

Term structures of discount rates for risky investments

Christian Gollier¹

Toulouse School of Economics (LERNA, University of Toulouse)

October 2012

Abstract

There is still much confusion about which discount rates should be used to evaluate actions having long-lasting impacts, as in the contexts of climate change, social security reforms or large public infrastructures for example. Contrary to the existing literature that focuses on the discount rate for safe projects, this paper characterizes the term structure of discount rates for investment projects and assets with a non-zero beta. We assume that the growth rate of aggregate consumption follows a Brownian motion with uncertain parameters. We show that the term structures of the risk free discount rate and of the aggregate risk premium are respectively decreasing and increasing. Overall, the slope of term structure to be used for a specific project depends upon whether its beta is smaller or larger than half the index of relative risk aversion. We also argue that the beta of actions to mitigate climate change is relatively large around 1.3, so that the term structure of the associated real discount rates is increasing, from 1.3% for short maturities to 4.6% for extra-long ones, given the current low growth of the economy.

Keywords: asset prices, term structure, risk premium, decreasing discount rates, parametric uncertainty, CO2 beta.

JEL Codes: G11, G12, E43, Q54.

¹ The research leading to these results has received funding from the Chairs “Risk Markets and Value Creation” and “Sustainable Finance and Responsible Investments” at TSE, and from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013) Grant Agreement no. 230589.

1. Introduction

Do we do enough for the distant future? This question is implicit in many policy debates, from the fight against climate change to the speed of reduction of public deficits, investments in research and education, or the protection of the environment and of natural resources for example. The discount rate used to evaluate investments is the key determinant of our individual and collective efforts in favor of the future. Since Weitzman (1998), an intense debate has emerged among economists about whether one should use different discount rates for different time horizons t . It is however well-known that the term structure of efficient discount rates is flat if we assume that the representative agent has a constant relative risk aversion and that the growth rate of consumption is a random walk. In this benchmark specification, if a rate of 3% is efficient to discount cash flows occurring in 12 months, it is also efficient to use that rate of 3% to discount cash flows occurring in 200 years. This yields an exponentially decreasing present value of a given benefit as a function of its maturity.

Compared to this benchmark, a decreasing term structure of discount rates would bias the economic evaluation of investments towards those with more distant positive impacts. Weitzman (1998, 2001) and Newell and Pizer (2003) justified such a decreasing structure by assuming that the socially efficient discount factor should equal to the expected discount factor when the discount rate is uncertain. This is because the discount factor is an increasingly convex function of the discount rate, so that its certainty equivalent is decreasing. Gollier (2002, 2012) and Weitzman (2007) used standard consumption-based asset pricing theory to conclude that the large uncertainty associated to the distant future should induce the prudent representative agent to use lower rates to discount more distant cash flows. Under constant relative risk aversion (CRRA), the various dynamic processes that support this result include for example mean-reversion, Markov regime-switches, and parametric uncertainty on the trend of a Brownian motion. Gollier (2008) demonstrates that the positive serial correlation of the growth rate of consumption that is inherent to these stochastic processes is the driving force of the result, together with prudence. Prudence is a concept defined by Kimball (1990) to characterize the willingness to save more when the future becomes more uncertain. For growth processes with persistent shocks, aggregate uncertainty accumulates faster with respect to longer time horizons

than in a pure random walk with the same instantaneous volatility. Prudent people want to bias their investments towards those which yield more sure benefits for these horizons. Because the term structure of socially efficient discount rates is flat under a random walk, this bias is implemented by using a decreasing term structure. Persistent movements in expected growth rates of aggregate consumption are documented for the U.S. by Bansal and Yaron (2004) for example.

With the notable exception of Weitzman (2012), this recent literature focuses on rates r_{ft} at which *safe* cash flows should be discounted. In reality, most investment projects yield uncertain future costs and benefits. For marginal projects, we know that idiosyncratic risks should not be priced, because they will be washed out in diversified portfolios. In public economics, this result is usually referred to as the Arrow-Lind Theorem (Arrow and Lind (1970)), but this is a well-known feature of the consumption-based capital asset pricing model (CCAPM, Lucas (1978)). More generally, the discount rate ρ_t to be used to evaluate risky projects depends upon their beta which measures the elasticity of net cash flows to changes in aggregate consumption. Under the benchmark specification described above, the risk premium $\pi_t(\beta) = \rho_t(\beta) - r_{ft}$ of a project is proportional to β . This implies that it is enough to estimate the aggregate risk premium, i.e., the risk premium associated to a project with a unit beta, to fully operationalize the evaluation procedure of risky projects. As is well-known in the CCAPM, in the benchmark specification with CRRA and a random walk for the growth rate of consumption, the term structures of the risk free rate and of the aggregate risk premium are flat.

These two crucial properties of the benchmark specification are not robust to the introduction of persistent shocks to the growth rate of aggregate consumption. In particular, the project-specific risk premium is generally not proportional to the beta of the project. Second, the arguments listed above in favor of a decreasing term structure of safe discount rates are compatible with an increasing term structure of the risk premium associated to projects with a positive beta. If we assume that the stochastic process of the growth rate of consumption exhibits positive serial correlation, the annualized measure of aggregate risk will have an increasing term structure. Under risk aversion, the term structure of the risk premium will inherit this property.

With positively serially correlated growth rates, the project-specific discount rate $\rho_t(\beta) = r_{ft} + \pi_t(\beta)$ for positive betas is thus the sum of a prudence-driven decreasing function r_{ft} and of a risk-aversion-driven increasing function $\pi_t(\beta)$ of the time horizon t . Thus, this term structure will be decreasing if the project-specific beta is small enough, otherwise it will be increasing. The economic intuition of these results is based on the increasing accumulation of uncertainties with time. For projects with a low beta, long-termism should be favored because of prudence. For projects with a large beta, short-termism should be favored because of risk aversion.

In this paper, we consider a specific specification exhibiting positive serial correlation of the growth rate of aggregate consumption. Namely, we assume that the growth of log consumption follows a Brownian motion, but the trend and the volatility of this process are uncertain. Observe that the uncertainty on the trend of growth implies that the unconditional growth rates are positively correlated, thereby generating the phenomena described above. In the last section of the paper, we also examine growth processes that combine parametric uncertainty and mean-reversion.

Our paper provides important new insights about how public policies should be evaluated around the world. It is a common practice to use a single discount rate to evaluate public investments independent of their riskiness and time horizons. In the U.S. for example, the Office of Management and Budget (OMB) recommend to use a flat discount rate of 7% since 1992. It was argued that the *“7% is an estimate of the average before-tax rate of return to private capital in the U.S. economy”* (OMB (2003)). In 2003, the OMB also recommended the use of a discount rate of 3%, in addition to the 7% mentioned above as a sensitivity. This new rate of 3% was justified by the *“social rate of time preference. This simply means the rate at which society discounts future consumption flows to their present value. If we take the rate that the average saver uses to discount future consumption as our measure of the social rate of time preference, then the real rate of return on long-term government debt may provide a fair approximation”* (OMB, (2003)). The 3% corresponds to the average real rate of return of the relatively safe 10-year Treasury notes between 1973 and 2003. Interestingly enough, the recommended use of 3% and 7% is not differentiated by the nature of the underlying risk, and is independent of the time horizon of the project. In the another field, guidelines established by the Government Accounting

Standards Board (GASB) recommend that state and local governments discount their pension liabilities at expected returns on their plan assets, which is usually estimated around 8%, independent of their maturities.² The absence of risk-and-maturity-based price signals has potentially catastrophic consequences for the allocation of capital in the economy.³ This paper provides clear recommendations about the changes in evaluation tools that should be implemented. This reform is made at the cost of requiring evaluators to estimate the project-specific beta.

We agree that determining the beta of actions with long-lasting societal impacts is a complex matter. Let us illustrate this point in the context of climate change. It is well-known that the choice of the rate at which future damages are discounted is crucial to determine the so-called “social cost of carbon” (SCC). Sandsmark and Vennemo (2007) claim that the beta of mitigation investments is negative, so that the term structure of discount rates should be low and decreasing, thereby yielding a large SCC. They consider a simplified version of the standard integrated assessment model by Nordhaus’ DICE model (Nordhaus and Boyer (2000)). They assumed that the only source of aggregate fluctuations originates from climate change, with an uncertain climate sensitivity.⁴ Under this assumption, a large climate sensitivity yields at the same time a low consumption (due to the climate damages) and a large social benefit from early mitigation. This explains the negative beta of their model. But suppose alternatively that the climate sensitivity is known, but the growth rate of aggregate consumption is unknown. Because emissions are increasing in consumption, a larger growth rate of consumption goes with a larger concentration of CO₂. Because the damage function is assumed to be convex with the concentration of greenhouse gases, it also goes with larger damages, and with a larger societal benefit from early mitigation. This justifies a positive beta. We show in this paper that any credible calibration of a model combining the two sources of aggregate fluctuation yields a positive and large beta of mitigation. From our discussion above, this is compatible with using increasing discount rates to measure SCC.

² The European Union is currently debating about the new solvency regulation of insurance companies (Solvency 2). In the most recent consultation paper (European Insurance and Occupational Pensions Authority (2012)), it is proposed to discount safe liabilities using the yield curve up to 20-year maturities, and a real discount rate tending to 2% (“Ultimate Forward Rate”) for longer maturities.

³ In 2005, France has adopted a decreasing real discount rate from 4% to 2% for safe projects. This rule has been complemented in 2011 by an aggregate risk premium of 3% (Gollier (2011)).

⁴ The climate sensitivity is a physical parameter that measures the relationship between the concentration of greenhouse gases in the atmosphere and the average temperature of the earth.

This paper is organized as follows. In section 2, we restate the classical pricing model with constant relative risk aversion and an arithmetic Brownian motion for the logarithm of aggregate consumption. The core of the paper is in section 3, in which we explore the term structure of discount rates for projects as a function of their beta when the parameters of the growth process are uncertain. In section 4, we show how these results allow us to evaluate a broad class of risky projects with non-constant betas. An application to climate change is presented in section 5.

2. *The benchmark CRRA-Normal model*

We evaluate a marginal investment project that reduces current consumption by some sure amount ε and that generates a flow of benefits $(\varepsilon F_1, \varepsilon F_2, \dots)$ in the future, which can be uncertain seen from today. In order to evaluate the social desirability of such a project, we measure its impact on the intertemporal social welfare

$$W = u(c_0) + \sum_{t=1} e^{-\delta t} E u(c_t), \quad (1.1)$$

where u is the increasing and concave utility function of the representative agent, δ is her rate of pure preference of the present, and c_t is the consumption level of the representative agent at date t , with domain in \mathbb{R}_+ . Because ε is assumed to be small, implementing the project increases intertemporal social welfare if and only if

$$-1 + \sum_{t=1} E \left[\frac{e^{-\delta t} F_t u'(c_t)}{u'(c_0)} \right] \geq 0. \quad (2)$$

This can be rewritten as a standard NPV formula:

$$-1 + \sum_{t=1} e^{-\rho_t(F_t)t} E F_t \geq 0, \quad (3)$$

where $\rho_t(F_t)$ is the rate at which the expected cash flow occurring in t years should be discounted. This discount rate is written as follows

$$\rho_t(F_t) = \delta - \frac{1}{t} \ln \frac{EF_t u'(c_t)}{u'(c_0)EF_t} = r_{ft} + \pi_t(F_t). \quad (4)$$

It is traditional in the CCAPM to decompose the project-specific discount rate $\rho_t(F_t)$ into a risk free discount rate r_{ft} and a project-specific risk premium $\pi_t(F_t)$. From (4), we define these two components of the discount rate as follows:

$$r_{ft} = \delta - \frac{1}{t} \ln \frac{Eu'(c_t)}{u'(c_0)}, \quad (5)$$

$$\pi_t(F_t) = -\frac{1}{t} \ln \frac{EF_t u'(c_t)}{EF_t Eu'(c_t)}. \quad (6)$$

Observe that the risk premium $\pi_t(F)$ is zero when the project is safe or when its future cash flow is independent of future aggregate consumption. This implies that r_{ft} is indeed the rate at which safe projects should be discounted. The CCAPM also characterizes the project-specific risk premium $\pi_t(F_t)$. Throughout the paper, we assume that $u'(c) = c^{-\gamma}$ and that

$$F_t = \xi_t c_t^\beta \quad (7)$$

where $(\xi_t)_{t=1,2,\dots}$ is a set of random variables independent of c_t , and β is the CCAPM beta of the project.⁵ Because the idiosyncratic risk ξ_t is not priced, we hereafter identify a project $\{F_t\}$ by its β . Under this specification, asset pricing formulas (5) and (6) can be rewritten as follows:

$$r_{ft} = \delta - \frac{1}{t} \ln E e^{-\gamma \ln c_t / c_0}, \quad (8)$$

⁵ The beta of a project is defined in the CCAPM as the coefficient of regression of the return of the project on the growth of aggregate consumption. Using (7), the continuously-compounded return of the project between $t=0$ and $t=1$ is equal to $\ln F_1 = a + \beta \ln(c_1 / c_0) + \varepsilon$, with $a = \beta \ln c_0$ and $\varepsilon = \ln \xi_1$. Because $\ln(c_1 / c_0)$ is the continuously-compounded growth rate of aggregate consumption, our definition of the beta of a project is thus consistent with the CCAPM. More generally, it is easy to check in the standard CCAPM framework that an asset yielding a flow of dividends (F_1, F_2, \dots) characterized by (7) will have an equilibrium price of the form $P_t = k c_t^\beta$, so that the continuously compounded return R_t of such an asset will be such that $R_t = \alpha + \beta g_t + \varepsilon_t$, with $g_t = \ln c_{t+1} / c_t$ being the growth rate of aggregate consumption.

$$\pi_t(\beta) = -\frac{1}{t} \ln \frac{Ee^{(\beta-\gamma)\ln c_t/c_0}}{Ee^{\beta\ln c_t/c_0} Ee^{-\gamma\ln c_t/c_0}}. \quad (9)$$

In the remainder of this paper, we calibrate these equations for different specifications of the stochastic process of $\ln c_t / c_0$. The benchmark process is such that log consumption follows an arithmetic Brownian motion with trend μ and volatility σ . This implies that $\ln c_t / c_0$ is normally distributed with mean μt and variance $\sigma^2 t$. In this benchmark case, one can compute the different expectations in the above equations by using the following well-known property:

$$x \sim N(\mu, \sigma^2) \text{ and } \alpha \in \mathbb{R} \Rightarrow Ee^{\alpha x} = e^{\alpha(\mu+0.5\alpha\sigma^2)}. \quad (10)$$

The reader can then easily check that equations (8) and (9) implies that

$$r_{ft} = \delta + \gamma\mu - 0.5\gamma^2\sigma^2, \quad (11)$$

and

$$\pi_t(\beta) = \beta\gamma\sigma^2 = \beta\pi_m, \quad (12)$$

Equation (11), which is often referred to as the extended Ramsey rule, holds independent of the maturity of the cash flow. In other words, the term structure of the safe discount rate is flat in that case. Its level is determined by three elements: impatience, a wealth effect and a precautionary effect. The wealth effect comes from the observation that investing for the future in a growing economy does increase intertemporal inequality. Because of inequality aversion (which is equivalent to risk aversion under the veil of ignorance), this is desirable only if the return of the project is large enough to compensate for this adverse effect on welfare. From (11), this wealth effect is equal to the product of the expected growth of log consumption by the degree γ of concavity of the utility function which measures inequality aversion. The precautionary effect comes from the observation that consumers want to invest more for the future when this future is more uncertain (Drèze and Modigliani (1972), Kimball (1990)). This tends to reduce the discount rate. The precautionary effect is proportional to the volatility of the growth of consumption.

Equation (12) tells us that the project-specific risk premium $\pi_t(\beta)$ is just equal to the product of the project-specific beta by the CCAPM aggregate risk premium $\pi_m = \gamma\sigma^2$. Under this standard

specification, the risk premium associated to benefit F_t is independent of its maturity t . The standard calibration of these two equations yields a too large risk free rate (risk free rate puzzle (Weil (1989))) and a too small risk premium (equity premium puzzle (Mehra and Prescott (1985))) compared to historical market data. Barro (2006) showed that these two puzzles can be solved by introducing a small probability of economic catastrophes in the stochastic growth process.

Because both the risk free rate and the risk premium of the project are independent of the maturity in this benchmark specification, their sum $\rho_t(\beta) = r_{ft} + \pi_t(\beta)$ is also independent of t . The term structure of risky discount rates is flat in this case. The risky discount rate equals

$$\rho_t(\beta) = \delta + \gamma\mu + \gamma(\beta - 0.5\gamma)\sigma^2. \quad (13)$$

Notice that the risky discount rate can be either increasing or decreasing in the aggregate uncertainty measured by σ^2 depending upon whether the β of the project is larger or smaller than $\gamma/2$. Two competing effects are at play here. First, a large aggregate risk induces the representative agent to save more for the future (precautionary saving motive). That reduces the risk free discount rate. Second, ceteris paribus, a larger aggregate risk increases the project-specific risk and the associated risk premium. This risk aversion effect is proportional to the beta of the project. The two effects counterbalance each other perfectly when $\beta = \gamma/2$. When β is smaller than $\gamma/2$, the risk aversion effect (which is increasing in β) is dominated by the precautionary effect (which is increasing in γ).

3. *The CRRA-Normal model with parametric uncertainty*

Following Weitzman (2007) and Gollier (2008), we now characterize the term structure of the risk free rate and the risk premium when there is some uncertainty about the true value of some of the parameters of the model. We assume that $\ln(c_t/c_0)$ conditional to θ is normally distributed with parameters $(\mu_{t\theta}, \sigma_{t\theta}^2)$, where the mean and the variance can depend upon a parameter θ that is uncertain. This uncertainty is characterized by the distribution function G of the random variable θ . Using property (10) in equations (8) and (9), we can write that

$$\begin{aligned}
r_{ft} &= \delta - \frac{1}{t} \ln \int E \left[e^{-\gamma \ln(c_t/c_0)} \mid \theta \right] dG(\theta) \\
&= \delta - \frac{1}{t} \ln \int e^{-\gamma(\mu_\theta - 0.5\gamma\sigma_\theta^2)} dG(\theta),
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
\pi_t(\beta) &= -\frac{1}{t} \ln \frac{\int E \left[e^{(\beta-\gamma) \ln(c_t/c_0)} \mid \theta \right] dG(\theta)}{\int E \left[e^{\beta \ln(c_t/c_0)} \mid \theta \right] dG(\theta) \int E \left[e^{-\gamma \ln(c_t/c_0)} \mid \theta \right] dG(\theta)} \\
&= -\frac{1}{t} \ln \frac{\int e^{(\beta-\gamma)(\mu_\theta + 0.5(\beta-\gamma)\sigma_\theta^2)} dG(\theta)}{\int e^{\beta(\mu_\theta + 0.5\beta\sigma_\theta^2)} dG(\theta) \int e^{-\gamma(\mu_\theta - 0.5\gamma\sigma_\theta^2)} dG(\theta)}.
\end{aligned} \tag{15}$$

We apply these pricing formulas to different structures of uncertainty about the trend and the volatility of the growth of the economy.

3.1. The trend of growth is uncertain

Suppose first that $\ln(c_t / c_0)$ follows a Brownian motion with a known constant volatility σ and an unknown constant trend μ_θ . Equation (14) can then be rewritten as follows:

$$r_{ft} = \delta - 0.5\gamma^2\sigma^2 - \frac{1}{t} \ln \int e^{-\gamma\mu_\theta t} dG(\theta), \tag{16}$$

Fully differentiating equation (16) with respect to t implies that

$$\frac{\partial r_{ft}}{\partial t} = -H_t(-\gamma\mu_\theta), \tag{17}$$

where functional H_t is defined on the set of random variables such that for any $t \in \mathbb{R}^+$, we have

$$H_t(x) = \frac{\partial}{\partial t} \left[\frac{1}{t} \ln E e^{xt} \right] = \frac{1}{t} \left[\frac{E x e^{tx}}{E e^{tx}} - \frac{1}{t} \ln E e^{tx} \right]. \tag{18}$$

The following lemma gives us a useful tool to determine the slope of the term structures under scrutiny in the case in which the uncertainty is limited.

Lemma 1: Consider the family of random variables $x_k = x_0 + ky$ where $(x_0, k) \in \mathbb{R}^2$ and y is a zero-mean random variable with finite variance σ_y^2 . We have that

$$H_t(x_k) = \frac{\partial}{\partial t} \left[\frac{1}{t} \ln E e^{x_k t} \right] = 0.5k^2 \sigma_y^2 + o(k^2), \quad (19)$$

with $\lim_{k \rightarrow 0} o(k^2) / k^2 = 0$.

Proof: See the Appendix. \square

This means that function H_t is zero when its argument is risk free, and is approximately proportional to its variance. To illustrate, suppose that the uncertainty on the trend of growth is small in the sense that $\mu_\theta = \mu_0 + ky_\theta$ with k small and $\text{Var}(y_\theta) = \sigma_0^2$. Applying the lemma to equation (17) implies that

$$\frac{\partial r_{ft}}{\partial t} = -0.5\gamma^2 \text{Var}(\mu_\theta) + o(k^2). \quad (20)$$

The slope of the term structure of the risk free discount rate is negative and approximately proportional to the parametric uncertainty measured by the variance of the trend of growth.

A similar exercise with equation (15) implies that

$$\pi_t(\beta) = \beta\gamma\sigma^2 - \frac{1}{t} \ln \frac{\int e^{(\beta-\gamma)\mu_\theta t} dG(\theta)}{\int e^{\beta\mu_\theta t} dG(\theta) \int e^{-\gamma\mu_\theta t} dG(\theta)}. \quad (21)$$

and

$$\frac{\partial \pi_t(\beta)}{\partial t} = H_t(\beta\mu_\theta) + H_t(-\gamma\mu_\theta) - H_t((\beta-\gamma)\mu_\theta). \quad (22)$$

Using Lemma 1 yields

$$\begin{aligned} \frac{\partial \pi_t(\beta)}{\partial t} &= 0.5\text{Var}(\mu_\theta) [\beta^2 + \gamma^2 - (\beta-\gamma)^2] + o(k^2) \\ &= \beta\gamma\text{Var}(\mu_\theta) + o(k^2). \end{aligned} \quad (23)$$

We summarize these findings in the following proposition.

Proposition 1: *Suppose that $\ln c_t / c_0$ follows an arithmetic Brownian motion with a known volatility $\sigma \in \mathbb{R}^+$ and with a trend μ_0 that entails some small uncertainty. It implies that the term structures of the risk free discount rate and of the aggregate risk premium are respectively decreasing and increasing. Moreover, the term structure of the discount rates $\rho_t(\beta)$ is decreasing, flat or increasing depending upon β is smaller, equal or larger than 0.5γ .*

In the small, the uncertainty about the trend of growth makes the term structure of the risk free discount rate decreasing. The sensitivity of this rate to the maturity is approximately proportional to the product of $\gamma^2 / 2$ by the variance of the trend of the economy. On the contrary, the risk premium has an increasing term structure if the β of the project is positive. The sensitivity of the risk premium to the maturity is proportional to the product of $\beta\gamma$ by the variance of the trend of the economy. Combining these results, we also see that the risky discount rate is decreasing or increasing depending upon whether the beta of the project is smaller or larger than $\gamma / 2$. It is flat for $\beta = 0.5\gamma$. The intuition of this result combines the observation that the parametric uncertainty magnifies long term risks, and the observation made in the previous section that risk decreases or increases the discount rate depending upon whether β is smaller or larger than $\gamma / 2$.

Assuming a Gaussian distribution for the trend of growth yields an analytical solution to the pricing equations. Suppose that the current uncertainty about the expected change in log consumption is normally distributed with mean μ_0 and variance σ_0^2 . It implies that $\ln(c_t / c_0)$ is $N(\mu_0 t, (\sigma^2 + \sigma_0^2 t))$. Using property (10) once again in equations (16) and (21), we obtain that

$$r_{ft} = \delta + \gamma\mu_0 - 0.5\gamma^2(\sigma^2 + t\sigma_0^2), \quad (24)$$

and

$$\pi_t(\beta) = \beta\gamma(\sigma^2 + t\sigma_0^2). \quad (25)$$

Under this specification, the risky discount rate $\rho_t(\beta)$ can thus be written as

$$\rho_t(\beta) = \delta + \gamma\mu + \gamma(\beta - 0.5\gamma)(\sigma^2 + t\sigma_0^2). \quad (26)$$

The term structure of the risk free discount rate is linearly decreasing, whereas the term structure of the risk premium is linearly increasing when the beta is positive. The intuition of these two results is based on the observation that the uncertainty related to the growth trend of the economy magnifies the long term macroeconomic uncertainty, and the corresponding uncertainty about the benefits of the project. Equation (26) shows that the term structure of risky discount rate is linearly decreasing or increasing if the project's β is smaller or larger than 0.5γ , as is the case for small risk on μ_θ (Proposition 1).

We hereafter relax the above specification by allowing for any distribution of the trend μ_θ . When the uncertainty on the trend is not small or normally distributed, an additional complexity arises in the evaluation strategy. Indeed, a striking observation that can be made from equations (15) and (21) is that the risk premium of a project is in general not proportional to its beta: $\pi_i(\beta) \neq \beta\pi_i(1)$. It implies that computing the aggregate risk premium for $\beta=1$ will not be helpful to determine the risk premium for another project with $\beta \neq 1$.

The next proposition describes the shape of the term structure of the risky discount rates for different betas. It relies on the following equation which combines equations (16) and (21) :

$$\rho_i(\beta) = \delta + \gamma\bar{\mu} + \gamma\sigma^2(\beta - \gamma/2) - \frac{1}{t} \ln \frac{\int e^{(\beta-\gamma)(\mu_\theta - \bar{\mu})t} dG(\theta)}{\int e^{\beta(\mu_\theta - \bar{\mu})t} dG(\theta)}, \quad (27)$$

with $\bar{\mu} = E\mu_\theta$. There are two simple cases that are worthy to examine. When $\beta=0$, $\rho_i(\beta=0) = r_{ft}$ is obviously decreasing in t , since $-(t^{-1}) \ln E \exp(-\gamma\mu_\theta t)$ is the certainty equivalent of $\gamma\mu_\theta$ for a concave exponential utility function with degree of concavity t . Similarly, when $\beta = \gamma$, $\rho_i(\beta = \gamma)$ is obviously increasing in t , since $(t^{-1}) \ln E \exp(\gamma\mu_\theta t)$ is the certainty equivalent of $\gamma\mu_\theta$ for a convex exponential utility function with degree of convexity t . The other cases are described in Proposition 2. It relies on the full differentiation of equation (27) with respect to t , which yields

$$\frac{\partial \rho_i(\beta)}{\partial t} = H_t(\beta(\mu_\theta - \bar{\mu})) - H_t((\beta - \gamma)(\mu_\theta - \bar{\mu})). \quad (28)$$

Proposition 2: Suppose that $\ln c_t / c_0$ follows an arithmetic Brownian motion with a known volatility $\sigma \in \mathbb{R}^+$ and with an unknown trend μ_θ with support $[\mu_{\min}, \mu_{\max}]$. When the distribution of μ_θ is symmetric, the term structure of the discount rate $\rho_t(\beta = 0.5\gamma)$ is flat at $\delta + \gamma\bar{\mu}$. It is increasing when β is smaller than 0.5γ , and it is decreasing when β is in $[0.5\gamma, \gamma]$.

Proof: See the Appendix. \square

Thus, we recover essentially the same characterization for the slope of the term structure if we replace the assumption of a small degree of uncertainty contained in Proposition 1 by the assumption that this uncertainty is just symmetric around the mean. In the next proposition, we characterize the asymptotic properties of the term structure of discount rates.

Proposition 3: Suppose that $\ln c_t / c_0$ follows an arithmetic Brownian motion with a known volatility $\sigma \in \mathbb{R}^+$ and with an unknown trend μ_θ with support $[\mu_{\min}, \mu_{\max}]$. The efficient discount rates $\rho_t(\beta)$ have the following properties:

- For short horizons, the discount rate tends to $\rho_0(\beta) = \delta + \gamma\sigma^2(\beta - 0.5\gamma) + \gamma\bar{\mu}$.
- For long horizons, the discount rate tends to

$$\rho_\infty(\beta) = \begin{cases} \delta + \gamma\sigma^2(\beta - 0.5\gamma) + \gamma\mu_{\min} & \text{if } \beta \leq 0 \\ \delta + \gamma\sigma^2(\beta - 0.5\gamma) + (\gamma - \beta)\mu_{\min} + \beta\mu_{\max} & \text{if } 0 < \beta \leq \gamma \\ \delta + \gamma\sigma^2(\beta - 0.5\gamma) + \gamma\mu_{\max} & \text{if } \beta > \gamma \end{cases} \quad (29)$$

Proof: See the Appendix. \square

This proposition provides some interesting insights about the risky discount rates. For short horizons, the ambiguity affecting the trend of the economy has no effect on the risky discount rate. Only the expected trend $\bar{\mu}$ of the economy matters to measure the wealth effect in the short term. On the contrary, for distant futures, the ambiguity affecting the trend is crucial for the determination of the discount rate. The long term wealth effect is equal to the product of γ by a growth rate of consumption belonging to its support $[\mu_{\min}, \mu_{\max}]$. Its selection depends here upon the beta of the project. When β is negative, the wealth effect should be computed on the basis of

the smallest possible growth rate μ_{\min} of the economy. On the contrary, when β is larger than γ , the wealth effect should be computed on the basis of the largest possible rate μ_{\max} . When the beta of the project is positive but smaller than γ , the selected growth rate is a weighted average of μ_{\min} and μ_{\max} , with weights $(\gamma - \beta) / \gamma$ and β / γ respectively.

In Figure 1, we illustrate some of the above findings through the following numerical example. We assume that $\gamma = 2$, $\sigma = 4\%$ and μ_{θ} is uniformly distributed on interval $[0\%, 3\%]$. The term structure is flat for $\beta = \gamma / 2 = 1$. The discrimination of the discount rate for different betas is increasing in the maturity of the cash flows. This numerical example also illustrates the property that project-specific risk premia are in general not proportional to the project-specific beta. For example, consider a time horizon of 200 years. For this maturity, the risk premia associated to $\beta = 1$ and $\beta = 0.5$ are respectively equal to $\pi_{200}(1) = 2.08\%$ and $\pi_{200}(0.5) = 0.94\% < 0.5\pi_{200}(1)$.

Proposition 3 also tells us that the condition of a symmetric distribution for μ_{θ} in Proposition 2 cannot be relaxed. Indeed, Proposition 3 implies that $\rho_0(\beta = \gamma / 2)$ and $\rho_{\infty}(\beta = \gamma / 2)$ are equal only if $\bar{\mu}$ and $(\mu_{\min} + \mu_{\max}) / 2$ coincide. Most asymmetric distributions will not satisfy this condition, which implies that the constancy of $\rho_t(\beta = \gamma / 2)$ with respect to t will be violated.

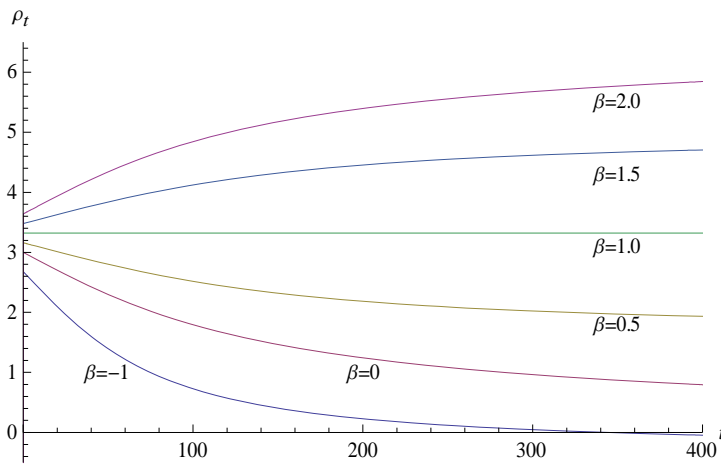


Figure 1: Term structure of the risky discount rate (in %) for different betas. We assume that

$$\gamma = 2, \sigma = 4\% \text{ and } \mu_\theta \text{ is uniformly distributed on interval } [0\%, 3\%].$$

3.2. The volatility of growth is uncertain

Let us now turn to the case in which the trend is known, but the volatility is ambiguous. Weitzman (2007) examined this question by assuming that the σ_θ^2 has an inverted Gamma distribution. This implies that $\ln(c_t / c_0)$ is a Student's t-distribution rather than a normal, yielding fat tails, a safe discount rate of $-\infty$ and a market risk premium of $+\infty$. Let us reexamine this question without specifying the distribution of σ_θ^2 , apart from assuming that its variance $\sigma_\theta^2 = \sigma^2 + ky_\theta$ is such that k is small. Equation (14) implies that

$$r_{ft} = \delta + \gamma\mu - \frac{1}{t} \ln \int e^{0.5\gamma^2\sigma_\theta^2 t} dG(\theta). \quad (30)$$

Lemma 1 implies that

$$\frac{\partial r_{ft}}{\partial t} = -H(0.5\gamma^2\sigma_\theta^2) = -\frac{\gamma^4}{8} \text{Var}(\sigma_\theta^2) + o(k^2). \quad (31)$$

Similarly, equation (15) implies that

$$\pi_t(\beta) = -\frac{1}{t} \ln \frac{\int e^{0.5(\beta-\gamma)^2\sigma_\theta^2 t} dG(\theta)}{\int e^{0.5\beta^2\sigma_\theta^2 t} dG(\theta) \int e^{0.5\gamma^2\sigma_\theta^2 t} dG(\theta)}. \quad (32)$$

Lemma 1 implies in turn that

$$\begin{aligned} \frac{\partial \pi_t(\beta)}{\partial t} &= H(0.5\beta^2\sigma_\theta^2) + H(0.5\gamma^2\sigma_\theta^2) - H(0.5(\beta-\gamma)^2\sigma_\theta^2) \\ &= \frac{1}{8} [\beta^4 + \gamma^4 - (\beta-\gamma)^4] \text{Var}(\sigma_\theta^2) + o(k^2) \\ &= \frac{\beta\gamma}{2} \left[\beta^2 - \frac{3}{2}\beta\gamma + \gamma^2 \right] \text{Var}(\sigma_\theta^2) + o(k^2). \end{aligned} \quad (33)$$

The bracketed term in the right-hand side of this equation is positive, which implies that the term structure of the risk premium is upward-sloping for all $\beta > 0$ if the uncertainty on the volatility

of growth is small enough. Finally, combining these two results implies that the discount rate $\rho_i(\beta)$ to be used for this project is such that

$$\begin{aligned}\frac{\partial \rho_i(\beta)}{\partial t} &= \frac{1}{8} [\beta^4 - (\beta - \gamma)^4] \text{Var}(\sigma_\theta^2) + o(k^2) \\ &= \frac{\gamma(2\beta - \gamma)(\beta^2 + (\beta - \gamma)^2)}{8} \text{Var}(\sigma_\theta^2) + o(k^2).\end{aligned}\tag{34}$$

Thus, when the uncertainty on the volatility is small enough, the term structure of the discount rate to be used for risky projects is downward-sloping if and only if β is smaller than $\gamma/2$. This result is identical to what we obtained earlier in this section when the uncertainty is about the trend of the economy. The following proposition summarizes the findings of this section.

Proposition 4: *Suppose that $\ln c_t / c_0$ follows an arithmetic Brownian motion with a known trend μ and with a variance σ_θ^2 that entails some small uncertainty. It implies that the term structures of the risk free discount rate and of the risk premium for $\beta > 0$ are respectively decreasing and increasing. Moreover, the term structure of the discount rates $\rho_i(\beta)$ is decreasing, flat or increasing depending upon β is smaller, equal or larger than 0.5γ .*

In other words, when the parametric uncertainty is small enough, the qualitative properties of the term structures of discount rates are independent of whether the uncertainty is about the trend of growth or about its volatility.

3.3. The trend and the volatility of growth are uncertain

The above propositions suggest that that the term structure of $\rho_i(\beta)$ is decreasing under small parametric uncertainty if and only if β is smaller than $\gamma/2$. This is not true in general, as shown in this section. To do this, let us suppose that μ_θ and σ_θ^2 be statistically dependent. From (14) and (15), we obtain that

$$\frac{\partial \rho_i(\beta)}{\partial t} = H_i(\beta(\mu_\theta + 0.5\beta\sigma_\theta^2)) - H_i((\beta - \gamma)(\mu_\theta + 0.5(\beta - \gamma)\sigma_\theta^2)).\tag{35}$$

Using Lemma 1, this implies that

$$\begin{aligned}
2 \frac{\partial \rho_t(\beta)}{\partial t} &\simeq \text{Var}(\beta(\mu_\theta + 0.5\beta\sigma_\theta^2)) - \text{Var}((\beta - \gamma)(\mu_\theta + 0.5(\beta - \gamma)\sigma_\theta^2)) \\
&= [\beta^2 - (\beta - \gamma)^2] \text{Var}(\mu_\theta) + 0.25[\beta^4 - (\beta - \gamma)^4] \text{Var}(\sigma_\theta^2) \\
&\quad + [\beta^3 - (\beta - \gamma)^3] \text{cov}(\mu_\theta, \sigma_\theta^2).
\end{aligned} \tag{36}$$

In the special case with $\beta = \gamma / 2$, this implies in turn that

$$\frac{\partial \rho_t(\beta)}{\partial t} \simeq \frac{\gamma^3}{8} \text{cov}(\mu_\theta, \sigma_\theta^2). \tag{37}$$

Thus, when $\beta = \gamma / 2$, the signs of the slope of the term structure of the discount rate and of $\text{cov}(\mu_\theta, \sigma_\theta^2)$ coincide. To illustrate this result, let us consider the following simple numerical example. Suppose that $\delta = 0$, $\gamma = 2$ and that there are two possible states of nature. In the first state, $\mu_1 = 0\%$ and $\sigma_1 = 1\%$. In the second state, $\mu_2 = 3\%$ and $\sigma_2 = 7\%$, so that the trend and the volatility of growth are positively correlated. Under this calibration of the model, the discount rate for short-term cash flows with $\beta = 1 = \gamma / 2$ is $\rho_0(1) = 3\%$, and the discount rate for distant cash flows is $\rho_\infty(1) = 3.24\% > \rho_0(1)$.

4. Term structures with parametric uncertainty and mean-reversion

In the benchmark specification with a CRRA utility function and a random walk for the growth rate of consumption, the term structures of discount rates are flat and constant through time. In the specification examined in Section 3 with some parametric uncertainty on the underlying random walk of the growth rate, they are monotone and move smoothly through time due to the revision of beliefs about the true values of the uncertain parameters. But these processes ignore the cyclicity of the economic activity. The introduction of predictable changes in the trend of growth introduces a new ingredient to the evaluation of investments. When expectations are diminishing, the discount rate associated to short horizons should be reduced to bias investment decisions toward projects that dampen the forthcoming recession. Long termism is a luxury that should be favored only in periods of economic prosperity with low expectations for the future. More generally, when expectations are cyclical, it is important to frequently adapt the price signals contained in the term structure of discount rates to the moving macroeconomic expectations. From this theoretical result, it is clearly inefficient to maintain the U.S. official discount rate unchanged since 1992.

In this section, we propose a simple model in which the economic growth is cyclical, with some uncertainty about the parameter governing this process. Following Bansal and Yaron (2004) for example, the change in log consumption follows an auto-regressive process:

$$\begin{aligned}\ln c_{t+1} / c_t &= x_t \\ x_t &= \mu_\theta + y_t + \varepsilon_{xt}, \\ y_t &= \phi y_{t-1} + \varepsilon_{yt},\end{aligned}\tag{38}$$

for some initial (potentially ambiguous) state characterized by $y_{-1\theta}$, where ε_{xt} and ε_{yt} are independent and serially independent with mean zero and variance σ_x^2 and σ_y^2 , respectively. Parameter ϕ , which is between 0 and 1, represents the degree of persistence in the expected growth rate process. When ϕ is zero, then the model returns to a pure random walk as in Section 3. We hereafter allow the trend of growth μ_θ to be uncertain.⁶ By forward induction of (38), it follows that:

$$\ln c_t / c_0 = \mu_\theta t + y_{-1\theta} \phi \frac{1 - \phi^t}{1 - \phi} + \sum_{\tau=0}^{t-1} \frac{1 - \phi^{t-\tau}}{1 - \phi} \varepsilon_{y\tau} + \sum_{\tau=0}^{t-1} \varepsilon_{x\tau}.\tag{39}$$

It implies that, conditional to θ , $\ln c_t - \ln c_0$ is normally distributed with annualized variance

$$t^{-1} \text{Var}(\ln c_t / c_0) = \frac{\sigma_y^2}{(1 - \phi)^2} \left[1 - 2\phi \frac{\phi^t - 1}{t(\phi - 1)} + \phi^2 \frac{\phi^{2t} - 1}{t(\phi^2 - 1)} \right] + \sigma_x^2.\tag{40}$$

We consider as before an investment project $\{F_t\}$ with a constant β , so that equation (7) holds.

The pricing formula (4) can therefore be rewritten as:

$$\begin{aligned}\rho_t(\beta) &= \delta - \frac{1}{t} \ln \frac{E e^{(\beta - \gamma) \ln c_t / c_0}}{E e^{\beta \ln c_t / c_0}} \\ &= \delta + \gamma y_{-1} \phi \frac{1 - \phi^t}{t(1 - \phi)} + \gamma(\beta - 0.5\gamma) \left(\frac{\sigma_y^2}{(1 - \phi)^2} \left[1 - 2\phi \frac{\phi^t - 1}{t(\phi - 1)} + \phi^2 \frac{\phi^{2t} - 1}{t(\phi^2 - 1)} \right] + \sigma_x^2 \right) \\ &\quad - \frac{1}{t} \ln \frac{E e^{(\beta - \gamma) \mu_\theta t}}{E e^{\beta \mu_\theta t}}\end{aligned}\tag{41}$$

⁶ A more general model entails a time-varying volatility of growth as in Bansal and Yaron (2004). Mean-reversion in volatility is useful to explain the cyclicity of the market risk premium.

Bansal and Yaron (2004) consider the following calibration of the model, using annual growth data for the United States over the period 1929-1998. Taking the month as the time unit, they obtained, $\mu = 0.0015$, $\sigma_x = 0.0078$, $\sigma_y = 0.00034$, and $\phi = 0.979$. Using this ϕ yields a half-life for macroeconomic shocks of 32 months. Let us assume that $\delta = 0$, and let us introduce some uncertainty about the historical trend of growth from the sure $\mu = 0.0015$ to the uncertain context with two equally likely trends $\mu_1 = 0.0005$ and $\mu_2 = 0.0025$. In figures 2 to 4, we draw the term structures of discount rates for three different positions y_{-1} in the business cycle. In Figure 2, the expected annual growth rate of the economy is 0.6% per year, well below its unconditional expectation of 1.8%. In this recession phase, the short term discount rate is a low 1%, but the expectation of a recovery makes the term structure steeply increasing for low maturities. For betas below unity, the term structure is non-monotone because of the fact that for very distant maturities, the effect of parametric uncertainty eventually dominates.

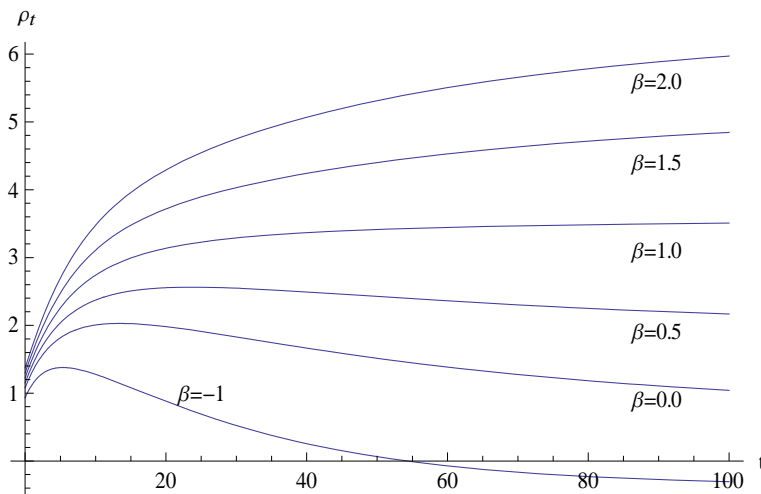


Figure 2: The discount rate (in % per year) as a function of the maturity (in years) in recession for different betas. Equation (41) is calibrated with $\delta = 0$, $\gamma = 2$, $\sigma_x = 0.0078$, $\sigma_y = 0.00034$, $\phi = 0.979$, two equally likely trends $\mu_1 = 0.0005$ and $\mu_2 = 0.0025$, and $y_{-1} = -0.001$.

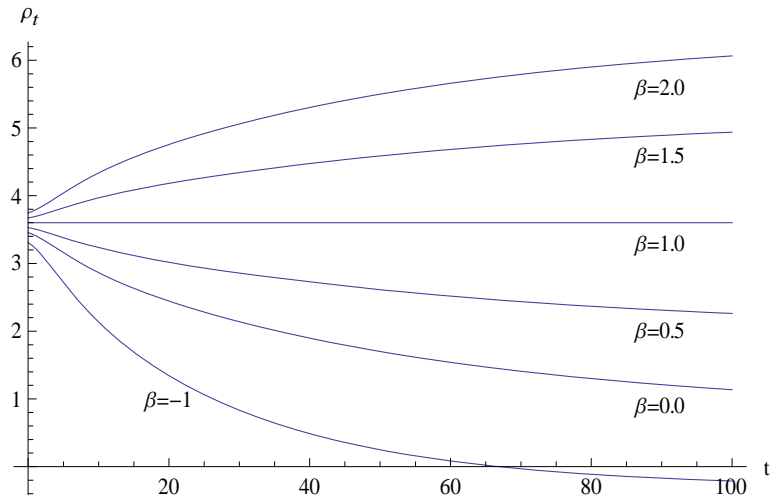


Figure 3: The discount rate (in % per year) as a function of the maturity (in years) in mid-cycle for different betas. Equation (41) is calibrated as in Figure 2, except for $y_{-1} = 0$.

In Figure 3, the expected instantaneous growth rate is at its unconditional expectation of 1.8%. It should be noticed that the term structure is flat for $\beta = 0.5\gamma = 1$ in this case with a symmetric distribution for μ_θ , as can be inferred from equation (41). Finally, in Figure 4, the expected instantaneous growth rate is 1.2% per year above its unconditional expectation. In this expansionary phase of the cycle, the short term discount rate is large at around 6%, but is steeply decreasing for short maturities because of the diminishing expectations.

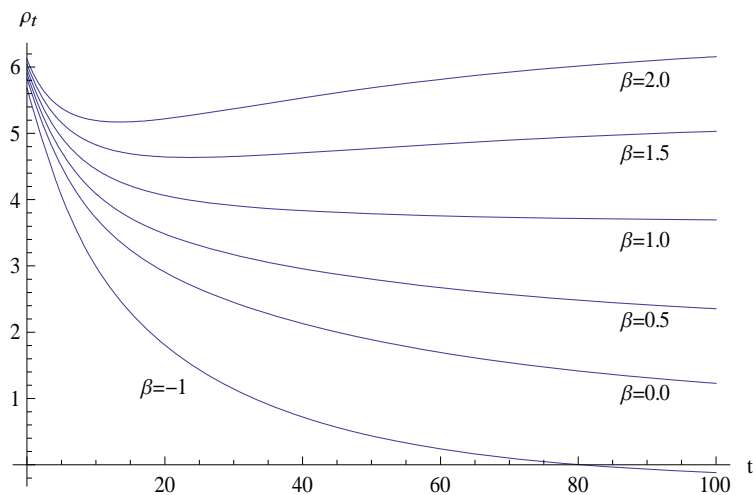


Figure 4: The discount rate (in % per year) as a function of the maturity (in years) in expansion for different betas. Equation (41) is calibrated as in Figure 2, except for $y_{-1} = 0.001$.

One can also examine a model in which the current state variable y_{-1} is uncertain. It is easy to generalize equation (41) to examine this ambiguous context. We obtain the following pricing formula:

$$\rho_t(\beta) = \delta + \gamma(\beta - 0.5\gamma) \left(\frac{\sigma_y^2}{(1-\phi)^2} \left[1 - 2\phi \frac{\phi^t - 1}{t(\phi - 1)} + \phi^2 \frac{\phi^{2t} - 1}{t(\phi^2 - 1)} \right] + \sigma_x^2 \right) - \frac{1}{t} \ln \frac{Ee^{(\beta-\gamma)\left(\mu_{\theta^t+y_{-1\theta}}\frac{1-\phi^t}{1-\phi}\right)}}{Ee^{\beta\left(\mu_{\theta^t+y_{-1\theta}}\frac{1-\phi^t}{1-\phi}\right)}} \quad (42)$$

Observe again that when $\beta = 0.5\gamma$, the term structure of discount rates is flat if both μ_{θ} and $y_{-1\theta}$ have a symmetric distribution function. We also observe that the ambiguity on $y_{-1\theta}$ plays a role similar to the ambiguity on μ_{θ} to shape the term structure. Our numerical simulations (available upon request) show that the hidden nature of the state variable does not modify the general characteristics of the term structures described above.

5. Pricing projects with a non-constant beta

Specification (7) is critical for our results. When β is positive, implementing the project raises the risk on aggregate consumption. When β is negative, the project has an insurance component since it pays more on average in the worse macroeconomic scenarios. Although specification (7) is restrictive, examining the pricing of a project satisfying it opens the path to examining the pricing of a much larger class of projects of the form

$$F_t = \sum_{i=1}^n \alpha_{it} F_{it}, \quad \text{with } F_{it} = \xi_{it} c_t^{\beta_i}, \quad (43)$$

where random variables ξ_{it} are independent of c_t . This project can be interpreted as a portfolio of n different projects, project i having a constant $\beta = \beta_i$, $i=1, \dots, n$. Of course, the resulting “portfolio project” has a non-constant beta. By a standard arbitrage argument, the value of this portfolio is the sum of the values of its β -specific components. We can thus rely on our results in this paper for the evaluation of projects with a non-constant beta. If $\rho_i(\beta_i)$ is the efficient discount rate to evaluate the β_i – component of the project, the global value of the project will be equal to

$$\sum_{i=1}^n e^{-\rho_i(\beta_i)t} \alpha_{it} E F_{it}, \quad (44)$$

where we assumed without loss of generality that $E \xi_{it} = 1$. In other words, the discount factors – rather than the discount rates – must be averaged to determine the discount factor to be used to evaluate the cash flows of the global portfolio. Weitzman (2012) examines the case in which the project under scrutiny is a portfolio of a risk free asset ($\beta_1 = 0$) and of a risky project with a unit beta ($\beta_2 = 1$).

6. *The beta of mitigation projects*

The purpose of this section is to heuristically derive a crude numerical estimate for term structure of discount rates to be used for the evaluation of climatic policies. To do this, we need to answer the following often overlooked question: What is the beta of investments whose main objective is to abate emissions of greenhouse gases? Let us consider a simple two-date version of the DICE model of Nordhaus (2008) and Nordhaus and Boyer (2000):

$$T = \omega_1 E \quad (45)$$

$$E = \omega_2 Y - I_0 \quad (46)$$

$$D = \theta_1 T^{\theta_2} \quad (47)$$

$$Q = e^{-D} Y \quad (48)$$

$$C = \alpha Q \quad (49)$$

All parameters of the model are assumed to be nonnegative. T is the increase in temperature and E is the emission of greenhouse gases from date 0 to date 1. It is assumed in equation (45) that the increase in temperature is proportional to the emission of these gases. By equation (46), emissions are proportional to the pre-damage production level Y , but they can be reduced by investing I_0 in a green technology at date 0. In equation (47), we assume that the damage D is an increasing power function of the increase in temperature. Equation (48) defines damage D as the logarithm of the ratio Y/Q of pre-damage and post-damage production levels.⁷ We hereafter refer to D as the relative damage. Finally, consumption C is proportional to the post-damage production Q . This model yields the following reduced form:

$$C = \alpha Y \exp\left[-\theta_1 (\omega_1 (\omega_2 Y - I_0))^{\theta_2}\right]. \quad (50)$$

We consider the beta of a green investment I_0 . Such an investment has the benefit to raise consumption in the future by

$$\frac{\partial C}{\partial I_0} = \alpha Y \theta_1 \theta_2 \xi^{\theta_2} (\omega_2 Y - I_0)^{\theta_2 - 1} \exp\left[-\theta_1 (\omega_1 (\omega_2 Y - I_0))^{\theta_2}\right]. \quad (51)$$

This is the future cash flow F of the investment. We assume that this investment is marginal, so that our model can be rewritten as:

$$\begin{cases} C = \alpha Y \exp\left[-\theta_1^* Y^{\theta_2}\right] \\ F = \frac{\theta_1^* \theta_2 \alpha}{\omega_2} Y^{\theta_2} \exp\left[-\theta_1^* Y^{\theta_2}\right], \end{cases} \quad (52)$$

where $\theta_1^* = \theta_1 \omega_1^{\theta_2} \omega_2^{\theta_2}$ can be interpreted as a synthetic climate-sensitivity parameter.

A critical parameter for this model is θ_2 . When θ_2 is equal to unity, the relative damage is just proportional to the change in temperature and in concentration of greenhouse gases. The absolute damage $Y - Q$ is thus convex in Y in that case. When θ_2 is larger than unity, the relative damage

⁷ Equation (48) is traditionally expressed as $Q_1 = (1 - D_1)Y_1$. However, for high temperatures, this specification could lead to a negative after-damage production.

is itself convex, thereby bringing even more convexity to the absolute damage as a function of Y . Let us first assume that $\theta_2 = 1$. In that case, we can derive from system (52) that

$$F = \theta_1 \omega_1 C. \quad (53)$$

Let us further assume that the only source of uncertainty is about the growth of pre-damage production. This implies that $\theta_1 \omega_1$ is a constant. We can thus conclude in this case that the green investment project under scrutiny in this section satisfies condition (7) with $\beta = 1$. This proves the following proposition.

Proposition 5: *Consider the simplified integrated assessment model (45)-(49) with $\theta_2 = 1$ and without uncertainty about the climatic parameter $\xi = \theta_1 \omega_1$. Under this specification, any project whose benefits are to reduce emissions of greenhouse gases has a constant beta equaling unity.*

The intuition of a positive beta in this model is as follows. When economic growth is high, more greenhouse gases are emitted in the atmosphere and the benefits of mitigation are large. Consumption and benefits covary positively in this model. However, this simple result raises two difficulties. First, although all experts in the field recognize the scarcity of evidence to infer θ_2 , most of them agree that the relation $D = f(T)$ should be convex, yielding $\theta_2 > 1$. Nordhaus and Boyer (2000) used $\theta_2 = 2$,⁸ whereas Cline (1992) used $\theta_2 = 1.3$. The Monte-Carlo simulations of the PAGE model used in the Stern (2007) Review draw θ_2 from an asymmetric triangular probability density function with support in $[1,3]$, giving a mean of about 1.8 (See Dietz, Hope and Patmore, 2007). Although there is no consensus on the value of this parameter, this suggests a more consensual beta somewhere between 1 and 2. Compared to the result in Proposition 5, a larger θ_2 tends to increase the benefits of reducing emissions in good states (large Y), and to reduce them in bad states (low Y). Intuitively, this should raise the beta of green projects above unity. To see this observe that a local estimation of the beta from (7) can be obtained by fully differentiating system (52) with respect to Y . We obtained

⁸ Nordhaus (2007) used a quadratic function, yielding a similar degree of convexity of the damage function in the relevant domain of increases in concentration.

$$\beta \approx \frac{d \ln F / dY}{d \ln C / dY} = \theta_2 \frac{1 - \theta_1^* Y^{\theta_2}}{1 - \theta_1^* Y} \quad (54)$$

which is close to θ_2 when θ_2 is close to unity. Observe that the beta of the project is not constant when θ_2 is not equal to unity. In other words, the expected benefit function conditional to C is not a power function of C .

Second, when $\xi = \theta_1 \omega_1$ is random, it will in general be correlated to C , as shown by the first equation in system (52). In that case, equation (53) cannot anymore be interpreted as describing a project with a unit beta. To illustrate this point, consider the extreme case where economic growth Y is certain, together with ω_1 , ω_2 and θ_2 , but the climate sensitivity parameter θ_1 is uncertain. In that case, eliminating θ_1 from system (52) yields

$$F = \frac{\theta_2}{\omega_2} \frac{C}{Y} \ln \frac{\alpha Y}{C}. \quad (55)$$

In this case, cash flow F is a deterministic function of C . Because it is not a power function, the beta of green projects is not constant in this specification. We can approximate it through the following formula:

$$\beta \approx \frac{d \ln F / d\theta_1}{d \ln C / d\theta_1} = 1 - \frac{1}{D}, \quad (56)$$

with $D = \theta_1^* Y^{\theta_2}$. This typically yields a negative beta, which is large in absolute value. Indeed, if we assume a range of damages between 5% and 20% of the aggregate production, we obtain a beta in the range between -4 and -19. In this story based on the uncertain climate sensitivity, a large sensitivity yields at the same time large damages, low consumption, and large benefits of mitigation. This explains the negative beta obtained under this specification. This story is similar to the one proposed by Sandsmark and Vennemo (2007) who claim that the beta of mitigation investments should be negative.⁹ Their argument is based on the climate variability as being the only source of fluctuation in the economy.

⁹ They obtain $\beta = -0.004$, which is much closer to zero than what we obtain here. However, notice that these authors consider another definition of the beta, which is equal to the ratio of the covariance of (C_I, F_I) to the variance of C_I . In our model, the beta is equal to the ratio of the covariance of $(\ln C_I, \ln F_I)$ to the variance of $\ln C_I$.

To sum up, the result presented in Proposition 5 suffers from two major deficiencies: The selected θ_2 is too small, and it does not recognize that there is still much uncertainty about the sensitivity of the climate to an increase in concentration of greenhouse gases in the atmosphere. The difficulty is that improving the model to allow for $\theta_2 \neq 1$ or for an uncertain climate sensitivity implies that the benefit F cannot be written anymore as a power function of C as in equation (7), or as a sum of power functions of C as in equation (43).

We can conclude from this discussion that the beta of investments whose main benefits are a reduction of emissions of greenhouse gases is non-constant. Its average level is determined by the relative intensity of two sources of uncertainty, the one coming from the future economic prosperity, and the one due to the unknown intensity of the climatic problem. We believe that the economic source of variability has an order of magnitude larger than the climatic source of variability. When the annual growth rate of the economy varies between 0% and 3%, aggregate consumption in 100 years is between 0% and 1800% larger than today. This should be compared to climate damages for this time horizon which are usually estimated between 0% and 5% (see for example the Stern Review, 2007). To make this argument more concrete, let us consider the calibration of the simple above model as described in Table 1.

<i>Variable</i>	<i>Value</i>	<i>Remark</i>
t	50 years	Time horizon between dates 0 and 1.
$Y = e^{\sum_{i=1}^t x_i}$	$x_i \text{ iid } \sim N(\mu, \sigma^2)$ $\mu = 1.5\%, \sigma = 4\%$	Y_0 is normalized to unity. The growth rate of production follows a normal random walk.
ω_2	1	Normalization
ω_1	0.45	This implies that the expected increase in temperature in the next 50 years equals $\xi EY = 1^\circ C$.
θ_2	1.5	Center of the “consensus interval” [1,2].
θ_1	$\sim U[0\%, 5\%]$	This means that the damage at the average temperature increase of $1^\circ C$ is uniformly distributed on $[0\%, 5\%]$ of pre-damage production.
α	0.75	Consumption equals 75% of post-damage production.

Table 1: Calibration of the two-date IAM model

Using the Monte-Carlo method, we generated 50 000 independent random selections of the pair (Y, θ_1) and the corresponding outcomes in terms of aggregate consumption C and benefits of mitigation F , as defined by system (52). Using these data, we regressed $\ln F$ on $\ln C$. The OLS estimation of the beta equals $\hat{\beta} = 1.32$ with a standard deviation of 0.016.¹⁰

In Figure 5, we draw the term structures of discount rates prevailing for $\beta = 1.32$ in three different phases of the macroeconomic cycle under the calibration used in the previous section. The discount rate to be used for super-long maturities is around 4.6%, whereas the short-term discount rate fluctuates along the business cycle from around 1.3% when the expected instantaneous trend is 1.2% per annum below its historical mean, to around 6% when the expected instantaneous trend is 1.2% above its historical mean.

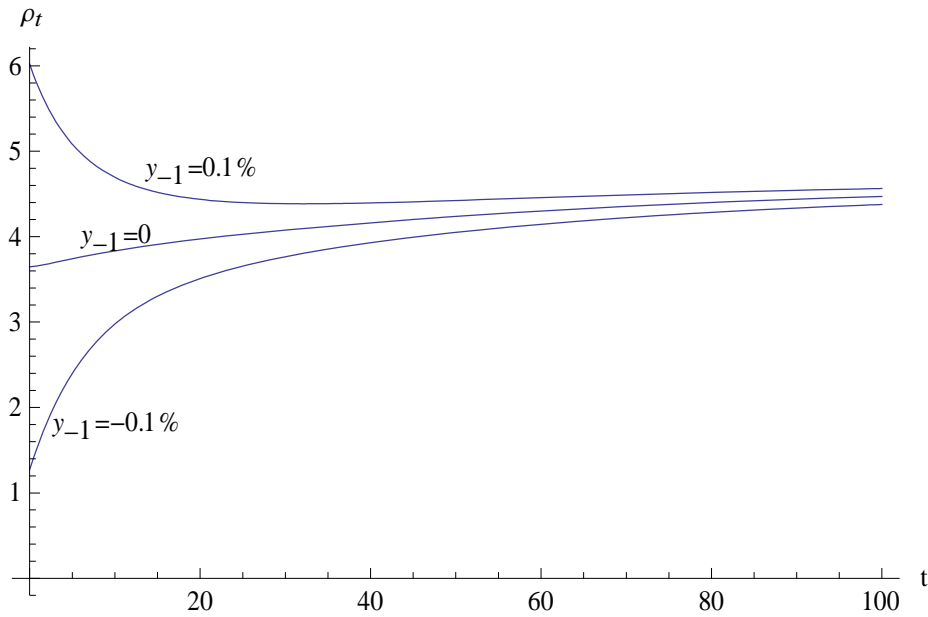


Figure 5: The discount rate (in % per year) as a function of the maturity (in years) for $\beta = 1.32$ in different phases of the cycle. Equation (41) is calibrated with $\delta = 0$, $\gamma = 2$, $\sigma_x = 0.0078$, $\sigma_y = 0.00034$, $\phi = 0.979$, and two equally likely trends $\mu_1 = 0.0005$ and $\mu_2 = 0.0025$.

¹⁰ Increasing the degree of uncertainty affecting θ_1 has a sizeable impact on this estimation. For example, if we replace the assumption that the interval $[0\%, 5\%]$ on which it is uniformly distributed by $[0\%, 10\%]$, the OLS estimation of the beta goes down to 1.14%. However, it is hard to imagine damages amounting to 10% of the world production due to a 1°C increase in temperature.

7. *Concluding remarks*

By focusing on the riskiness of future benefits and costs, this paper contributes to the debate on the discount rate for climate change in several directions. Following Weitzman (2007) and Gollier (2008), we examined a model with a Brownian motion for the growth of the economy, some of the parameters of this underlying stochastic process being uncertain. We then extended this model to include business cycles into the picture. Our main messages in this framework are as follows. First, we showed that the shape of the term structure of discount rates for risky projects is determined by the relative intensity of a precautionary effect that pushes towards a decreasing term structure, and of a risk aversion effect that pushes towards an increasing term structure. Under some weak restrictions on the distribution of the uncertain parameters, the term structure is decreasing or increasing depending upon whether the beta of the project is respectively smaller or larger than half the relative risk aversion of the representative agent.

Second, we showed that the risk premium associated to a project is generally not proportional to its beta, which implies that knowing the aggregate risk premium and the project's beta is not enough to compute the project-specific risk premium. We derived simple formulas to compute the project-specific risk premium as a function of the project's beta. Third, when shocks on the growth rate of the economy are persistent, short-term and medium-term discount rates should be frequently revised as a function of changes in the state variable. The term structures of discount rates are generally not monotone in that context.

Finally, we have shown that there are reasons to believe that the beta of projects whose main benefits are to reduce emissions of greenhouse gases is relatively large, around 1.3. This allows us to conclude that the discount rates to be used to evaluate public policies to fight climate change should be increasing with respect to maturities. Given the current global economic crisis in the western world, following Figure 5, we are in favor of using a real discount rate for climate change around 1.3% for short horizons, up to 4.6% for maturities exceeding 100 years.

A word of caution should be added to this conclusion. The use of price signals like discount rates and risk premia is possible only for investment projects that are marginal, i.e., for actions that do not affect expectations about the growth of the economy. The reader should be aware that this assumption does not hold when considering the *global* strategy to fight climate change. When thinking globally, one needs to take into account the general equilibrium effects that the chosen

strategy will have on the stochastic growth process, hence on the discount rates that are used to evaluate this strategy. The right evaluation approach for global projects relies on the direct measure of the impact of the global action on the intergenerational social welfare function, as done for example in Stern (2007) and Nordhaus (2008).

Appendix A: Proof of Lemma 1

Define function h_t such that

$$h_t(k) = H_t(x_0 + ky) = \frac{1}{t} \frac{E(x_0 + ky)e^{t(x_0 + ky)}}{Ee^{t(x_0 + ky)}} - \frac{1}{t^2} \ln Ee^{t(x_0 + ky)}. \quad (57)$$

We have that $h_t(0) = 0$ and

$$h_t'(k) = \frac{Ey(x_0 + ky)e^{t(x_0 + ky)}}{Ee^{t(x_0 + ky)}} - \frac{E(x_0 + ky)e^{t(x_0 + ky)}Ey e^{t(x_0 + ky)}}{(Ee^{t(x_0 + ky)})^2}. \quad (58)$$

It implies that

$$h_t'(0) = x_0Ey - x_0Ey = 0. \quad (59)$$

We also obtain that

$$\begin{aligned} h_t''(k) &= \frac{Ey^2e^{t(x_0 + ky)}}{Ee^{t(x_0 + ky)}} + \frac{Ety^2(x_0 + ky)e^{t(x_0 + ky)}}{Ee^{t(x_0 + ky)}} - \frac{Ey(x_0 + ky)e^{t(x_0 + ky)}Etye^{t(x_0 + ky)}}{(Ee^{t(x_0 + ky)})^2} \\ &\quad - \left(\frac{Etye^{t(x_0 + ky)}}{Ee^{t(x_0 + ky)}} \right)^2 - \frac{Ety(x_0 + ky)e^{t(x_0 + ky)}Ey e^{t(x_0 + ky)}}{(Ee^{t(x_0 + ky)})^2} \\ &\quad - \frac{E(x_0 + ky)e^{t(x_0 + ky)}Ety^2e^{t(x_0 + ky)}}{(Ee^{t(x_0 + ky)})^2} + 2 \frac{tE(x_0 + ky)e^{t(x_0 + ky)}(Etye^{t(x_0 + ky)})^2}{(Ee^{t(x_0 + ky)})^3}. \end{aligned} \quad (60)$$

It yields

$$h_t''(0) = \sigma_y^2 + tx_0\sigma_y^2 - tx_0\sigma_y^2 = \sigma_y^2. \quad (61)$$

The Taylor expansion of $h_t(k)$ yields in turn

$$\begin{aligned} H_t(x_k) &= h_t(k) = h_t(0) + kh_t'(0) + 0.5k^2h_t''(0) + o(k^2) \\ &= 0.5k^2\sigma_y^2 + o(k^2). \end{aligned} \quad (62)$$

This concludes the proof. \square

Appendix B: Proof of Proposition 2

When $\beta = \gamma / 2$, equation (28) implies that

$$-t \frac{\partial \rho_t(\beta = \gamma / 2)}{\partial t} = H_t(-\beta(\mu_\theta - \bar{\mu})) - H_t(\beta(\mu_\theta - \bar{\mu})) = 0. \quad (63)$$

This is a direct consequence of the assumption that μ_θ has a symmetric distribution. Thus, the term structure of ρ_t is flat for $\beta = \gamma / 2$.

Fully differentiating equation (28) with respect to γ yields

$$\begin{aligned} -t \frac{\partial^2 \rho_t(\beta)}{\partial \gamma \partial t} &= -\frac{E(\mu_\theta - \bar{\mu})e^{t(\beta-\gamma)(\mu_\theta - \bar{\mu})}}{Ee^{t(\beta-\gamma)(\mu_\theta - \bar{\mu})}} - t(\beta - \gamma) \frac{E(\mu_\theta - \bar{\mu})^2 e^{t(\beta-\gamma)(\mu_\theta - \bar{\mu})}}{Ee^{t(\beta-\gamma)(\mu_\theta - \bar{\mu})}} \\ &\quad + t(\beta - \gamma) \left(\frac{E(\mu_\theta - \bar{\mu})e^{t(\beta-\gamma)(\mu_\theta - \bar{\mu})}}{Ee^{t(\beta-\gamma)(\mu_\theta - \bar{\mu})}} \right)^2 + \frac{E(\mu_\theta - \bar{\mu})e^{t(\beta-\gamma)(\mu_\theta - \bar{\mu})}}{Ee^{t(\beta-\gamma)(\mu_\theta - \bar{\mu})}} \\ &= -t(\beta - \gamma) \left[\int (\mu_\theta - \bar{\mu})^2 d\hat{G}(\theta) - \left(\int (\mu_\theta - \bar{\mu}) d\hat{G}(\theta) \right)^2 \right], \end{aligned} \quad (64)$$

where

$$d\hat{G}(\theta) = \frac{e^{t(\beta-\gamma)(\mu_\theta - \bar{\mu})}}{Ee^{t(\beta-\gamma)(\mu_\theta - \bar{\mu})}} dG(\theta) \quad (65)$$

characterizes a distorted probability distribution. Because the bracketed term in the RHS of (64) is always positive, we have that $\partial \rho_t(\beta) / \partial t$ is decreasing in γ for all $\beta \in]-\infty, \gamma]$. Because it vanishes at $\beta = \gamma / 2$, we obtain that $\partial \rho_t(\beta) / \partial t$ is negative for all $\beta < \gamma / 2$, and it is positive for all $\beta \in [\gamma / 2, \gamma]$. \square

Appendix C: Proof of Proposition 3

Consider equation (27). When t tends to zero, using L'Hospital's rule implies that the limit of $\rho_t(\beta)$ when t tends to zero equals

$$\lim_{t \rightarrow 0} \left[\delta + \gamma \sigma^2 (\beta - \gamma / 2) - (\beta - \gamma) \frac{\int \mu_\theta e^{(\beta - \gamma) \mu_\theta t} dG(\theta)}{\int e^{(\beta - \gamma) \mu_\theta t} dG(\theta)} + \beta \frac{\int \mu_\theta e^{\beta \mu_\theta t} dG(\theta)}{\int e^{\beta \mu_\theta t} dG(\theta)} \right]. \quad (66)$$

This implies that

$$\lim_{t \rightarrow 0} \rho_t(\beta) = \delta + \gamma \bar{\mu} + \gamma \sigma^2 (\beta - \gamma / 2). \quad (67)$$

When t tends to infinity, the same technique implies that the term structure of $\rho_t(\beta)$ tends asymptotically to

$$\delta + \gamma \sigma^2 (\beta - \gamma / 2) + \lim_{t \rightarrow \infty} \left[-(\beta - \gamma) \frac{\int \mu_\theta e^{(\beta - \gamma) \mu_\theta t} dG(\theta)}{\int e^{(\beta - \gamma) \mu_\theta t} dG(\theta)} + \beta \frac{\int \mu_\theta e^{\beta \mu_\theta t} dG(\theta)}{\int e^{\beta \mu_\theta t} dG(\theta)} \right]. \quad (68)$$

When $\beta > \gamma > 0$, both ratios in the limit tend to μ_{\max} , so that we obtain that

$$\lim_{t \rightarrow \infty} \rho_t(\beta) = \delta + \gamma \mu_{\max} + \gamma \sigma^2 (\beta - \gamma / 2). \quad (69)$$

When $0 < \beta < \gamma$, the first ratio tends to μ_{\min} and the second ratio tends to μ_{\max} . This implies that

$$\lim_{t \rightarrow \infty} \rho_t(\beta) = \delta + (\gamma - \beta) \mu_{\min} + \beta \mu_{\max} + \gamma \sigma^2 (\beta - \gamma / 2). \quad (70)$$

The third case with $\beta < 0$ follows in a similar fashion. \square

References

- Arrow, K.J., and R.C. Lind, (1970), Uncertainty and the evaluation of public investment decision, *American Economic Review*, 60, 364-378.
- Bansal, R., and A. Yaron, (2004), Risks For the Long Run: A Potential Resolution of Asset Pricing Puzzles, *Journal of Finance* 59, 1481–1509.
- Barro, R.J., (2006). “Rare Disasters and Asset Markets in the Twentieth Century,” *Quarterly Journal of Economics*, 121, 823-866.
- Cline, W.R., (1992), *The Economics of Global Warming*, Institute for International Economics, Washington.
- Dietz, S., C. Hope and N. Patmore, (2007), Some economics of ‘dangerous’ climate change: Reflections on the Stern Review, *Global Environmental Change*, 17, 311-325.
- Drèze, J.H. and F. Modigliani, (1972), Consumption decisions under uncertainty, *Journal of Economic Theory*, 5, 308-335.
- European Insurance and Occupational Pensions Authority, (2012), *Draft technical specifications QIS of EIOPA’s advice on the review of the IORP directive: Consultation paper*, EIOPA-CP-12-003, 15 June 2012.
- Gollier, C., (2002), Time horizon and the discount rate, *Journal of Economic Theory*, 107, 463-473.
- Gollier, C., (2008), Discounting with fat-tailed economic growth, *Journal of Risk and Uncertainty*, 37, 171-186.
- Gollier, C., (2011), *Le calcul du risque dans les investissements publics*, Centre d’Analyse Stratégique, Rapports & Documents n°36, La Documentation Française.
- Gollier, C., (2012), *Pricing the planet’s future: The economics of discounting in an uncertain world*, Princeton University Press, Princeton, forthcoming (October 2012).
- Kimball, M.S., (1990), Precautionary savings in the small and in the large, *Econometrica*, 58, 53-73.
- Lucas, R., (1978), Asset prices in an exchange economy, *Econometrica*, 46, 1429-46.

- Mehra, R. and E. Prescott, (1985), The Equity Premium: A Puzzle, *Journal of Monetary Economics*, 10, 335-339.
- Newell, R. and W. Pizer, (2003), Discounting the Benefits of Climate Change Mitigation: How Much Do Uncertain Rates Increase Valuations? *Journal of Environmental Economics and Management*, 46 (1), 52-71.
- Nordhaus, W.D., (2008), *A Question of Balance: Weighing the Options on Global Warming Policies*, New Haven: Yale University Press.
- Nordhaus, W.D., and J. Boyer, (2000), *Warming the World: Economic Models of Global Warming*, Cambridge, MA: MIT Press.
- Office of Management and Budget, (2003), Circular N. A-4 To the Heads of Executive Department Establishments, Subject: Regulatory Analysis. Washington: Executive Office of the President.
- Sandsmark, M., and H. Vennemo, (2007), A portfolio approach to climate investments: CAPM and endogeneous risk, *Environmental and Resource Economics*, 37, 681-695.
- Stern, N., (2007), *The Economics of Climate Change: The Stern Review*, Cambridge University Press, Cambridge.
- Weil, P., (1989), The equity premium puzzle and the risk free rate puzzle, *Journal of Monetary Economics*, 24, 401-21.
- Weitzman, M.L., (1998), Why the far-distant future should be discounted at its lowest possible rate?, *Journal of Environmental Economics and Management*, 36, 201-208.
- Weitzman, M.L., (2001), Gamma discounting, *American Economic Review*, 91, 260-271.
- Weitzman, M. L., (2007), Subjective expectations and asset-return puzzle, *American Economic Review*, 97, 1102-1130.
- Weitzman, M. L., (2009), On Modeling and Interpreting the Economics of Catastrophic Climate Change, *Review of Economics and Statistics*, 91 (1), 1-19.
- Weitzman, M.L., (2010), Risk-adjusted gamma discounting, *Journal of Environmental Economics and Management*, 60, 1-13.

Weitzman, M.L., (2012), On the risk-adjusted discount rate for public investments, mimeo, Harvard University.