Optimal illusions and the simplification of beliefs

Christian Gollier[1]
Toulouse School of Economics (LERNA and IDEI)

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Abstract

Following Brunnermeier and Parker (2005), we examine a decision problem in which the agent can manipulate her beliefs to rationalize her behavior and to extract more benefit from her anticipatory feelings. The optimal beliefs are a best compromise between this benefit and the cost of the risk-taking inefficiency that these optimistic beliefs will generate. We show that the optimal beliefs will be degenerated, with a number of states with a positive probability that does not exceed one plus the number of degrees of freedom in the decision problem. For example, in the portfolio problem with n independent assets, the optimal beliefs will have no more than n states of nature with a positive subjective probability. In the one-safe-one-risky-asset model, we also show that the two possible excess returns with a positive subjective probability are concentrated at the two boundaries of the support of the objective probability distribution. Therefore, we claim that the transformation of probabilities described in Cumulative Prospect Theory, rather than being a genetic characteristic of human beings, corresponds to a natural tendency of rational agents to select beliefs that maximize their intertemporal welfare.

Keywords: Anticipatory feelings, rationalization, positive thinking, endogenous beliefs, cumulative prospect theory, optimism.

1 Introduction

Festinger (1957) is a psychologist who is mostly known for his theory of cognitive dissonance, which suggests that inconsistency between behaviors and the rational expectations about the utility consequences of these behaviors will cause an uncomfortable psychological tension. He has been among the first to suggest that cognitive dissonance may lead people to change their beliefs to fit their actual behavior, rather than the other way around, as popular wisdom may suggest. To illustrate, let us consider three obvious sources of cognitive dissonance: smoking, gambling and portfolio choice. It is known that smoking is a risky welfare-reducing habit for most people, in particular those who value the future at a sensible discount rate. The objective risk of lung cancer is therefore dissonant with the act to smoke. Some rationalize their behavior by looking on the optimistic side, using all sources of information that can bias their subjective beliefs towards a low impact of smoking on health. Most smokers are clever enough to come up with ad hoc rationalizations to smoke without stressing too much about the consequences. Manipulating one's beliefs is useful to fight cognitive dissonance but can also be dangerous for one's lifetime well-being. The question for smokers is then to determine the best combination of cognitive manipulation, rationalization and risk-taking.

The florishing markets for lottery games provide another illustration of the model presented in this paper. In his famous book entitled "The Gambler", Dostoevsky describes a young middle-class man who dreams that he will become wealthy by gambling one day at the casino. However, he perfectly knows that the odds at the casino are unfair, as he forcefully advises other people not to gamble. When gambling, he is confronted to a cognitive dissonance that he resolves by distorting his beliefs. This illustrates what Sigmund Freud will describe sixty years later as illusions, i.e., beliefs that establish themselves by the will of one's desires. The gambler's optimistic dream allows him to survive in a world of pretentious wealthy Russian expatriates. However, relying on his subjective beliefs, the gambler decides to take a chance, and eventually loses everything.

In a portfolio context, positive thinking implies a mental manipulation of the objective probability distribution of assets returns. If the agent invests in stocks, it is tempting to raise the subjective probability of the large excess returns, in particular if the portfolio will be liquidated in a distant time. This may be intended to rationalize a large demand for these stocks. However, this manipulation of beliefs will have a negative indirect impact on welfare through its adverse effect on the portfolio allocation, which will be incompatible with the objective distribution of returns. This in turn affects negatively the investor's future felicity. Whether this manipulation will have a positive impact on lifetime welfare will depend upon the intensity of anticipatory feelings. Many other illustrations of our model could be discussed, from the determination of efforts to prepare an exam or a marathon, the decision to declare one's revenues to the tax authority, the choice of religious beliefs, or the investment in one's human capital. Because the optimistic beliefs raise current felicity from dreaming and savoring but reduce future felicity due to inefficient risk decision, the problem of cognitive dissonance is to determine the best compromise between these two opposite forces.

In this paper, we consider a model that combines these different ingredients: anticipatory feelings, manipulation of beliefs, cognitive dissonance, and rationalization of inefficient decision under risk. In order to fit with the ideas developed by Dostoevsky, Freud and Festinger, among others, we use the crucial concept of "optimal beliefs" introduced by Brunnermeier and Parker (2005). An important implicit ingredient of the above stories is the idea that, consciously, unconsciously or through a Darwinian selection process, the degree of optimism and the associated rationalized behavior are determined in order to maximize the agent's lifetime utility. Following Brunnermeier and Parker (2005), this maximization process takes into account all psychological and socioeconomic costs and benefits generated by the manipulation of beliefs. Because psychological manipulations have objective socioeconomic costs generated by inefficient risk management, the agent faces at that early stage in life some cognitive dissonance due to the coexistence of potentially incompatible subjective and objective beliefs. However, once this selection has been made, one's eliminates all sources of information that are incompatible with one's selected beliefs, thereby eliminating cognitive dissonance. Later in life but prior to the resolution of uncertainty, the agent extracts felicity from anticipatory feelings. We assume that this felicity can be measured by the subjective future expected utility of the agent. We also assume that one's behavior is compatible with one's subjective beliefs. In other words, the manipulation of beliefs rationalizes behaviors.

Consider the one-safe-one-risky-asset model that has only one degree of freedom, which is a special case of the model presented in this paper.[1] Exploiting the linearity of subjective expected utility with respect to state probabilities, we prove that the optimal subjective probability distribution must be degenerated with at most two atoms, i.e., optimal beliefs are binary. When the true probability distribution has more than states, the optimal subjective probability distribution is thus a simplification of the real world. This result is true for any von Neumann-Morgenstern preference functional, any intensity of anticipatory feelings, and any objective distribution of the risky asset. This means in particular that this degeneracy result is independent of the notion of risk aversion, or that it does not rely on Jensen's inequality. In a second step, we show under weak restrictions on the utility function that investors select the two atoms that are at the bounds of the set of possible asset returns. In other words, optimally controlling beliefs leads individuals to believe that only the smallest possible return and the largest possible return can have a positive probability to occur. This strong result is compatible with Hurwicz's criterion in which only the worst and the best possible outcomes matter for the decision under risk.[2] It is also related to smoother version of this idea introduced by Tversky and Kahneman (1992) that the decision maker modifies the objective probability distribution in favor of the more extreme outcomes. Cumulative prospect theory takes this into account by assuming an inverse S-shaped transformation function of the objective cumulative distribution function. This is equivalent to transferring the probability mass from the interior of the support of the distribution to its lower and upper bounds. Our work suggests that the transformation of probabilities described in cumulative prospect theory, rather than being a genetic characteristic of human beings, corresponds to a natural tendency of rational agents to optimize their intertemporal welfare.

This work departs from the long tradition in economics to measure lifetime utility as a discounted sum of the flow of instantaneous felicity gen-

[1]Alternative interpretations of our choice problem can be found in insurance economics and in the theory of investment. A consumer faces a risk of loss for which there exists an insurance market offering proportional insurance contracts with an actuarially unfair tariff. The problem of the consumer is to select the rate of insurance coverage for the risk. In the theory of investment, a risk-averse entrepreneur with a linear technology must determine the optimal capacity of production under uncertainty about the output price.

[2]See Hurwicz (1951) and extensions by Gilboa (1988), Jaffray (1988), Cohen (1992) and Essid (1997).

erated by immediate consumption, as described for example by Samuelson (1937). This tradition is incompatible with the intuition that happiness is extracted not only from the immediate consumption of goods and services, but also from feelings from past and future consumption (Caplin and Leahy (2004)). This is particularly the case for thoughts related to savoring the possibility of future pleasant events, or to the fear associated to the consequences of adverse ones. Anticipatory feelings have been incorporated in preferences by Caplin and Leahy (2001). Kopczuk and Slemrod (2005) considered a model in which instantaneous felicity is decreasing in the intensity of anticipation of death in the future. In the economic literature, Akerlof and Dickens (1982) were the first to assume that subjective beliefs are derived from a welfare-maximizing process. This work is also related to the more recent literature on self-control and willpower (see for example Carrillo and Mariotti (2000) and Benabou and Tirole (2004)), in which one rationalizes the natural tendency of human beings to succumb to short-term impulses at the cost of their long-run interests. This is usually justified through the existence of time-inconsistent preferences. In this paper, we alternatively explain the weakness of self-control by one's ability to manipulate one's beliefs to limit the long-term adverse impact that short-termist behavior may yield.

In the next section, we describe our general model with n degrees of freedom in the decision problem under risk. In section 3, we prove the main propositions of the paper. Section 4 is devoted to the special case of the one-safe-one-risky-asset model. In section 5, we illustrate the model with a simple numerical example, whereas we conclude in section 6.

2 The model

We consider a simple decision problem under uncertainty. There are S possible states of nature indexed by $s = 1, ..., S$. The decision variables are characterized by a vector $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^n$, where $n \geq 1$ is the number of degrees of freedom in risk management. Final consumption c is a function from $\mathbb{R}^{n+1}$ to $\mathbb{R}$, where $c(\alpha, s)$ is final consumption in state s when decision α has been made ex ante. We suppose that $c(\alpha, s) \geq 0$ is differentiable and concave in α for all $s = 1, ..., S$. We also assume that the agent's preferences under risk satisfy the von Neumann-Morgenstern axioms of rationality. This implies that there exists an increasing utility function u on final consumption such that welfare is measured ex ante by the expected utility of final consumption. We assume that this utility function is differentiable and concave. We define $A(z) = -u''(z)/u'(z)$ as the Arrow-Pratt index of absolute risk aversion.

We suppose that there is an objective probability distribution $Q = (q_1, ..., q_S) \in \mathcal{L}^S$ on the set of possible states of nature, where $\mathcal{L}^S$ is the simplex of $\mathbb{R}^{\mathbb{S}}$. However, the agent may hide this information in day-to-day life and adopt alternative beliefs. These subjective beliefs are characterized by the subjective probability distribution $P = (p_1, ..., p_S)$. The existence of discrepancies between P and Q illustrates a cognitive manipulation.

Before the resolution of the uncertainty, the agent extracts welfare from anticipating felicity from future consumption. These anticipatory feelings generate felicity that is assumed to be measured by the subjective expected utility, i.e., the expected utility under distribution P:

$$\sum_{s=1}^{S} p_s u(c(\alpha, s)). \tag{1}$$

To be consistent with these subjective beliefs in her day-to-day life, the agent selects the strategy α that maximizes her subjective expected utility:

$$S(P) = \max_{\alpha} \sum_{s=1}^{S} p_s u(c(\alpha, s)). \tag{2}$$

An interpretation of equation (2) is that subjective beliefs P rationalize choice $\alpha(P)$. $S(P)$ measures the agent's felicity prior to the resolution of the uncertainty. Let $\alpha(P)$ denote the argument of the maximum. Because the objective function of the above program is concave in vector α, the first-order conditions on $\alpha(P)$ are necessary and sufficient:

$$\sum_{s=1}^{S} p_s u'(c(\alpha(P), s)) \frac{\partial c}{\partial \alpha_i}(\alpha(P), s) = 0 \quad i = 1, ..., n. \tag{3}$$

We assume that a bounded solution $\alpha(P)$ exists to system (3) of equations.

Ex ante, one can measure the objective expected utility $O(P, Q)$ of the agent by using the objective probability distribution Q over the set of possible states of nature. It yields

$$O(P, Q) = \sum_{s=1}^{S} q_s u(c(\alpha(P), s)). \tag{4}$$

Observe that the objective expected utility O is a function of the subjective probability distribution because the later determines the optimal risk exposure $\alpha(P)$. Because of the use of manipulated beliefs when determining $\alpha(P)$, the objective expected utility $O(P, Q)$ will generically be smaller than the maximum objective expected utility that can be obtained by using the objective probability distribution:

$$O(P, Q) \leq \max_{\alpha} \sum_{s=1}^{S} q_s u(c(\alpha, s)). \tag{5}$$

This inequality expresses the welfare cost of manipulating beliefs. This should be compared to the welfare benefit of this manipulation, which comes from the ability to extract more utility from anticipatory feelings like dreaming and savoring.

The lifetime welfare W of the agent is a function of the ex ante felicity generated by anticipatory feelings and of the objective expectation of the felicity generated by final consumption:

$$W(P, Q) = \phi(S(P), O(P, Q)). \tag{6}$$

The aggregator function ϕ is assumed to be increasing in its two arguments.[3] Following Brunnermeier and Parker (2005), we assume that the subjective beliefs P used by the agent is the outcome of an optimization process. This selection is made early in the decision process. At the time of this selection, the agent takes into account all costs and benefits of manipulating beliefs. At that time, she is aware of the fact that making P more optimistic than Q will raise the ex ante benefit of anticipatory feelings, but will induce her to make bad decisions – in terms of the objective expected utility – in the day-to-day management of the risk. Following the literature about cognitive dissonance initiated by Festinger (1957), the agent will transform Q into P by manipulating his search and memorization of information about the risk.

[3] Brunnermeier and Parker (2005) consider the special case with $\phi = S + O$.

Alternatively, we may justify this optimization procedure by a Darwinian selection process in which only those who are endowed with the lifetime-welfare-maximizing beliefs do survive in the long run. In this alternative justification of the model, consumers are never aware of the objective probability distribution. To sum up, we assume that P maximizes the agent's ex ante aggregate welfare, which is an increasing function of the ex ante utility extracted from anticipatory feelings, and of the objective expected utility extracted from consumption ex post:

$$P^* = \arg\max_{P \in \mathcal{L}^S} W(P, Q). \tag{7}$$

The optimal risk choice is denoted $\alpha^* = \alpha(P^*)$. Once the opimal subjective belief P^* is selected at that early stage of life, the objective probability distribution Q is hidden.

The timing of the model is thus as follows:

- At date 0, the agent is aware of the objective probability distribution Q. She selects the subjective probability distribution P^* that maximizes her aggregate welfare W evaluated at that date. Alternatively, P^* is the outcome of a Darwinian selection process. Distribution Q is forgotten for the remaining lifetime.

- At date 1, the agent selects the vector α^* that maximizes her subjective expected utility. She extracts welfare S from her anticipatory feelings.

- At date 2, the state of nature s is revealed. The agent consumes $c(\alpha^*, s)$, and extracts felicity $u(c(\alpha^*, s))$.

The main objective of the paper is to characterize P^*.

3 A basic property of optimal beliefs

As stated in the previous section, the objective expected felicity O generated by final consumption depends upon beliefs P only through its effect on the earlier choice of the optimal choice $\alpha^* = \alpha(P^*)$. In general, there is more than one probability distribution that yields the optimal choice α^*. Let $B(\alpha)$

be the non-empty set of subjective cumulative probability distributions that yield the same optimal choice α:

$$B(\alpha) = \left\{ P \in \mathcal{L}^{\mathcal{S}} \mid \alpha(\mathcal{P}) = \alpha \right\}. \tag{8}$$

It implies that $O(P, Q) = O(P^*, Q)$ for all P in $B(\alpha^*)$. This observation has an important consequence on the structure of optimal beliefs. From the various subjective probability distributions P that yield this choice α^*, the one that is selected by the consumer prior to date 0 must maximize the anticipatory felicity $S(P)$, since they all yield the same date-1 felicity $O(P^*, Q)$. In other words, it must be true that

$$P^* \in \arg \max_{P \in B(\alpha^*)} S(P). \tag{9}$$

Observe that this property of optimal beliefs holds independent of the characteristics of the objective probability distribution Q. It allows us to derive the following useful properties of optimal beliefs.

Proposition 1 *Let n denote the dimension of the vector α of decision variables. The optimal subjective probability distribution P^* has at most $n+1$ atoms.*

Proof: We can rewrite problem (9) as follows:

$$(p_1^*, ..., p_S^*) \in \arg \max_{(p_1, ..., p_S) \geq 0} \quad \sum_{s=1}^{S} p_s u(c(\alpha^*, s)) \tag{10}$$

$$\begin{aligned} s.t. \quad \sum_{s=1}^{S} p_s u'(c(\alpha^*, s)) \frac{\partial c}{\partial \alpha_i}(\alpha^*, s) &= 0 \qquad i = 1, ..., n. \\ \sum_{s=1}^{S} p_s &= 1. \end{aligned}$$

The first constraint states that P belongs to $B(\alpha^*)$, i.e., that beliefs P yield the optimal risk exposure α^*. Because the feasible set of this program is a subset of $\mathcal{L}^{\mathcal{S}}$, it is compact, which implies that this problem has a solution. Observe that the above program is a linear programming problem on a compact set with $n+1$ equality constraints. From the fundamental theorem of linear programming (Luenberger (1984)), its solution has at most $n+1$ states s with $p_s > 0$. ■

This result is quite robust. For example, it is independent of the individual degree of risk aversion. It is also independent of the representation of the long-term welfare O that could be measured through a non-expected utility criterion. Moreover, throughout this paper, we assume that beliefs can be distorted without any constraint. A more realistic assumption would be to assume that only a proportion $1-\kappa \in [0,1]$ of the probability mass can be manipulated. As shown in the proof of the following proposition,[4] the same logic as above can be extended to a model where, for any $s = 1, ..., S$, p_s is constrained to be larger or equal to κq_s rather than to zero. Under this constrained model, the optimal beliefs must be such that $p_s^* = \kappa q_s$ for all $s \in \{1, ..., S\}$ but $n+1$ states.

Proposition 2 *Suppose that the agent is constrained in its beliefs to preserve a minimal share $1-\kappa$ of each objective state probability: $p_s \geq (1-\kappa)q_s$ for $s = 1, ..., S$. Then, the optimal subjective probability distribution P^* has at most $n+1$ states s in which the subjective probability p_s is larger than $(1-\kappa)q_s$.*

Proof: Let r_s be defined as $r_s = p_s - (1-\kappa)q_s$. Replacing constraints $p_s \geq 0$ by constraints $r_s \geq 0$, we can rewrite program (10) as follows

$$(r_1^*, ..., r_S^*) \in \arg \max_{(r_1,...,r_S)\geq 0} \quad (1-\kappa)O + \sum_{s=1}^{S} r_s u(c(\alpha^*, s)) \tag{11}$$

$$\begin{aligned} s.t. \quad \sum_{s=1}^{S} r_s u'(c(\alpha^*, s)) \frac{\partial c}{\partial \alpha_i}(\alpha^*, s) &= b_i \quad i = 1, ..., n. \\ \sum_{s=1}^{S} r_s &= \kappa, \end{aligned}$$

with $O = O(P^*, Q)$ and

$$b_i = -(1-\kappa) \sum_{s=1}^{S} q_s u'(c(\alpha^*, s)) \frac{\partial c}{\partial \alpha_i}(\alpha^*, s).$$

[4]This proposition was suggested to us by a reviewer.

Using the fundamental theorem of linear programming, we obtain that the solution of the above program has at most $n+1$ positive components. ■

Of course, these results are not robust to the generalization of the measurement of anticipatory feelings by a non-expected utility criterion. The use of an expected utility functional for S is essential to derive this class of results, since they rely on the linearity of the objective function with respect to probabilities.

Proposition 1 is the central result of this paper. Whatever the preferences and the objective probability distribution, the optimal beliefs have no more atoms of positive probability than one plus the number of degrees of freedom in risk management. Suppose for example that the decision problem be a portfolio choice with n risky assets and one safe asset. Let $x_i(s)$ denote the excess return of asset i in state s, R is the gross risk free rate, and α_i is the investment in risky asset i. If w_0 denote initial wealth, final wealth in state s equals

$$c(\alpha, s) = \left(w_0 - \sum_{i=1}^{n} \alpha_i \right) R + \sum_{i=1}^{n} \alpha_i x_i(s). \tag{12}$$

This is concave in α. In spite of the existence of $n+1$ assets, there are only n degree of freedom because of the ex ante budget constraint. Applying Proposition 1 implies that the optimal beliefs contain at most $n+1$ states of nature with a positive subjective probability. If financial markets are incomplete, $n+1 < S$ and this proposition provides a useful characterization of optimal beliefs.[5] These beliefs are a simplication of reality, since some states of nature are ignored.

4 The 2-asset portfolio problem

In the remainder of this paper, we consider the standard model with one risky asset and one safe asset, in which case the optimal beliefs must be binary. This special case has many real world applications. Financial decisions are often represented by determining the share of wealth that should be invested in a risky fund. Moreover, the one-risky-one-safe-asset model is equivalent

[5] Brunnermeier, Gollier and Parker (2007) examine the portfolio choice problem under complete markets. They show in particular that this model may explain why people have skewed beliefs.

to the coinsurance problem in which a risk-averse agent must determine the share of an uncertain loss to cover by an actuarially unfair insurance contract. Because there is only one degree of freedom (the coverage rate) here also, the policyholders' optimal beliefs must be binary.

Thus, suppose that $\alpha \in \mathbb{R}$, and that $c(\alpha, s) = w_0 + \alpha x_s$. This can be interpreted as a 2-asset portfolio problem with intitial wealth $w_0 > 0$, a safe asset whose return $R-1$ is normalized to zero, and a risky asset whose return in state s is denoted x_s. We consider the limit case of an infinite number of states so that the support of the distribution of excess returns is the interval $[a, b]$ with $a < 0 < b$. Proposition 1 implies that the optimal subjective beliefs take the form $P^* = (x_-, 1-p^*; x_+,p^*)$ for some pair $(x_-, x_+) \in [a, 0] \times [0, b]$ and for some probability $p^* \in [0, 1]$ of the good state. The risk exposure α^* is rationalized by the subjective probability p^* via the following the first-order condition:

$$p^* x_+ u'(w_0 + \alpha^* x_+) + (1 - p^*) x_- u'(w_0 + \alpha^* x_-) = 0 \tag{13}$$

Proposition 1 is useful because it replaces the problem of finding a probability distribution with support in $[a, b]$ into a problem of finding a triplet (x_-, x_+, p^*) that maximizes $W(P)$. In the next proposition, we claim that the two subjectively possible returns are at the boundaries of the support of the objective probability distribution under some mild additional assumptions on the utility function. Let $R(z) = zA(z) = -zu''(z)/u'(z)$ denote the relative risk aversion of u evaluated at z. It is weakly increasing if $R'(z)$ is non-negative for all $z > 0$.

Proposition 3 *Suppose that absolute risk aversion is weakly decreasing and that relative risk aversion is weakly increasing. In the one-safe-one-risky-asset model, only the two extreme returns have a positive subjective probability:* $\exists p^* \in [0, 1]$ *such that* $P^* = (a, 1-p^*; b, p^*)$.

Proof: Suppose by contradiction that $x_- > a$ or $x_+ < b$. Suppose for example that x_+ is less than b. We consider a marginal increase in x_+ that is compensated by a change in p^* in such a way that α^* be unaffected by the change. Totally differentiating the definition of subjective utility using optimal beliefs, we have that

$$\left.\frac{dp^*}{dx_+}\right|_{\alpha^*} = -\frac{p^* u'(w_0 + \alpha^* x_+) \left[1 - \alpha^* x_+ A(w_0 + \alpha^* x_+)\right]}{x_+ u'(w_0 + \alpha^* x_+) - x_- u'(w_0 + \alpha^* x_-)}. \tag{14}$$

By definition of the subjective expected utility, we have that

$$\left.\frac{dS}{dx_+}\right|_{\alpha^*} = p^*\alpha^* u'(w_0+\alpha^* x_+) + \left[u(w_0+\alpha^* x_+) - u(w_0+\alpha^* x_-)\right] \left.\frac{dp^*}{dx_+}\right|_{\alpha^*}.$$

Using (14), S is increasing in x_+ if

$$\begin{aligned} K(x_+,x_-) &= \alpha^* x_+ u'(w_0+\alpha^* x_+) - \alpha^* x_- u'(w_0+\alpha^* x_-) \\ &\quad - \left[1-\alpha^* x_+ A(w_0+\alpha^* x_+)\right]\left[u(w_0+\alpha^* x_+) - u(w_0+\alpha^* x_-)\right] \end{aligned}$$

is positive. Observe that, by risk aversion,

$$K(0,x_-) = u(w_0+\alpha^* x_-) - \alpha^* x_- u'(w_0+\alpha^* x_-) - u(w_0)$$

is positive for all x_-. Notice also that

$$\begin{aligned} \frac{dK}{dx_+}(x_+,x_-) &= \alpha^*\left[u(w_0+\alpha^* x_+) - u(w_0+\alpha^* x_-)\right]\left[A(w_0+\alpha^* x_+) + \alpha^* x_+ A'(w_0+\alpha^* x_+)\right] \\ &= \alpha^*\left[u(w_0+\alpha^* x_+) - u(w_0+\alpha^* x_-)\right]\left[R'(w_0+\alpha^* x_+) - w_0 A'(w_0+\alpha^* x_+)\right]. \end{aligned}$$

We hereafter show that the right-hand side of this equality is positive. Obviously, $\alpha^*\left[u(w_0+\alpha^* x_+) - u(w_0+\alpha^* x_-)\right]$ is positive. The second bracketed term in the right-hand side of the above equality is also positive since, by assumption, R' is non-negative and A' is non-positive. We conclude that K is positive for all nonnegative x_+. Therefore, this change in beliefs raises the lifetime well-being of the decision maker, a contradiction. A parallel proof can be made when x_- is larger than a. ■

The familiar set of power utility functions $u(z) = z^{1-\eta}/(1-\eta)$ exhibits constant relative risk aversion and decreasing absolute risk aversion. Therefore, it satisfies the condition of the above proposition. More generally, decreasing absolute risk aversion is commonly accepted by the profession as a reasonable assumption. Non-decreasing relative risk aversion is compatible with the observation that, conditional on holding a portfolio, wealthier consumers invest a smaller share of their wealth in stocks.[6] Under these reasonable assumptions, it is optimal for the agent to believe that there are only two possible outcomes, and that outcomes are extreme in the sense that only the smallest and the largest plausible outcomes are possible.

[6] See for example Guiso, Jappelli and Terlizzese (1996).

In the remainder of the paper, we will assume that the optimal subjective probability distribution is of the form $(a, 1-p^*; b, p^*)$, where p^* denotes the probability of the state with the highest possible return $x = b$. This probability p^* is the only remaining degree of freedom to be determined. It depends upon the objective function Q and the utility function u.

4.1 Link with non-expected utility models

The most obvious link of our result in Proposition 3 to decision theory goes back to Hurwicz (1951). Hurwicz's criterion measures ex ante welfare by a weighted sum of the worst and the best possible outcome. The weight attached to the best utility corresponds to our variable p^*, which is known as the index of optimism in that literature. Thus, contrary to Hurwicz's criterion, the index of optimism is endogeneous in our model, and it is affected by the characteristics of the problem under scrutiny. Several extensions to Hurwicz's model have been proposed over the last decades. In his review of the psychology of decision making under risk, Lopes (1987) suggested that a decision maker takes into account three factors while evaluating risk: the expected utility, the worst outcome, and the best outcome. Contrary to the EU theory, the decision maker is sensitive to the "security level" and to the "potential level" of the lottery. Relying on these ideas, Gilboa (1988), Jaffray (1988), Cohen (1992) and Essid (1997) developed criteria that incorporates these additional aspects into the expected utility model. By extension, our work is also related to the literature on decision with multiple priors. In the $a-$MEU criterion proposed by Girardato, Maccheroni and Marrinacci (2004), the ex ante welfare is measured by a weighted sum of the worst and of the best possible expected utility. This criterion simplifies to the Hurwicz's one if the possible priors are all degenerated, and to the well-known Gilboa and Schmeidler (1989) maxmin criterion when the weight on the minimum expected utility goes to unity. In the smooth ambiguity aversion model of Klibanoff, Marrinacci and Mukerji (2005), the decision maker also transforms her beliefs towards either the worst or the best probability distribution compared to an ambiguity-neutral agent.

In Rank-Dependent Expected Utility (RDEU, Quiggin (1982)) and in Cumulative Prospect Theory (CPT, Tversky and Kahneman (1992)), it is assumed that agents maximize their expected utility by using a (subjective) probability distribution P that is a non-linear increasing function f of the

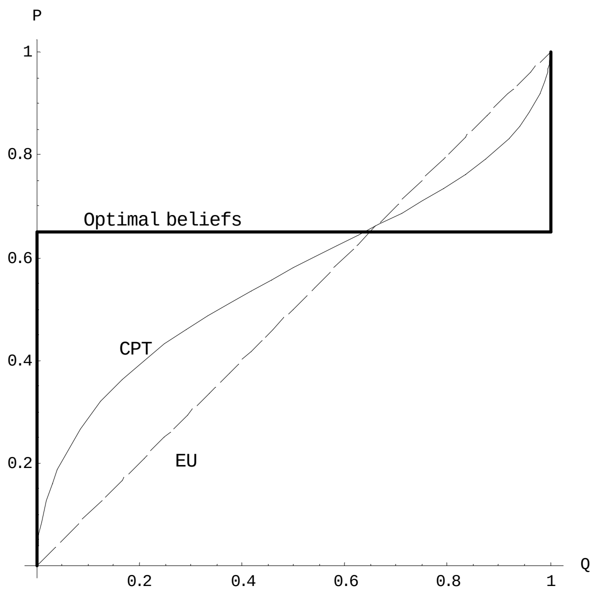

Figure 1: Typical inverse S-shaped transformation function of the cumulative probability function Q.

objective function Q: $P(x) = f(Q(x))$. By definition, it must be that $f(0) = 0$ and $f(1) = 1$. These theories make the assumption that the weighting function f is intrinsic to the preferences of the agents, exactly as is the utility function u. Together with Brunnermeier and Parker (2005), we take a different road by assuming that agents endogenously select the weighting function f in order to maximize their lifetime utility. At this stage, it is thus interesting to see whether the estimated function f of RDEU and CPT exhibits properties that are shared by the optimal beliefs characterized in this paper.

CPT is consistent with the psychological principle of diminishing sensitivity, the two endpoints of the support of the distribution of returns serving as reference points. It has been observed that increments near these endpoints have more impact than increments in the middle of the support. The transformation function estimated by Tversky and Kahneman (1992) in the gain domain, as depicted by the smooth inverse S-shaped curve in Figure

1, satisfies this property. Tversky and Kahneman (1992) used the following specification:

$$f(q) = \frac{q^\gamma}{[q^\gamma + (1-q)^\gamma]^{1/\gamma}}. \tag{15}$$

The inverted S-shaped probability weighting function, concave for low probabilities and convex for large probabilities, is compatible with $\gamma < 1$. Most estimations of γ from experimental studies are compatible with this assumption (Gonzalez and Wu (1999)). For example, Tversky and Kahneman (1992) estimated γ to be equal to 0.61 (as in Figure 1) . In a similar experiment, Abdellaoui, L'Haridon and Zank (2010) confirmed the inverse S-shape of the probability transformation for a specification of f that is more general than (15). Abdellaoui (2000) obtained similar results by using a parameter-free method to estimate the weigthing function.

Compared to the 45° line corresponding to Expected Utility, we see that the probability transformation function in CPT transfers much of the probability mass from the center of the support to the two extreme possible returns a and b. To illustrate this, suppose that the objective distribution of excess return is uniformly distributed on interval $[-1, 1]$. In Figure 2, we have drawn the subjective density function of excess return for different values of the distortion parameter γ. This picture illustrates the important transfer of prbability mass to the extreme values of the distribution. For example, for $\gamma = 0.3$, the subjective distribution has 73% of the probability mass for event $x \geq 0.8$, compared to the 10% objective probability. The optimal beliefs in our model push this kind of transformations to the limit by transferring the entire probability mass from the interior of the support to its two endpoints. Therefore, this work suggests that the transformation of probabilities described in Cumulative Prospect Theory, rather than being a genetic characteristic of human beings, corresponds to a natural tendency of rational agents to optimize their intertemporal welfare.

5 A numerical example

In this section, we illustrate this model by a simple numerical example. The safe asset has a zero return, whereas the risky asset has a return that is objectively uniformly distributed in interval $[-1, 1]$. We assume that the investor has a CRRA utility function $u(z) = z^{1-\eta}/(1-\eta)$ with $\eta = 4$. We

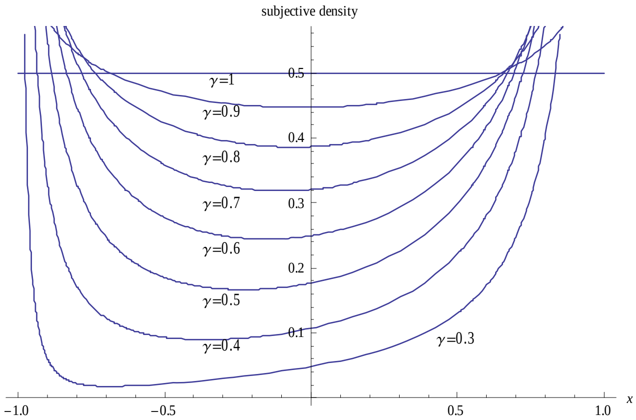

Figure 2: Subjective density under the probability transformation function (15) when the objective distribution is uniformly distributed on $[-1, 1]$.

also normalize initial wealth w_0 to unity, so that α can be interpreted as the share of wealth invested in the risky asset. Because of risk aversion and the zero expected excess return, the investor with rational expectation would not invest in the risky asset. Suppose alternatively that the lifetime welfare of the agent is the following special case of our model:

$$W(P, Q) = kS(P) + (1 - k)O(P, Q).$$

We can interpret parameter $k \in [0, 1]$ as the intensity of anticipatory feelings. From Proposition 3, the subjective beliefs take the form $(-1, 1 - p^*; +1, p^*)$, where p^* denotes the probability of the good outcome. When $p^* = 1/2$, the subjective excess return is zero and the agent will select the objectively optimal portfolio allocation $\alpha^* = 0$. More generally, the demand for the risky asset will have the same sign as $p^* - 0.5$. In Figure 3, we have drawn the lifetime welfare of the investor as a function of the subjective probability p of the good outcome. We have estimated numerically the solution of the model for 4 different intensities of anticipatory feelings, from $k = 0$ to $k = 0.5$. In Figure 4, we represent the optimal subjective probability p^* of the best return as a function of k.

Several observations can be made from this figure. Notice first that the objective function W is generally not concave in the decision variable p. This

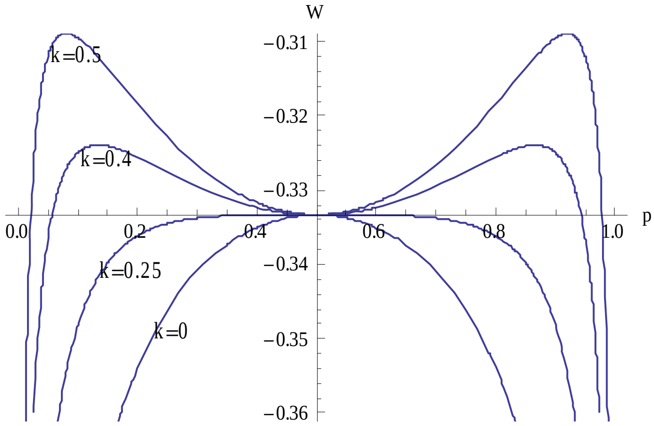

Figure 3: Lifetime utility W as a function of the probability p of the best return. The objective probability distribution of excess returns is uniformly distributed on $[-1, 1]$. The investor is characterized by $u(z) = -z^{-3}/3$ and $w_0 = 1$.

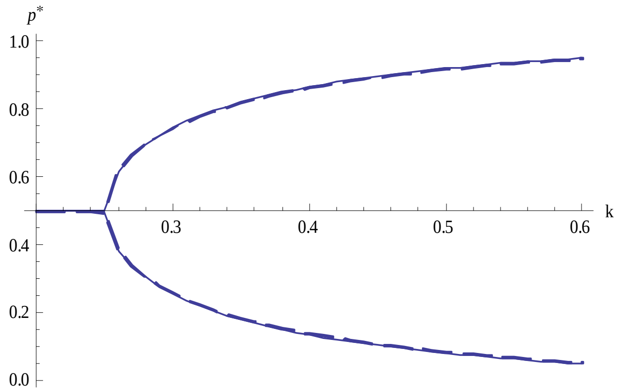

Figure 4: Optimal probability of the best state as a function of the intensity k of anticipatory feelings. The calibration is the same than in Figure 3. The doted curve is for the alternative calibration in which $\gamma = 40$ rather than $\gamma = 4$.

is due to the fact that $S(p)$ is the maximum of a sum of linear functions of p, as seen from equation (2). Therefore, $S(p)$ is a convex function of the subjective probability p. If the intensity k of anticipatory feelings is large enough, W will not be globally concave. More specifically, W is concave with a maximum at $p^* = 0.5$ whenever the intensity k of anticipatory feelings is less than 0.26. In that case, in spite of her desire to extract felicity from dreams of large portfolio gains, the investor prefers to behave rationally by selecting $\alpha^* = 0$.

For an intensity of anticipatory feelings larger than 0.26, the investor cannot resist to the manipulation of her beliefs. This can be done in two alternative ways. The first strategy is to purchase some of the risky asset and to rationalize this demand by increasing the subjective probability of the large asset return above 1/2. The second strategy is to go short on the risky asset and to rationalize this behavior by reducing this probability below 1/2. Suppose for example that $k = 0.5$. In that case, the two optimal strategies are as follows:

- The share of wealth invested in the risky asset is $\alpha^* = 29.6\%$. This is rationalized with subjective beliefs $(-1, 8\%; +1, 92\%)$ about the net return of the risky asset.
- The investor goes short on the risky asset with $\alpha^* = -29.6\%$. This is rationalized with subjective beliefs $(-1, 92\%; +1, 8\%)$.

Notice that these cognitive manipulations of beliefs are optimistic. In both strategies, the agent overestimates the expected gain from taking risk. It is also noteworthy that creating a pari-mutuel on this basis will be Pareto-efficient. If the risky asset is in zero net supply, there is an equilibrium in which half of the population will invest 29.6% of their wealth to sell this asset, and the other half of the population will invest 29.6% of their wealth to purchase it.

An important question is to determine whether the heterogeneity in risk aversion can explain the heterogeneity of subjective beliefs in the population. In order to examine this question, let us perform the comparative statics analysis of an increase in risk aversion. In Figure 4, the dashed curves correspond to the optimal subjective probabilities when relative risk aversion is increased from its benchmark level $\eta = 4$ to $\eta = 40$. Observe that the impact of this huge increase in risk aversion is marginal. For example, for $k = 0.5$, the optimal subjective probability goes down from $p^* = 91.96\%$ to $p^* = 91.64\%$. This numerical example suggests that the heterogeneity of risk aversion has an effect on the distribution of beliefs, but also that this impact is very limited.[7]

6 Concluding remarks

We have shown that the selection of optimal beliefs is governed by very precise rules. Our main result is that the number of states on which the agent puts a positive probability cannot exceed one plus the number of degrees of freedom in the choice problem. For example, in the portfolio allocation problem with n independent assets, there are $n-1$ degrees of freedom in the choice problem, so there will be at most n states with a positive subjective probability. In an uncertain environment with many possible states of nature, optimal beliefs are a simplification of reality. In the special case of the one-safe-one-risky-asset model, we have shown that investors will optimally believe that only the worst and the best possible returns may occur. This suggests that the observed probability transformations that have been estimated in the Rank-Dependent-Expected-Utility and Cumulative-Prospect-Theory frameworks are the outcome of an optimization process. The optimal subjective probability distribution and the corresponding portfolio allocation that is rationalized by these beliefs depends upon the degree of risk aversion, the intensity of anticipatory feelings, and the objective distribution of the risk.

This work calls for further investigations in various directions. First, it would be interesting to examine a more general model in which more risk-taking opportunities are available. This would be useful in order to examine the effect of anticipatory feelings on the optimal diversification of individual asset portfolios. Second, the ex ante optimism that emerges from this model can generate disappointment ex post. Gollier and Muermann (2010) examine an alternative model in which the optimal beliefs trade off ex ante savoring and ex post dissapointment, but in a model where the manipulation of beliefs does not affect behavior. Third, this work suggests that delegating

[7] We provide more insights on the properties of the optimal beliefs in this framework in Gollier (2005).

the selection of the individual asset portfolios to an independent agent can be efficient. This would neutralize the negative effect on portfolio choices of distorting individual beliefs.

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