frac{(\frac{|\gamma|}{(n+1)!}+\varepsilon)(1-x)^{n+1}}{\frac{2^{2n+2}\gamma^2}{(2n+2)!}-\varepsilon}\right)^{\frac{2n+1}{2n+2}}}{(\frac{|\gamma|}{n!}-\varepsilon)(1-x)^n} \\ &= \ \frac{(\frac{2^{2n+2}\gamma^2}{(2n+1)!}+\varepsilon)\left(\frac{\frac{|\gamma|}{(n+1)!}+\varepsilon}{\frac{2^{2n+2}\gamma^2}{(2n+2)!}-\varepsilon}\right)^{\frac{2n+1}{2n+2}}}{(\frac{|\gamma|}{n!}-\varepsilon)}\sqrt{1-x} \qquad \text{for each } x\in(1-\delta,1)\end{aligned}$$

which implies $\lim_{x\uparrow 1}\frac{\tilde{f}(y(x))}{f(x)}=0$.

Step 3.2 Proof that $\tilde{f}(x^{**}(\alpha))<f(x^{*}(\alpha))$ if $f(1)=0$ and α is large.

Given x close to 1, we select $z(x)$ as the unique $z\in(\frac{1}{2},x)$ such that $\tilde{f}(z)=f(x)$. We prove that $\tilde{W}(z(x))<W(x)$ for x close to 1, hence $\tilde{W}(z(x^{*}(\alpha)))<W(x^{*}(\alpha))=\frac{1}{2}+\sigma\alpha$ for a large α, which implies $x^{**}(\alpha)>z(x^{*}(\alpha))$ and thus $\tilde{f}(x^{**}(\alpha))<\tilde{f}(z(x^{*}(\alpha)))=f(x^{*}(\alpha))$. The inequality $\tilde{W}(z(x))<W(x)$ reduces to $f(x)(z(x)-x)<2[F(x)-\tilde{F}(z(x))]$, and since $z(x)<x$ it suffices to prove that $F(x)>F(z(x))$. We know from the proof of Step 3.1 that $\tilde{F}(z(x))=F(x)$ for x close to 1 implies $\tilde{f}(z(x))<f(x)$. In order to obtain $\tilde{f}(z(x))=f(x)$ it is necessary to decrease $z(x)$, which implies $\tilde{F}(z(x))<F(x)$.

6.4 Proof of Proposition 6

We fix an arbitrary mixed bundling strategy (p_1,p_2,P) and prove that if $J_1^m>0$, $J_2^m>0$, then it is profitable for M to reduce the price of the bundle to $\min\{p_1+\underline{v}_2,p_2+\underline{v}_1\}$, which means that M plays a pure bundling strategy; therefore no mixed bundling strategy maximizes π. In order to fix the ideas, we consider p_1,p_2 such that $p_2+\underline{v}_1\leq p_1+\underline{v}_2$.[45] Hence, if (p_1,p_2,P) is a mixed bundling strategy then $p_2+\underline{v}_1<P$. Step 1 proves that $\frac{\partial\pi}{\partial P}<0$ if $p_1+\underline{v}_2<P$. Step 2 proves that $\frac{\partial\pi}{\partial P}<0$ if $p_2+\underline{v}_1<P=p_1+\underline{v}_2$. Note that we showed in Section 4 that we can consider $p_2\leq\min\{\overline{v}_1,P-\underline{v}_2\}$ w.l.o.g.

Step 1 If $J_1^m>0$ and (p_1,p_2,P) is a mixed bundling strategy such that $p_1+\underline{v}_2<p$, then $\frac{\partial\pi}{\partial P}<0$.

Proof of Step 1. Since $\frac{\partial\pi}{\partial P}=\frac{\partial\mu_1}{\partial P}p_1+\frac{\partial\mu_2}{\partial P}p_2+\frac{\partial\mu_{12}}{\partial P}P+\mu_{12}$ and

$$\begin{aligned}\frac{\partial\mu_1}{\partial P}&=[1-G_1(p_1)]g_2(P-p_1),\qquad \frac{\partial\mu_2}{\partial P}=[1-G_2(p_2)]g_1(P-p_2)\\ \frac{\partial\mu_{12}}{\partial P}&=-[1-G_1(p_1)]g_2(P-p_1)-[1-G_2(p_2)]g_1(P-p_2)-\int_{P-p_1}^{p_2}g_1(P-v_2)g_2(v_2)dv_2\end{aligned}$$

[45]Notice that given $p_2+\underline{v}_1\leq p_1+\underline{v}_2$, the inequality $J_1^m>0$ suffices to prove $\frac{\partial\pi}{\partial P}<0$. If $p_2+\underline{v}_1>p_1+\underline{v}_2$, then we can prove $\frac{\partial\pi}{\partial P}<0$ by using the inequality $J_2^m>0$.

after rearranging we obtain

$$\begin{aligned}
\frac{\partial \pi}{\partial P} &= -(P-p_1)[1-G_1(p_1)]g_2(P-p_1)-(P-p_2)[1-G_2(p_2)]g_1(P-p_2)+\\
&\quad +\int_{P-p_1}^{p_2}[1-G_1(P-v_2)]g_2(v_2)dv_2+[1-G_1(P-p_2)][1-G_2(p_2)]\\
&\quad -P\int_{P-p_1}^{p_2} g_1(P-v_2)g_2(v_2)dv_2\\
&= -(P-p_1)[1-G_1(p_1)]g_2(P-p_1)+[1-G_2(p_2)][1-G_1(P-p_2)-(P-p_2)g_1(P-p_2)]+\\
&\quad +\int_{P-p_1}^{p_2}[1-G_1(P-v_2)-Pg_1(P-v_2)]g_2(v_2)dv_2
\end{aligned}$$

We know that $J_1(x)>0$ for each $x\in[\underline{v}_1,\bar{v}_1]$, therefore $1-G_1(P-p_2)-(P-p_2)g_1(P-p_2)<0$ and $\int_{P-p_1}^{p_2}[1-G_1(P-v_2)-Pg_1(P-v_2)]g_2(v_2)dv_2 \le \int_{P-p_1}^{p_2}[1-G_1(P-v_2)-(P-v_2)g_1(P-v_2)]g_2(v_2)dv_2\le 0$. In order to rule out the possibility of $\frac{\partial\pi}{\partial P}=0$, we recall that (p_1,p_2,P) is a mixed bundling strategy and therefore $p_1<\bar{v}_1$ and/or $p_2<\bar{v}_2$. This implies $1-G_1(p_1)>0$ and/or $1-G_2(p_2)>0$. ■

Step 2 If $J_1^m>0$ and (p_1,p_2,P) is a mixed bundling strategy such that $p_2+\underline{v}_1<P=p_1+\underline{v}_2$, then $\frac{\partial\pi}{\partial P}<0$.

Proof of Step 2. Given $P=p_1+\underline{v}_2$, we obtain $\mu_1=0$, $\mu_2=[1-G_2(p_2)]G_1(P-p_2)$, $\mu_{12}=[1-G_1(P-p_2)][1-G_2(p_2)]+\int_{\underline{v}_2}^{p_2}[1-G_1(P-v_2)]g_2(v_2)dv_2$. Since $\frac{\partial\pi}{\partial P}=\frac{\partial\mu_2}{\partial P}p_2+\frac{\partial\mu_{12}}{\partial P}P+\mu_{12}$ and

$$\frac{\partial\mu_2}{\partial P}=[1-G_2(p_2)]g_1(P-p_2),\qquad \frac{\partial\mu_{12}}{\partial P}=-[1-G_2(p_2)]g_1(P-p_2)-\int_{\underline{v}_2}^{p_2}g_1(P-v_2)g_2(v_2)dv_2$$

after rearranging we obtain

$$\begin{aligned}
\frac{\partial \pi}{\partial P} &= -(P-p_2)[1-G_2(p_2)]g_1(P-p_2)+[1-G_1(P-p_2)][1-G_2(p_2)]\\
&\quad +\int_{\underline{v}_2}^{p_2}[1-G_1(P-v_2)-Pg_1(P-v_2)]g_2(v_2)dv_2\\
&= [1-G_2(p_2)][1-G_1(P-p_2)-(P-p_2)g_1(P-p_2)]\\
&\quad +\int_{\underline{v}_2}^{p_2}g_2(v_2)[1-G_1(P-v_2)-Pg_1(P-v_2)]dv_2
\end{aligned}$$

We can argue as in the proof of Step 1 to infer that $1-G_1(P-p_2)-(P-p_2)g_1(P-p_2)<0$ and $\int_{\underline{v}_2}^{p_2}[1-G_1(P-v_2)-Pg_1(P-v_2)]g_2(v_2)dv_2\le 0$. In order to rule out the possibility of $\frac{\partial\pi}{\partial P}=0$, we recall that (p_1,p_2,P) is a mixed bundling strategy and $\mu_1=0$, which implies $\mu_2>0$. Hence, $p_2<\bar{v}_2$ holds, which implies $1-G_2(p_2)>0$. ■

6.5 Example

First notice that if $p_2+6 \le p_1$, then we can argue as in the proof of Proposition 6 to infer that pure bundling is better than any other strategy. In order to find the optimal pure bundling price, we need to derive the demand function for the pure bundle, that is $\Pr\{v_1+v_2 \ge P\}$ as a function of P. We obtain

$$\Pr\{v_1+v_2 \ge P\} = \begin{cases} 1-\frac{1}{20}(P-6)^2 & \text{if } 6 \le P \le 7 \\ 1-\frac{1}{20}(2P-13) & \text{if } 7 < P \le 16 \\ \frac{1}{20}(17-P)^2 & \text{if } 16 < P \le 17 \end{cases}$$

Since $\pi(P) = P\Pr\{v_1+v_2 \ge P\}$, it follows that

$$\pi'(p) = \begin{cases} \frac{6}{5}P - \frac{3}{20}P^2 - \frac{4}{5} & \text{if } 6 \le P \le 7 \\ \frac{33}{20} - \frac{1}{5}P & \text{if } 7 < P \le 16 \\ \frac{3}{20}P^2 - \frac{17}{5}P + \frac{289}{20} & \text{if } 16 < P \le 17 \end{cases}$$

and $\pi'(P) > 0$ for $P \in [6, 8.25)$, $\pi'(P) < 0$ for $P \in (8.25, 17]$. Thus $P^* = 8.25$ and $\pi(8.25) = 6.806$.

If $p_1 < p_2+6$, then we can argue like in proof of Step 1 (in the proof of Proposition 6) to prove that $\frac{\partial\pi}{\partial P} < 0$ if $P > p_2+6$, and thus $\mu_2 = 0$ in the optimal strategy. However, it is not necessarily the case that $\frac{\partial\pi}{\partial P} < 0$ if $p_1 < P = p_2+6$. In fact, consider the strategy $p_1 = 6$ and $P = 11$. Then $S_1 = \{(v_1,v_2) \in V : v_2 < 5\}$ and $S_{12} = \{(v_1,v_2) \in V : v_2 \ge 5\}$, with $\mu_1 = \frac{1}{2}$, $\mu_{12} = \frac{1}{2}$ and $\pi = 8.5$.

6.6 Proof of Proposition 7

Using symmetry in the distributions and $p_1 = p_2 = p$, we obtain $\mu_1 = \mu_2 = [1-G(p)]G(P-p-s)$ and $\mu_{12} = \int_{P-p-s}^{p}[1-G(P-s-v_2)]g(v_2)dv_2 + [1-G(P-p-s)][1-G(p)]$. Therefore, from $\pi = 2p\mu_1 + P\mu_{12}$, we obtain

$$\frac{\partial\pi}{\partial P} = [1-G(p)][1-G(P-p-s)-2(P-p)g(P-p-s)] + \int_{P-p-s}^{p}[1-G(P-s-v_2)-Pg(P-s-v_2)]g(v_2)dv_2$$

and (i) $1-G(P-p-s)-2(P-p)g(P-p-s) < 0$ is equivalent to $(P-p-s)+2s+J(P-p-s) > 0$, which is satisfied since $\underline{v}+s+J^m > 0$; (ii) $1-G(P-s-v_2)-Pg(P-s-v_2) < 0$ is equivalent to $v_2+s+J(P-s-v_2) > 0$, which is satisfied since $\underline{v}+s+J^m > 0$.

6.7 Proof of Proposition 8

In order to describe M's design of the optimal mechanism it is possible to apply the Revelation Principle to describe a generic mechanism in terms of two functions, $q = (q_1, q_2) : V \to [0,1] \times [0,1]$ and $T : V \to \mathbb{R}$, such that a buyer reporting valuations $v' = (v_1', v_2')$ receives object 1 with probability $q_1(v')$, receives object 2 with probability $q_2(v')$, and pays $T(v')$. The objective of M is to choose q and T in order to maximize the expectation of $T(v)$ subject to participation and incentive constraints:

$$v_1 q_1(v) + v_2 q_2(v) - T(v) \geq \max\{0, v_1 q_1(v') + v_2 q_2(v') - T(v')\} \text{ for each } v \text{ and } v' \text{ in } V$$

Although this is typically a complicated problem, the results in Pavlov (2011) allow to gain some insights on the optimal mechanism. First, since v_1, v_2 are identically distributed, we can focus on determining the optimal q_1, q_2, T for the case of $v_2 \geq v_1$ (symmetric results are obtained when $v_2 < v_1$). Second, Pavlov (2011) shows that under the condition

$$3 + v_1 \frac{g'(v_1)}{g(v_1)} + v_2 \frac{g'(v_2)}{g(v_2)} \geq 0 \quad \text{for each } (v_1, v_2) \in V \tag{16}$$

it is optimal for M to restrict to mechanisms in which the buyer either gets no good, or gets his most preferred good for sure, and his less preferred good with some probability.[46] Formally, given $v_2 \geq v_1$, either $q_1(v_1, v_2) = q_2(v_1, v_2) = 0$, or $q_2(v_1, v_2) = 1$ and $q_1(v_1, v_2) \in [0,1]$. Furthermore, there is no loss for M in performing the screening only on the valuation v_1 for good 1, and in optimizing over mechanisms characterized by two functions, $q_1 : [\underline{v}, \bar{v}] \to [0,1]$ and $t : [\underline{v}, \bar{v}] \to \mathbb{R}$, and in which the buyer chooses a message v_1' in $[\underline{v}, \bar{v}] \cup \{\varnothing\}$.[47] If $v_1' = \varnothing$, then the buyer does not participate: she receives no object and pays zero. If conversely $v_1' \in [\underline{v}, \bar{v}]$, then the buyer receives object 2 with probability 1, receives object 1 with probability $q_1(v_1')$, and pays $t(v_1')$. Let $u(v_1) \equiv v_1 q_1(v_1) - t(v_1)$. Then, the payoff of a type $v = (v_1, v_2)$ reporting $v_1' = v_1$ is $u(v_1) + v_2$, thus type v participates if and only if $u(v_1) + v_2 \geq 0$.

As a consequence, the profit of M from type v is $t(v_1) = v_1 q_1(v_1) - u(v_1)$ if $u(v_1) + v_2 \geq 0$,

[46] Condition (16) is a sort of hazard condition that is widespread in the literature on multidimentional mechanism design: see for instance McAfee and MacMillan (1988), or Manelli and Vincent (2006).

[47] We are somewhat abusing notation here, using again q_1 to denote a function defined on $[\underline{v}, \bar{v}]$, whereas q_1 introduced at the beginning of this proof is defined on V. However, since only $q_1 : [\underline{v}, \bar{v}] \to [0,1]$ is used from now on, there is no concern about ambiguity.

but is 0 otherwise. The expected profit is

$$\int_{\underline{v}}^{\bar{v}}[v_1q_1(v_1)-u(v_1)]g(v_1)[1-G(\max\{v_1,-u(v_1)\})]dv_1 \tag{17}$$

in which the term $1-G(\max\{v_1,-u(v_1)\})$ takes into account that we are considering types such that $v_2 \geq v_1$, and only types such that $v_2 \geq -u(v_1)$ participate. The incentive constraints in this problem are satisfied if and only if q_1 is weakly increasing and $u(v_1) = u(\underline{v}) + \int_{\underline{v}}^{v_1} q_1(x)dx$ for each $v_1 \in [\underline{v},\bar{v}]$; it turns out it is convenient to define $\gamma \equiv -u(\underline{v})$, and therefore from now on we use $u(v_1) = -\gamma + \int_{\underline{v}}^{v_1} q_1(x)dx$. The problem of M is reduced to maximizing (17) with respect to γ (within the set $\mathbb{R}$) and with respect to q_1 (within the set of weakly increasing functions with domain $[\underline{v},\bar{v}]$). We use $\pi(\gamma,q_1)$ to denote the profit of M in (17), as a function of (γ,q_1). In this setting pure bundling is obtained if $q_1(v_1) = q_1^{pb}(v_1) \equiv 1$ for each $v_1 \in [\underline{v},\bar{v}]$, since then each type either receives no object, or receives both objects.[48]

In the following we prove that π is maximized with respect to q_1 at $q_1 = q_1^{pb}$ if there exists $\beta > 1$ such that both the following conditions are satisfied:

$$G(\frac{1}{2}\underline{v}+\frac{1}{2}\bar{v}) \leq \beta - 1 \tag{18}$$

$$xg(x) \geq \beta \qquad \text{for each } x \in [\underline{v},\bar{v}]. \tag{19}$$

In particular, it is immediate to see that if $\beta = \frac{3}{2}$ and g is increasing, then (i) (16) is satisfied; (ii) (18) holds since g increasing implies G convex, thus $G(\frac{1}{2}\underline{v}+\frac{1}{2}\bar{v}) \leq \frac{1}{2}G(\underline{v})+\frac{1}{2}G(\bar{v}) = \frac{1}{2}$; (iii) (19) is equivalent to $\underline{v}g(\underline{v}) \geq \frac{3}{2}$. If (19) is satisfied for $\beta \geq 2$, then (18) holds even though g is not increasing, but notice that g not increasing does not guarantee that (16) holds.[49]

The proof is split in three steps. The first two steps establish that M can restrict his attention to values of γ in the interval $[\underline{v},\bar{v}]$. The third step proves that the optimal γ is relatively close to $\underline{v}$, and this implies that the optimal q_1 is q_1^{pb}.

Step 1: It is suboptimal to choose $\gamma < \underline{v}$

[48] In this case the bundle price is $\underline{v}+\gamma$, and is selected by M through the choice of γ.

[49] For instance, suppose that (i) H is the c.d.f. of a random variable with support $[\underline{\nu},\bar{\nu}]$ and with a strictly positive and differentiable density; (ii) v_1, v_2 are i.i.d., each with support $[\underline{v},\bar{v}]$ such that $\underline{v} = \underline{\nu}+\omega$, $\bar{v} = \bar{\nu}+\omega$ for some $\omega > 0$, and with c.d.f. G which is a rightward shitf of H. Given $\beta \geq 2$, (19) is satisfied if ω is large, but if the density's derivative is negative at some point then (16) fails to hold for a large ω.

If $\gamma < \underline{v}$, then $-u(v_1) < v_1$ and $G(\max\{v_1, -u(v_1)\}) = G(v_1)$ for any $v_1 \in [\underline{v}, \bar{v}]$. Hence

$$\pi(\gamma, q_1) = \int_{\underline{v}}^{\bar{v}} [v_1 q_1(v_1) + \gamma - \int_{\underline{v}}^{v_1} q_1(x)dx] g(v_1)[1 - G(v_1)]dv_1$$

which is increasing with respect to γ. Therefore no γ smaller than $\underline{v}$ is optimal.

Step 2: Given (γ, q_1) such that $\gamma > \bar{v}$, there exists $\dot{q}_1$ such that $\pi(\bar{v}, \dot{q}_1) = \pi(\gamma, q_1)$

First notice that if $-u(\bar{v}) \geq \bar{v}$, then no type participates, that is $G(\max\{v_1, -u(v_1)\}) = 1$ for any $v_1 \in [\underline{v}, \bar{v}]$. Therefore $\pi(\gamma, q_1) = 0 = \pi(\bar{v}, \dot{q}_1)$ with $\dot{q}_1$ such that $\dot{q}_1(v_1) = 0$ for each $v_1 \in [\underline{v}, \bar{v}]$. If $-u(\bar{v}) < \bar{v}$, then pick $\dot{v} \in (\underline{v}, \bar{v})$ such that $-u(\dot{v}) = \bar{v}$ and let $\dot{q}_1(v_1) \equiv \begin{cases} 0 & \text{if } v_1 \in [\underline{v}, \dot{v}] \\ q_1(v_1) & \text{if } v_1 \in (\dot{v}, \bar{v}] \end{cases}$. The equality $\pi(\bar{v}, \dot{q}_1) = \pi(\gamma, q_1)$ holds because the set of participating types and the payment of each participating type are the same in the two cases.

Step 3: If (18) and (19) are satisfied for some $\beta > 1$, then π is maximized at (γ, q_1) such that $q_1 = q_1^{pb}$

The proof of this step is split in three substeps. First we define a function $\tilde{\pi}(\gamma, q_1)$ such that $\pi(\gamma, q_1) \leq \tilde{\pi}(\gamma, q_1)$ and $\pi(\gamma, q_1^{pb}) = \tilde{\pi}(\gamma, q_1^{pb})$, and then we prove that $\tilde{\pi}$ is maximized with respect to q_1 at $q_1 = q_1^{pb}$. Since $\pi(\gamma, q_1) \leq \tilde{\pi}(\gamma, q_1)$ and $\pi(\gamma, q_1^{pb}) = \tilde{\pi}(\gamma, q_1^{pb})$, it follows that also π is maximized with respect to q_1 at $q_1 = q_1^{pb}$.

Step 3.1: The definition of $\tilde{\pi}(\gamma, q_1)$ In view of Steps 1-2, we assume that γ belongs to $[\underline{v}, \bar{v}]$, and we let $\hat{v} \in [\underline{v}, \bar{v}]$ be such that $-u(v_1) \geq v_1$ for $v_1 \in [\underline{v}, \hat{v}]$, $-u(v_1) < v_1$ for $v_1 \in (\hat{v}, \bar{v}]$; hence $G(\max\{v_1, -u(v_1)\}) = G(-u(v_1))$ if $v_1 \in [\underline{v}, \hat{v}]$, and $G(\max\{v_1, -u(v_1)\}) = G(v_1)$ if $v_1 \in (\hat{v}, \bar{v}]$. Therefore

$$\pi(\gamma, q_1) = \int_{\underline{v}}^{\hat{v}} [v_1 q_1(v_1) - u(v_1)] g(v_1)[1 - G(-u(v_1))]dv_1 + \int_{\hat{v}}^{\bar{v}} [v_1 q_1(v_1) - u(v_1)] g(v_1)[1 - G(v_1)]dv_1$$

Let $v^* \equiv \frac{1}{2}(\underline{v} + \gamma)$ and notice that $v^* = \hat{v}$ if $q_1(v_1) = 1$ for each $v_1 \in (\underline{v}, \hat{v}]$, but $v^* < \hat{v}$ if $q_1(v_1) < 1$ for some $v_1 \in (\underline{v}, \hat{v}]$. A related remark is that $-u(v_1) \geq \gamma - (v_1 - \underline{v}) = 2v^* - v_1$ for $v_1 \in [\underline{v}, v^*]$, and $-u(v_1) \geq v_1$ for $v_1 \in (v^*, \hat{v}]$. Since $v_1 q_1(v_1) - u(v_1) = v_1 q_1(v_1) + \gamma -$

$\int_{\underline{v}}^{v_1} q_1(x)dx > 0$,[50] it follows that $\pi(\gamma, q_1) \le \tilde{\pi}(\gamma, q_1)$, with

$$\begin{aligned}\tilde{\pi}(\gamma, q_1) \;\equiv\; & \int_{\underline{v}}^{v^*} [v_1 q_1(v_1) + \gamma - \int_{\underline{v}}^{v_1} q_1(x)dx] g(v_1)[1 - G(2v^* - v_1)]dv_1 \\ & + \int_{v^*}^{\bar{v}} [v_1 q_1(v_1) + \gamma - \int_{\underline{v}}^{v^*} q_1(x)dx - \int_{v^*}^{v_1} q_1(x)dx] g(v_1)[1 - G(v_1)]dv_1\end{aligned}$$

Step 3.2: If (19) is satisfied for some $\beta > 1$ and $1 - \frac{1}{2\beta} + \frac{1}{2\beta}G^2(v^*) \ge G(\gamma)$, then $\tilde{\pi}$ is maximized with respect to q_1 at $q_1 = q_1^{pb}$ Consider $q_1(v_1)$ for $v_1 \in [v^*, \bar{v}]$, which affects $\tilde{\pi}$ only through the term

$$\int_{v^*}^{\bar{v}} [v_1 q_1(v_1) - \int_{v^*}^{v_1} q_1(x)dx] g(v_1)[1 - G(v_1)]dv_1 \tag{20}$$

Integration by parts yields $\int_{v^*}^{\bar{v}} \int_{v^*}^{v_1} q_1(x)dx g(v_1)[1-G(v_1)]dv_1 = \left[-\frac{1}{2}[1 - G(v_1)]^2 \int_{v^*}^{v_1} q_1(x)dx\right]_{v^*}^{\bar{v}} + \int_{v^*}^{\bar{v}} \frac{1}{2}[1 - G(v_1)]^2 q_1(v_1)dv_1 = \int_{v^*}^{\bar{v}} \frac{1}{2}[1 - G(v_1)]^2 q_1(v_1)dv_1$, thus (20) is equal to $\int_{v^*}^{\bar{v}} \{v_1 g(v_1)[1 - G(v_1)] - \frac{1}{2}[1-G(v_1)]^2\} q_1(v_1)dv_1$. From (19) it follows that $v_1 g(v_1)[1-G(v_1)] - \frac{1}{2}[1-G(v_1)]^2 > \frac{1}{2}(1 - G(v_1))(2\beta - 1 + G(v_1)) \ge 0$, and therefore it is optimal to set $q_1(v_1) = 1$ for any $v_1 \in [v^*, \bar{v}]$.

Now consider $q_1(v_1)$ for $v_1 \in [\underline{v}, v^*)$, which affects $\tilde{\pi}$ only through the term

$$\int_{\underline{v}}^{v^*} [v_1 q_1(v_1) - \int_{\underline{v}}^{v_1} q_1(x)dx] g(v_1)[1 - G(2v^* - v_1)]dv_1 - \int_{\underline{v}}^{v^*} q_1(x)dx \frac{[1 - G(v^*)]^2}{2} \tag{21}$$

Let $\Psi(v_1) \equiv \int_{\underline{v}}^{v_1} g(x)[1 - G(2v^* - x)]dx$, for $v_1 \in [\underline{v}, v^*]$, and find

$$\begin{aligned}\int_{\underline{v}}^{v^*} \int_{\underline{v}}^{v_1} q_1(x)dx g(v_1)[1 - G(2v^* - v_1)]dv_1 &= \left[\Psi(v_1) \int_{\underline{v}}^{v_1} q_1(x)dx\right]_{\underline{v}}^{v^*} - \int_{\underline{v}}^{v^*} \Psi(v_1) q_1(v_1)dv_1 \\ &= \int_{\underline{v}}^{v^*} [\Psi(v^*) - \Psi(v_1)] q_1(v_1)dv_1 \\ &= \int_{\underline{v}}^{v^*} \int_{v_1}^{v^*} g(x)[1 - G(2v^* - x)]dx q_1(v_1)dv_1\end{aligned}$$

Therefore (21) is equal to $\int_{\underline{v}}^{v^*} \{v_1 g(v_1)[1 - G(2v^* - v_1)] - \int_{v_1}^{v^*} g(x)[1 - G(2v^* - x)]dx - \frac{[1-G(v^*)]^2}{2}\} q_1(v_1)dv_1$. From (19) it follows that $v_1 g(v_1)[1 - G(2v^* - v_1)] - \int_{v_1}^{v^*} g(x)[1 - G(2v^* - x)]dx - \frac{[1-G(v^*)]^2}{2} \ge \beta[1 - G(2v^* - v_1)] - \int_{v_1}^{v^*} g(x)[1 - G(2v^* - x)]dx - \frac{1}{2}[1 - G(v^*)]^2 \equiv \lambda(v_1)$, and λ is an increasing function. Since $\lambda(\underline{v}) = \beta[1 - G(\gamma)] - \int_{\underline{v}}^{v^*} g(x)[1 - G(2v^* -$

[50] Precisely, $v_1 q_1(v_1) + \gamma - \int_{\underline{v}}^{v_1} q_1(x)dx = \int_{\underline{v}}^{v_1} [q_1(v_1) - q_1(x)]dx + \underline{v} q_1(v_1) + \gamma$, which is positive since q_1 is increasing and $\gamma \ge \underline{v} > 0$.

$x)]dx - \int_{v^*}^{\bar{v}} g(x)[1 - G(x)]dx$ and $-\int_{\underline{v}}^{v^*} g(x)[1 - G(2v^* - x)]dx - \int_{v^*}^{\bar{v}} g(x)[1 - G(x)]dx = -1 + \int_{\underline{v}}^{v^*} g(x)G(2v^* - x)]dx + \int_{v^*}^{\bar{v}} g(x)G(x)dx \geq -1 + \int_{\underline{v}}^{v^*} g(x)G(v^*)dx + \frac{1}{2} - \frac{1}{2}\left[G(v^*)\right]^2 = \frac{1}{2}\left[G(v^*)\right]^2 - \frac{1}{2}$, we infer that $\lambda(\underline{v}) \geq \beta - \frac{1}{2} - \beta G(\gamma) + \frac{1}{2}\left[G(v^*)\right]^2$; therefore $q_1 = q_1^{pb}$ maximizes $\tilde{\pi}$ as long as $\kappa(\gamma) \equiv 1 - \frac{1}{2\beta} + \frac{1}{2\beta}\left[G(\frac{1}{2}\underline{v} + \frac{1}{2}\gamma)\right]^2 - G(\gamma)$ is positive or zero. It is immediate that $\kappa(\underline{v}) = 1 - \frac{1}{2\beta} > 0 > \kappa(\bar{v}) = -\frac{1}{2\beta}\{1 - [G(\frac{1}{2}\underline{v} + \frac{1}{2}\bar{v})]^2\}$.

Step 3.3: If (18) and (19) are satisfied for some $\beta > 1$, then the optimal γ is such that $1 - \frac{1}{2\beta} + \frac{1}{2\beta}\left[G(v^*)\right]^2 \geq G(\gamma)$ We prove that if γ is such that $\kappa(\gamma) < 0$, then $\frac{\partial\tilde{\pi}}{\partial\gamma} < 0$. This reveals that the optimal γ satisfies $\kappa(\gamma) \geq 0$.[51] After rearranging we find

$$\begin{aligned}
\frac{\partial\tilde{\pi}}{\partial\gamma} &= \frac{1}{2}[1 - G(v^*)]^2 - \int_{\underline{v}}^{v^*} \{[v_1 q_1(v_1) + \gamma - \int_{\underline{v}}^{v_1} q_1(x)dx]g(2v^* - v_1) + G(2v^* - v_1) - 1\}g(v_1)dv_1 \\
&\leq \frac{1}{2}[1 - G(v^*)]^2 - \int_{\underline{v}}^{v^*} [\gamma g(2v^* - v_1) + G(2v^* - v_1) - 1]g(v_1)dv_1
\end{aligned}$$

and $\int_{\underline{v}}^{v^*} \gamma g(2v^* - v_1)g(v_1)dv_1 > \beta\int_{\underline{v}}^{v^*} g(2v^* - v_1)dv_1 = \beta\int_{v^*}^{\gamma} g(x)dx = \beta G(\gamma) - \beta G(v^*) > \beta - \frac{1}{2} - \beta G(v^*) + \frac{1}{2}\left[G(v^*)\right]^2$;[52] moreover, $G(2v^* - v_1) \geq G(v^*)$ for $v_1 \in [\underline{v}, v^*]$. Therefore

$$\begin{aligned}
\frac{\partial\tilde{\pi}}{\partial\gamma} &< \frac{1}{2}[1 - G(v^*)]^2 - \left[\beta - \frac{1}{2} - \beta G(v^*) + \frac{1}{2}\left[G(v^*)\right]^2 + [G(v^*)]^2 - G(v^*)\right] \\
&= -\left(1 - G(v^*)\right)\left(\beta - 1 - G(v^*)\right)
\end{aligned}$$

The last expression is negative or zero since $G(v^*) \leq G(\frac{1}{2}\underline{v} + \frac{1}{2}\bar{v}) \leq \beta - 1$ by (18).

6.8 Proof of Lemma 4

Since f is symmetric around $\frac{1}{2}$, the equality $F(x) + F(1 - x) = 1$ holds for each $x \in [0, 1]$. Moreover, the synergy s has the same effect of a reduction in the price of the bundle. Therefore, given $p = p_A$ and $P = P_A$ we find

$$\begin{aligned}
\mu_1 &= \mu_2 = G(P_A - p_A - s)[1 - G(p_A)] = F(\frac{P_A - p_A - s - b_1 + t}{2t})[1 - F(\frac{p_A - b_1 + t}{2t})] \\
&= [1 - F(1 - \frac{P_A - p_A - s - b_1 + t}{2t})]F(1 - \frac{p_A - b_1 + t}{2t}) \\
&= [1 - F(\frac{1}{2} + \frac{b_2 - P_A + p_A}{2t})]F(\frac{1}{2} + \frac{b_1 - p_A}{2t}) = [1 - F(x'')]F(x')
\end{aligned}$$

[51] In order to evaluate $\frac{\partial\tilde{\pi}}{\partial\gamma}$ we use the result that the optimal $q_1(v_1)$ is equal to 1 for v_1 close to v^*. This follows from the proof of Step 3.2, since $\lambda(v_1) > 0$ for v_1 close to v^*.

[52] The first inequality in this chain follows from (19) and $\gamma > v^*$; the first equality is obtained using the substitution $x = \underline{v} + \gamma - v_1$; the last inequality holds since $\kappa(\gamma) < 0$.

which coincides with μ_{AB} in (4).

Regarding the sales of the bundle we find

$$\begin{aligned}
\mu_{12} &= 1-2G(P_A-p_A-s)+G(P_A-p_A-s)G(p_A)-\int_{P_A-p_A-s}^{p_A} G(P_A-s-v_1)h(v_1)dv_1 \\
&= 1-2F(\frac{P_A-p_A-s-b_1+t}{2t})+F(\frac{P_A-p_A-s-b_1+t}{2t})F(\frac{p_A-b_1+t}{2t}) \\
&\quad -\frac{1}{2t}\int_{P_A-p_A-s}^{p_A} F(\frac{P_A-s-v_1-b_1+t}{2t})f(\frac{v_1-b_1+t}{2t})dv_1
\end{aligned}$$

Now use the substitution $z=\frac{1}{2}+\frac{b_1-v_1}{2t}$ to obtain

$$\begin{aligned}
\mu_{12} &= 1-2F(1-x'')+F(1-x'')F(1-x')-\frac{1}{2t}\int_{x''}^{x'} f(1-z)F(\frac{P-b_2-b_1}{2t}+z)(-2t)dz \\
&= 1-2[1-F(x'')]+[1-F(x'')][1-F(x')]+\int_{x''}^{x'} f(z)[1-F(1-z-\frac{P-b_2-b_1}{2t})dz \\
&= F(x')F(x'')+\int_{x'}^{x''} f(z)F(x'+x''-z)dz
\end{aligned}$$

which coincides with μ_{AA} in (4).

6.9 Proof of Proposition 9(i)

Given that $G(x)=F(\frac{x-b_1+t}{2t})$ for $x\in[b_1-t,b_1+t]$, we obtain $J(x)=x-\frac{1-F(\frac{x-b_1+t}{2t})}{\frac{1}{2t}f(\frac{x-b_1+t}{2t})}$. This is an increasing function of x since f is log-concave, thus $\underline{v}+s+J^m=(b_1-t)+(b_2-b_1)+(b_1-t-\frac{2t}{f(1)})=b_1+b_2-2t-\frac{2t}{f(1)}=P_B+2(\alpha-t-\frac{t}{f(1)})$, which is positive since $\alpha\geq t+\frac{t}{f(1)}$ by assumption.

6.10 Proof of Proposition 9(ii)

Given $b_1\equiv P_B-p_B+\alpha$, $b_2\equiv p_B+\alpha$, we say that firm A plays a *pure bundling strategy* if $p_A\geq b_1+t$ and/or $P_A\leq b_2-t+p_A$ because $\mu_{AB}=0$ in either of these cases.[53] Given b_1,b_2, we define M_A as the set of (p_A,P_A) such that $\mu_{AB}>0$, that is

$$M_A=\{(p_A,P_A):0\leq P_A\leq 2p_A,\quad p_A<b_1+t,\quad P_A>b_2-t+p_A\}.$$

We say that A plays a *mixed bundling strategy* if $(p_A,P_A)\in M_A$. Notice that M_A is non-empty if and only if $b_1>-t$ and $b_2<2t+b_1$: see Figure 4.

[53] If $p_A\geq b_1+t$, then $x'\leq 0$; if $P_A\leq b_2-t+p_A$, then $x''\geq 1$.

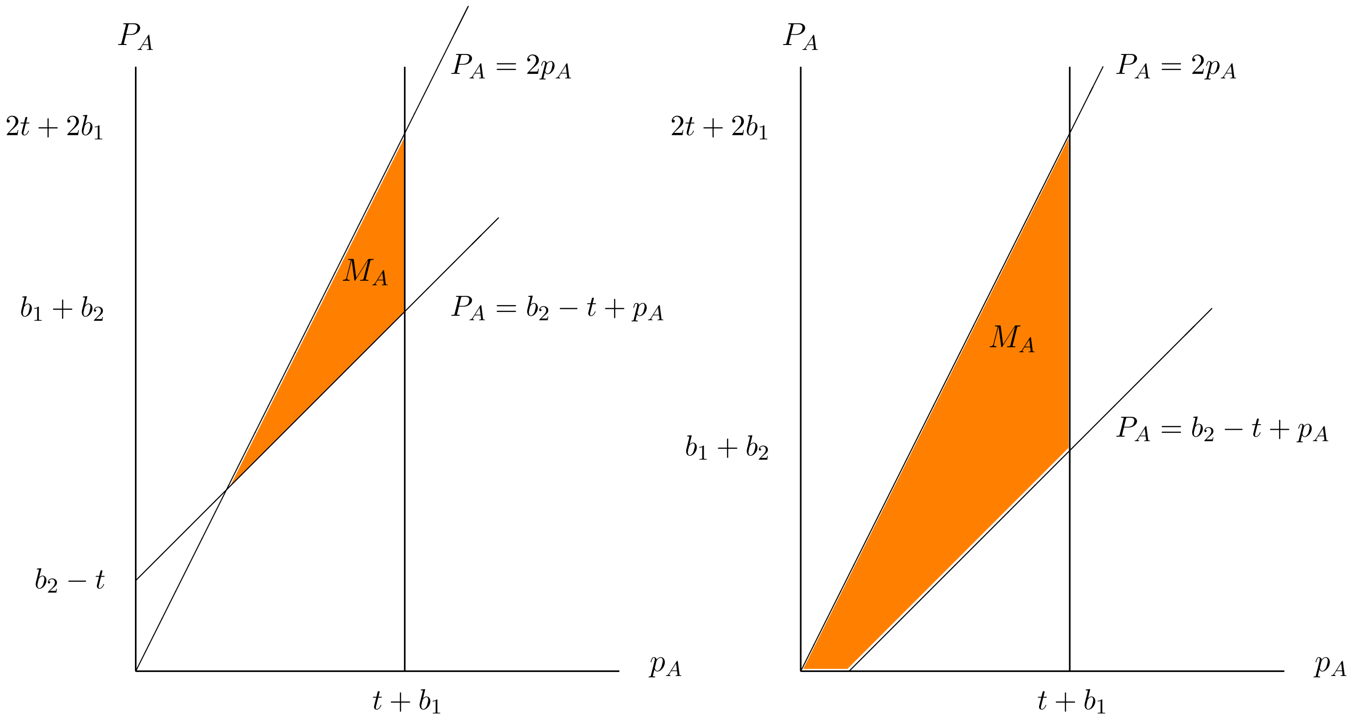

Figure 4: Mixed bundling strategies with $b_2 > t$ (left panel) and $b_2 < t$ (right panel).

Likewise, for firm B we define $a_1 \equiv P_A - p_A - \alpha$, $a_2 \equiv p_A - \alpha$ (with $a_1 \leq a_2$ from $P_A \leq 2p_A$) and the set M_B (analogous to M_A) of (p_B, P_B) such that $\mu_{AB} > 0$:

$$M_B = \{(p_B, P_B) : 0 \leq P_B \leq 2p_B, \qquad p_B < a_1 + t, \qquad P_B > a_2 - t + p_B\}.$$

One way to express Proposition 9(ii) is that if $\alpha \geq t$ and x_1, x_2 are uniformly distributed, then there is no NE $(p_A^*, P_A^*, p_B^*, P_B^*)$ satisfying $(p_A^*, P_A^*) \in M_A$ and $(p_B^*, P_B^*) \in M_B$.

Using (4), for each $(p_A, P_A) \in M_A$ we have

$$\pi_A = \frac{1}{8t^2}\begin{pmatrix} P_A^3 + 4p_A^3 - 2\,(2t + b_1 + b_2)\,P_A^2 - 6p_A^2 P_A - 4\,(2t - b_2 + b_1)\,p_A^2 + 8\,(b_1 + t)\,P_A p_A \\ +(2t^2 + 4tb_2 + b_2^2 + 2b_2 b_1 - b_1^2)P_A + 4(t - b_2)(t + b_1)p_A \end{pmatrix}$$

and

$$\begin{aligned} \frac{\partial \pi_A}{\partial p_A} &= \frac{1}{8t^2}\left(12p_A^2 - 4\,(3P_A + 4t - 2b_2 + 2b_1)\,p_A + 8\,(b_1 + t)\,P_A + 4(t - b_2)(t + b_1)\right) \\ \frac{\partial \pi_A}{\partial P_A} &= \frac{1}{8t^2}\left(3P_A^2 - 4\,(2t + b_1 + b_2)\,P_A - 6p_A^2 + 8\,(b_1 + t)\,p_A + 2t^2 + 4tb_2 + b_2^2 + 2b_2 b_1 - b_1^2\right). \end{aligned}$$

Likewise, for each $(p_B, P_B) \in M_B$ we have

$$\pi_B = \frac{1}{8t^2}\left(\begin{array}{c} P_B^3 + 4p_B^3 - 6P_Bp_B^2 - 2\left(2t + a_1 + a_2\right)P_B^2 - 4\left(2t - a_2 + a_1\right)p_B^2 + 8\left(a_1 + t\right)P_Bp_B \\ + \left(2t^2 + 4ta_2 + a_2^2 + 2a_1a_2 - a_1^2\right)P_B + 4\left(t - a_2\right)\left(a_1 + t\right)p_B \end{array}\right)$$

and

$$\begin{aligned} \frac{\partial \pi_B}{\partial p_B} &= \frac{1}{8t^2}(12p_B^2 - 4(3P_B + 4t - 2a_2 + 2a_1)p_B + 8\left(a_1 + t\right)P_B + 4\left(t - a_2\right)\left(a_1 + t\right)) \\ \frac{\partial \pi_B}{\partial P_B} &= \frac{1}{8t^2}\left(3P_B^2 - 4\left(2t + a_1 + a_2\right)P_B - 6p_B^2 + 8\left(a_1 + t\right)p_B + 2t^2 + 4ta_2 + a_2^2 + 2a_2a_1 - a_1^2\right) \end{aligned}$$

Since $\alpha \geq t$ implies $b_1 > t$, we consider the following set $\mathcal{B}$ of possible values for (b_1, b_2) for firm A: $\mathcal{B} = \{(b_1, b_2) : b_1 > t, b_1 \leq b_2 < 2t + b_1\}$.

Our first result refers to the case of $\alpha \geq 1.125t$, which implies $b_2 \geq 1.125t$. We prove that in this case the best reply of firm A is a pure bundling strategy.

Lemma 4. *Suppose that $(b_1, b_2) \in \mathcal{B}$ and $b_2 \geq 1.125t$. Then it is never a best reply for firm A to play (p_A, P_A) in M_A.*

Proof The proof of this lemma is organized in three steps. In Step 1 we prove that for firm A playing independent pricing (that is, $P_A = 2p_A$)in M_A is suboptimal. A mixed bundling strategy for firm A can thus only be optimal if it lies in the interior of M_A, which implies that the first (and second) order conditions must be satisfied. However, in Step 2 we show that if $(p_A, P_A) \in M_A$ is such that that $\frac{\partial \pi_A}{\partial p_A} = 0$, then P_A must be above some threshold level, while in Step 3 we show that $\frac{\partial \pi_A}{\partial P_A} = 0$ implies that P_A must be strictly below that some threshold level. Hence, it must be optimal for firm A to play a pure bundling strategy whenever $b_2 \geq 1.125t$.

Step 1 Suppose that $(b_1, b_2) \in \mathcal{B}$. Playing $(p_A, P_A) \in M_A$ such that $P_A = 2p_A$ is not a best reply for firm A.

We start by evaluating $\frac{\partial \pi_A}{\partial p_A}$ and $\frac{\partial \pi_A}{\partial P_A}$ at $P_A = 2p_A$ and we find

$$\begin{aligned} \frac{\partial \pi_A}{\partial p_A} &= \frac{1}{t^2}\left(-\frac{3}{2}p_A^2 + (b_2 + b_1)p_A - \frac{1}{2}\left(b_2 - t\right)\left(t + b_1\right)\right) \equiv z(p_A), \\ \frac{\partial \pi_A}{\partial P_A} &= \frac{1}{t^2}\left(\frac{3}{4}p_A^2 - (t + b_2)p_A + \frac{1}{8}\left(2b_2b_1 + b_2^2 + 4tb_2 + 2t^2 - b_1^2\right)\right) \equiv Z(p_A). \end{aligned}$$

Notice that if $(p_A, P_A) \in M_A$, then $p_A \in (b_2 - t, b_1 + t)$. Let p_A^* denote the larger solution to $z(p_A) = 0$, that is $p_A^* = \frac{1}{3}(b_1 + b_2 + \sqrt{(b_2 - t)^2 + (b_1 + t)\left(2t + b_1 - b_2\right)})$, which satisfies

$b_2 - t < p_A^* < b_1 + t$ since $z(b_2 - t) = \frac{1}{2t^2}(b_2 - t)(2t - b_2 + b_1) > 0$ and $z(b_1 + t) = -\frac{1}{2t^2}(b_1 + t)(2t - b_2 + b_1) < 0$ in $\mathcal{B}$. In fact, from $z(b_2 - t) > 0 = z(p_A^*)$ we infer that $z(p_A) > 0$ for $p_A \in (b_2 - t, p_A^*)$. This implies that, starting from (p_A, P_A) such that $P_A = 2p_A$ and $p_A \in (b_2 - t, p_A^*)$, for A it is convenient to increase p_A.

For $p_A \in [p_A^*, b_1 + t)$ we prove that $Z(p_A) < 0$. This implies that, starting from (p_A, P_A) such that $P_A = 2p_A$ and $p_A \in [p_A^*, b_1 + t)$, for A it is convenient to reduce P_A. We find $Z(b_1 + t) = -\frac{1}{8t^2}(b_2 - b_1)(2t + b_1 - b_2 + 2t + 4b_1) \leq 0$ in $\mathcal{B}$ and

$$Z(p_A^*) = -\frac{(2t + b_2 - b_1)\left(b_2 + b_1 + 4\sqrt{(b_2 - t)^2 + (b_1 + t)(2t + b_1 - b_2)}\right) - 12t^2}{24t^2}$$

which now we prove to be negative in $\mathcal{B}$. Precisely, we define $\xi_1(b_1, b_2) \equiv (2t + b_2 - b_1)(b_2 + b_1 + 4\sqrt{(b_2 - t)^2 + (b_1 + t)(2t + b_1 - b_2)})$ and show that

$$\xi_1(b_1, b_2) > 12t^2 \qquad \text{for any} \quad (b_1, b_2) \in \mathcal{B}. \tag{22}$$

To this purpose we prove below that $\frac{\partial \xi_1}{\partial b_2} > 0$ in $\mathcal{B}$, and $\xi_1(b_1, b_1) = 4t(b_1 + 2\sqrt{b_1^2 + 3t^2}) > 12t^2$ for any $b_1 \geq t$ implies (22). Precisely, $\frac{\partial \xi_1}{\partial b_2} = 2b_2 + 2t + \frac{6b_1^2 + 8b_2^2 - 10b_2 b_1 + 14b_1 t - 10tb_2}{\sqrt{(b_2 - t)^2 + (b_1 + t)(2t + b_1 - b_2)}}$ and $\frac{\partial \xi_1}{\partial b_2} > 0$ in $\mathcal{B}$ since $\xi_2(b_1, b_2) \equiv 6b_1^2 + 8b_2^2 - 10b_2 b_1 + 14b_1 t - 10tb_2 > 0$ in $\mathcal{B}$.[54] ■

Step 2 Suppose that $(b_1, b_2) \in \mathcal{B}$. If $(p_A, P_A) \in M_A$ is such that $\frac{\partial \pi_A}{\partial p_A} = 0$, then $P_A \geq \frac{2}{3}(b_1 + b_2 + \sqrt{(b_1 + t)(b_2 - t)})$.

The equation $\frac{\partial \pi_A}{\partial p_A} = 0$ in the unknown p_A has at least a real solution if and only if $P_A \leq \frac{2}{3}(b_1 + b_2 - \sqrt{(b_1 + t)(b_2 - t)})$ or $P_A \geq \frac{2}{3}(b_1 + b_2 + \sqrt{(b_1 + t)(b_2 - t)})$. We prove that if (p_A, P_A) is such that $\frac{\partial \pi_A}{\partial p_A} = 0$ and $P_A \leq \frac{2}{3}(b_1 + b_2 - \sqrt{(b_1 + t)(b_2 - t)})$, then $(p_A, P_A) \notin M_A$. First notice that $\frac{2}{3}(b_1 + b_2 - \sqrt{(b_1 + t)(b_2 - t)}) < b_1 + b_2$ and then (i) at $p_A = P_A - b_2 + t$ (i.e., along the south-east boundary of M_A) we find $\frac{\partial \pi_A}{\partial p_A} = \frac{1}{2}(b_2 - t)(b_1 + b_2 - P_A)$, which is positive given $P_A \leq \frac{2}{3}(b_1 + b_2 - \sqrt{(b_1 + t)(b_2 - t)})$; (ii) $\frac{\partial \pi_A}{\partial p_A}$ is decreasing with respect to p_A for $p_A \leq \frac{1}{2}P_A + \frac{1}{3}(b_1 - b_2) + \frac{2}{3}t$, and $P_A - b_2 + t < \frac{1}{2}P_A + \frac{1}{3}(b_1 - b_2) + \frac{2}{3}t$ given $P_A \leq \frac{2}{3}(b_1 + b_2 - \sqrt{(b_1 + t)(b_2 - t)})$. Therefore $\frac{\partial \pi_A}{\partial p_A} > 0$ for each $(p_A, P_A) \in M_A$ such that $P_A \leq \frac{2}{3}(b_1 + b_2 - \sqrt{(b_1 + t)(b_2 - t)})$, and in fact for each $(p_A, P_A) \in M_A$ such that $P_A < \frac{2}{3}(b_1 + b_2 + \sqrt{(b_1 + t)(b_2 - t)})$. ■

Step 3 Suppose that $(b_1, b_2) \in \mathcal{B}$ and that $b_2 \geq 1.125t$. If $(p_A, P_A) \in M_A$ is a best reply for firm A, then $P_A < \frac{2}{3}(b_1 + b_2 + \sqrt{(b_1 + t)(b_2 - t)})$.

[54] Minimizing ξ_2 over the closure of $\mathcal{B}$ yields $b_1 = t$, $b_2 = \frac{5}{4}t$, with $\xi_2(t, \frac{5}{4}t) = \frac{15}{2}t^2 > 0$.

The equation $\frac{\partial \pi_A}{\partial P_A} = 0$ is quadratic and convex in P_A. In order to satisfy the second order condition, the best reply for firm A must thus have P_A being equal to the smaller solution of this equation.

We now show that if $b_2 \geq 1.125t$ the smaller solution to $\frac{\partial \pi_A}{\partial P_A} = 0$ is strictly smaller than $\frac{2}{3}(b_1 + b_2 + \sqrt{(b_1 + t)(b_2 - t)})$.

It suffices to prove that $\frac{\partial \pi_A}{\partial P_A} < 0$ at $P_A = \frac{2}{3}(b_1 + b_2 + \sqrt{(b_1 + t)(b_2 - t)})$ for $b_2 \geq 1.125t$. At $P_A = \frac{2}{3}(b_1 + b_2 + \sqrt{(b_1 + t)(b_2 - t)})$ we find

$$\frac{\partial \pi_A}{\partial P_A} = -\frac{3}{4t^2}p_A^2 + \frac{b_1 + t}{t^2}p_A + \frac{2b_2b_1 - 7b_1^2 - b_2^2 - 20tb_1 + 2t^2 - 16t\sqrt{(b_2 - t)(b_1 + t)}}{24t^2} \equiv W(p_A)$$

and we prove that $W(p_A) < 0$ for each $p_A \in (b_2 - t, b_1 + t)$. W is maximized with respect to p_A at $p_A = \max\{\frac{2}{3}t + \frac{2}{3}b_1, b_2 - t\} = \begin{cases} b_2 - t & \text{if } b_2 > \frac{2}{3}b_1 + \frac{5}{3}t \\ \frac{2}{3}t + \frac{2}{3}b_1 & \text{if } b_2 \leq \frac{2}{3}b_1 + \frac{5}{3}t \end{cases}$.

- If $b_2 \leq \frac{2}{3}b_1 + \frac{5}{3}t$, then $b_1 \leq 5t$ needs to hold in order to satisfy $b_2 \geq b_1$, and $W(\frac{2}{3}t + \frac{2}{3}b_1) = \frac{1}{t^2}(\frac{5}{12}t^2 - \frac{1}{6}b_1t - \frac{1}{24}b_2^2 + \frac{1}{12}b_2b_1 + \frac{1}{24}b_1^2 - \frac{2}{3}t\sqrt{(b_1 + t)(b_2 - t)}) \equiv \xi_3(b_1, b_2)$, which is decreasing in b_2. For $b_1 \in (t, 1.125t]$, $\xi_3(b_1, 1.125t) = \frac{1}{t^2}(\frac{5}{12}t^2 - \frac{1}{6}b_1t - \frac{1}{24}(1.125t)^2 + \frac{1}{12}(1.125t)b_1 + \frac{1}{24}b_1^2 - \frac{2}{3}t\sqrt{(t + b_1)(1.125 - 1)t})$ is negative; for $b_1 \in (1.125t, 5t]$, $\xi_3(b_1, b_1) = \frac{1}{12t^2}(5t^2 - 2tb_1 + b_1^2 - 8t\sqrt{b_1^2 - t^2})$ is negative.

- If $b_2 > \frac{2}{3}b_1 + \frac{5}{3}t$, then we evaluate $W(b_2 - t) = \frac{1}{24t^2}(60b_2t + 26b_2b_1 - 40t^2 - 19b_2^2 - 44tb_1 - 7b_1^2 - 16t\sqrt{(b_2 - t)(b_1 + t)})$ and we prove it is negative. Precisely, we show that

  $$\xi_4(b_1, b_2) \equiv 16t\sqrt{(b_2 - t)(b_1 + t)} - 60b_2t - 26b_2b_1 + 40t^2 + 19b_2^2 + 44tb_1 + 7b_1^2 > 0$$

  We show below that $\frac{\partial \xi_4}{\partial b_2} > 0$, and it is simple to verify that $\xi_4(b_1, \frac{2}{3}b_1 + \frac{5}{3}t) = -\frac{17}{9}b_1^2 + (\frac{16}{3}\sqrt{6} + \frac{26}{9})tb_1 + (\frac{16}{3}\sqrt{6} - \frac{65}{9})t^2 > 0$ for $b_1 \in [t, 5t]$ and $\xi_4(b_1, b_1) = 8t\frac{20b_1t - 29t^2}{2\sqrt{b_1^2 - t^2 + 2b_1 - 5t}} > 0$ for $b_1 > 5t$. Regarding $\frac{\partial \xi_4}{\partial b_2} = 8t\sqrt{\frac{b_1 + t}{b_2 - t}} + 38b_2 - 26b_1 - 60t$, since $b_2 \geq b_1$ it follows that $\frac{\partial \xi_4}{\partial b_2} \geq 8t\sqrt{\frac{b_1 + t}{b_2 - t}} + 12b_1 - 60t > 0$ if $b_1 \geq 5t$. For $b_1 < 5t$ notice that $\frac{\partial^2 \xi_4}{\partial b_2^2} = 38 - \frac{4t(b_1 + t)^{1/2}}{(b_2 - t)^{3/2}}$, which is positive given $b_2 > \frac{2}{3}b_1 + \frac{5}{3}t$. Since at $b_2 = \frac{2}{3}b_1 + \frac{5}{3}t$ we have $\frac{\partial \xi_4}{\partial b_2} = 4t\sqrt{6} + \frac{10}{3}t - \frac{2}{3}b_1 > 0$ given $b_1 < 5t$, we conclude that $\frac{\partial \xi_4}{\partial b_2} > 0$. ■

The next two lemmas refer to the case of $\alpha \in [t, 1.125t)$. In this case we cannot rule out that the best reply of firm A is a mixed bundling strategy, but taking into account the behavior of firm B we can still prove that no mixed bundling NE exists.

Lemma 5. *Suppose that* $(b_1, b_2) \in \mathcal{B}$ *and* $b_2 < 1.125t$. *If* $(p_A, P_A) \in M_A$ *is a best reply for* A, *then*

$$\frac{4}{3}t \leq p_A \leq 1.589t \tag{23}$$
$$\frac{4}{3}t \leq P_A \leq 1.832t \tag{24}$$

PROOF **Step 1** Suppose that $(p_A, P_A) \in M_A$ is a best reply for A. Then

$$p_A^* \leq p_A \leq \frac{1}{3}(2t + 2b_1 + \sqrt{(b_2 - t)(b_1 + t)}) \tag{25}$$

with $p_A^* = \frac{1}{3}(b_1 + b_2 + \sqrt{(b_2 - t)^2 + (b_1 + t)(2t + b_1 - b_2)})$.
First notice that $\frac{\partial \pi_A}{\partial p_A}$ is a convex second degree polynomial in p_A and therefore only the smaller solution of $\frac{\partial \pi_A}{\partial p_A} = 0$ may be an optimum for A. In order to prove (25) we verify that $\frac{\partial \pi_A}{\partial p_A} \geq 0$ at $p_A = p_A^*$ and $\frac{\partial \pi_A}{\partial p_A} \leq 0$ at $p_A = \frac{1}{3}(2t + 2b_1 + \sqrt{(b_2 - t)(b_1 + t)})$.
At $p_A = p_A^*$ we find

$$\begin{aligned} \frac{\partial \pi_A}{\partial p_A} &= \frac{1}{2t^2}\left(2t + b_1 - b_2 - \sqrt{(b_2 - t)^2 + (b_1 + t)(2t + b_1 - b_2)}\right) P_A \\ &\quad - \frac{1}{3}(b_2 - t)\frac{3t + b_1 - 2b_2 - 2\sqrt{(b_2 - t)^2 + (b_1 + t)(2t + b_1 - b_2)}}{t^2} \end{aligned}$$

This expression is nonnegative since it is decreasing in P_A (given that $2t + b_1 - b_2 < \sqrt{(b_2 - t)^2 + (b_1 + t)(2t + b_1 - b_2)}$ in $\mathcal{B}$) and it is zero at $P_A = 2p_A^*$, the highest value for P_A given $p_A = p_A^*$.
At $p_A = \frac{1}{3}(2t + 2b_1 + \sqrt{(b_2 - t)(b_1 + t)})$ we find

$$\frac{\partial \pi_A}{\partial p_A} = -\frac{1}{2t^2}\sqrt{(b_2 - t)(b_1 + t)} P_A + \frac{1}{3}\frac{(b_2 - t)(b_1 + t) + (b_2 + b_1)\sqrt{(b_2 - t)(b_1 + t)}}{t^2}$$

This expression is negative or zero since it decreasing in P_A and it is zero at $P_A = \frac{2}{3}(b_1 + b_2 + \sqrt{(b_1 + t)(b_2 - t)})$ [recall from Step 2 in the proof of Lemma 4 that $\frac{\partial \pi_A}{\partial p_A} = 0$ implies $P_A \geq \frac{2}{3}(b_1 + b_2 + \sqrt{(b_1 + t)(b_2 - t)})$]. ■

Step 2 Suppose that $(p_A, P_A) \in M_A$ is a best reply for A. Then

$$\frac{2}{3}(b_1 + b_2 + \sqrt{(b_1 + t)(b_2 - t)}) \leq P_A \leq \frac{1}{3}(4t + 2b_1 + 2b_2 - \sqrt{2t^2 + 4b_2 t + b_2^2 + 2b_2 b_1 - b_1^2}) \tag{26}$$

We know that $\frac{2}{3}(b_1 + b_2 + \sqrt{(b_1 + t)(b_2 - t)}) \leq P_A$ from Step 2 in the proof of Lemma 4. Furthermore, from Step 3 in the proof of Lemma 4 we know that if $(p_A, P_A) \in M_A$ is a best reply, then P_A is the smaller solution to the equation $\frac{\partial \pi_A}{\partial P_A}=0$. Such a solution is maximized with respect to p_A if the expression $-6p_A^2+8(b_1+t)p_A$ in $\frac{\partial \pi_A}{\partial P_A}$ is maximized, which is at $p_A=\frac{2}{3}t+\frac{2}{3}b_1$. For this value of p_A the smaller solution of $\frac{\partial \pi_A}{\partial P_A}=0$ is $\frac{1}{3}(4t+2b_2+2b_1-\sqrt{2t^2+4b_2t+b_2^2+2b_2b_1-b_1^2})$ and therefore (26) is proved. ■

Step 3 Proof of (23)-(24).
The lower bound and the upper bound for p_A in (25) are both increasing in b_1, b_2, thus $(b_1,b_2)\in\mathcal{B}$ and $b_2<1.125t$ imply (23). The same argument applies to (26) and yields (24). ■

Lemma 6. *If $\alpha\in[t,1.125t)$, then no mixed bundling NE exists.*

Proof If a mixed bundling NE exists, when $\alpha\in[t,1.125t)$, then we can use Lemma 5 to derive upper and lower bounds for a_1 and a_2 and we obtain

$$-1.381t\le a_1\le -0.5t, \qquad 0.208t\le a_2\le 0.589t$$

If $a_1\le -t$, then $M_B=\emptyset$ and firm B necessarily plays a pure bundling strategy, thus we define $\mathcal{A}=\{(a_1,a_2): a_1\in(-t,-0.5t],\ a_2\in[0.208t,0.589t]\}$ and we prove that for any $(a_1,a_2)\in\mathcal{A}$, the best reply of firm B is either a pure bundling strategy, or $P_B=2p_B$ with $p_B>1.125t$. This implies that $b_2>0.125t$, and therefore Lemma 4 applies to rule out that a best reply for A belongs to M_A.

Step 1 Suppose that $(a_1,a_2)\in\mathcal{A}$. If $(p_B,P_B)\in M_B$ and $p_B>p_B^*\equiv\frac{1}{3}(a_1+a_2+\sqrt{(a_2-t)^2+(a_1+t)(2t+a_1-a_2)})$, then $\frac{\partial \pi_B}{\partial p_B}<0$. Thus no interior point in M_B such that $p_B>p_B^*$ is a best reply for B.
We prove that (i) $\frac{\partial \pi_B}{\partial p_B}<0$ at $p_B=p_B^*$, for any $P_B<2p_B^*$; (ii) $\frac{\partial \pi_B}{\partial p_B}<0$ at $p_B=a_1+t$ for any $P_B\in(a_1+a_2,2a_1+2t]$; (iii) $\frac{\partial \pi_B}{\partial p_B}<0$ at $p_B=P_B+t-a_2$, for $P_B\in[0,a_1+a_2)$.
At $p_B=p_B^*$ we find $\frac{\partial \pi_B}{\partial p_B}=\frac{1}{2t^2}\left(2t+a_1-a_2-\sqrt{(a_2-t)^2+(a_1+t)(2t+a_1-a_2)}\right)P_B+\frac{1}{3}(t-a_2)\frac{3t-2a_2+a_1-2\sqrt{(a_2-t)^2+(a_1+t)(2t+a_1-a_2)}}{t^2}$, which is negative or zero as it is increasing in P_B (because $2t+a_1-a_2-\sqrt{(a_2-t)^2+(a_1+t)(2t+a_1-a_2)}>0$ in $\mathcal{A}$) and is zero at $P_B=2p_B^*$. At $p_B=a_1+t$, $\frac{\partial \pi_B}{\partial p_B}=-\frac{1}{2t^2}(a_1+t)(P_B-a_1-a_2)\le 0$ since $P_B\ge a_1+a_2$. At $p_B=P_B+t-a_2$, $\frac{\partial \pi_B}{\partial p_B}=-\frac{1}{2t^2}(t-a_2)(a_1+a_2-P_B)<0$ since $P_B<a_1+a_2$. ■

Step 2 Suppose that $(a_1,a_2)\in\mathcal{A}$. If $(p_B,P_B)\in M_B$ and $p_B\le p_B^*$, then $\frac{\partial \pi_B}{\partial P_B}>0$. Thus no interior point in M_B such that $p_B\le p_B^*$ is a best reply for B.

At $P_B = 2p_B$ we have $\frac{\partial \pi_B}{\partial P_B} = \frac{3}{4t^2}p_B^2 - \frac{t+a_2}{t^2}p_B + \frac{1}{8t^2}(a_2+t)^2 + \frac{1}{8t^2}(a_1+t)(t+2a_2-a_1) \equiv U(p_B)$ and we prove below that $U(p_B) > 0$ for $p_B \leq p_B^*$. Since $\frac{\partial \pi_B}{\partial P_B}$ is decreasing with respect to P_B for $P_B \leq \frac{2}{3}(a_1+a_2+2t)$ and $2p_B^* < \frac{2}{3}(a_1+a_2+2t)$, the result below suffices to prove the claim of Step 2.

It is simple to see that U is decreasing in p_B for $p_B < \frac{2}{3}(a_2+t)$ and $p_B^* < \frac{2}{3}(a_2+t)$. Hence it suffices to show that $U(p_B^*) = \frac{1}{24t^2}(12t^2 - (2t+a_2-a_1)\left(a_1+a_2+4\sqrt{(a_2-t)^2+(a_1+t)(2t+a_1-a_2)}\right))$.

We define $\xi_5(a_1,a_2) \equiv 12t^2 - (2t+a_2-a_1)\left(a_1+a_2+4\sqrt{(a_2-t)^2+(a_1+t)(2t+a_1-a_2)}\right)$ and prove that

$$\xi_5(a_1,a_2) > 0 \text{ for each } (a_1,a_2) \in \mathcal{A} \tag{27}$$

In order to prove (27) we show below that $\frac{\partial \xi_5}{\partial a_1}$ is negative in $\mathcal{A}$. This implies that ξ_5 is decreasing with respect to a_1 and therefore, for each $(a_1,a_2) \in \mathcal{A}$, $\xi_5(a_1,a_2) \geq \xi_5(-0.5t, a_2) = \frac{53}{4}t^2 - 2a_2t - a_2^2 - (5t+2a_2)\sqrt{4a_2^2 - 10a_2t + 7t^2}$, which is positive for each $a_2 \in [0.208t, 0.589t]$.

Regarding $\frac{\partial \xi_5}{\partial a_1} = \frac{10a_1t - 14a_2t + 8a_1^2 + 6a_2^2 - 10a_1a_2}{\sqrt{(a_2-t)^2+(a_1+t)(2t+a_1-a_2)}} - 2(t-a_1)$, we prove it is negative in $\mathcal{A}$ by showing that $\xi_6(a_1,a_2) \equiv 10a_1t - 14a_2t + 8a_1^2 + 6a_2^2 - 10a_1a_2 < 0$ in $\mathcal{A}$. Since ξ_6 is a convex function and $\mathcal{A}$ is a rectangle, ξ_6 is maximized at one of the corner points of the rectangle. Given that $\xi_6(-t, 0.208t) = -2.572t^2$, $\xi_6(-t, 0.588t) = -2.278t^2$, $\xi_6(-0.502t, 0.208t) = -4.612t^2$, and $\xi_6(-0.502t, 0.588t) = -6.21t^2$ we infer that $\xi_6(a_1,a_2) < 0$ in $\mathcal{A}$. ■

Step 3 Suppose that $(a_1,a_2) \in \mathcal{A}$. Then B's best reply is either a pure bundling strategy or $p_B = \frac{2a_1t+2a_2t-a_1^2+a_2^2+4t^2}{8t-4a_1+4a_2}$ and $P_B = 2p_B$, with $p_B > 0.125t$.

From Steps 1-2 it follows that if B's best reply is in M_B, then it is such that $P_B = 2p_B$. Under this equality $\pi_B = -\frac{1}{2t^2}(2t+a_2-a_1)p_B^2 + \frac{1}{4t^2}(2a_1t+4t^2+2a_2t+a_2^2-a_1^2)p_B$ for $p_B \in [0, a_1+t]$. If $\xi_7(a_1,a_2) \equiv 4t^2+2a_1t+2a_2t-3a_1^2-a_2^2+4a_1a_2 \leq 0$, then π_B is maximized at $p_B = a_1+t$, which means that B plays a pure bundling strategy. If instead $\xi_7(a_1,a_2) > 0$, then π_B is maximized at $p_B = \frac{2a_1t+2a_2t-a_1^2+a_2^2+4t^2}{4(2t+a_2-a_1)} > 0$, included in $(0, a_1+t)$. In the latter case we show that $\frac{2a_1t+2a_2t-a_1^2+a_2^2+4t^2}{4(2t+a_2-a_1)} > 0.125t$, which is equivalent to $\xi_8(a_1,a_2) \equiv -2a_1^2+5a_1t+2a_2^2+3a_2t+6t^2 > 0$. We prove that, given (a_1,a_2) in $\mathcal{A}$, $\xi_7(a_1,a_2) > 0$ implies $\xi_8(a_1,a_2) > 0$. Precisely, $\xi_7(a_1,a_2) > 0$ requires $a_1 > -\frac{9}{10}t$ because ξ_7 is increasing in a_1 and $\xi_7(-\frac{9}{10}t, a_2) = -a_2^2 - \frac{8}{5}a_2t - \frac{23}{100}t^2 < 0$. However, also ξ_8 is increasing in a_1 and $a_1 > -\frac{9}{10}t$ implies $\xi_8(a_1,a_2) > \xi_8(-\frac{9}{10}t, a_2) = 2a_2^2 + 3a_2t - \frac{3}{25}t^2$, which is positive for any $a_2 \in [0.208t, 0.589t]$. ■