Does flexibility enhance risk tolerance?

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Abstract

The optimal decision under risk depends upon the ability of the decision makers to adapt their actions to the state of nature ex post. We examine two choice problems, one in which agents select their action ex post, and one in which they must commit on their action ex ante. Contrary to the intuition, it is not true in general that agents are more tolerant to risk in the flexible context than in the rigid one. We provide some sufficient conditions to guarantee that the optimal risk exposure is larger in the flexible context. We apply these results to examine various questions. In particular, we examine the effect of housing and labour markets rigidities and of rigid long-term saving plans on the demand for equity. We also compare the optimal portfolios in continuous and discrete time.

Keywords: dynamic portfolio choice, risk sharing, continuous versus discrete time

1 Introduction

Arrow (1963) and Pratt (1964) were the first to relate the optimal decision under risk to the shape of the decision marker's utility function on consumption. They assumed that the only decision is about which lottery to accept, and that the outcome of the lottery is immediately consumed. Drèze and Modigliani (1966, 1972), Mossin (1969) and Spence and Zeckhauser (1972) noticed that this is rarely a realistic assumption. Most agents will react to the outcome of the lottery by making additional decisions. If they find it optimal ex post, households holding stocks will react to a crash on financial markets by moving to a smaller house, by working more, by reducing their stock holding, or by reducing their saving. In other words, they will rarely compensate their immediate financial loss by a corresponding reduction in their consumption. The expectation of these ex post actions affects the optimal attitude towards risk ex ante. Mossin (1968), Merton (1969) and Samuelson (1969) used backward induction to determine the optimal portfolio when investors are fully flexible both on their saving decision and on their portfolio rebalancement strategy.

In this paper, we examine a quite general dynamic choice problem to determine the effect of flexibility on the optimal risk attitude. We compare the choices under risk in two different contexts. In the flexible context, the agent first chooses a lottery in a choice set, and then takes an action x after observing the outcome of the lottery. In the rigid context, the agent must commit on an action before observing the state of nature. The intuition suggests that the agent should be more risk-prone in the flexible context than in the rigid one. Spence and Zeckhauser (1972) showed this in a numerical example in which the Cobb-Douglass agent's ex post choice problem is a standard consumption choice with two goods. Using the same framework in continuous time, Bodie, Merton and Samuelson (1992) and Chetty and Szeidl (2003) showed that flexibility enhances risk tolerance for a subset of the parameter values of the Cobb-Douglass specification. The same result holds when the agent's utility function is log linear rather than Cobb-Douglass. Thus, rigidities on housing and labor markets can potentially explain the equity premium puzzle that arises when we assume full flexibility.

However, Machina (1982) used an exponential-quadratic utility function to show numerically that it is not true in general that flexibility induces a more risk-prone behavior. It may thus be possible that the loss of flexibility on housing and labor decisions raises the demand for risky assets. Our contribution to this literature is twofold. We first derive some sufficient conditions that yield an unambiguous effect on the risky choice. In the second part of the paper, we discuss various applications of these general results.

As explained by Drèze and Modigliani (1972), the intuition that flexibility enhances risk tolerance is based on the well-known result that, in the flexible EU context, the value of information is always non-negative. It implies that the certainty equivalent of any lottery is larger in the flexible context than in the rigid one. If the choice problem is to choose between a risky prospect or a risk free prospect, any lottery that is acceptable in the rigid context is also acceptable in the flexible one. However, as for example explained in Gollier (2001, chapter 6), the enlargement of the lottery acceptance set due to flexibility does not mean that risk aversion is globally reduced. For example, it does not imply that the demand for risky asset is reduced. The enlargement of acceptable lottery just implies that flexibility reduces risk aversion locally around the initial safe wealth. This is because the direction of the adaptation of the ex post action always reduces the sensitiveness of the marginal utility to change in wealth. This direct effect of flexibility may however be dominated by an "action" effect. It comes from the fact that the action selected ex ante in the rigid context may yield a low aversion to risk on wealth.

We are mostly interested in determining the effect of various rigidities on the optimal one-risky-one-risk-free-asset portfolio. More generally, we examine a class of ex ante decision problems under uncertainty in which the payoff is linear: $\widetilde{z} = z_0 + \alpha\widetilde{r}$, as is the case in the portfolio problem, the coinsurance problem, or the capacity problem of the firm under price uncertainty. For this class of ex ante choice problems, if the objective function is supermodular or submodular, the optimal risk exposure α in the flexible context is larger than in a rigid context in which the action is selected as if wealth would equal z_0 with certainty. Thus, when the option to invest in stocks has a negligible effect on the optimal action in the rigid context, super/submodularity is sufficient to imply that flexibility enhances risk tolerance. Because we also show that the option to invest is stocks only yields a second-order effect on the optimal rigid action, this result is useful in many applications.

The most obvious applications of this theory are when the second stage choice problem is a standard consumption allocation problem under certainty. After observing their portfolio return, consumers decide how to allocate their wealth into consumption for various goods. The existence of durable goods like housing, or the rigidities existing on the labor market, have an effect on the consumers' optimal portfolio allocation. When preferences are Cobb-Douglass, the optimal action in the rigid context is independent of the option to invest in the risky asset. If the parameters of the Cobb-Douglass utility function yield a value function that is less risk-averse than the log, the utility function is supermodular. Therefore, we can use the general result summarized above to claim that flexibility always raises the demand for stocks under these conditions, as shown by Bodie, Merton and Samuelson (1992). We also show that when the parameters of the Cobb-Douglass preferences yield a value function that is more risk-averse than the log, it is always possible to find an initial wealth and a distribution of excess returns such that flexibility reduces the demand for the risky asset.

Following Gabaix and Laibson (2002), we also examine the effect of reducing the frequency at which the consumption plan is reoptimized given the observation of the portfolio return since the previous reset date. There are several potential reasons for why consumers are unable to adapt their consumption level to shocks on their wealth in continuous time, as the existence of transaction costs or the willingness to commit on a specific saving plan to solve a time inconsistency problem. We examine a dynamic choice problem in which the second stage problem is a consumption/saving problem under certainty. Time-additive intertemporal utility functionals are supermodular. We show that the reduction of the frequency of reoptimizations of the consumption plan yields a reduction in the demand for stocks when the utility function exhibits harmonic absolute risk aversion (HARA), a class of preferences that includes power, exponential and logarithmic functions. The same model can be used to determine the effect of the households' inability to internally share risk efficiently on the demand for stocks.

We also apply our general model to the analysis of the reduction of the frequency at which the portfolio is rebalanced. We show that it reduces the demand for stocks when relative risk aversion is constant and less than unity. When relative risk aversion is larger than unity, we can always find a distribution of excess returns such that the reduction of rebalancement frequency raises the demand for stocks. This analysis sheds some light on the relationship between discrete time and continuous time in the modern theory of finance.

2 Some general results

The utility U of the decision-maker depends upon wealth $z \in \mathbb{R}$ and a decision x belonging to a decision set A. Function $U : \mathbb{R} \times \mathbb{A} \rightarrow \mathbb{R}$ is increasing in z and concave in x. We consider two dynamic choice problems. In the flexible framework, the agent first selects a risk-taking decision. Once the outcome z of the risk exposure is observed, the agent selects the $x \in A$ that maximizes U :

$$v(z) = \max_{x \in A} U(z, x). \tag{1}$$

Because U is concave in x, the solution $x^*(z)$ of the above program is unique. In this flexible framework, the attitude on risk in the first stage of the decision problem is fully characterized by the indirect utility function v. In other words, the first-stage choice among various lotteries can be rationalized by an expected utility functional

$$V^*(\widetilde{z}) = Ev(\widetilde{z}),$$

where $\widetilde{z}$ is the random variable describing the distribution of wealth. Because the operator max is convex, the value function v is not necessarily concave.

The alternative framework is rigid in the sense that the choice of $x \in A$ cannot be made sensitive to the realization of $\widetilde{z}$. In this rigid framework, the choice of x must be made before observing the realization of z. The choice of among various lotteries is rationalized in the rigid framework by the following functional:

$$\widehat{V}(\widetilde{z}) = \max_{x \in A} EU(\widetilde{z}, x). \tag{2}$$

Because concavity is preserved by summation, the solution $\widehat{x}(\widetilde{z})$ of program (2) is unique. As first stated by Mossin (1969), Drèze and Modigliani (1972) and Spence and Zeckhauser (1972), $\widehat{V}$ is not an expected utility functional. In particular, $\widehat{V}$ violates the independence axiom. It implies that contrary to what we have in the flexible framework, the risk attitude in the rigid world cannot be expressed by the concavity of a von Neumann-Morgenstern utility function. This is an important source of complexity of the analysis.

The main objective of the paper is to determine the conditions under which flexibility enhances risk tolerance. To be more precise, consider a specific first-stage choice problem under uncertainty that is specified by a risk opportunity set S, a subset of real-valued random variables. Let $\widehat{\widetilde{z}} \in S$ and $\widehat{x} = \widehat{x}(\widehat{\widetilde{z}}) \in A$ be respectively the optimal risk on wealth and the optimal choice in the rigid context. The problem is to determine whether the optimal risk in the flexible framework is riskier than $\widehat{\widetilde{z}}$? To answer this question, one should compare the concavity of functions $v(.)$ and $U(., \widehat{x})$. Notice that it is essential here to specify not only function U, but also the risk opportunity set S. A change in S potentially modifies the optimal $\widehat{x}$ and the reference function $U(., \widehat{x})$. In the remainder of this section, we consider two first-stage choice problems: the acceptance of a lottery, and a portfolio allocation problem.

2.1 The first stage problem is a binary choice

In the following Proposition, we consider the simplest risk opportunity set, which is limited to two random variables, one of which being degenerated. We show that flexibility always enlarges the set of acceptable risks. It is a direct consequence of the well-known property that the value of information is nonnegative. This value of information raises the willingness to accept risk.

Proposition 1 *For any pair* $(z_0, \widetilde{z})$, z_0 *being degenerated,* $\widehat{V}(\widetilde{z}) \geq \widehat{V}(z_0)$ *implies* $V^*(\widetilde{z}) \geq V^*(z_0)$.

Proof: Suppose that $\widehat{V}(\widetilde{z}) \geq \widehat{V}(z_0)$. The proof is directly obtained by the following sequence of relations:

$$\begin{aligned} V^*(\widetilde{z}) &= E \max_{x \in A} U(\widetilde{z}, x) \\ &\geq EU(\widetilde{z}, \widehat{x}(\widetilde{z})) = \widehat{V}(\widetilde{z}) \\ &\geq \widehat{V}(z_0). \quad \blacksquare \end{aligned}$$

In this binary riskfree-risky choice problem, we can evaluate the effect of flexibility by estimating the difference in the risk premia. Let consider a risk-free wealth z_0 and the introduction of a small risk $k\widetilde{\varepsilon}$ with $E\widetilde{\varepsilon} = 0$. In the flexible context, the maximum premium $\pi^*(k)$ that the decision-maker is ready to pay to eliminate risk $k\widetilde{\varepsilon}$ satisfies the following condition:

$$Ev(z_0 + k\widetilde{\varepsilon}) = v(z_0 - \pi^*(k)). \tag{3}$$

In the rigid context, the risk premium $\widehat{\pi}(k)$ associated to risk $k\widetilde{\varepsilon}$ is defined by

$$\max_{x\in A} EU(z_0 + k\widetilde{\varepsilon}, x) = \max_{x\in A} U(z_0 - \widehat{\pi}(k), x). \tag{4}$$

In Proposition 2, we show that the aversion to small risk is larger in the rigid context than in the flexible one.

Proposition 2 *Suppose that U is twice differentiable and that $A = \mathbb{R}$. The risk premia associated to zero-mean risk $k\widetilde{\varepsilon}$ respectively in the rigid and flexible contexts satisfy the following properties:*

$$\widehat{\pi}(k) = \frac{1}{2}k^2\sigma_\epsilon^2 \left[\frac{-U_{zz}(z_0, x^*(z_0))}{U_z(z_0, x^*(z_0))}\right] + o(k^2) \tag{5}$$

$$\pi^*(k) = \frac{1}{2}k^2\sigma_\epsilon^2 \left[-\frac{v''(z_0)}{v'(z_0)}\right] + o(k^2), \tag{6}$$

with

$$-\frac{v''(z)}{v'(z)} = -\frac{U_{zz}(z, x^*(z))}{U_z(z, x^*(z))} + \frac{U_{zx}^2(z, x^*(z))}{U_z(z, x^*(z))U_{xx}(z, x^*(z))}. \tag{7}$$

Consequently, in the case of small risk, flexibility always reduces risk aversion.

Proof: See the Appendix. ■

In the rigid context, we see that the risk premium for small risk is not different from what one would obtain with a completely exogenous x that would be fixed equal to $x^*(z_0)$, the optimal action when $z = z_0$ with certainty. Indeed, equation (5) tells us that the premium associated to small risk $k\widetilde{\varepsilon}$ is the one that a von Neumann-Morgenstern agent with utility $\widehat{v}_0(.) = U(., x^*(z_0))$ would be ready to pay. The fact that the choice of action x is affected by the existence of risk $k\widetilde{\varepsilon}$ has only a second-order effect on $\widehat{\pi}$. This is because, as noticed in the proof, $d\widehat{x}/dk = 0$ when evaluated at $k = 0$: the introduction of a small risk does not affect the optimal action at the margin. In consequence, as clearly explained by Machina (1984), the expected utility approximation (5) holds in the rigid context. The preference functional $\widehat{V}$ is Fréchet differentiable at z_0 with local utility function $U(., x^*(z_0))$.

The picture is completely different in the flexible context. The risk aversion on wealth is determined by the Arrow-Pratt index $-v''/v'$ of the indirect utility function v . As shown by Machina (1984), the absolute aversion to temporal risk is the sum of two terms that are expressed in the right-hand side of (7). Evaluated at $z = z_0$, the first term is the absolute risk aversion of $\widehat{v}_0(.) = U(., x^*(z_0))$, as in the rigid context. The second term in the right-hand side of (7) originates from the flexibility of action $x^*(z)$. It is always negative. Thus, flexibility always reduces local risk aversion.

The above two propositions suggest that flexibility always leads to a globally less risk-averse behavior. However, as noticed by Machina (1982), this is not true in general. To see this, notice that $v(.)$ is less concave than $U(., \widehat{x})$ in the sense of Arrow-Pratt if

$$\frac{-zU_{zz}(z, x^*(z))}{U_z(z, x^*(z))} [1 - F(z)] \leq \frac{-zU_{zz}(z, \widehat{x})}{U_z(z, \widehat{x})} \tag{8}$$

for all z, where $F(z)$ is an index of "flexibility tolerance" that is defined by:

$$F(z) = \frac{-U_{zx}^2(z, x^*(z))}{U_{zz}(z, x^*(z))U_{xx}(z, x^*(z))} \geq 0. \tag{9}$$

We see that flexibility has two effects on risk aversion. The direct effect of flexibility is expressed by the multiplicative term $(1 - F) < 1$ in the left-hand side of (8), as explained above. We qualify this effect to be "direct" because it directly comes from the fact that the action x^* is sensitive to the outcome z. It is useful to see why the direct flexibility effect always reduces risk aversion. Suppose that U_{zx} is positive. It implies that x^* is increasing in z. It implies that an increase in z raises x^* and U_z in the flexible context. Similarly, if U_{zx} is negative, an increase in z reduces x^* and it raises U_z. In both cases, the direct effect flexibility on the marginal utility of wealth is to make it less sensitive to wealth. In short, it reduces risk aversion.

The second effect comes from the fact that the absolute risk aversion of U at z is evaluated at $x^*(z)$ in the flexible context, and at $\widehat{x}$ in the rigid one. Let us refer to this second effect as the "action effect". It can be either positive or negative. In Figure 1, these two effects can easily be isolated. Considering a specific z, the action effect is obtained by comparing the concavity indexes of the two plain curves corresponding to $U(., \widehat{x})$ and $U(., x^*(z))$. The direct flexibility effect is obtained by comparing the concavity indexes of the plain curve $U(., x^*(z))$ and the dashed curve $v(.)$.

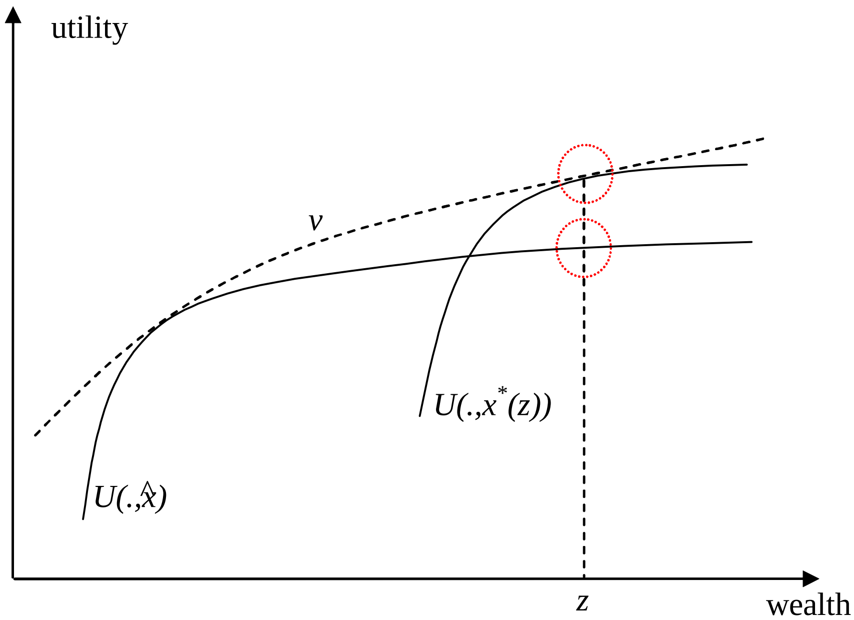

Figure 1: The value function and utility functions conditional to choice $x^*(z)$ and $\widehat{x}$.

Flexibility would raise risk aversion if the unambiguously risk-prone direct effect of flexibility is more than compensated by the action effect. Whereas Proposition 2 states that this cannot be the case for small risks because $\widehat{x} = x^*(z)$, we cannot exclude this possibility for larger risks.

2.2 The first stage problem is a portfolio choice

If v is not globally risk-averse than $U(., \widehat{x})$, we cannot sign the effect of flexibility on the optimal portfolio allocation. We illustrate this point by the following example. We consider a two-stage decision problem under uncertainty. In the first stage, the agent endowed with wealth $z_0 = 1$ invests in a portfolio of two assets, one of which is risk-free with a zero return. The other asset has a return distributed as $\widetilde{r} \sim (-1, 1/10; 10, 9/10)$. In the second stage, after observing the outcome of this investment, the agent can either accept or reject a lottery $\widetilde{y} \sim (-0.2, 1/2; 1, 1/2)$. The agent has constant relative risk aversion $\gamma = 4$. It can be shown that the optimal strategy in the first stage consists in investing $\alpha = 14.98\%$ of z_0 in the risky asset. In the second stage, the agent accepts lottery $\widetilde{y}$ only if the risky asset return is positive. Suppose alternatively that the two decisions must be made simultaneously, i.e., that the lottery decision cannot be made after observing the portfolio return. In this rigid context, it can be shown that the optimal decision is to reject the lottery and to invest a share $\alpha = 15.90\%$ of z_0 in the risky asset. From the observation of the demand for the risky asset, it appears that the investor is more risk-averse in the flexible context than in the rigid one. Flexibility does not enhance risk tolerance. Notice that this model is characterized by an objective function $U(z, x) = E\left[z + x\widetilde{y}\right]^{1-\gamma}/(1-\gamma)$ and a choice set $A = \{0, 1\}$.

As in the above example, we will hereafter focus much of our attention on the one-risk-free-one-risky-asset choice problem in the first stage. In the flexible context, this first stage problem can be written as

$$\alpha^* = \arg\max_{\alpha} Ev(z_0 R + \alpha\widetilde{r}), \tag{10}$$

where z_0 is initial wealth, $\widetilde{r}$ is the excess return of the risky asset, R is the gross risk-free rate and α is the demand for this asset.[1] We hereafter suppose without loss of generality that the $E\widetilde{r} > 0$, which implies that α^* is positive.

[1] Many other economic problems can be written in that way, as the coinsurance prob-

In the following Lemma, we show that the demand for the risky asset in the flexible context is larger than when the action x is exogenously fixed at $x^*(z_0)$.

Proposition 3 *Consider the first stage portfolio problem described by program (10). Suppose that U is concave in its first argument. If U is either supermodular or submodular, then α^* is larger than $\alpha_0 = \arg\max_\alpha EU(z_0R+\alpha\widetilde{r}, x^*(z_0))$, i.e., the demand for the risky asset in the flexible context is larger than when action a is exogenously fixed at $x^*(z_0)$, its optimal level without the risky asset.*

Proof: Suppose that U is supermodular (submodular). By Topkis' theorem (Topkis (1978)), $x^*(z)$ is nondecreasing (nonincreasing) in z. Because U_z is nondecreasing (nonincreasing) in x, it implies that

$$rU_z(z_0R+\alpha^*r, x^*(z_0)) \leq rU_z(z_0R+\alpha^*r, x^*(z_0+\alpha^*r)) = rv'(z_0R+\alpha^*r)$$

for all r. Taking the expectation, we obtain that

$$E\widetilde{r}U_z(z_0R+\alpha^*\widetilde{r}, x^*(z_0)) \leq E\widetilde{r}v'(z_0R+\alpha^*\widetilde{r}) = 0.$$

Because U is concave in z, this inequality implies that the optimal demand for risky asset is less than α^* when x is exogenously fixed at $x^*(z_0)$. ■

This Proposition sheds some light on the above numerical example. It can be shown that the optimal choice when $z = z_0$ with certainty is to reject lottery $\widetilde{y}$, i.e., $a_0 = 0$. The optimal decision to invest $\alpha = 15.90\%$ of initial wealth in the risky asset does not affect the decision to reject this lottery. The fact that $U(z,x) = E\left[z + x\widetilde{y}\right]^{1-\gamma}/(1-\gamma)$ is neither supermodular nor submodular explains why the demand $\alpha^* = 14.98\%$ for the risky asset is not larger than in the rigid context.

Proposition 3 has some interest on its own, but it does solve our problem only when the optimal action in the rigid context is close to $x^*(z_0)$. But in general, the option to invest in the risky asset also has an effect on the optimal rigid action: $\widehat{x}$ will in general differ from $x^*(z_0)$. As we will see in the applications, the action effect may be stronger than the direct flexibility effect, inducing the demand for the risky asset to be reduced by flexibility. This can be the case only for large portfolio risks, i.e., when the equity premium is large relative to the return volatility.

lem, the capacity problem of the risk-averse firm under price uncertainty, or various risk prevention problems.

3 Applications

3.1 Cobb-Douglass: Labor, health and housing

We start the list of applications of the above theory with a second stage choice represented by the standard consumption problem under certainty, whereas the first stage choice is the portfolio problem examined in the previous section. Wealth z is allocated to the consumption of $n+1$ goods indexed from $i = 0$ to $i = n$.[2] The price of good $i > 0$ relative to the numeraire good $i = 0$ is denoted p_i. The utility of bundle $(x_0, x_1, ..., x_n)$ is measured by $u(x_0, x_1, ..., x_n)$. The choice set is $A = R^n_+$ and the objective function is written as

$$U(z, x_1, ..., x_n) = u\left(z - \sum_{i=1}^{n} p_i x_i, x_1, ..., x_n\right). \tag{11a}$$

Following for example Bodie, Merton and Samuelson (1992), let us consider the following Cobb-Douglass specification:

$$u(x_0, x_1, ..., x_n) = \gamma_0^{-1} \prod_{i=0}^{n} x_i^{\gamma_i}, \tag{12}$$

where the elements in vector $(\gamma_0, ..., \gamma_n)$ are restricted to have the same sign and where $\Gamma = \sum_{i=0}^{n} \gamma_i$ is less than unity. In the rigid context, the consumption of the goods other than good $i = 0$ must be chosen before observing the outcome of the risk on z, so that the only way to adapt to a shock on wealth is to modify the consumption of the numeraire good x_0 accordingly.

Using Proposition 2, we first examine the case of small risks. Under specification (11a) and (12), the relative risk aversion locally at z_0 respectively in the rigid and flexible contexts are measured by

$$-\frac{zU_{zz}(z_0, \widehat{x}_1, ..., \widehat{x}_n)}{U_z(z_0, \widehat{x}_1, ..., \widehat{x}_n)} = \Gamma(\gamma_0^{-1} - 1) \text{ and } -\frac{z_0 v''(z_0)}{v'(z_0)} = 1 - \Gamma. \tag{13}$$

[2]Strictly speaking, this model is a particular case of the above general model only when $n = 1$. In the case of Cobb-Douglass, it is easy to show that the choice problem with $n > 1$ is equivalent to the one with $n = 1$ for a composite bundle of the goods $(x_1, ..., x_n)$ with shares $(\gamma_1/p_1, ..., \gamma_n/p_n)$.

As stated in Proposition 2, the local risk aversion is reduced by flexibility. The effect is particularly powerful when the γ_is are positive and Γ is close to unity, yielding a risk behavior close to risk neutrality in the flexible context. For example, when $\gamma_0 = \Sigma_{i=1}^n \gamma_i = 0.5$, the share of wealth invested in equity is infinite in the flexible case, whereas it approximately equals $E\widetilde{r}/\sigma^2_{\widetilde{r}}$ in the rigid context.

We then show that the optimal bundle $(\widehat{x}_1, \widehat{x}_2, ..., \widehat{x}_n)$ in this rigid context is not affected by the option to invest in the risky asset.

Lemma 1 *Suppose that the decision maker must determine her demand for consumption goods $i = 1, ..., n$ before observing the return of her portfolio. Suppose also that the utility function $u : R_+^{n+1} \to R$ satisfies (12). It implies that the optimal bundle of these consumption goods is not affected by the option to invest in equity.*

Proof: Consider an arbitrary bundle $(x_1, ..., x_n) \in R_+^{n+1}$ with $z_0 - \Sigma_{i=1}^n p_i x_i > 0$. It is easy to check that the portfolio α that is optimal conditional to this bundle is characterized by

$$\alpha(x_1, ..., x_n) = b\left(z_0 R - \sum_{i=1}^n p_i x_i\right)$$

where scalar b is implicitly defined by $E\widetilde{r}(1 + b\widetilde{r})^{\gamma_0 - 1} = 0$. It implies that the optimal bundle in the rigid context must maximize the following objective function

$$\gamma_0^{-1} E(1 + b\widetilde{r})^{\gamma_0}\left(z_0 R - \sum_{i=1}^n p_i x_i\right)^{\gamma_0} \prod_{i=1}^n x_i^{\gamma_i}.$$

The optimal solution of this program is independent of $\widetilde{r}$. ■

It implies that the optimal demand for the n goods in the rigid framework is $\widehat{x}_i = \gamma_i z_0 R / p_i \Gamma$. Because the optimal second stage choice is independent of the investment opportunity, we can use Proposition 3 to determine the effect of flexibility in this framework. The objective function (11a) cannot be submodular. It is easy to check that it is supermodular if and only if the γs are positive. The following Corollary is thus a direct consequence of the combination of Proposition 3 and Lemma 1.

Corollary 1 *Consider the two stage problem with a portfolio choice and a budget allocation with multiple goods, assuming a Cobb-Douglass utility function (12) with $\gamma_i > 0$, $i = 0, ..., n$. Under this assumption, the demand for equity is larger in the flexible context than in the rigid one.*

The assumption the γ_is are positive is equivalent to the flexible indirect utility function being less risk-averse than the log. In fact, this assumption implies that the indirect utility function $U(., \widehat{x}_1, ..., \widehat{x}_n)$ in the rigid context is globally more concave than the indirect utility function v in the flexible one. To check this, we measure the relative risk aversion at an arbitrary wealth level z by

$$-\frac{zU_{zz}(z, \widehat{x}_1, ..., \widehat{x}_n)}{U_z(z, \widehat{x}_1, ..., \widehat{x}_n)} = \frac{(1-\gamma_0)\Gamma z}{z\gamma_0 + (z - z_0)(\Gamma - \gamma_0)} \tag{14}$$

in the rigid context, and

$$-\frac{zv''(z)}{v'(z)} = 1 - \Gamma \tag{15}$$

in the flexible one. In fact, the flexibility index $F(z)$ defined by (9) is constant and equal to $F = (\Gamma - \gamma_0)/\Gamma(1 - \gamma_0)$. Risk aversion in the rigid context is uniformly larger than risk aversion in the flexible context if and only if $z\Gamma$ is uniformly larger than $z_0(\Gamma - 1)$. This is the case only when the γ_is are positive. It is the case only locally around $z = z_0$ when the γ_is are negative. In this latter case, the local aversion to risk at large wealth levels is larger in the flexible context! In the following Proposition, we prove that the effect of flexibility is intrinsically ambiguous for this set of utility functions.

Proposition 4 *Consider the two stage problem with a portfolio choice and a budget allocation with multiple goods, assuming a Cobb-Douglass utility function (12) with $\gamma_i < 0$, $i = 0, ..., n$. We can always find an initial wealth level z_0 and a distribution of the equity return $\widetilde{r}$ such that the demand for equity is smaller in the flexible context than in the rigid one.*

Proof: See the Appendix. ■

To illustrate, suppose that $n = 1$, $p_1 = 1$, $\gamma_0 = \gamma_1 = -2$, $z_0R = 1$ and $\widetilde{r} \sim (-0.2, \pi; 5, 1 - \pi)$ with $\pi = 0.00404$. In the rigid context, it is optimal to take $\widehat{x}_1 = 0.5$ and $\widehat{\alpha} = 100\%$. The demand for the risky asset in the flexible context is reduced to $\alpha^* = 77\%$. Observe that this counterexample is based on a very skewed distribution of $\widetilde{r}$ in order to take advantage of the larger local risk aversion of $v(.)$ compared to $U(., \widehat{x})$ at large wealth levels.

Bodie, Merton and Samuelson (1992) examined the consequences of the flexibility of labor supply on the optimal portfolio. Relying on some specifications of the consumption/leisure utility function as the one presented in Corollary 1, they claimed that the ability to vary labor supply as a function of the portfolio return should induce young individuals to raise their demand for risky assets. All their numerical simulations yield the same conclusion. We have shown above that this result does not hold in general, except for small portfolio risks. We have also shown above that other variables like health and housing can be included in the utility function without changing the structure of the results. The presence of high transaction costs and taxes in the housing market together with a rigid health insurance system in many countries of continental Europe may also explain the relative resistance of European households to invest in stocks.

3.2 Additive model: Saving and risk-sharing

We now examine a simple portfolio-saving problem under certainty. At the beginning of period $t = 0$, individuals initially endowed with wealth z_0 determine their α for the risky asset with return $\widetilde{r}$. Wealth z_0 includes the human capital, which is the net present value of the flow of future labor incomes. In the flexible context, they observe their portfolio return before determining their consumption plan $(x_0, x_1, ..., x_n)$ over the remaining $n + 1$ dates. In the rigid context, the future consumption plan $(x_1, ..., x_n)$ must be determined before observing the portfolio return in the first period, i.e., the portfolio risk must be entirely allocated to the consumption in the first period. Assuming a time-additive utility function, the objective function for this problem can be written as

$$U(z, x_1, ..., x_n) = u_0\left(z - \sum_{i=1}^{n} p_i x_i\right) + \sum_{i=1}^{n} u_i(x_i), \tag{16}$$

where $z = z_0 + \alpha\widetilde{r}$ is the wealth accumulated in the first period, and p_i is the price of consumption at date i relative to consumption at date 0. In the special case of a flat yield curve, $p_i = R^{-i}$. The utility functions u_i on consumption at $i = 0, ..., n$ are assumed to be increasing and concave.

During the last three decades, tax incentives have been developed for long-term saving schemes, as the 401k in the US, or the PEL or PERP in France. Following Laibson (1997), these tax incentives can be justified on the basis that individual have difficulties to commit themselves to save for their retirement. This may be due for example because of a time-inconsistency problem due to the hyperbolic structure of their time preferences. However, there is a clear cost associated to the system, since it does not allow individuals to use their savings as a buffer stock to smooth shocks on their incomes. We hereafter focus on another indirect cost of rigid saving schemes. When the contributions to the long-term saving plan are fixed ex-ante, households cannot time-diversify their portfolio risks. This induces them to be less tolerant to portfolio risks, thereby reducing the benefit that they can extract from the positive equity premium. We examine the condition under which the demand for equity is reduced by the rigidity of the saving plan.

Function u is neither supermodular nor submodular. However, a rewriting of this choice problem using backward induction allows us to use Proposition 3. To see this, let us define the value function J such that

$$J(x) = \max_{x_1,\ldots,x_n} \sum_{i=1}^{n} u_i(x_i) \text{ s.t. } \sum_{i=1}^{n} p_i x_i = x,$$

where x is the amount saved from z in the first period. Using this definition, we can rewrite the objective function (16) as

$$U^{\circ}(z, x) = u_0(z - x) + J(x).$$

Because U° is supermodular in (z, x), we know from Proposition 3 that the demand α^* for the risky asset in the flexible context is larger than the demand α_0 that is optimal when saving is exogenously fixed at $x^*(z_0)$, the optimal saving when the individual has no access to the equity market. However, except in the case where u_0 would be logarithmic, we know that the access to the equity market affects the optimal saving $\widehat{x}$ in the rigid context. We would be done if we would have that $\widehat{\alpha}$ be smaller than α_0. Because the optimal portfolio risk $\widehat{\alpha}$ in this context must maximize $Eu_0(z_0 - \widehat{x} + \alpha\widetilde{r})$, whereas α_0 maximizes $Eu_0(z_0 - x^*(z_0) + \alpha\widetilde{r})$, we would have $\widehat{\alpha} \leq \alpha_0$ if u_0 exhibits decreasing absolute risk aversion and $\widehat{x} > x_0$. To sum up, the question boils down to determining the conditions under which the access to the equity market raises the optimal saving. This is the case if

$$Eu_0'(w_0 + \alpha_0 \widetilde{r}) \leq u_0'(w_0), \tag{17}$$

with $w_0 = z_0 - x^*(z_0)$, and where α_0 must satisfy condition $E\widetilde{r}u_0'(w_0 + \alpha_0 \widetilde{r}) = 0$. Gollier and Kimball (1996) and Gollier (2001, Proposition 75) solved this question. Using the concept of risk tolerance $T_0(z) = -u_0'(z)/u_0''(z)$, they showed that the necessary and sufficient condition to guarantee that inequality (17) holds for all z_0 and all acceptable distributions of $\widetilde{r}$ is that the derivative of risk tolerance T_0 be uniformly larger than unity: $T_0'(z) \geq 1$ for all z. This condition is stronger than decreasing absolute risk aversion, which is equivalent to $T_0'(z) \geq 0$ for all z. This proves the first part of the following proposition.

Proposition 5 *Consider the two stage problem with a portfolio choice and a saving decision. The demand for equity is larger when the saving plan is flexible than when it must be fixed before observing the portfolio return if one of the two conditions is satisfied:*

1. *The derivative of risk tolerance T_0 is uniformly larger than unity;*
2. *The derivative of risk tolerance T_0 is uniformly negative.*

Proof: It just remains to prove the sufficiency of condition 2, which is equivalent to increasing absolute risk aversion. Following Gollier and Kimball (1996), this condition implies that inequality (17) is reversed, yielding $\widehat{x} \leq x_0$. Increasing absolute risk aversion implies in turn that the maximum of $Eu_0(z_0 - \widehat{x} + \alpha\widetilde{r})$ is smaller than the one of $Eu_0(z_0 - x^*(z_0) + \alpha\widetilde{r})$, or $\widehat{\alpha} \leq \alpha_0$. Because U is supermodular, we also know that $\alpha_0 \leq \alpha^*$. This concludes the proof of the sufficiency of condition 2. ■

Both conditions 1 and 2 imply that $\widehat{\alpha}$ be smaller than α_0, which we know is smaller than α^*. In the intermediate case of weakly decreasing absolute risk aversion, i.e., when absolute prudence is in between one and two time the absolute risk aversion, $\widehat{\alpha}$ is larger than $\alpha_0 \leq \alpha^*$, and the Gollier and Kimball's result does not allow us to conclude whether $\widehat{\alpha}$ is larger or smaller than α^*.

We can use an alternative approach by explicitly measuring the risk tolerance indexes of $v(.)$ and $U(., x_1, ..., x_n)$. Fully differentiating the first-order condition $v'(z) = u_0'(z - x^*(z)) = p_i^{-1} u_i'(x_i^*(z))$ yields

$$T_v(z) = T_0(z - x^*(z)) + \sum_{i=1}^{n} p_i T_i(x_i^*(z)), \tag{18}$$

where $T_v(z) = -v'(z)/v''(z)$ and $T_i(z) = -u_i'(z)/u_i''(z)$ are the indexes of absolute risk tolerance of v and u_i respectively. Suppose that functions u_0 and u_i are proportional to each other ($u_i = \beta_i u_0$), so that $T_0 \equiv T_i \triangleq T$. Under the additional assumption that T is convex, we obtain that

$$\begin{aligned} T_v(z) &= (1+P)\left[\frac{1}{1+P}T(z - x^*(z)) + \sum_{i=1}^{n} \frac{p_i}{1+P} T(x_i^*(z))\right] \\ &\geq (1+P)T\left(\frac{z - x^*(z) + \sum_{i=1}^{n} p_i x_i^*(z)}{1+P}\right) = (1+P)T\left(\frac{z}{1+P}\right), \end{aligned}$$

with $P = \Sigma_{i=1}^{n} p_i$. Suppose moreover that the absolute tolerance of risk on consumption is subhomogeneous, which implies that the right-hand side of the above inequality is larger than $T(z)$.[3] Under decreasing absolute risk aversion, this is in turn larger than $T(z - x)$ for all $x \geq 0$. This proves the following proposition.

Proposition 6 *Consider the two stage problem with a portfolio choice and a saving decision. Suppose that $u_i(.) = \beta_i u_0(.)$. The demand for equity is larger when the saving plan is flexible than when it is fixed ex-ante at an arbitrary level $x \geq 0$ if the absolute risk tolerance is increasing, convex and subhomogeneous.*

Under this condition, the indirect utility function $v(.)$ is less concave than $U(., x)$ for all x. This condition is satisfied for the important set of HARA utility functions

$$u(z) = \zeta(\eta + \frac{z}{\gamma})^{1-\gamma} \tag{19}$$

[3] A function T is subhomogeneous if and only if $kT(z) \geq T(kz)$ for all z and all $k \geq 1$.

with $\zeta(1-\gamma)\gamma^{-1} > 0$ to ensure monotonicity and concavity. The corresponding absolute risk tolerance $T(z) = \eta + z/\gamma$ is increasing, convex and subhomogeneous when η and γ are both nonnegative. This covers the classical cases of constant relative risk aversion ($\eta = 0$) and of constant absolute risk aversion ($\gamma \to \infty$). Chiappori (1999) and Bodie, Merton and Samuelson (1992) have examined in a consumption/leisure context the special case of log linear preferences where $\eta = 0$ and $\gamma \to 1$, implying $u(z) = \log z$.

Lynch (1996) and Gabaix and Laibson (2002) examined a model of delayed adjustments in consumption. Investors can rebalance their portfolio every period, but they adapt their consumption to change in portfolio wealth only every D periods. They showed that the reduced flexibility of consumption plans due to these delays can potentially explain the equity premium puzzle. Because they assume constant relative risk aversion, we know from Proposition 6 that this loss of flexibility reduces the demand for equity, thereby increasing the equity premium. Their decision problem can be written as

$$J(z) = \max_{\substack{(c_0,...,c_{D-1}) \\ (\alpha_0,...,\alpha_{D-1})}} \sum_{t=0}^{D-1} \beta^t u(c_t) + \beta^D EJ\left(\left(z - \sum_{t=0}^{D-1} \frac{c_t}{R^t}\right) R^D + \sum_{t=0}^{D-1} \alpha_t \widetilde{r}_t\right), \tag{20}$$

where the consumption plan $(c_0, ..., c_{D-1})$ between two reset dates must be chosen at the previous reset date, whereas the portfolio choices $(\alpha_0, ..., \alpha_{D-1})$ can depend upon the history of returns since that date. We assume that the excess returns $(\widetilde{r}_0, ..., \widetilde{r}_{D-1})$ are independent and identically distributed. When $D = 1$, this model is the standard portfolio-saving model of Merton (1969) with full flexibility. Assuming that $u(z) = z^{1-\gamma}/(1-\gamma)$, it is easy to check that the optimal demand for equity at each reset date is proportional to z, the total wealth of the investor at that date. The optimal share of wealth invested in equity equals

$$\frac{\alpha(z; D)}{z} = b\left(R\beta E(1 + b\widetilde{r})^{-\gamma}\right)^{D/\gamma}, \tag{21}$$

where b solves $E\widetilde{r}(1 + b\widetilde{r})^{-\gamma} = 0$. Because $R\beta E(1 + b\widetilde{r})^{-\gamma}$ must be less than unity for the solution to be bounded, we obtain that the share of wealth invested in equity is exponentially decreasing with the length D of time between two reset dates. This illustrates Proposition 6. If the equity premium $E\widetilde{r}$ is small compared to the volatility $\sigma^2_{\widetilde{r}}$, we obtain the following approximation for the portfolio allocation rule:

$$\frac{\alpha(z;D)}{z} \simeq b \exp - \left[\frac{\delta - \mu}{\gamma} + \frac{\gamma - 1}{\gamma} \frac{(E\widetilde{r})^2}{2\gamma\sigma^2_{\widetilde{r}}} \right] D, \tag{22}$$

where $\delta = -\log \beta$ is the rate of pure preference for the present and $\mu = \log R$ is the risk-free rate. To illustrate, suppose that investors rebalance their portfolio every year, but they adapt their consumption to their portfolio wealth only every D years. For $\gamma = 4$, $\delta - r = 2\%$, $E\widetilde{r} = 7\%$ and $\sigma_{\widetilde{r}} = 17\%$, we obtain that the share of wealth invested in equity is approximately equal to $0.61 \exp -0.021D$. When there is no delay in the adjustment of consumption ($D = 1$, flexible context), the optimal share invested in equity is 59.3%. It goes down to 58.1% (49.1%) if consumption is planned every two (ten) years.[4]

Going back to the original problem expressed by (16), observe that it can be reinterpreted as a risk-sharing problem between $n + 1$ agents with utility functions u_i, $i = 0, ..., n$. The transfer of one unit of the consumption good from agent i yields p_i units to the reference agent 0. The intuition suggests that syndicates that are able to internally share risks efficiently should accept more portfolio risk, compared to syndicates in which individual 0 bears the entire portfolio risk, whereas agents $i \neq 0$ get a fixed income x_i. Propositions 5 and 6 provides sufficient conditions for this property to hold. An immediate extension of these results are obtained by combining risk sharing with one's future self and risk sharing with another individual.

3.3 Dynamic portfolio management

In the previous section, we examined a portfolio-saving model in which we introduced some rigidities in the way consumption can be adjusted to financial shocks. In this section, we alternatively examine the effect of introducing some rigidities in the way the portfolio allocation can be adjusted to these shocks. We consider the pure investment problem in which consumption takes place only at the end of the investment period. There are three dates.

[4] Gabaix and Laibson (2002) get an effect of the low frequency of consumption adjustments on the equity premium that is much stronger than what this analysis suggests. This is because a (rotating) small fraction $1/D$ of investors participates to the sharing of the market risk in each period.

At date $t = 0$, the agent is endowed with wealth z_0. This wealth can be invested in two assets, one of which is risk-free. At date $t = 1$, after observing the portfolio return, the investor can rebalance her portfolio in the flexible context. At date $t = 2$, the portfolio is liquidated and the accumulated wealth is consumed. The gross risk-free rate in each subperiod is denoted R. The return of the risky asset in the first and second subperiods are denoted respectively $\widetilde{r}$ and $\widetilde{r}_1$. We assume that $\widetilde{r}$ and $\widetilde{r}_1$ are independent. This application is a special case of our general model with

$$U(z, x) = Eu(zR + x\widetilde{r}_1), \tag{23}$$

where z is the wealth accumulated at the intermediary date, and x is the euro investment in the risky asset in the second subperiod. The optimal investment in the risky asset in this flexible context at date $t = 0$ solves the following backward induction problems:

$$\alpha^* = \max_{\alpha} Ev(z_0R + \alpha\widetilde{r}) \quad \text{with} \quad v(z) = \max_{x} Eu(zR + x\widetilde{r}_1). \tag{24}$$

In the rigid context, the demand for the risky asset in the two subperiods must be determined at date $t = 0$:

$$(\widehat{\alpha}, \widehat{x}) = \arg\max_{\alpha,x} Eu(z_0R^2 + \alpha\widetilde{r}R + x\widetilde{r}_1). \tag{25}$$

When $\widetilde{r}$ and $\widetilde{r}_1$ are identically distributed, it must be that $\widehat{x} = \widehat{\alpha}R$. This corresponds to a "buy-and-hold" strategy in which the excess return of the risky portfolio during the first subperiod is automatically reinvested in the risk-free asset during the second subperiod.

The objective function U defined in (23) is neither supermodular nor submodular, so that the only general result that we can use from section 2 is that the ability of the investor to rebalance her portfolio induces her to raise her demand for the risky asset at date $t = 0$ ($\alpha^* > \widehat{\alpha}$) when the expected excess return $E\widetilde{r}$ in the first subperiod is small compared to its standard deviation. In order to compare α^* and $\widehat{\alpha}$ for larger portfolio risk in the first subperiod, let us assume that the utility function u exhibits constant relative risk aversion γ. It is well-known since Mossin (1968) that $v(z) = ku(Rz)$ for all z, which means that myopia is optimal. In the flexible context described by (24), the demand α^* at $t = 0$ is independent of the distribution of the return of the risky asset in the second subperiod. It implies that

$$\alpha^* = \max_{\alpha} Eu(z_0 R^2 + \alpha \widetilde{r} R). \tag{26}$$

Thus, in the CRRA case, the question can be restated in a completely static framework. It boils down to determining the effect on the demand for the risky asset $\widetilde{r}$ of the introduction of another risky asset $\widetilde{r}_1$. Gollier (2001, Proposition 36) showed that independent static risks $\widetilde{r}$ and r_1 are substitutes, i.e., $\widehat{\alpha} < \alpha^*$, if absolute risk aversion is decreasing and absolute prudence is decreasing and larger than twice the absolute risk aversion. In the CRRA case, it is easy to check that these conditions hold when relative risk aversion is less than unity. Gollier, Lindsey and Zeckhauser (1997) showed that this condition is necessary to guarantee the result when returns are identically distributed.

Proposition 7 *Consider the two-period investment problem with constant relative risk aversion and independent and identically distributed returns. If relative risk aversion is less than unity, the initial demand for the risky asset is larger when the portfolio can be flexibly rebalanced at the end of the first period than when the investor follows a rigid buy-and-hold strategy. When relative risk aversion is larger than unity, it is always possible to find a distribution of excess returns so that the initial demand is smaller in the flexible context than in the more rigid one.*

To understand the necessity of relative risk aversion less than unity, consider the case of utility function $u(z) = -z^{-3}$, which exhibits constant relative risk aversion $\gamma = 4$. Normalize the risk free rate to zero. If the yearly excess return of the risky asset is distributed as $(-0.1, 2/3, 10, 1/3)$, the optimal investment in the risky asset in the flexible context equals 16.16% of initial wealth, whereas it goes up to 16.27% in the rigid context for a time horizon of two years. Observe that this counterexample is built on a very skewed distribution of excess returns. For more symmetric distributions, the standard intuition that an increase of the frequency at which the portfolio can be rebalanced raises the demand for the risky asset. To illustrate, consider the more realistic assumption that the distribution of excess return is normally distributed with mean 4.6% per year and a standard deviation of 14.2%.[5] In

[5]We calibrate our model on the real excess return of SP500 over the period 1963-1995.

this case, the optimal portfolio allocation is to invest 56.35% of the investor's wealth if the investor rebalances her portfolio every year. This share of wealth invested in the risky asset goes sligthly down to 56.01% when the investor rebalances her portfolio only every two years.[6]

4 Conclusion

Because the value of information is nonnegative in the expected utility model, flexibility always raises the expected utility of the decision maker. In this paper, we examine the comparative statics consequence of flexibility on the attitude towards risk ex ante. At the exception of Machina (1982), the general tone of the literature is that flexibility enhances risk tolerance. Workers who can adjust their labor supply to their financial wealth should take more risk on financial markets. Spouses should take more risk on their job if these risks are efficiently shared within their household. Investors should invest more in risky assets if they can rebalance their portfolio more frequently. Consumers who have access to the credit market should take more risk than those who cannot adjust their saving/credit to the shocks on their incomes.

When the decision under risk is a choice between a risky prospect or a safe one, or when it entails only small risks, the intuition that flexibility enhances risk tolerance is correct. For more general choice problems under uncertainty, additional restrictions on preferences or on the characteristics of the available ex post adjustments are necessary to guarantee the comparative statics result. When the choice under uncertainty is a portfolio allocation, we have shown that the submodularity or the supermodularity of the objective function is useful to derive sufficient conditions in various applications.

[6] The demand for the risky asset goes down to 55.66%, 55.28% and 54.87% when the frequency of portfolio rebalancement is respectively every 3, 4 and 5 years. We solve the problem numerically by using a discrete approximation for the normal distribution of $\log \widetilde{r}$ with $n = 20$ equally distant points between $\mu - 5\sigma$ and $\mu + 5\sigma$. An increase in n over 20 has no effect on the first four digits of α.

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Appendix: Proof of Proposition 2

We first examine the flexible context. The first-order condition of program 1 is written as

$$U_a(z, x^*(z)) = 0. \tag{27}$$

Fully differentiating this condition yields

$$x^{*\prime}(z) = -\frac{U_{zx}(z, x^*(z))}{U_{xx}(z, x^*(z))}. \tag{28}$$

Using the envelop theorem, we have that

$$v'(z) = U_z(z, x^*(z)). \tag{29}$$

Fully differentiating this condition implies that

$$v''(z) = U_{zz}(z, x^*(z)) + x^{*\prime}(z)U_{zx}(z, x^*(z)) \tag{30}$$

Combining equations (28), (29) and (30) yields property (7). Condition (6) is in Pratt (1964).

We now turn to the analysis of the rigid context. It is obvious that $\widehat{\pi}(0) = 0$. Using the envelop theorem, fully differentiating condition (4) yields

$$E\widetilde{\varepsilon}U_z(z_0 + k\widetilde{\varepsilon}, \widehat{x}(k)) = -\widehat{\pi}'(k)U_z(z_0 - \widehat{\pi}(k), \overline{x}(k)), \tag{31}$$

where functions $\widehat{x}(k)$ and $\overline{x}(k)$ satisfy the following conditions:

$$EU_x(z_0 + k\widetilde{\varepsilon}, \widehat{x}(k)) = 0 \tag{32}$$

$$U_x(z_0 - \widehat{\pi}(k), \overline{x}(k)) = 0. \tag{33}$$

We have that $\widehat{x}(0) = \overline{x}(0) = x^*(z_0)$ and $\widehat{\pi}'(0) = 0$. Fully differentiating equations (31), (32) and (33) shows that $\widehat{x}'(0) = \overline{x}'(0) = 0$ and

$$\begin{aligned}
& E\widetilde{\varepsilon}^2U_{zz}(z_0 + k\widetilde{\varepsilon}, \widehat{x}(k)) + \widehat{x}'(k)E\widetilde{\varepsilon}U_{zx}(z_0 + k\widetilde{\varepsilon}, \widehat{x}(k)) \\
= \; & (\widehat{\pi}'(k))^2U_{zz}(z_0 - \widehat{\pi}(k), \overline{x}(k)) - \widehat{\pi}''(k)U_z(z_0 - \widehat{\pi}(k), \overline{x}(k)) + \overline{x}'(k)U_{zx}(z_0 - \widehat{\pi}(k), \overline{x}(k)).
\end{aligned}$$

Evaluating this at $k = 0$ yields

$$\widehat{\pi}''(0) = \sigma_\epsilon^2 \frac{-U_{zz}(z_0, x^*(z_0))}{U_z(z_0, x^*(z_0))}.$$

It implies that

$$\begin{aligned}\widehat{\pi}(k) &= \widehat{\pi}(0) + k\widehat{\pi}'(0) + \frac{1}{2}k^2\widehat{\pi}''(0) + o(k^2) \\ &= \frac{1}{2}k^2\sigma_\epsilon^2 \frac{-U_{zz}(z_0, x^*(z_0))}{U_z(z_0, x^*(z_0))} + o(k^2).\end{aligned}$$

This concludes the proof.■

Proof of Proposition 4

In the rigid context, the optimal bundle is characterized by $\widehat{x}_i = \gamma_i z_0 R/p_i\Gamma$, $i = 1, ..., n$, yielding a total expenditure equaling $z_0 R(1 - (\gamma_0/\Gamma))$. It implies that the portfolio choice problem in this context can be written as

$$\widehat{\alpha} = \arg\max_\alpha E\widehat{u}(z_0 R + \alpha\widetilde{r}),$$

where $\widehat{u}(z) == \gamma_0^{-1}\left(z - z_0\left(1 - \gamma_0\Gamma^{-1}\right)\right)^{\gamma_0}$. In the flexible context, the optimal portfolio solves

$$\alpha^* = \arg\max_\alpha Ev(z_0 R + \alpha\widetilde{r}),$$

with $v(z) = k\ z^\Gamma$. Gollier and Kimball (1996)[7] have shown that the necessary and sufficient condition for α^* to be larger than $\widehat{\alpha}$ for all acceptable distributions of the equity return $\widetilde{r}$ is that v be "centrally less risk-averse" than $\widehat{u}$ around z_0, i.e., that

$$h(r; z_0) = r\left(\frac{v'(z_0R + r)}{v'(z_0R)} - \frac{\widehat{u}'(z_0R + r)}{\widehat{u}'(z_0R)}\right) \geq 0 \qquad (34)$$

[7] See also Gollier (2001), section 6.3.3.

for all r. For r positive, this condition can be rewritten as

$$\left(1+\frac{r}{z_0 R}\right)^{\Gamma-1} > \left(1+\frac{\Gamma}{\gamma_0}\frac{r}{z_0 R}\right)^{\gamma_0-1}.$$

When r tends to infinity, the two sides of this inequality tend to zero. However, because $\Gamma - 1 < \gamma_0 - 1 < 0$, the left-hand side converges quicker to zero than the right-hand side. This implies that the necessary condition (34) is violated for large r. ■