Term structures of discount rates for risky investments

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Abstract
There is still much confusion about which discount rates should be used to evaluate actions having long-lasting impacts, as in the contexts of climate change, social security reforms or large public infrastructures for example. Contrary to the existing literature that focuses on the discount rate for safe projects, this paper characterizes the term structure of discount rates for investment projects and assets with a non-zero beta. We assume that the growth rate of aggregate consumption follows a Brownian motion with uncertain parameters. We show that the term structures of the risk free discount rate and of the aggregate risk premium are respectively decreasing and increasing. Overall, the slope of term structure to be used for a specific project depends upon whether its beta is smaller or larger than half the relative risk aversion. We also argue that the beta of actions to mitigate climate change is relatively large, so that the term structure of the associated real discount rates is increasing, from 1% for short maturities to 6% for extra-long ones.

Keywords: asset prices, term structure, risk premium, decreasing discount rates, parametric uncertainty, climatic beta.

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1. Introduction

Do we do enough for the distant future? This question is implicit in many policy debates, from the fight against climate change to the speed of reduction of public deficits, investments in research and education, or the protection of the environment and of natural resources for example. The discount rate used to evaluate investments is the key determinant of our individual and collective efforts in favor of the future. Since Weitzman (1998), an intense debate has emerged among economists about whether one should use different discount rates for different time horizons $\tau$. It is however well-known that the term structure of efficient discount rates is flat if we assume that the representative agent has a constant relative risk aversion and that the growth rate of consumption is a random walk. In this benchmark specification, if a rate of 3% is efficient to discount cash flows occurring in 12 months, it is also efficient to use that rate of 3% to discount cash flows occurring in 200 years.

Compared to this benchmark, a decreasing term structure of discount rates would bias the economic evaluation of investments towards those with more distant positive impacts. Weitzman (1998, 2001) and Newell and Pizer (2003) justified such a decreasing structure by assuming that the socially efficient discount factor should equal to the expected discount factor when the discount rate is uncertain. Gollier (2002, 2012) and Weitzman (2007) used standard consumption-based asset pricing theory to conclude that the large uncertainty associated to the distant future should induce the prudent representative agent to use lower rates to discount more distant cash flows. The various dynamic processes that support this result include for example mean-reversion, Markov regime-switches, and parametric uncertainty on the trend of a Brownian motion. Gollier (2008) demonstrates that the positive serial correlation of the growth rate of consumption that is inherent to these stochastic processes is the driving force of the result, together with prudence. Prudence is a concept defined by Kimball (1990) to characterize the willingness to save more when the future becomes more uncertain. For growth processes with persistent shocks, aggregate uncertainty accumulates faster with respect to longer time horizons than in a pure random walk with the same short-term volatility. Prudent people want to bias their investments towards those which yield more sure benefits for these horizons. Because the term structure of socially efficient discount rates is flat for this latter process, this is done by using a
decreasing term structure. Persistent movements in expected growth rates of aggregate consumption are documented for the U.S. by Bansal and Yaron (2004) for example.

With the notable exception of Weitzman (2012), this recent literature focuses on rates \( r_\beta \) at which \textit{safe} cash flows should be discounted. In reality, most investment projects yield uncertain future costs and benefits. For marginal projects, we know that idiosyncratic risks should not be priced, because they will be washed out in diversified portfolios. In public economics, this result is usually referred to as the Arrow-Lind Theorem (Arrow and Lind (1970)), but this is a well-known feature of the consumption-based capital asset pricing model (CCAPM, Lucas (1978)). More generally, the discount rate \( \rho_t \) to be used to evaluate risky projects depends upon their beta which measures the elasticity of net cash flows to changes in aggregate consumption. Under the benchmark specification described above, the risk premium \( \pi_t(\beta) = \rho_t(\beta) - r_\beta \) of a project is proportional to \( \beta \). This implies that it is enough to estimate the aggregate risk premium, i.e., the risk premium associated to a project with a unit beta, to characterize the evaluation procedure for all risky projects. As is well-known, the benchmark specification also implies that the term structure of the aggregate risk premium is flat.

These two crucial properties of the benchmark specification are not robust to the introduction of persistent shocks to the growth rate of aggregate consumption. In particular, the project-specific risk premium is generally not proportional to the beta of the project. Second, the arguments provided in the recent literature in favor of a decreasing term structure of safe discount rates are compatible with an increasing term structure of the risk premium associated to projects with a positive beta. If we assume that the stochastic process of the growth rate of consumption exhibits positive serial correlation, the annualized measure of aggregate risk will have an increasing term structure. By risk aversion, the term structure of the risk premium will inherit this property.

With positively serially correlated growth rates, the project-specific discount rate \( \rho_t(\beta) = r_\beta + \pi_t(\beta) \) for positive betas is thus the sum of a prudence-driven decreasing function and of a risk-aversion-driven increasing function of the time horizon. Thus, this term structure will be decreasing if the project-specific beta is small enough, and it will be increasing if the beta is large enough. The economic intuition of these results is based on the increasing accumulation of uncertainties with time. For projects with a low beta, long-termism should be favored because of
prudence. For projects with a large beta, short-termism should be favored because of risk aversion. In this paper, we assume that the growth of log consumption follows a Brownian motion, but the trend and the volatility of this process are uncertain. Observe that the uncertainty on the trend of growth implies that the unconditional growth rates are positively correlated, thereby generating the phenomena described above. In the last section of the paper, we also examine growth processes that combine parametric uncertainty and mean-reversion.

Our paper provides important new insights about how public policies are evaluated around the world. It is a common practice to use a single discount rate to evaluate public investments independent of their riskiness and time horizons. In the U.S. for example, the Office of Management and Budget (OMB) recommend to use a discount rate of 7% since 1992. It was argued that the “7% is an estimate of the average before-tax rate of return to private capital in the U.S. economy” (OMB (2003)). In 2003, the OMB also recommended the use of a discount rate of 3%, in addition to the 7% mentioned above as a sensitivity. This new rate of 3% was justified by the “social rate of time preference. This simply means the rate at which society discounts future consumption flows to their present value. If we take the rate that the average saver uses to discount future consumption as our measure of the social rate of time preference, then the real rate of return on long-term government debt may provide a fair approximation” (OMB, (2003)). The 3% corresponds to the average real rate of return of the relatively safe 10-year Treasury notes between 1973 and 2003. Interestingly enough, the recommended use of 3% and 7% is not differentiated by the nature of the underlying risk, and is independent of the time horizon of the project. In the another field, guidelines established by the Government Accounting Standards Board (GASB) recommend that state and local governments discount their pension liabilities at expected returns on their plan assets, which is usually estimated around 8%, independent of their maturities. The absence of risk-and-maturity-based price signals has potentially catastrophic consequences for the allocation of capital in the economy. This paper provides clear recommendations about the changes in evaluation tools that should be

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2 The European Union is currently debating about the new solvency regulation of insurance companies (Solvency 2). In the most recent consultation paper (European Insurance and Occupational Pensions Authority (2012)), it is proposed to discount safe liabilities using the yield curve up to 20-year maturities, and a real discount rate tending to 2% (“Ultimate Forward Rate”) for longer maturities.

3 In 2005, France has adopted a decreasing real discount rate from 4% to 2% for safe projects. This rule has been complemented in 2011 by an aggregate risk premium of 3% (Gollier’s Report (2010)).
implemented. This reform is made at the cost of requiring evaluators to estimate the project-specific beta.

We agree that determining the beta of actions with long-lasting societal impacts is a complex matter. Let us illustrate this point in the context of climate change. It is well-known that the choice of the rate at which future damages are discounted is crucial to determine the so-called “social cost of carbon” (SCC). Sandsmark and Vennemo (2007) claim that the beta of mitigation investments is negative, so that the term structure of discount rates should be low and decreasing, thereby yielding a large SCC. They assume that the only source of aggregate fluctuations originate from climate change, with an uncertain climate sensitivity. Under this assumption, a large climate sensitivity yields at the same time a low consumption and a large social benefit from early mitigation. This explains the negative beta of their model. But consider alternatively a more standard integrated assessment model as Nordhaus’ DICE model (Nordhaus and Boyer (2000)) in which the climate sensitivity is known, emissions are increasing in aggregate consumption, the damage function is convex, and the growth rate of consumption has some intrinsic volatility. In that alternative modeling, a larger growth rate of consumption goes with a larger concentration of CO2, a larger damage, and a larger societal benefit from early mitigation. This justifies a positive beta. I show in this paper that any credible calibration of a model combining the two sources of aggregate fluctuation yields a positive and large beta of mitigation. This is compatible with using increasing discount rates to measure SCC.

2. The benchmark CRRA-Normal model

We evaluate a marginal investment project that reduces current consumption by some sure amount and that generates a flow of benefits \( \epsilon F_1, \epsilon F_2, \ldots \) in the future, which can be uncertain seen from today. In order to evaluate the social desirability of such a project, we measure its impact on the intertemporal social welfare

\[
W = u(c_0) + \sum_{i=1}^{\infty} e^{-\delta t} E u(c_t),
\]

(1.1)

The climate sensitivity is a physical parameter that measures the relationship between the concentration of greenhouse gases in the atmosphere and the average temperature of the earth.
where \( u \) is the increasing and concave utility function of the representative agent, \( \delta \) is her rate of pure preference of the present, and \( c_t \) is the consumption level of the representative agent at date \( t \). Because \( \varepsilon \) is assumed to be small, implementing the project increases intertemporal social welfare if and only if

\[
-1 + \sum_{t=1}^{T} E \left[ \frac{e^{-\delta t} F_t u'(c_t)}{u'(c_0)} \right] \geq 0. \tag{2}
\]

This can be rewritten as a standard NPV formula:

\[
-1 + \sum_{t=1}^{T} e^{-\rho_t F_t} E F_t \geq 0, \tag{3}
\]

where \( \rho_t(F_t) \) is the rate at which the expected cash flow occurring in \( t \) years should be discounted. This discount rate is written as follows

\[
\rho_t(F_t) = \delta - \frac{1}{t} \ln \frac{E F_t u'(c_t)}{u'(c_0) E F_t} = r_{ft} + \pi_t(F_t). \tag{4}
\]

It is traditional in the CCAPM to decompose the project-specific discount rate \( \rho_t(F_t) \) into a risk free discount rate \( r_{ft} \) and a project-specific risk premium \( \pi_t(F_t) \). From (4), we define these two components of the discount rate as follows:

\[
r_{ft} = \delta - \frac{1}{t} \ln \frac{E u'(c_t)}{u'(c_0)}, \tag{5}
\]

\[
\pi_t(F_t) = -\frac{1}{t} \ln \frac{E F_t u'(c_t)}{E F_t E u'(c_t)}. \tag{6}
\]

Observe that the risk premium \( \pi_t(F) \) is zero when the project is safe or when its future cash flow is independent of future aggregate consumption. This implies that \( r_{ft} \) is indeed the rate at which safe projects should be discounted. The CCAPM also characterizes the project-specific risk premium \( \pi_t(F_t) \). Throughout the paper, we suppose that \( u'(c) = c^{-\gamma} \) and that

\[
F_t = \xi_t c_t^\beta \tag{7}
\]
where $\xi_t$ is a random variable independent of $c_t$ with $E\xi_t = 1$, and $\beta$ is the beta of the project. \(^5\) Because the idiosyncratic risk $\xi_t$ is not priced, we hereafter identify a project $\{F_t\}$ by its $\beta$.

Under this specification, asset pricing formulas (5) and (6) can be rewritten as follows:

$$r_{\beta} = \delta - \frac{1}{t} \ln e^{-\gamma \ln c_t / c_0},$$

$$\pi_\gamma(\beta) = -\frac{1}{t} \ln \frac{E e^{(\beta-\gamma) \ln c_t / c_0}}{E e^{\beta \ln c_t / c_0} E e^{-\gamma \ln c_t / c_0}}.$$  \hspace{1cm} (8)

$$\pi_\gamma(\beta) = -\frac{1}{t} \ln \frac{E e^{(\beta-\gamma) \ln c_t / c_0}}{E e^{\beta \ln c_t / c_0} E e^{-\gamma \ln c_t / c_0}}.$$  \hspace{1cm} (9)

In this paper, we calibrate these equations for different specifications of the stochastic process of $\ln c_t / c_0$. The benchmark process is such that log consumption follows an arithmetic Brownian motion with trend $\mu$ and volatility $\sigma$. This implies that $\ln c_t / c_0$ is normally distributed with mean $\mu t$ and variance $\sigma^2 t$. In this benchmark case, one can compute the different expectations in the above equations by using the following well-known property:

$$x \sim N(\mu, \sigma^2) \text{ and } \alpha \in \mathbb{R} \implies E e^{\alpha x} = e^{\alpha(\mu + 0.5\alpha \sigma^2)}. \hspace{1cm} (10)$$

The reader can then easily check that equations (8) and (9) implies that

$$r_{\beta} = \delta + \gamma \mu - 0.5 \gamma^2 \sigma^2,$$

$$\pi_\gamma(\beta) = \beta \gamma \sigma^2 = \beta \pi_m.$$ \hspace{1cm} (11)

Equation (11) which is often referred to as the extended Ramsey rule, holds independent of the maturity of the cash flow. In other words, the term structure of the safe discount rate is flat in that case. Its level is determined by three elements: impatience, a wealth effect and a precautionary effect. The wealth effect comes from the observation that investing for the future in a growing economy does increase intertemporal inequality. Because of inequality aversion, this is desirable only if the return of the project is large enough to compensate for this adverse effect on welfare. The wealth effect is equal to the product of the expected growth of log consumption by the degree $\gamma$ of concavity of the utility function which measures inequality aversion. The

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\(^5\) There is no difficulty to generalize the analysis to time-varying betas.
precautionary effect comes from the observation that consumers want to invest more for the future when this future is more uncertain (Drèze and Modigliani 1972), Kimball (1990)). This tends to reduce the discount rate. The precautionary effect is proportional to the volatility of the growth of consumption.

Equation (12) tells us that the project-specific risk premium $\pi_t(\beta)$ is just equal to the product of the project-specific beta by the CCAPM aggregate risk premium $\pi_w = \gamma \sigma^2$. Under this standard specification, the risk premium associated to benefit $F_t$ is independent of its maturity $t$. The standard calibration of these two equations yields a too large risk free rate (risk free rate puzzle (Weil (1989))) and a too small risk premium (equity premium puzzle (Mehra and Prescott (1985))) compared to historical market data. Barro (2006) showed that these two puzzles can be solved by introducing in the stochastic growth process a small probability of economic catastrophes.

Because both the risk free rate and the risk premium of the project are independent of the maturity in this benchmark specification, their sum $\rho_t(\beta) = r + \pi_t(\beta)$ is also independent of $t$. The term structure of risky discount rates is flat in this case. The risky discount rate equals

$$\rho_t(\beta) = \delta + \gamma \mu + \gamma (\beta - 0.5 \gamma ) \sigma^2. \quad (13)$$

Notice that the risky discount rate can be either increasing or decreasing in the aggregate uncertainty measured by $\sigma^2$ depending upon whether the $\beta$ of the project is larger or smaller than $\gamma / 2$. Two competing effects are at play here. First, a large aggregate risk induces the representative agent to save more for the future (precautionary saving motive). That reduces the risk free discount rate. Second, ceteris paribus, a larger aggregate risk increases the project-specific risk and the associated risk premium. This risk aversion effect is proportional to the beta of the project. The two effects counterbalance each other perfectly when $\beta = \gamma / 2$. When $\beta$ is smaller than $\gamma / 2$, the precautionary effect dominates the risk aversion effect.

3. The CRRA-Normal model with parametric uncertainty
Following Weitzman (2007) and Gollier (2008), we now characterize the term structure of the risk free rate and the risk premium when there is some uncertainty about the true value of some of the parameters of the model. Let us assume that \(\ln(c_t / c_0)\mid \theta \sim N(\mu_\theta, \sigma_\theta^2)\), where the mean and the variance can depend upon a parameter \(\theta\) that is uncertain. This uncertainty is characterized by the distribution function \(G\) of the random variable \(\theta\). Using property (10) in equations (8) and (9), we can write that

\[
r_{\beta t} = \delta - \frac{1}{t} \ln \left[ E[e^{-\gamma \ln(c_t / c_0)} | \theta] \right] dG(\theta) = \delta - \frac{1}{t} \ln \left[ e^{-\gamma (\mu_\theta - 0.5 \sigma_\theta^2)} dG(\theta) \right],
\]

and

\[
\pi_t(\beta) = -\frac{1}{t} \ln \left[ \frac{\left[ E\left[ e^{\beta \ln(c_t / c_0)} | \theta \right] dG(\theta) \left[ E\left[ e^{-\gamma \ln(c_t / c_0)} | \theta \right] dG(\theta) \right]}{\left[ E\left[ e^{(\beta - \gamma) \ln(c_t / c_0)} | \theta \right] dG(\theta) \right]} \right] - \frac{1}{t} \ln \left[ e^{\beta (\mu_\theta + 0.5 (\beta - \gamma) \sigma_\theta^2)} dG(\theta) \right] e^{-\gamma (\mu_\theta - 0.5 \sigma_\theta^2)} dG(\theta).
\]

We apply these pricing formulas to different structures of uncertainty about the trend and the volatility of the growth of the economy.

3.1. The trend of growth is uncertain

Suppose first that \(\ln(c_t / c_0)\) follows a Brownian motion with a known constant volatility \(\sigma\) and an unknown constant trend \(\mu_\theta\). Equation (14) can then be rewritten as follows:

\[
r_{\beta t} = \delta - 0.5 \gamma^2 \sigma^2 - \frac{1}{t} \ln \left[ e^{-\gamma \mu_\theta} dG(\theta) \right],
\]

Fully differentiating equation (16) with respect to \(t\) implies that

\[
\frac{\partial r_{\beta t}}{\partial t} = -H_t(-\gamma \mu_\theta),
\]

where functional \(H_t\) is defined on the set of random variables such that for any \(t \in \mathbb{R}_+\), we have
The following lemma gives us a useful tool to determine the slope of the term structures under scrutiny in the case in which the uncertainty is limited.

**Lemma 1:** Consider the family of random variables

\[ x_k = x_0 + ky \] where \( x_0 \in \mathbb{R} \) and \( y \) is a zero-mean random variable with finite variance \( \sigma_y^2 \). We have that

\[ H_i(x_k) = \frac{\partial}{\partial t} \left[ \frac{1}{t} \ln Ee^{xt} \right] = \frac{1}{t} \left[ \frac{Ee^{xt}}{Ee^{xt} - \frac{1}{t} \ln Ee^{xt}} \right] . \] (18)

with \( \lim_{k \to 0} o(k^2) / k^2 = 0 \).

**Proof:** See the Appendix. □

Suppose that the uncertainty on the trend of growth is small in the sense that \( \mu_y = \mu_0 + ky, \) with \( k \) small and \( Var(y_0) = \sigma_0^2 \). Applying the lemma to equation (17) implies that

\[ \frac{\partial \tilde{r}_\theta}{\partial t} = -0.5 \gamma^2 Var(\mu_0) + o(k^2). \] (20)

A similar exercise with equation (15) implies that

\[ \pi_i(\beta) = -\gamma^2 - \frac{1}{t} \ln \left[ \frac{\int e^{(\beta - \gamma)\theta} dG(\theta)}{\int e^{\theta} dG(\theta)} e^{-\gamma \theta} dG(\theta) \right] . \] (21)

and

\[ \frac{\partial \pi_i(\beta)}{\partial t} = H_i(\beta \mu_0) + H_i(-\gamma \mu_0) - H_i(\beta - \gamma) \mu_0 . \] (22)

Using Lemma 1 yields

\[ \frac{\partial \pi_i(\beta)}{\partial t} = 0.5 Var(\mu_0) \left[ \beta^2 + \gamma^2 - (\beta - \gamma)^2 \right] + o(k^2) \]

\[ = \beta \gamma Var(\mu_0) + o(k^2). \] (23)

We summarize these findings in the following proposition.
Proposition 1: Suppose that $\ln c_t / c_0$ follows an arithmetic Brownian motion with a known volatility $\sigma \in \mathbb{R}^+$ and with a trend $\mu_0$ that entails some small uncertainty. It implies that the term structures of the risk free discount rate and of the aggregate risk premium are respectively decreasing and increasing. Moreover, the term structure of the discount rates $\rho_t(\beta)$ is decreasing, flat or increasing depending upon $\beta$ is smaller, equal or larger than $0.5\gamma$.

In the small, the uncertainty about the trend of growth makes the term structure of the risk free discount rate decreasing. The sensitivity of this rate to the maturity is approximately proportional to the product of $\gamma^2 / 2$ by the variance of the trend of the economy. On the contrary, the risk premium has an increasing term structure if the $\beta$ of the project is positive. The sensitivity of the risk premium to the maturity is proportional to the product of $\beta\gamma$ by the variance of the trend of the economy. We also see that the risky discount rate is decreasing or increasing depending upon whether the beta of the project is smaller or larger than $\gamma / 2$. It is flat for $\beta = 0.5\gamma$. The intuition of this result combines the observation that the parametric uncertainty magnifies long term risks, and the observation made in the previous section that risk decreases or increases the discount rate depending upon whether $\beta$ is smaller or larger than $\gamma / 2$.

Assuming a Gaussian distribution for the trend of growth yields an analytical solution to the pricing equations. Suppose that the current uncertainty about the expected change in log consumption is normally distributed with mean $\mu_0$ and variance $\sigma^2_0$. It implies that $\ln(c_t / c_0)$ is $N(\mu_0t, (\sigma^2 + \sigma^2_0)t)$. Using property (10) once again in equations (16) and (21), we obtain that

$$r_\beta = \delta + \gamma \mu_0 - 0.5\gamma^2 (\sigma^2 + t\sigma^2_0),$$

and

$$\pi_\beta(\beta) = \beta \gamma (\sigma^2 + t\sigma^2_0).$$

Under this specification, the risky discount rate $\rho_t(\beta)$ can thus be written as

$$\rho_t(\beta) = \delta + \gamma \mu + \gamma (\beta - 0.5\gamma)(\sigma^2 + t\sigma^2_0).$$
The term structure of the risk free discount rate is linearly decreasing, whereas the term structure of the risk premium is linearly increasing when the beta is positive. The intuition of these two results is based on the observation that the uncertainty related to the growth trend of the economy magnifies the long term macroeconomic uncertainty, and the corresponding uncertainty about the benefits of the project. Equation (26) shows that the term structure of risky discount rate is linearly decreasing or increasing if the project’s $\beta$ is smaller or larger than $0.5\gamma$, as is the case for small risk on $\mu_0$ (Proposition 1).

We hereafter relax the above specification by allowing for any distribution of the trend $\mu_0$. When the uncertainty on the trend is not small or normally distributed, an additional complexity arises in the evaluation strategy. Indeed, a striking observation that can be made from equations (15) and (21) is that the risk premium of a project is in general not proportional to its beta. It implies that computing the aggregate risk premium for $\beta = 1$ will not be helpful to determine the risk premium for another project with $\beta \neq 1$, or that $\pi_t(\beta) \neq \beta \pi_t(1)$.

The next proposition describes the shape of the term structure of the risky discount rates for different betas. It relies on the following equation which combines equations (16) and (21):

$$\rho_t(\beta) = \delta + \gamma \overline{\mu} + \gamma \sigma^2 \left( \beta - \frac{\gamma}{2} \right) - \frac{1}{t} \ln \int e^{\beta \gamma \mu_0 - \pi} dG(\theta) - \int e^{\beta \gamma \mu_0 - \pi} dG(\theta),$$

(27)

with $\overline{\mu} = E \mu_0$. There are two simple cases that are worthy to examine. When $\beta = 0$, $\rho_t(\beta = 0) = r_\beta$ is obviously decreasing in $t$, since $-(t^{-1}) \ln E \exp(-\gamma \mu_t)$ is the certainty equivalent of $\gamma \mu_0$ for a concave exponential utility function with degree of concavity $t$. Similarly, when $\beta = \gamma$, $\rho_t(\beta = \gamma)$ is obviously increasing in $t$, since $(t^{-1}) \ln E \exp(\gamma \mu_t)$ is the certainty equivalent of $\gamma \mu_0$ for a convex exponential utility function with degree of convexity $t$. The other cases are described in Proposition 2. It relies on the full differentiation of equation (27) with respect to $t$, which yields

$$\frac{\partial \rho_t(\beta)}{\partial t} = H_t(\beta(\mu_0 - \overline{\mu})) - H_t((\beta - \gamma)(\mu_0 - \overline{\mu})).$$

(28)
Proposition 2: Suppose that $\ln c_t / c_0$ follows an arithmetic Brownian motion with a known volatility $\sigma \in \mathbb{R}^+$ and with an unknown trend $\mu_0$ with support $[\mu_{\min}, \mu_{\max}]$. When the distribution of $\mu_0$ is symmetric, the term structure of the discount rate $\rho_t(\beta = 0.5\gamma)$ is flat at $\delta + \gamma \bar{\mu}$. It is increasing when $\beta$ is smaller than $0.5\gamma$, and it is decreasing when $\beta$ is in $[0.5\gamma, \gamma]$.

Proof: See the Appendix. □

Thus, we recover the essentially the same characterization for the slope of the term structure if we replace the assumption of a small degree of uncertainty contained in Proposition 1 by the assumption that this uncertainty is just symmetric around the mean. In the next proposition, we characterize the asymptotic properties of the term structure of discount rates.

Proposition 3: Suppose that $\ln c_t / c_0$ follows an arithmetic Brownian motion with a known volatility $\sigma \in \mathbb{R}^+$ and with an unknown trend $\mu_0$ with support $[\mu_{\min}, \mu_{\max}]$. The efficient discount rates $\rho_t(\beta)$ have the following properties:

- For short horizons, the discount rate tends to $\rho_0(\beta) = \delta + \gamma \sigma^2 (\beta - 0.5\gamma) + \gamma \bar{\mu}$.
- For long horizons, the discount rate tends to

$$\rho_o(\beta) = \begin{cases} 
\delta + \gamma \sigma^2 (\beta - 0.5\gamma) + \gamma \mu_{\min} & \text{if } \beta \leq 0 \\
\delta + \gamma \sigma^2 (\beta - 0.5\gamma) + (\gamma - \beta) \mu_{\min} + \beta \mu_{\max} & \text{if } 0 < \beta \leq \gamma \\
\delta + \gamma \sigma^2 (\beta - 0.5\gamma) + \gamma \mu_{\max} & \text{if } \beta > \gamma
\end{cases} \quad (29)$$

Proof: See the Appendix. □

This proposition provides some interesting insights about the risky discount rates. For short horizons, the ambiguity affecting the trend of the economy has no effect on the risky discount rate. Only the expected trend $\bar{\mu}$ of the economy matters to measure the wealth effect in the short term. For distant futures, the ambiguity affecting the trend is crucial for the determination of the discount rate. The long term wealth effect is equal to the product of $\gamma$ by a growth rate of consumption belonging to its support $[\mu_{\min}, \mu_{\max}]$. Its selection depends here upon the beta of the project. When $\beta$ is negative, the wealth effect should be computed on the basis of the smallest
possible growth rate $\mu_{\text{min}}$ of the economy. On the contrary, when $\beta$ is larger than $\gamma$, the wealth effect should be computed on the basis of the largest possible rate $\mu_{\text{max}}$. When the beta of the project is positive but smaller than $\gamma$, the selected growth rate is a weighted average of $\mu_{\text{min}}$ and $\mu_{\text{max}}$, with weights $(\gamma - \beta) / \gamma$ and $\beta / \gamma$ respectively.

Proposition 3 also tells us that the condition of a symmetric distribution for $\mu_\theta$ in Proposition 2 cannot be relaxed. Indeed, Proposition 3 implies that $\rho_t(\beta = \gamma / 2)$ and $\rho_t(\beta = \gamma / 2)$ are equal only if $\beta$ and $(\mu_{\text{min}} + \mu_{\text{max}}) / 2$ coincide. Most asymmetric distributions will not satisfy this condition, which implies that the constancy of $\rho_t(\beta = \gamma / 2)$ with respect to $t$ will be violated.

In Figure 1, we illustrate some of the above findings through the following numerical example. We assume that $\gamma = 2$, $\sigma = 4\%$ and $\mu_\theta$ is uniformly distributed on interval $[0\%, 3\%]$. The term structure is flat for $\beta = \gamma / 2 = 1$. The discrimination of the discount rate for different betas is increasing in the maturity of the cash flows. This numerical example also illustrates the property that project-specific risk premia are in general not proportional to the project-specific beta. For example, consider a time horizon of 200 years. For this maturity, the risk premia associated to $\beta = 1$ and $\beta = 0.5$ are respectively equal to $\pi_{200}(1) = 2.08\%$ and $\pi_{200}(0.5) = 0.94\% < 0.5\pi_{200}(1)$.  

![Figure 1: Term structure of the risky discount rate (in %) for different betas. We assume that $\gamma = 2$, $\sigma = 4\%$ and $\mu_\theta$ is uniformly distributed on interval $[0\%, 3\%]$.](image-url)
3.2. The volatility of growth is uncertain

Let us now turn to the case in which the trend is known, but the volatility is ambiguous. Weitzman (2007) examined this question by assuming that the $\sigma_{\phi}^2$ has an inverted Gamma distribution. This implies that $\ln(c_t / c_{t-1})$ is a Student’s t-distribution rather than a normal, yielding fat tails, a safe discount rate of $-\infty$ and a market risk premium of $+\infty$. Let us reexamine this question without specifying the distribution of $\sigma_{\phi}^2$, apart from assuming that its variance $\sigma_{\phi}^2 = \sigma^2 + ky_{\phi}$ is such that $k$ is small. Equation (14) implies that

$$ r_{\phi} = \delta + \gamma t - \frac{1}{t} \ln \int e^{0.5 \gamma \sigma_{\phi}} dG(\theta). \quad (30) $$

Lemma 1 implies that

$$ \frac{\partial r_{\phi}}{\partial t} = -H(0.5 \gamma^2 \sigma_{\phi}^2) = -\frac{\gamma^4}{8} Var(\sigma_{\phi}^2) + o(k^2). \quad (31) $$

Similarly, equation (15) implies that

$$ \pi_t(\beta) = -\frac{1}{t} \ln \left[ \int e^{0.5 \beta^2 \sigma_{\phi}^2} dG(\theta) \int e^{0.5 \gamma \sigma_{\phi}} dG(\theta) \right]. \quad (32) $$

Lemma 1 implies in turn that

$$ \frac{\partial \pi_t(\beta)}{\partial t} = H(0.5 \beta^2 \sigma_{\phi}^2) + H(0.5 \gamma^2 \sigma_{\phi}^2) - H(0.5 (\beta - \gamma)^2 \sigma_{\phi}^2) $$

$$ = \frac{1}{8} \left[ \beta^4 + \gamma^4 - (\beta - \gamma)^4 \right] Var(\sigma_{\phi}^2) + o(k^2) $$

$$ = \frac{\beta \gamma}{2} \left[ \beta^2 - \frac{3}{2} \beta \gamma + \gamma^2 \right] Var(\sigma_{\phi}^2) + o(k^2). \quad (33) $$

The first term in the right-hand side of this equation is positive, which implies that the term structure of the risk premium is upward-sloping if the uncertainty on the volatility of growth is small enough. Finally, combining these two results implies that the discount rate $\rho_t(\beta)$ to be used for this project is such that
\[
\frac{\partial \rho_t(\beta)}{\partial t} = \frac{1}{8}\left[\beta^4 - (\beta - \gamma)^4\right]\text{Var}(\sigma_\theta^2) + o(k^2)
= \frac{\gamma(2\beta - \gamma)(\beta^2 + (\beta - \gamma)^2)}{8}\text{Var}(\sigma_\theta^2) + o(k^2).
\] (34)

Thus, when the uncertainty on the volatility is small enough, the term structure of the discount rate to be used for risky projects is downward-sloping if and only if \(\beta\) is smaller than \(\gamma/2\). This result is identical to what we obtained earlier in this section when the uncertainty is about the trend of the economy. The following proposition summarizes the findings of this section.

Proposition 4: Suppose that \(\ln c_t / c_0\) follows an arithmetic Brownian motion with a known trend \(\mu_\theta\) and with a variance \(\sigma_\theta^2\) that entails some small uncertainty. It implies that the term structures of the risk free discount rate and of the aggregate risk premium are respectively decreasing and increasing. Moreover, the term structure of the discount rates \(\rho_t(\beta)\) is decreasing, flat or increasing depending upon \(\beta\) is smaller, equal or larger than \(0.5\gamma\).

In other words, when the parametric uncertainty is small enough, the qualitative properties of the term structures of discount rates are independent of whether the uncertainty is about the trend of growth or about its volatility.

3.3. The trend and the volatility of growth are uncertain

However, it is not true in general that the term structure of \(\rho_t(\beta)\) is decreasing under small parametric uncertainty if and only if \(\beta\) is smaller than \(\gamma/2\). To show this, let \(\mu_\theta\) and \(\sigma_\theta^2\) be statistically dependent. From (14) and (15), we obtain that

\[
\frac{\partial \rho_t(\beta)}{\partial t} = H_t(\beta(\mu_\theta + 0.5\beta\sigma_\theta^2)) - H_t((\beta - \gamma)(\mu_\theta + 0.5(\beta - \gamma)\sigma_\theta^2)).
\] (35)

Using Lemma 1, this implies that
$$2 \frac{\partial \rho_\gamma (\beta)}{\partial t} = Var(\beta(\mu_\delta + 0.5 \beta \sigma_\delta^2)) - Var((\beta - \gamma)(\mu_\delta + 0.5(\beta - \gamma)\sigma_\delta^2))$$

$$= \left[ \beta^2 - (\beta - \gamma)^2 \right] Var(\mu_\delta) + 0.25 \left[ \beta^4 - (\beta - \gamma)^4 \right] Var(\sigma_\delta^2)$$

$$+ \left[ \beta^3 - (\beta - \gamma)^3 \right] cov(\mu_\delta, \sigma_\delta^2).$$

(36)

In the special case with $\beta = \gamma / 2$, this implies in turn that

$$\frac{\partial \rho_\gamma (\beta)}{\partial t} = \frac{\gamma^3}{8} \text{cov}(\mu_\delta, \sigma_\delta^2).$$

(37)

Thus, when $\beta = \gamma / 2$, the signs of the slope of the term structure of the discount rate and of $\text{cov}(\mu_\delta, \sigma_\delta^2)$ coincide. To illustrate this result, let us consider the following simple numerical example. Suppose that $\delta = 0$, $\gamma = 2$ and that there are two possible states of nature. In the first state, $\mu_1 = 0$% and $\sigma_1 = 1$%. In the second state, $\mu_2 = 3$% and $\sigma_2 = 7$%, so that the trend and the volatility of growth are positively correlated. Under this calibration of the model, the discount rate for short-term cash flows with $\beta = 1 = \gamma / 2$ is $\rho_0(1) = 3\%$, and the discount rate for distant cash flows is $\rho_\infty(1) = 3.24\% > \rho_0(1)$.

4. **Term structures with parametric uncertainty and mean-reversion**

In the benchmark specification with a CRRA utility function and a random walk for the growth rate of consumption, the term structures of discount rates are flat and constant through time. In the specification examined in Section 3 with some parametric uncertainty on the underlying random walk of the growth rate, they are monotone and move slowly through time due to the revision of beliefs about the true values of the uncertain parameters. But these processes ignore the cyclicality of the economic activity. The introduction of predictable changes in the trend of growth introduces a new ingredient to the evaluation of investments. When expectations are diminishing, the discount rate associated to short horizons should be reduced to bias investment decisions toward projects that dampen the forthcoming recession. Long termism is a luxury that should be favored only in periods of improving expectations. More generally, when expectations are cyclical, it is important to frequently adapt the price signals contained in the term structure of discount rates to the moving macroeconomic expectations. From this theoretical result, it is clearly inefficient to maintain the U.S. official discount rate unchanged since 1992.
In this section, we propose a simple model in which the economic growth is cyclical, with some uncertainty about the parameter governing this process. Following Bansal and Yaron (2004) for example, the change in log consumption follows an auto-regressive process:

\[
\ln c_{t+1} / c_t = x_t, \\
x_t = \mu_0 + y_t + \varepsilon_{xt}, \\
y_t = \phi y_{t-1} + \varepsilon_{yt},
\]

for some initial state characterized by \( y_{-1} \), where \( \varepsilon_{xt} \) and \( \varepsilon_{yt} \) are independent and serially independent with mean zero and variance \( \sigma_x^2 \) and \( \sigma_y^2 \), respectively. Parameter \( \phi \), which is between 0 and 1, represents the degree of persistence in the expected growth rate process. When \( \phi \) is zero, then the model returns to a pure random walk as in Section 3. We hereafter allow the trend of growth \( \mu_0 \) to be uncertain. By forward induction of (38), it follows that:

\[
\ln c_t / c_0 = \mu_0 t + y_{-1} \phi + \frac{1 - \phi^t}{1 - \phi} + \sum_{t=0}^{t-1} \frac{1 - \phi^{t-\tau}}{1 - \phi} \varepsilon_{yt} + \sum_{t=0}^{t-1} \varepsilon_{yt}.
\]

It implies that, conditional to \( \theta \), \( \ln c_t - \ln c_0 \) is normally distributed with annualized variance

\[
t^{-1}Var(\ln c_t / c_0) = \frac{\sigma_y^2}{(1 - \phi)^2} \left[ 1 - 2 \phi \frac{\phi^t - 1}{t(\phi - 1)} + \phi^2 \frac{\phi^{2t} - 1}{t(\phi^2 - 1)} \right] + \sigma_x^2.
\]

We consider as before an investment project \( \{F_t\} \) with a constant \( \beta \), so that equation (7) holds. The pricing formula (4) can therefore be rewritten as:

\[
\rho(\beta) = \delta - \frac{1}{t} \ln \frac{E e^{(\beta - \gamma) \ln c_t / c_0} / E e^{\beta \ln c_t / c_0}}{E e^{(\beta - \gamma) \mu_0} / E e^{\beta \mu_0}}
\]

\[
= \delta + \gamma y_{-1} \phi \frac{1 - \phi}{t(1 - \phi)} + \gamma (\beta - 0.5 \gamma) \left( \frac{\sigma_y^2}{(1 - \phi)^2} \left[ 1 - 2 \phi \frac{\phi^t - 1}{t(\phi - 1)} + \phi^2 \frac{\phi^{2t} - 1}{t(\phi^2 - 1)} \right] + \sigma_x^2 \right)
\]

\[
= \delta - \frac{1}{t} \ln \frac{E e^{(\beta - \gamma) \mu_0} / E e^{\beta \mu_0}}
\]

---

6 A more general model entails a time-varying volatility of growth as in Bansal and Yaron (2004). Mean-reversion in volatility is useful to explain the cyclicality of the market risk premium.
Bansal and Yaron (2004) consider the following calibration of the model, using annual growth data for the United States over the period 1929-1998. Taking the month as the time unit, they obtained, $\mu = 0.0015$, $\sigma_x = 0.0078$, $\sigma_y = 0.00034$, and $\phi = 0.979$. Using this $\phi$ yields a half-life for macroeconomic shocks of 32 months. Let us assume that $\delta = 0$, and let us introduce some uncertainty about the historical trend of growth from the sure $\mu = 0.0015$ to the uncertain context with two equally likely trends $\mu_1 = 0.0005$ and $\mu_2 = 0.0025$. In figures 2 to 4, we draw the term structures of discount rates for three different positions $y_{-1}$ in the business cycle. In Figure 2, the expected annual growth rate of the economy is 1.2% per year below its unconditional expectation of 1.8%. In this recession phase, the short term discount rate is small at around 1%, but the expectation of a recovery makes the term structure steeply increasing for low maturities. For betas below unity, the term structure is non-monotone because of the fact that for very distant maturities, the effect of parametric uncertainty eventually dominates.

In Figure 3, the expected instantaneous growth rate is at its unconditional expectation of 1.8% ($y_{-1} = 0$). It should be noticed that the term structure is flat for $\beta = 0.5\gamma = 1$ in this case with a symmetric distribution for $\mu_0$, as can be inferred from equation (41). Finally, in Figure 4, the expected instantaneous growth rate is 1.2% per year above its unconditional expectation. In this
expansionary phase of the cycle, the short term discount rate is large at around 6%, but is steeply decreasing for short maturities because of the diminishing expectations.

Figure 3: The discount rate (in % per year) as a function of the maturity (in years) in mid-cycle for different betas. Equation (41) is calibrated as in Figure 2, except for $y_{-1} = 0$.

Figure 4: The discount rate (in % per year) as a function of the maturity (in years) in expansion for different betas. Equation (41) is calibrated as in Figure 2, except for $y_{-1} = 0.001$. 
One can also examine a model in which the current state variable $y_{-1}$ is uncertain. It is easy to generalize equation (41) to examine this ambiguous context. We obtain the following pricing formula:\footnote{Weitzman (2012) considers the special case with permanent shocks ($\phi = 1$).}

\begin{equation}
\rho_t(\beta) = \delta + \gamma (\beta - 0.5 \gamma) \left( \frac{\sigma_y^2}{(1-\phi)^2 \left[ 1 - 2(\phi^t - 1) + \phi^t \phi^{\gamma - 1} \right] + \sigma^2} \right)
\end{equation}

Observe again that when $\beta = 0.5 \gamma$, the term structure of discount rates is flat if both $\mu_\theta$ and $y_{-10}$ have a symmetric distribution function. We also observe that the ambiguity on $y_{-10}$ plays a role similar to the ambiguity on $\mu_\theta$ to shape the term structure. Our numerical simulations available upon request show that the hidden nature of the state variable does not modify the general characteristics of the term structures described above.

5. The beta of mitigation projects

The purpose of this section is to motivate heuristically, and to derive a crude numerical estimate for term structure of discount rates to be used for the evaluation of climatic policies. To do this, we need to answer the following often overlooked question: What is the beta of investments whose main objective is to abate emissions of greenhouse gases? Let us consider a simple two-date version of the DICE model of Nordhaus (2008) and Nordhaus and Boyer (2000):

\begin{align*}
T_1 &= \xi E_1 \\
E_i &= \omega Y_i - I_0 \\
D_1 &= \theta T_i^{\omega_1} \\
Q_i &= (1 - D_i) Y_i \\
C_i &= \alpha Q_i
\end{align*}
All parameters of the model are assumed to be nonnegative. $T_i$ is the increase in temperature and $E_i$ is the emission of greenhouse gases from date 0 to date 1. It is assumed in equation (43) that the increase in temperature is proportional to the emission of these gases. By equation (44), emissions are proportional to the pre-damage production level $Y_i$, but they can be reduced by investing $I_0$ in a green technology at date 0. In equation (45), we assume that the damage $D_i$ expressed as a percentage of pre-damage production is an increasing power function of the increase in temperature. Finally, consumption $C_i$ is proportional to the post-damage production $Q_i$. The damage to the economy at date 1 is thus equal to

$$D_i Y_i = \theta_i (\xi_i (\omega_i - I_0))^{\phi_i} Y_i.$$  
(48)

We consider the beta of a marginal investment $I_0$. Such an investment has the benefit to reduce damages by

$$F_i = - \frac{\partial D_i Y_i}{\partial I_0} \bigg|_{I_0=0} = \theta_i \theta_i \xi_i^{\phi_i} \omega_i^{\phi_i-1} Y_i^{\phi_i}.$$  
(49)

Notice that the benefit $F_i$ cannot be written as a power function of $C_i$ as assumed in equation (7) to derive the CCAPM formulas. This is because aggregate consumption $C_i$ is a non-linear function of the pre-damage production $Y_i$. Thus, the pricing equations derived earlier in this paper can only be used as approximations.

Using this simple model, two possible stories can be developed to determine the beta of mitigation projects depending upon which source of variability is considered. In the story based on the macroeconomic variability, there is some uncertainty about the growth $Y_i$ of the economy between the two dates. When economic growth is high, more greenhouse gases are emitted in the atmosphere and the benefits of mitigation are large. This is thus an argument for a positive beta. In the story based on the climate variability, there is some uncertainty about the intensity of damages, because $\xi_i$, $\omega_i$ or $\theta_i$ is variable. When damages are high, aggregate consumption is small and the benefits of mitigation are large. This is an argument for a negative beta. Let us examine in more details each of these two stories separately.
Suppose first that the only source of variability is the growth of pre-damage production $Y$. From equation (49), the elasticity of the benefits of mitigation to change in future pre-damage production level is a constant $\theta_1$. Observe also that consumption is linked to production through the following relation:

$$\Delta C_i \frac{1-(1+\theta_1)D_i}{1-D_i} \frac{\Delta Y_i}{Y_i}.$$  \hspace{1cm} (50)

We therefore obtain that the beta of mitigation equals

$$\beta = \frac{d \ln F_i}{d \ln C_i} = \frac{d \ln F_i}{d \ln Y_i} \frac{d \ln Y_i}{d \ln C_i} = \theta_2 \frac{1-D_i}{1-(1+\theta_2)D_i}.$$ \hspace{1cm} (51)

Again, because $\beta$ is not constant in $C$, our pricing equations should be interpreted as approximations when the uncertainty on $Y$ is small enough. The range of damage rates $D_i$ in most IAM models is between 0 and 10%, so that the ratio in the right-hand side of the above equality is close to $1+\theta_2D_i$, which is itself close to unity. Thus we have in first approximation that the beta of mitigation investments is equal to $\theta_2$, the elasticity of the monetary impacts of climate change to changes in concentration of greenhouse gases in the atmosphere (or in temperature). Although all authors in the field recognize the scarcity of evidence to infer $\theta_2$, most of them agree that the relation $D_i = f(T_i)$ should be convex, yielding $\theta_2 > 1$. Nordhaus and Boyer (2000) used $\theta_2 = 2$, whereas Cline (1992) used $\theta_2 = 1.3$. The PAGE model used in the Stern (2007) Review draws $\theta_2$ from a triangular probability density function with support in $[1,3]$, giving a mean of about 1.8 (See Dietz, Hope and Patmore, 2007). Although there is no consensus on the value of this parameter, a beta between 1 and 2 seems to emerge as a reasonable range of plausible values. Thus, this story based on economic variability favors a beta that is larger than the average beta in the economy.

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8 Nordhaus (2007) used a quadratic function, yielding a similar degree of convexity of the damage function in the relevant domain of increases in concentration.
Sandsmark and Vennemo (2007) claim that the beta of mitigation investments is negative. Their argument is based on the climate variability as the only source of fluctuation. In their model, the only source of variability is the uncertain damage from climate change. In our simple two-date model above, this would correspond for example to making $\theta_i$ random, thereby generating negatively correlated fluctuations of $F_i$ and $C_i$. Under this assumption, we obtain that

$$\frac{\Delta F_i}{F_i} = \frac{\Delta \theta_i}{\theta_i}, \quad (52)$$

and

$$\frac{\Delta C_i}{C_i} = \frac{\Delta Q_i}{Q_i} = -\frac{\Delta D_i}{1-D_i} = -\frac{D_i}{1-D_i} \frac{\Delta \theta_i}{\theta_i}. \quad (53)$$

This yields

$$\beta = \frac{d \ln F_i}{d \ln C_i} = \frac{d \ln F_i}{d \ln \theta_i} \frac{d \ln \theta_i}{d \ln C_i} = -\frac{1-D_i}{D_i}. \quad (54)$$

This is a large beta in absolute value. Indeed, if we assume a range of damages between 5% and 20% of the aggregate production, we obtain a beta in the range between -4 and -19. This is because the benefits of mitigation are much more volatile than consumption when damages are a small fraction of aggregate consumption.

Obviously in that case, benefits from mitigation are large when damages are large and aggregate consumption is small. In our model, the only source of variability comes from the uncertain growth rate of production. In this case, the benefits from mitigation are large when production and aggregate consumption are large, yielding a positive beta. We believe that this economic source of variability has an order of magnitude larger than the climatic source of variability. When the annual growth rate of the economy varies between 0% and 3%, aggregate consumption in 100 years varies between 0% and 2000%. This should be compared to climate damages for this time horizon which are usually estimated between 0% and 5% (see for example the Stern

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9 They obtain $\beta = -0.004$, which is much closer to zero than what we obtain here. However, notice that these authors consider another definition of the beta, which is equal to the ratio of the covariance of $(C_i,F_i)$ to the variance of $C_i$. In our model, the beta is equal to the ratio of the covariance of $(\ln C_i, \ln F_i)$ to the variance of $\ln C_i$. 

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Review, 2007). To make this argument more concrete, let us consider the calibration of the simple above model as described in Table 1.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>50 years</td>
<td>Time horizon between dates 0 and 1.</td>
</tr>
<tr>
<td>$Y_i = e^{\sum_{i=1}^{n} x_i 1(\theta_i)}$</td>
<td>$\theta_i \sim N(\theta, \sigma^2)$</td>
<td>$\theta$ is normalized to unity. Conditional to $\theta$, the growth rate of production follows a normal random walk. The expected change in log production has a mean that is uniformly distributed on [0%, 3%].</td>
</tr>
<tr>
<td>$\omega$</td>
<td>1</td>
<td>Normalization</td>
</tr>
<tr>
<td>$\xi$</td>
<td>0.43</td>
<td>This implies that the expected increase in temperature in the next 50 years equals $\xi EY_i = 1^\circ C$.</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>2</td>
<td>Nordhaus and Boyer (2000)</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>$\sim U[0.2%, 1.8%]$</td>
<td>This means that the damage at the average temperature increase of 1°C is uniformly distributed on [0.2%, 1.8%] of pre-damage production.</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.75</td>
<td>Consumption equals 75% of post-damage production.</td>
</tr>
</tbody>
</table>

Table 1: Calibration of the two-date IAM model

In Figure 5, we document the outcome of 10,000 independent random selections of the pair $(Y_i, \theta_i)$ and the corresponding outcomes in terms of aggregate consumption $C_i$ and benefits of mitigation $F_i$. Using these data, we regressed $\ln F_i$ on $\ln C_i$. The OLS estimation of the beta equals $\hat{\beta} = 2.04$ with a standard deviation of 0.012 and an adjusted $R^2$ of 0.75. The plain curve in Figure 5 describes this best fit of the model. An estimated beta around $\theta_z(1 + \theta_2 \bar{D}) \approx 2.04$ as anticipated by equation (51) demonstrates the dominance of the economic story over the climatic story to characterize the sensitiveness of the benefits of mitigation with respect to changes in aggregate consumption.\(^{10}\)

\(^{10}\) Increasing the degree of uncertainty affecting $\theta_i$ has a limited impact on this estimation. For example, if we replace the assumption that the interval [0.2%, 1.8%] on which it is uniformly distributed by [0%, 10%], the OLS estimation of the beta goes down to 1.99%.
Figure 5: Monte-Carlo experience with 10 000 draws of the model described in Table 1. The plain curve describe the OLS regression of $\ln F_t = \alpha + \beta \ln C_t + \epsilon$.

In Figure 6, we draw the term structures of discount rates prevailing for $\beta = 2$ in three different phases of the macroeconomic cycle under the calibration used in the previous section. The discount rate to be used for super-long maturities is around 6.8%, whereas the short-term discount rate fluctuates along the business cycle from around 1.5% when the expected instantaneous trend is 1.2% per annum below its historical mean, to around 6% when the expected instantaneous trend is 1.2% above its historical mean.
Figure 6: The discount rate (in % per year) as a function of the maturity (in years) for $\beta = 2$ in different phases of the cycle. Equation (41) is calibrated with $\delta = 0$, $\gamma = 2$, $\sigma = 0.0078$, $\sigma_y = 0.00034$, $\phi = 0.979$, and two equally likely trends $\mu_1 = 0.0005$ and $\mu_2 = 0.0025$.

6. Concluding remarks

This paper contributes to the debate on the discount rate for climate change in several directions. We developed a model with a Brownian motion for the growth of the economy, with some uncertainty about the parameter of this underlying stochastic process. We then extended this model to include business cycles into the picture. Our main messages in this framework are as follows. First, the shape of the term structure of discount rates to be used to evaluate risky projects is determined by the relative intensity of a precautionary effect that pushes towards a decreasing term structure, and of a risk aversion effect that pushes towards an increasing term structure. Under some weak restrictions on the distribution of the uncertain parameters, the term structure is decreasing or increasing depending upon whether the beta of the project is respectively smaller or larger than half the relative risk aversion of the representative agent. Second, the risk premium associated to a project is generally not proportional to its beta, which implies that knowing the aggregate risk premium and the project’s beta is not enough to compute the project-specific risk premium. Third, when shocks on the growth rate of the economy are persistent, short-term and medium-term discount rates should be frequently revised as a function of changes in the state variable. The term structures of discount rates are generally not monotone in that context. Finally, we have shown that there are reasons to believe that the beta of projects to mitigate climate change is relatively large, around 2. This allows us to conclude that the discount rates to be used to evaluate public policies to fight climate change should be increasing with respect to maturities. Given the current global economic crisis in the western world, following Figure 6, we are in favor of using a real discount rate for climate change around 1% for short horizons, up to 6% for maturities exceeding 100 years.

A word of caution should be added to this conclusion. The use of price signals like discount rates and risk premia is possible only for investment projects that are marginal, i.e., for actions that do not affect expectations about the growth of the economy. The reader should be aware that this assumption does not hold when considering the global strategy to fight climate change. The right
evaluation approach for global projects relies on the direct measure of the impact of the global action on the intergenerational social welfare function, as done for example in Nordhaus (2008) and Stern (2007).
Appendix A: Proof of Lemma 1

Define function \( h_t \) such that

\[
h_t(k) = H_t(x_0 + ky) = \frac{1}{t} \frac{E(x_0 + ky)e^{t(x_0 + ky)}}{Ee^{t(x_0 + ky)}} - \frac{1}{t^2} \ln Ee^{t(x_0 + ky)}.
\] (55)

We have that \( h_t(0) = 0 \) and

\[
h_t'(k) = \frac{Ey(x_0 + ky)e^{t(x_0 + ky)}}{Ee^{t(x_0 + ky)}} - \frac{E(x_0 + ky)e^{t(x_0 + ky)}Ee^{t(x_0 + ky)}}{(Ee^{t(x_0 + ky)})^2}.
\] (56)

It implies that

\[ h_t'(0) = tx_0 Ey - tx_0 Ey = 0. \] (57)

We also obtain that

\[
h_t''(k) = \frac{Ey^2 e^{t(x_0 + ky)}}{Ee^{t(x_0 + ky)}} + \frac{Ety^2 (x_0 + ky)e^{t(x_0 + ky)}}{Ee^{t(x_0 + ky)}} - \frac{Ey(x_0 + ky)e^{t(x_0 + ky)}Ete^{t(x_0 + ky)}}{(Ee^{t(x_0 + ky)})^2} - \frac{(Ete^{t(x_0 + ky)})^2}{Ee^{t(x_0 + ky)}} - \frac{Et(x_0 + ky)e^{t(x_0 + ky)}Ey^{t(x_0 + ky)}}{(Ee^{t(x_0 + ky)})^2} - \frac{2Et(x_0 + ky)e^{t(x_0 + ky)}(Ete^{t(x_0 + ky)})^2}{(Ee^{t(x_0 + ky)})^3}.
\] (58)

It yields

\[ h_t''(0) = \sigma_y^2 + tx_0 \sigma_y^2 - tx_0 \sigma_y^2 = \sigma_y^2. \] (59)

The Taylor expansion of \( h_t(k) \) yields in turn

\[
H_t(x_k) = h_t(k) = h_t(0) + kh_t'(0) + 0.5k^2 h_t''(0) + o(k^2)
\] (60)

This concludes the proof. □
Appendix B: Proof of Proposition 2

When $\beta = \gamma / 2$, equation (28) implies that

$$-t \frac{\partial \rho_t(\beta = \gamma / 2)}{\partial t} = H_t(-\beta(\mu_0 - \bar{\mu})) - H_t(\beta(\mu_0 - \bar{\mu})) = 0.$$  \hspace{1cm} (61)

This is a direct consequence of the assumption that $\mu_0$ has a symmetric distribution. Thus, the term structure of $\rho_t$ is flat for $\beta = \gamma / 2$.

Fully differentiating equation (28) with respect to $\gamma$ yields

$$-t \frac{\partial^2 \rho_t(\beta)}{\partial \gamma \partial t} = \frac{E(\mu_0 - \bar{\mu})e^{t(\beta - \gamma)(\mu_0 - \bar{\mu})}}{Ee^{t(\beta - \gamma)(\mu_0 - \bar{\mu})}} - t(\beta - \gamma) \frac{E(\mu_0 - \bar{\mu})^2 e^{t(\beta - \gamma)(\mu_0 - \bar{\mu})}}{Ee^{t(\beta - \gamma)(\mu_0 - \bar{\mu})}}$$

$$+t(\beta - \gamma) \left( \frac{E(\mu_0 - \bar{\mu})e^{t(\beta - \gamma)(\mu_0 - \bar{\mu})}}{Ee^{t(\beta - \gamma)(\mu_0 - \bar{\mu})}} \right)^2 + \frac{E(\mu_0 - \bar{\mu})e^{t(\beta - \gamma)(\mu_0 - \bar{\mu})}}{Ee^{t(\beta - \gamma)(\mu_0 - \bar{\mu})}}$$

$$= -t(\beta - \gamma) \left[ \int (\mu_0 - \bar{\mu})^2 d\hat{G}(\theta) - \left( \int (\mu_0 - \bar{\mu}) d\hat{G}(\theta) \right)^2 \right],$$

where

$$d\hat{G}(\theta) = \frac{e^{t(\beta - \gamma)(\mu_0 - \bar{\mu})}}{Ee^{t(\beta - \gamma)(\mu_0 - \bar{\mu})}} dG(\theta)$$  \hspace{1cm} (63)

characterizes a distorted probability distribution. Because the bracketed term in the RHS of (62) is positive, we have that $\partial \rho_t(\beta) / \partial t$ is decreasing in $\gamma$ for all $\beta \in [-\infty, \gamma]$. Because it vanishes at $\beta = \gamma / 2$, we obtain that $\partial \rho_t(\beta) / \partial t$ is negative for all $\beta < \gamma / 2$, and it is positive for all $\beta \in [\gamma / 2, \gamma]$. □

Appendix C: Proof of Proposition 3

Consider equation (27). When $t$ tends to zero, using L’Hospital’s rule implies that the limit of $\rho_t(\beta)$ when $t$ tends to zero equals
\[
\lim_{t \to 0} \left[ \delta + \gamma \sigma^2 \left( \beta - \gamma / 2 \right) - \left( \beta - \gamma \right) \frac{\mu \varphi_{\beta \gamma} dG(\theta)}{e^{(\beta - \gamma) \mu \varphi} dG(\theta)} + \beta \frac{\mu \varphi_{\beta} dG(\theta)}{e^{(\beta - \gamma) \mu \varphi} dG(\theta)} \right].
\] (64)

This implies that

\[
\lim_{t \to 0} \rho_{t}(\beta) = \delta + \gamma \bar{\sigma} + \gamma \sigma^2 \left( \beta - \gamma / 2 \right).
\] (65)

When \( t \) tends to infinity, the same technique implies that the term structure of \( \rho_{t}(\beta) \) tends asymptotically to

\[
\delta + \gamma \sigma^2 \left( \beta - \gamma / 2 \right) + \lim_{t \to \infty} \left[ \left( \beta - \gamma \right) \frac{\mu \varphi_{\beta \gamma} dG(\theta)}{e^{(\beta - \gamma) \mu \varphi} dG(\theta)} + \beta \frac{\mu \varphi_{\beta} dG(\theta)}{e^{(\beta - \gamma) \mu \varphi} dG(\theta)} \right].
\] (66)

When \( \beta > \gamma > 0 \), both ratios in the limit tend to \( \mu_{\max} \), so that we obtain that

\[
\lim_{t \to \infty} \rho_{t}(\beta) = \delta + \gamma \mu_{\max} + \gamma \sigma^2 \left( \beta - \gamma / 2 \right).
\] (67)

When \( 0 < \beta < \gamma \), the first ratio tends to \( \mu_{\min} \) and the second ratio tends to \( \mu_{\max} \). This implies that

\[
\lim_{t \to \infty} \rho_{t}(\beta) = \delta + \left( \gamma - \beta \right) \mu_{\min} + \beta \mu_{\max} + \gamma \sigma^2 \left( \beta - \gamma / 2 \right).
\] (68)

The third case with \( \beta < 0 \) follows in a similar fashion.

Consider now the case with \( \beta = \gamma / 2 \). It yields

\[
\rho_{t}(\beta = \gamma / 2) = \delta - \frac{1}{t} \ln \frac{e^{-\beta \varphi_{\gamma \gamma} dG(\theta)}}{e^{\beta \varphi_{\gamma} dG(\theta)}}.
\] (69)

This can be rewritten as follows:

\[
\rho_{t}(\beta = \gamma / 2) = \delta + \gamma \mu_{\max} + \mu_{\min} - \frac{1}{t} \ln \frac{\int e^{\left( \frac{\mu_{\max} + \mu_{\min}}{2} \right)} dG(\theta)}{\int e^{\left( \frac{\mu_{\gamma} + \mu_{\min}}{2} \right)} dG(\theta)}.
\] (70)

Obviously, if \( \mu_{\theta} \) has a symmetric distribution, the ratio in the last term of the right-hand side of this equation equals unity for all \( t \). This yields the result. □
References


Kimball, M.S., (1990), Precautionary savings in the small and in the large, *Econometrica*, 58, 53-73.


