The consumption-based determinants of the term structure of discount rates: Corrigendum

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In Gollier (2007), I examine the effect of serially correlated growth rates on the efficient long discount rate. I develop the idea that positive serial correlation tends to magnify the long term risk. This tends to induce the prudent representative agent to accumulate more precautionary savings. This raises the efficient discount rate for long horizons. Various illustrations of this intuition were explored in the paper, with dynamic growth processes such as mean-reversion, Markov, and random walks with parametric uncertainty. The discount rate r_t associated to time horizon *t* is written as

$$r_t = \delta - \frac{1}{t}\ln \frac{Eu'(c_t)}{u'(c_0)}, \tag{1}$$

where δ is the rate of impatience, c_t is consumption at date *t*, and u is the increasing and concave utility function of the representative agent. Let x_i denote the continuous-time growth rate of consumption between date *i-1* and *i*. This implies that we can rewrite the above equation for *t=2* as follows:

$$r_2 = \delta - \frac{1}{2}\ln Eh(x_1, x_2), \tag{2}$$

with

$$h(x_1, x_2) = \frac{u'\left(c_0 \exp(x_1 + x_2)\right)}{u'(c_0)}. \tag{3}$$

Obviously, any change in the distribution of (x_1, x_2) that raises the expectation of $h(x_1, x_2)$ reduces the long discount rate r_2. In the general expected utility model, the coefficient of correlation between two random variables as x_1 and x_2 is usually insufficient to characterize the

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role of the statistical relationship on an expectation as $Eh(x_1,x_2)$. The full joint distribution function is generally required to determine the forward discount rate. In Gollier (2007), I defined the notion of "positive first-degree stochastic dependence" to solve this problem. To do this, consider a distribution function F for the pair of random variables (x_1,x_2), with $F(t_1,t_2)=\mathrm{P}[x_1\le t_1 \cap x_2 \le t_2]$. Let also $F_i(t)$ denote the marginal distribution of x_i, *i=1,2*.

Definition 1 (Gollier (2007): *There is positive first-degree stochastic dependence between* x_1 *and* x_2 *if* $P\left[x_2\le t_2 \middle| x_1=t_1\right]$ *is decreasing in* t_1 *for all* t_2.

In Lemma 1 of Gollier (2007), I mistakenly claimed that this condition was necessary and sufficient to raise $Eh(x_1,x_2)$ for all supermodular functions *h*. It is in fact sufficient but not necessary, as I show now. In fact, the necessary and sufficient condition for a change in the joint distribution of (x_1,x_2) with fixed marginals to raise the expectation of $h(x_1,x_2)$ for all supermodular functions *h* already exists in the literature. This is the concept of concordance introduced by Tchen (1980) and Epstein and Tanny (1980).

Definition 2 (Tchen (1980)): *Consider two pairs of random variables* (x_1,x_2) *and* $(\hat{x}_1,\hat{x}_2)$ *respectively with distribution functions F and* $\hat{F}$ *with the same marginal. We say that* (x_1,x_2) *is more concordant than* $(\hat{x}_1,\hat{x}_2)$ *if*

$$\forall (t_1,t_2)\in\mathbb{R}^2,\ F(t_1,t_2)\ge \hat{F}(t_1,t_2). \tag{4}$$

(x_1,x_2) *is concordant if it is more concordant than its corresponding pair of independent random variables with the same marginals than* (x_1,x_2)*, i.e., if* $F(t_1,t_2)\ge F_1(t_1)F_2(t_2)$ *for all* (t_1,t_2) *.*[2]

Observe that positive first-degree stochastic dependence implies concordance, as shown in the following Proposition.

[2] This is equivalent to the notion of "positive quadrant dependence" proposed by Lehmann (1966).

Proposition 1: *If* (x_1, x_2) *exhibits positive first-degree stochastic dependence, then* (x_1, x_2) *is concordant.*

Proof: Suppose that (x_1, x_2) exhibits positive first-degree stochastic dependence, so that $P\left[x_2 \le t_2 \middle| x_1 = t_1\right]$ is decreasing in t_1 for all t_2. I have to prove that this implies that $F(t_1, t_2) \ge F_1(t_1)F_2(t_2)$ for all (t_1, t_2). After dividing by $F_1(t_1)$, this condition can be rewritten as follows:

$$G(t_1) = \frac{\int^{t_1} P\left[x_2 \le t_2 \middle| x_1 = \tau\right] dF_1(\tau)}{F_1(t_1)} \ge \frac{\int^{\infty} P\left[x_2 \le t_2 \middle| x_1 = \tau\right] dF_1(\tau)}{F_1(\infty)} = G(\infty). \tag{5}$$

I would be done if *G* would be decreasing. Observe that *G'* is negative if and only if

$$\frac{\int^{t_1} P\left[x_2 \le t_2 \middle| x_1 = \tau\right] dF_1(\tau)}{F_1\ (t_1)} \ge P\left[x_2 \le t_2 \middle| x_1 = t_1\right]. \tag{6}$$

Observe that the left-hand side of this inequality is a weighted mean of $P\left[x_2 \le t_2 \middle| x_1 = \tau\right]$ for $\tau \in \left[-\infty, t_1\right]$. Because this function is decreasing in τ, this mean is larger than its value at the upper bound $\tau = t_1$ of this semi-interval. Because this is true for all t_2, this concludes the proof. □

Positive first-degree stochastic dependence is sufficient for concordance, but it is clearly not necessary. Here is a counterexample: The support of x_1 is $\{-1, 0, 1\}$ and the support of x_2 is $\{0,1\}$. Suppose that $P(-1,0) = P(1,1) = 3/12$, $P(-1,1) = P(1,0) = 1/12$, $P(0,0) = 4/12$ and $P(0,1) = 0$. It is easy to check that this pair of random variables exhibits concordance but not positive first-degree stochastic dependence.

I now show that concordance is necessary and sufficient for a change in statistical relation in (x_1, x_2) to yield an increase in $Eh(x_1, x_2)$ for all supermodular functions *h*. The formal proof of the following lemma is in Tchen (1980), Epstein and Tanny (1980) or Meyer and Strulovici (2011).

Proposition 2: *Consider a bivariate function h. The following conditions are equivalent:*

- *For any two pairs of random variables* (x_1, x_2) *and* $(\hat{x}_1, \hat{x}_2)$ *such that* (x_1, x_2) *is more concordant than* $(\hat{x}_1, \hat{x}_2)$, $Eh(x_1, x_2) \geq Eh(\hat{x}_1, \hat{x}_2)$.
- *h is supermodular.*

The proof of this result is based on the observation that

$$Eh(x_1, x_2) - Eh(\hat{x}_1, \hat{x}_2) = \int\int h_{12}(t_1, t_2)\left[F(t_1, t_2) - \hat{F}(t_1, t_2)\right]dt_1 dt_2. \qquad (7)$$

This can be obtained by a double integration by parts. It implies that our results in Gollier (2007) are valid modulo the switch of all occurrences of the terms “positive first-degree stochastic dependence” by the term “concordance”. For example, here is the correct version of our Proposition 3 in Gollier (2007):

Proposition 3: *The presence of any concordance in changes in log consumption reduces the long-term risk-free rate if and only if relative prudence is larger than unity.*

Positive first-degree stochastic dependence is sufficient for the result, but not necessary. In Gollier (2007), I also developed the notion of positive second-degree stochastic dependence. The associated results suffer from the same deficiency. Denuit, Eeckhoudt, Testlin and Winkler (2010) and Heinzel (2012) provide a correct and complete characterization of this case.

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