EMPIRICAL GAME THEORETIC MODELS: CONSTRAINED EQUILIBRIUM & SIMULATION

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Abstract

We propose an operational concept of Constrained Strategic Equilibrium (CSE) applicable to a broad class of empirical game theoretic models. By restricting the players' strategic sets, we can compute solutions based upon auxiliary Monte Carlo (MC) simulations. Kernel methods are used to produce smooth estimates of the players' expected utility functions. In combination with the generic estimation principle proposed by Florens et al. (1997), our algorithm o¤ers an integrated methodology for the estimation of empirical game theoretic models. An application to a small data set from procurements in the French aerospace industry illustrates the ‡exibility of our approach.

Keywords : Auctions, Constrained Equilibrium, Simulation.

JEL Classi…cation :C63, C70, D44, C15.

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1. Introduction

The recent years have witnessed an explosive development in empirical applications of game theory with special emphasis on auctions (see e.g. La¤ont (1997) for a recent survey). Games of incomplete information have two key characteristics: …rstly, strategy functions transform unobserved random private 'types' or 'signals' into observed actions; secondly the strategic nature of the game implies that strategies depends upon the underlying probability distribution function of the types. Consequently, one cannot estimate jointly the functional form of players' strategies and the distribution of types from the sole observation of actions. This speci…cation problem is traditionally solved by imposing that strategies are a 'solution' of the game.[1] This solution concept is assumed to represent adequately agents' behavior. The most commonly used theoretical solution concept is that of Nash Equilibrium.[2] However, the concept of NE strategies assumes an exceptionally high level of rationality from the participants who need to have access to all relevant information (distribution of types, ...) and must be able to apply sophisticated mathematical tools. Actually, it turns out that, except under fairly restrictive assumptions whose empirical validity often is questionable, many games cannot be solved for NE solutions. So, not only the NE concept may no t represent actual behavior, but also we may not be able to derive the strategy functions which is an essential requirement for any inference method.

As an alternative to NE we introduce in this paper the concept of Constrained Strategic Equilibrium (hereafter CSE). Essentially, we propose to restrict our attention to appropriate subsets of strategies, typically indexed by an auxiliary parameter vector, and to search for an equilibrium solution within such subsets. CSE o¤er a major computational advantage, which turns out to be critical for empirical work, in that they can be solved at a high level of generality by strategic form analysis of the game based upon auxiliary Monte Carlo (hereafter MC) simulations. The CSE appeared to be relevant under at least two scenarios: the …rst one is directly related to the general notion of 'bounded rationality' and more speci…cally to the concept of Rules of Thumb ; in the second scenario, one would use the computational advantage of the CSE with the intent to approximate an analytically or a numerically untractable NE solution.

Our paper is organized as follows: In section 2, we de…ne the model, section 3 introduces the CSE as an alternative to the NE concept to be used in empirical application, in section 4 we combine Monte Carlo simulations and Kernel esti-

[1] See La¤ont, Ossard, Vuong (1995), as well as Hendricks and Paarsh (1995).

[2] Depending on the information available to players we shall consider Nash Equilibrium or Bayesian Nash Equilibrium. Hereafter both concepts are denoted NE.

mates to produce an operational numerical algorithm to calculate CSE, section 5 presents potential applications of the CSE, we illustrate our approach in section6 with an example from the French aerospace industry, finally, section 7 concludes.

2. The Model

For the ease of exposition as well as notation, CSE's will be presented here in the context of a single play of a strategic form game of incomplete information with a pure strategy solution. It ought to become obvious that the methodology developed below applies to a broader class of games (repeated games, mixed strategies, learning, ...).

There are N players each of which is endowed with a privately known 'type' or 'signal' $\xi_i \in \Omega_i$ with $\Omega_i \subset \mathbb{R}^p$: The types $\xi = (\xi_1; \dots; \xi_N)$ are drawn from a joint distribution with cumulative distribution function (hereafter c.d.f.) $F(\xi \mid \theta)$ and density $f(\xi \mid \theta)$, where $\theta \in \Theta$ denotes a vector of parameters (known to the players but not to the observer).[3] Let $F_i(\xi_i \mid \theta)$ denote the marginal c.d.f. of ξ_i and $f_i(\xi_i \mid \theta)$ the corresponding density. This general framework includes as special cases of interest i.i.d. types, affiliated types and asymmetric independently distributed types.

Unobserved signals are transformed into action by means of a transformation $\varphi_i : \Omega_i \to X_i(\theta)$ which depends upon θ,

$$x_i = \varphi_i(\xi_i; \theta); \qquad i : 1 \to N \quad : \tag{2.1}$$

When dealing with empirical applications we shall require that these decision rules or strategies be invertible[4] in ξ_i for any given $\theta \in \Theta$.

Player i is endowed with an individual utility function $U_i(\varphi(\xi;\theta); \xi)$.[5] The rules of the game serve to define the set of admissible strategies H_i for player i. The number of players N (depending upon the situation, the decision to participate may be endogenous or exogenous), the joint distribution F, the utility functions $\{U_i\}_{i=1;\dots;n}$ and the sets of admissible strategies $\{H_i\}_{i=1;\dots;n}$ are common knowledge to all players. Symmetry assumes that the joint distribution F is exchangeable (i.e. F is invariant under a permutations of players), $(U_i; H_i; \Omega_i) = (U_j; H_j; \Omega_j)$ and the equilibrium strategies (subject to existence) are such that

[3] We do not assume here that θ is of finite dimmension. In particular, θ could represent the actual density of the types.

[4] See Florens, Protopopescu and Richard (1997) for a discussion of potential violations of that condition in finite sample inference.

[5] For the ease of exposition we adopt the usual notation: $\xi = (\xi_i; \xi_{-i}) = (\xi_1; \dots; \xi_N)$ and $\varphi(\xi;\theta) = (\varphi_i(\xi_i;\theta); \varphi_{-i}(\xi_{-i};\theta)) = (\varphi_1(\xi_1;\theta); \dots; \varphi_N(\xi_N;\theta))$:

$\varphi_i = \varphi_j; \forall i \neq j$. Note that exchangeability reduces to the equality of marginal distributions when types are univariate or independent.

The strategic form of the game is based upon the set of individual expected utility functions

$$\tilde{U}_i(\varphi;\theta) = E_{\xi|\theta}\left[U_i(\varphi(\xi;\theta);\xi)\right] \quad . \tag{2.2}$$

The extensive form of the game is based upon the set of conditional individual expected utility functions

$$\hat{U}_i(\varphi;\xi_i;\theta) = E_{\xi_{-i}|\xi_i;\theta}\left[U_i\left(x_i;\varphi_{-i}\left(\xi_{-i};\theta\right);\xi_i;\xi_{-i}\right)\right] \quad . \tag{2.3}$$

3. Solution Concepts

3.1. Unconstrained NE solutions

For any given $\theta \in \Theta$; and subject to existence, a Bayesian Nash Equilibrium in pure strategy in the set of strategies $H = \prod_{i=1} H_i$ is defined by Harsanyi (1967) as a strategy profile $\left(\varphi_1^{NE};\ldots;\varphi_N^{NE}\right)$ of mutually best responses strategies in the extensive form game:

$$\hat{U}_i\left(\varphi_i^{NE}(\xi_i;\theta);\varphi_{-i}^{NE};\xi_i;\theta\right) \geq \hat{U}_i\left(x_i;\varphi_{-i}^{NE};\xi_i;\theta\right);$$

$$\forall x_i \in X_i(\theta);\ \forall \xi_i \in \Omega_i \text{ and } \forall i : 1 \to n \quad . \tag{3.1}$$

The set H is assumed to be such that it is equivalent to consider the extensive or the strategic form of the game to derive the NE solution. Therefore, φ^{NE} also verifies

$$\tilde{U}_i\left(\varphi_i^{NE};\varphi_{-i}^{NE};\theta\right) \geq \tilde{U}_i\left(\varphi_i;\varphi_{-i}^{NE};\theta\right);$$

$$\forall \varphi_i \in H_i \text{ and } \forall i : 1 \to n \quad . \tag{3.2}$$

Note that no general theorem insures the existence of a NE solution in a game of incomplete information with continuous types and actions. In practice, the problems of existence and uniqueness are solved by the direct determination of an analytical equilibrium solution. This solution obtains from the following optimization and fixed point problems,

$$\varphi_i^{NE}(\xi_i) = \underset{x_i \in X_i(\theta)}{\operatorname{ArgMax}}\ \hat{U}_i\left(x_i;\varphi_{-i}^{NE};\xi_i;\theta\right);\ \forall \xi_i \in \Omega_i \text{ and } \forall i : 1 \to n \quad . \tag{3.3}$$

The corresponding First Order Conditions (FOCs) often are reformulated as

$$\frac{d}{dx_i}\hat{U}_i\left(x_i;\varphi_{-i}^{NE};\xi_i;\theta\right)\Big|_{x_i=\varphi_i^{NE}(\xi_i;\theta)} = 0 \quad \forall \xi_i \in \Omega_i \text{ and } \forall i : 1 \to n; \tag{3.4}$$

which typically produce a set of differential equations in the φ_i's whose solution depends on θ. Except under fairly restrictive assumptions (such as symmetry, risk neutrality, ...) it is often impossible to find an analytical or even numerical solution to such problems. Noticeable exceptions to that statement are simple, mostly single item, auction games. Again, the complexity of unconstrained NE solutions raises obvious questions as to their empirical relevance.

3.2. Constrained Strategic Equilibrium

Constrained sets of strategies are implicitly defined here as subsets $H_i^{(k)} \subset H_i$. The definition of CSE now parallels that of a NE in strategic form, except that strategies are now restricted to $H_i^{(k)}$.

Definition 3.1. A CSE in the set of strategies $H^{(k)} = \prod_{i=1}^{N} H_i^{(k)}$ is a strategic implementation of the game $\varphi_{CSE}^{(k)} = \left(\varphi_{1;CSE}^{(k)}, \ldots, \varphi_{N;CSE}^{(k)}\right)$, whereby the $\varphi_{i;CSE}^{(k)}$'s are mutually best responses in the strategic form game

$$\tilde{U}_i\left(\varphi_{i;CSE}^{(k)}(\theta), \varphi_{-i;CSE}^{(k)}(\theta); \theta\right) \geq \tilde{U}_i\left(\varphi_i^{(k)}, \varphi_{-i;CSE}^{(k)}(\theta); \theta\right); \forall \varphi_i^{(k)} \in H_i^{(k)}; \forall i: 1 \to N. \tag{3.5}$$

The game of incomplete information $\Gamma = (N, U, H, F, \Omega)$ can be interpreted as an equivalent game of complete information $\tilde{\Gamma} = \left(N, \tilde{U}, H\right)$ since $\tilde{U}_i\left(\varphi_i, \varphi_{-i}, \theta\right) = E_{\xi|\theta}\left[U_i\left(\varphi(\xi, \theta), \xi\right)\right]$ is not function of a random variable. Then, the existence theorem of Nash Equilibrium in infinite games of complete information with continuous utility function (see Debreu, 1952) can be applied to $\tilde{\Gamma}$. Consider the following assumptions:

i) $H_i^{(k)}$ is compact and convex $\forall i = 1, \ldots, N$.

ii) the function $\tilde{U}_i\left(\varphi_i, \varphi_{-i}, \theta\right)$ is continuous in φ, $\forall i: 1 \to N$; $\forall \varphi \in H$.

iii) the function $\tilde{U}_i\left(\varphi_i, \varphi_{-i}, \theta\right)$ is quasi concave in φ_i, $\forall i: 1 \to N$; $\forall \varphi_i \in H_i$.

Under assumptions i) to iii) the game $\tilde{\Gamma}^{(k)} = \left(N, \tilde{U}, H^{(k)}\right)$ satisfies the conditions for the existence of a NE in pure strategy, and there exists a CSE in $H^{(k)}$. Assumption i) is easy to satisfy since it depends only upon the selection of an appropriate constrained set $H_i^{(k)}$. The continuity of $\tilde{U}_i\left(\varphi_i, \varphi_{-i}, \theta\right)$ in φ is guaranteed when $U_i(.)$ is Hölder continuous in φ (see Appendix 1). Finally, $\tilde{U}_i\left(\varphi_i, \varphi_{-i}, \theta\right)$ is quasi concave in φ_i if $U_i(.)$ or any conditional expectation $E_{\xi_{-(s)}|\xi_{(s)};\theta}\left[U_i\left(\varphi_{(s)}\left(\xi_{(s)};\theta\right), \varphi_{-(s)}\left(\xi_{-(s)};\theta\right); \xi_{(s)}, \xi_{-(s)}\right)\right]$ ($\forall s : 1 \to N$; $\forall \xi_{(s)} = \xi_1, \ldots, \xi_{(s)}$) is concave (see Appendix 2).

In the remainder we consider that assumptions i) to iii) are verified. Then, the CSE can be defined as a fixed point of the constrained best response correspondence

$$\varphi^{(k)}_{i;CSE}(\theta) = \underset{\varphi_i^{(k)} \in H_i^{(k)}}{\mathrm{ArgMax}}\ \Theta_i\left(\varphi_i^{(k)}, \varphi^{(k)}_{-i;CSE}(\theta); \theta\right) \quad \forall i : 1 \to n \quad : \tag{3.6}$$

The determination of this fixed point is greatly simplified with a parametrization of the strategies in $H_i^{(k)}$ by a vector of $d_i^{(k)} \in \mathbb{R}^k$. Such parametrization is always possible since $H_i^{(k)}$ is compact. This approach provides a major computational advantage since it requires to optimize over a finite set of parameters rather than an infinite set of functions as it is the case with NE.

4. The CSE in practice

We motivate the practical use of the CSE concept under two non exclusive scenario: as "Rule of Thumb" and as potential approximation of NE.

4.1. CSE as Rules of Thumb

Some authors have criticized the empirical relevance of 'perfect rationality' (e.g. Binmore (1987) or Simon (1987)). As an alternative, the notion of 'bounded rationality' has been developed and the ensuing literature has rapidly expanded in recent years (see e.g. Lipman (1994) for a survey). Among the models proposed, we are particularly interested in the concept of Rules of Thumb, as recently developed by Rosenthal (1993 a,b). In this model agents are assumed to have limited knowledge of their strategic sets and/or limited computational capabilities which prevent them from making perfectly rational choices. Instead they develop simple decision rules, based on intuition or on previous plays of the game, which happen to perform well for them. The relation between CSE's and Rules of Thumb equilibrium is then obvious since, by selecting simple Rules of Thumb agents are actually constraining their strategic sets. The practical relevance of the Rules of Thumb finds additional support in recent contributions in experimental economics. For example, Levin et al. (1996) and Kagel and Richard (1997) find that players use simple decision rules instead of more sophisticated NE bid strategies. Interestingly enough, these simpler rules produce payoffs that are (potentially) quite close to those that would obtain under NE strategies. Note that we do not intend to develop a 'theory' meant to rationalize agents' actual choices of constrained strategic sets. Actually very few authors have attempted to address this specific issue in the literature on bounded rationality. Our paper being primarily application oriented we can think of at least two ways to select appropriate constrained strategy sets in the context of specific real-life applications. Ideally, one would like to interview players as to their actual choice of decision rules and to use such 'revealed' rules as the central components of the inference procedure. Note, however, that in view of fairly obvious strategic considerations, the players might not be willing to participate to such interviews, neither would they necessarily accurately describe their actual strategic behavior. Alternatively, one might consider selecting constrained (Rules of Thumb) strategies that appear to be 'sensible' or 'common sense' on heuristic grounds, estimate an empirical model based upon these rules and then construct an ex-post 'specification test' aimed at validating that model. Such tests are by no means trivial to construct since the choice of a specific functional form for the strategies partially serves to 'identify' the empirical model and, therefore, is not fully empirically testable. There remain, nevertheless, aspects of the model specification that are 'overidentifying' and could serve as the basis for a specification test. The search for such specification tests belongs to our immediate research agenda.

4.2. CSE as approximation of NE

NE in games of incomplete informations typically have complex analytical forms, however, those few cases where NE strategies can be computed, their 'smooth' graphs clearly suggest that it ought to be possible to approximate them by simpler functional forms, such as low degree polynomials, piecewise linear and/or exponential functions (see e.g. some of the graphs of NE strategies found in Marshall et al. (1994)). Different approximation techniques can be considered when the NE cannot be calculated. For instance, assume that the union of a sequence of increasing constrained sets $\left\{ \left\{ H^{(k)} \right\}_{k=1 \to \infty} \mid H^{(k)} \subset H^{(k+1)} \ \forall k \in \mathbb{N}^* \right\}$ is dense in H with respect to an appropriate topology. The intuitive idea of an approximation theorem is that a sequence of CSE $\left\{ \varphi^{(k)}_{CSE} \right\}_{k=1 \to \infty}$ converges toward a NE under some regularity conditions. Such approximation have the double advantage not to rely on the FOC's of the NE, and the approximation $\varphi^{(k)}_{CSE}$ has a direct game theoretic interpretation in finite distance. Armantier et al. (2000) show that when a sequence of CSE $\left\{ \varphi^{(k)}_{CSE} \right\}_{k=1 \to \infty}$ has an accumulation point M, then M is a NE in H. Consequently, if H is a compact set, such as the set of functions of uniformly bounded variation on $[a; b]$ and bounded at a, then any sequence of

$\mathrm{CSE}\left\{\varphi^{(k)}_{CSE}\right\}_{k=1 \to \infty}$ has a subsequence that converges toward a NE. This result is particularly interesting in the light of recent works (e.g. Athey (1997)) that show that when t simple analytical structures and it is possible to derive an analytical expression for $\frac{\partial}{\partial d_i} U_i(d_i; d_{-i}; \xi; \theta)$, which reduces considerably the computational burden. Finally, we can use standard numerical techniques to solve the system

$$\frac{1}{S}\sum_{s=1} \frac{\partial}{\partial d_i} U_i\left(d_i; d_{-i}; \tilde{\xi}_s^{\theta}; \theta\right) = 0 \quad \forall i: 1 \to n \quad . \tag{5.1}$$

Note that some of these techniques might use second order derivatives which would require $U_i(d_i; d_{-i}; \xi; \theta)$ to be C_2 in d_i. A critical remark applies to this MC implementation: the strategic nature of the game is fundamentally captured by its strategic implementation functional $(d_i(\theta); i: 1 \to N)$. In many problems and, in particular, to apply an inference procedure we need to produce smooth estimates of the d_i's themselves as functions of θ. Readers familiar with the MC estimation of functional - see. e.g. Richard (1997, section 23.2.3) - know that, in order to do so, one has to rely upon a technique known as that of Common Random Numbers (CRN's). Therefore, we have to generate initially a single set of CRN's, say $(\varepsilon_{is}; i: 1 \to N; ; s: 1 \to S)$ which will be transformed into $\tilde{\xi}_s^{\theta}$ for any θ considered.

Case 2 : $U_i(d_i; d_{-i}; \xi; \theta)$ is not continuous in d_i but $\bar{U}_i(d_i; d_{-i}; \theta)$ is C_1 in d_i: This situation arises with games of the type 'winner takes all' such as auction. Such games are characterized by the fact that actions are ranked according to a scalar rule $\nu(x_i)$, the highest score wins and takes all and

$$U_i(d_i; d_{-i}; \xi; \theta) = V_i(d_i; \xi_i; \theta)\, \mathbb{1}\left[\nu\left(\varphi_i^{(k)}(\xi_i; d_i)\right) \geq b_{-i}\left(\xi_{-i}; d_{-i}\right)\right] \quad \forall i: 1 \to n \quad ; \tag{5.2}$$

where

$$b_{-i}\left(\xi_{-i}; d_{-i}\right) = \underset{j \neq i}{Max}\, \nu\left(\varphi_j^{(k)}\left(\xi_j; d_j\right)\right) \quad ; \tag{5.3}$$

$\mathbb{1}\left[. \geq b_{-i}\left(\xi_{-i}; d_{-i}\right)\right]$ is the characteristic function and $V_i(.)$ is the utility function of player i when she wins the game. In this game, an infinitesimal variation of the d_i may produce a change in the identity of the winner and a discrete "jump" in the utility function. However, the conditional expectation

$$\tilde{U}_i(d_i; d_{-i}; \xi_i; \theta) = E\left[U_i(d_i; d_{-i}; \xi; \theta)/\xi_i\right] = V_i(d_i; \xi_i; \theta)\, G_i\left(\nu\left(\varphi_i^{(k)}(\xi_i; d_i)\right); d_{-i}/\xi_i\right) \quad ; \tag{5.4}$$

where $G_i(.)$ is the conditional c.d.f of the highest score among player i rivals, is C_1 in d_i otherwise the problem would be ill defined from the start. The computational problem arises from the fact that the c.d.f $G_i(.)$ can be expressed analytically only in the simplest games. To solve this problem numerically, we propose to produce a smooth estimate of the joint distribution of $\left(\xi_i, b_{-i}\left(\xi_{-i}, d_{-i}\right)\right)$, $G_i(t, b \mid d_{-i}, \theta) = P\left(\xi_i < t, b_{-i}\left(\xi_{-i}, d_{-i}\right) < b\right)$. The empirical distribution is not a good candidate since it will produce some discontinuities. Instead we propose for any given strategic choice d_{-i}

$$\hat{G}_i(t, b \mid d_{-i}; \theta) = \frac{1}{S}\sum_{s=1}^{S} 1\left(\tilde{\xi}^{\theta}_{i,s} \le t\right) K_h\left(b - \tilde{b}^{\sigma}_{-i}\left(\tilde{\xi}^{\theta}_{-i,s}; d_{-i}\right)\right); \tag{5.5}$$

where $\tilde{b}^{\sigma}_{-i}\left(\tilde{\xi}^{\theta}_{-i,s}; d_{-i}\right) = \mathrm{Max}_{j \neq i}\, \nu\varphi^{(k)}_j\left(\tilde{\xi}^{\theta}_{j,s}; d_j\right)$ [6]; $\tilde{\xi}^{\theta}_s = \left(\tilde{\xi}^{\theta}_{1,s}, \ldots, \tilde{\xi}^{\theta}_{N,s}\right)$ is a vector of random numbers generated from $F(. \mid \theta)$, K_h denotes an arbitrary c.d.f., labeled 'kernel' and h a 'bandwidth' which controls the smoothness of the kernel estimate.[7] It has been well established that when h tends toward 0 $\hat{G}_i(.,. \mid d_{-i}; \theta)$ converges asymptotically in S toward $G_i(.,. \mid d_{-i}; \theta)$. Then we can write

$$\hat{U}_i(d_i; d_{-i}; \theta) = \int V_i(d_i; d_{-i}; \xi_i; \theta)\, 1\left[\left(\nu\varphi^{(k)}_i(\xi_i; d_i)\right) \ge b_{-i}\left(\xi_{-i}; d_{-i}\right)\right] \tag{5.6}$$

$$dG_i\left(\xi_i, b_{-i}\left(\xi_{-i}; d_{-i}\right) \mid d_{-i}; \theta\right) \quad i: 1 \to N \tag{5.7}$$

$$\simeq \frac{1}{S}\sum_{s=1}^{S} V_i\left(d_i; d_{-i}; \tilde{\xi}^{\theta}_{i,s}; \theta\right) K_h\left(\left(\nu\varphi^{(k)}_i\left(\tilde{\xi}^{\theta}_{i,s}; d_i\right) - \tilde{b}^{\sigma}_{-i}\left(\tilde{\xi}^{\theta}_{-i,s}; d_{-i}\right)\right)\right). \tag{5.8}$$

Note that this smoothing technique preserve the continuity of $G_i(.,. \mid d_{-i}; \theta)$ in d_i and considerably reduces the dimensionality of integration. Finally, we have to solve the system of N equations

$$\frac{1}{S}\sum_{l=1} \frac{\partial}{\partial d_i}\left(V_i\left(d_i; d_{-i}; \tilde{\xi}^{\theta}_{i,s}; \theta\right) K_h\left(\left(\nu\varphi^{(k)}_i\left(\tilde{\xi}^{\theta}_{i,s}; d_i\right) - \tilde{b}^{\sigma}_{-i}\left(\tilde{\xi}^{\theta}_{-i,s}; d_{-i}\right)\right)\right)\right) = 0 \quad . \tag{5.9}$$

[6] By construction $\varphi^{(k)}_j(.)$ is continuous in d_j and $\nu(.)$ is typically continuous, therefore, $\tilde{b}^{\sigma}_{-i}(.)$ is also continuous in d_{-i}. If however, $\tilde{b}^{\sigma}_{-i}(.)$ had to be C^1 in d_{-i} then one might use a differential approximation of the Max such as $\left[\sum_{j \neq i}\left(\nu\left(\varphi^{(k)}_j\left(\tilde{\xi}^{\theta}_{j,s}; d_j\right)\right)\right)^{1/\sigma}\right]^{\sigma}$ for sufficiently small values of σ. The authors would like to thank the referee who suggested this smoothing technique.

[7] The choice of a bandwidth is extensively discussed in the literature on nonparametric estimation - see e.g. Hardle (1990). One can select such optimal bandwidth. Here however we control the MC size and optimality considerations are less crucial. Visual inspection to determine an appropriate combination of h and S proves extremely useful in that respect.

Provided that analytical derivatives of $K(\cdot)$, $V_i(d_i, d_{-i}, \xi, \theta)$ and $\varphi_i^{(k)}\left(\tilde{\xi}_{i,l}^{\theta}, d_i\right)$ with respect to d_1 are available we can derive an analytical expression for (5.5) and considerably reduce computational time.

It ought to be emphasized here that, although finding solution(s) to a set of non linear equations is a well known problem for which there exist several methods, it is by no means a trivial numerical problem. In other words, even though our CSE-MC methodology enables us to analyze empirical game theoretic models that would otherwise be analytically and numerically intractable, there is no free lunch! We refer our reader to the discussion in Press et al. (1986, chap. 9) from which the following highly relevant quote is extracted:

"Once, however, you identify the neighborhood of a root, or a place where there might be a root, then the problem firms up considerably: It is time to turn to Newton-Raphson, which readily generalizes to multiple dimensions. This method gives you a very efficient means of converging to the root, if it exists, or of spectacularly failing to converge, indicating (though not proving) that your putative root does not exist nearby."

Actually, the situation might not be as bad as it sounds. There are many empirical applications of game theory, such as auctions, for which theory as well as institutional features of the game under consideration provide much insight as to reasonable values for the parameters of assumed Rules of Thumb decision rules. As we shall illustrate next, it is critical that such information be incorporated in the numerical search for CSE's in the form of explicit restrictions upon admissible parameter values. See Armantier et al. (1997) for detailed algorithms and additional numerical considerations.

6. An application from the French aerospace industry

In this section we introduce an application which will be used in the sequel of our paper to illustrate the solution concepts we propose. This application exploits a data set relative to tenders in the French aerospace industry. A detailed description of that industry together with earlier empirical results are found in Florens et al. (1997) and Armantier et al. (1997). We derive here a previously unavailable analytical expression for the unconstrained NE solution under a scenario whereby participants are ranked according to a quality/price ratio criterion. That expression will be used as a benchmark to evaluate the alternative CSE solutions which are computed below.

In a nutshell the organization of the French aerospace industry can be described as follows: in order to subcontract a piece of equipment the project manager selects a number of firms and provides them with a list of technical specifications. The number of consulted firms N is small (between 2 and 7) and known. Tenders consist of technical proposals together with financial plans. After evaluation by an independent committee, the tender of firm i ($i = 1, ..., N$) is summarized into a quality grade Q_i ($Q_i \in [0, 1]$) and a price P_i, standardized across tenders. Tenders with a quality below a threshold Q_0 are eliminated. Among the qualified firms ($Q_i \geq Q_0$) the one with the highest quality/price ratio is awarded the subcontract. Our model is based upon the following set of assumptions:

A.1 The Independent Private Value (IPV) paradigm applies, whereby participants' cost (C_i) and quality (Q_i) pairs are privately known to them and the random variables (C_i, Q_i) are (jointly) independently and identically distributed, with a c.d.f. $F(c, q)$[8] with support $\left[0, \overline{C}\right] \times [0, 1]$;

A.2 Q_0 is common knowledge and only qualified firms submit bids since preparation of a tender is costly;[9]

A.3 If only one firm qualifies, then it receives a pre-negotiated amount $\overline{P}$.[10] Therefore, the strategic analysis which follows is conditional upon two or more firms qualifying;

A.4 Firms bid their true quality.[11] Furthermore, firms being ex-ante symmetric by assumption (A.1), we restrict our attention to symmetric solutions. Therefore, a bid by firm i consists of a pair $(P_i, Q_i) = (\pi(C_i, Q_i), Q_i)$;

A.5 The boundary condition $\pi\left(\overline{C}, Q_0\right) = \overline{C}$ applies. For the ease of derivation we also assume that $\pi(C, Q)$ is continuous and strictly increasing in both arguments and that $\pi(C, Q) \geq C$.

In terms of the notation introduced in Section 5.1, firm i draws a private signal $\xi_i = (C_i, Q_i)$ from the c.d.f. F and transforms it into an observable action $x_i = \varphi(\xi_i) = (\pi(C_i, Q_i), Q_i)$. Firms are ranked according to the selection criterion $\mu(x_i) = Q_i/P_i$ and their utility function coincide with their actual payoff. The unconstrained set of admissible strategies H^N is implicitly defined by assumption (A.5).

A unique unconstrained NE solution obtains under assumptions (A.1) to (A.5). Its derivation is given in Appendix 3. Though it follows from a stan-

[8] This distinguishes our model from Yeon-Koo Che (1993). Yeon-Koo Che assumes that the firms can select any level of quality and that the cost of production is a function of the quality and a unique random variable representing the heterogeneity between firms. Here, we assume two types of heterogeneities: in cost and in quality.

[9] Note that our econometric model has to account for this censoring phenomenon.

[10] If firms stood a sufficiently high probability of being sole qualifier, then they might have an incentive to submit arbitrarily large prices.

[11] We are implicitly assuming that if a firm chooses to misrepresent its quality and wins the procurement, then it would be detected and banned from subsequent procurements.

dard line of argumentation it is significantly complicated by the following issues: firstly, participants know the number of firms being consulted but, at the time of submitting their bids, do not know how many firms will actually qualify ; secondly, tenders are bivariate and the selection criterion takes the form of a ratio of random variables. The unconstrained (symmetric) NE bid function for a qualifying firm ($Q_i \geq Q_0$) may be written as

$$P_i = \pi(C_i; Q_i) = Q_i . \omega\left(\frac{C_i}{Q_i}\right) \quad ; \tag{6.1}$$

with

$$\omega(x) = x + \frac{1}{\Delta(x; Q_0)} \int_x^{\overline{C}/Q_0} \Delta(u; Q_0) du \quad ; \tag{6.2}$$

and

$$\Delta(u; Q_0) = \left[1 - \int_{Q_0}^{1} f_Q(v) F_{C/Q}(uv/v) dv\right]^n - [F_Q(Q_0)]^n \quad ; \tag{6.3}$$

where $n = N - 1$, F_Q denotes the marginal c.d.f. of Q_i and $F_{C/Q}(\cdot \mid Q)$ the c.d.f. of C_i conditional on $Q_i = Q$. This NE bid function cle arly is a non-trivial function of C_i and Q_i (graphs for the application under consideration are provided below). The econometric specification of the model is completed by the following assumptions:

A.6 The marginal distribution of the quality index Q is a beta distribution with density function

$$f_Q(Q \mid a_Q; b_Q) \propto Q^{a_Q - 1} \cdot (1 - Q)^{b_Q - 1} \tag{6.4}$$

with $a_Q > 0$ and $b_Q > 0$;

A.7 The marginal distribution of C is a Weibull distribution[12] with c.d.f.

$$F_C(C \mid a_C; b_C) = 1 - \exp\left(- a_C C^{b_c}\right) \quad ; \tag{6.5}$$

with $a_c > 0$ and $b_c > 0$;

A.8 The joint distribution of $(C; Q)$ is a member of the Morgenstern class of bivariate distributions with preset marginals - see Johnson and Kotz (1972, chap. 34) - and is given by

$$F(C; Q) = F_C(C) \cdot F_Q(Q) \cdot [1 + \gamma(1 - F_C(C))(1 - F_Q(Q))] \quad ; \tag{6.6}$$

[12] Actually we truncate F_c at a value $\overline{C}$ far in the tail. We choose $\overline{C} = 2.5$ which in practice correspond at minimum to the fractile 0.983.

with $\gamma \in]-1;1[$. For $\gamma = 0$, C and Q are mutually independent.

We note that the regression functions associated F are non linear. All together, we have to estimate parameter vector $\theta' = (a_C; b_C; a_Q; b_Q; \gamma) \in \mathbb{R}^4 \times]-1;1[$. We apply the inference method developed by Florens and al. (1997) with the unfeasible estimator, being given by the censored ML estimator associated with the c.d.f. (6.6). We have a sample corresponding to 15 procurements where a total of 87 firms were consulted but only 50 firms qualified. For each procurements we observe the number of consulted firms and the tenders $(P_i; Q_i)$ for the qualified firms. The quality threshold is the same across procurements $Q_0 = .45$. The main empirical results are summarized in Table 6.1:[13]

Parameter	a_C	b_C	a_Q	b_Q	γ
Estimated Value	2.185	0.953	5.265	5.259	0.109
Standard deviation	(0.101)	(0.112)	(0.134)	(0.148)	(0.429)

Table 6.1: Parameter estimates in the model with NE strategies

Corresponding moments for C, Q and the profit margin $\left(\frac{P-C}{C}\right)$ are reported in Table 6.2. The most directly interpretable figure is the expectation of the profit margin. Interestingly enough, it is about twice as large as the corresponding figures reported by Armantier et al. (1997) for similar procurements where participants were ranked according to price only. This difference does not appear to originate from model misspecification since, in particular, the estimates in Table 6.1 are quite similar to those obtained by Armantier et al. (1997). It actually raises an interesting problem in the design of these tenders. By using a quality/price criterion instead of just price, the project manager does not appear to gain much in quality but might be more significantly penalized in terms of price.

Variable	C	Q	Profit Margin
Expectation	0.905	0.500	0.494
Standard deviation	0.437	0.147	0.195

Table 6.2: Moments in the model with NE strategies

Graphs of the corresponding NE bid functions for 2 to 5 participants are found in figure 1. Note that in sharp contrast with its complex analytical form, the NE bid function has a very smooth graph.

[13] The standard deviations in Table 1 are calculated based on Monte Carlo simulations. See Florens et al. (1997) for general results regarding the asymptotic distribution of the estimator.

6.1. CSE's for the French aerospace industry

In view of the quality/price scoring rule, it appears reasonable to assume the following functional form for the constrained strategies,

$$\pi = q \cdot \omega_k \text{ and } \omega_k \left(\frac{c}{q} \mid d \right) = \sum_{\ell=0}^{k} d_{\ell+1} \left(\frac{c}{q} \right)^{\ell} \quad . \tag{6.7}$$

In order to make sure that the corresponding bidding rule remains sensible, which is essential for numerical reasons, we impose the following restrictions on π:

$$\frac{\partial \pi}{\partial c} > 0 \quad ; \quad \frac{\partial \pi}{\partial q} > 0 \quad ; \quad \pi(c, q) \geq c \; ; \tag{6.8}$$

$$\pi \left(\overline{C}, Q_0 \right) = \overline{C} \quad ; \quad \frac{\partial \pi}{\partial c} \Big|_{(\overline{C}, Q_0)} = 1 \quad \forall (c, q) \in [0, \overline{c}] \times [Q_0, \infty] \quad . \tag{6.9}$$

The two equality constraints in (6.9) are used to eliminate d_1 and d_2 from equation (6.7). The inequalities in (6.8) are then resolved in the form of inequality constraints on the remaining coefficients.

To illustrate a possible Rule of Thumb, we consider $k = 2$ which is the simplest case allowing for an 'interaction' between cost and quality. The bid function is then given by

$$\pi(c, q) = d_1 q + d_2 c + d_3 \frac{c^2}{q} \quad . \tag{6.10}$$

The constraints in equations (6.8) and (6.9) imply that the CSE solution operates on a single coefficient d_3 within a narrow band $[0, 0.09]$. The general method (inference and CSE evaluations) takes of the order of 17 minutes of CPU time for a MC size $S = 10000$ versus 160 minutes under the NE solution (See Armantier and Richard (1997) for details regarding computation). Point estimates of the parameter of the model are found in Table 6.3 and the corresponding moments for C, Q and profit margin in Table 6.4. In Table 6.5 we reproduce the estimated coefficients $d_i^*(\theta, N)$ of the CSE bid function (6.10) for number of participants from 2 to 7. Graphs of the corresponding bid functions for 2 to 5 participants are reproduced in figure 2. Comparison of these results with those obtained under NE solutions in Tables 6.1 and 6.2 and figure 1 indicate that estimated cost, quality and profit margin are very similar, and that the graphs of the Rule of Thumb and NE are close to each other on the interval where players are likely to draw costs (roughly $[0.5, 1.4]$). The largest difference (which, nevertheless, remains within one standard deviation) is found in the estimates of γ; a parameter which is notoriously difficult to estimate in small samples.

Parameter	a_C	b_C	a_Q	b_Q	γ
Estimated Value	2.133	0.970	5.268	5.263	0.301
Standard deviation	0.188	0.179	0.237	0.256	0.531

Table 6.3: Parameter estimates with CSE solution (k=2)

Variable	C	Q	Profit Margin
Expectation	0.921	0.500	0.478
Standard deviation	0.454	0.147	0.202

Table 6.4: Moments with CSE solution (k=2)

To illustrate the possible use of CSE to approximate NE, we consider also the case where $k = 4$. The constraints implied by (6.8) and (6.9) are significantly more tedious to elicit and program and the numerical search for a CSE solution now operates on the three coefficients d_3, d_4, and d_5. Computing time is of the order of 64 minutes of CPU time, still 2.5 times faster than for NE solutions. Results are reproduced in Table 6.6 to 6.8 and in figure 3. The CSE solution for $k = 4$ clearly provides a very close approximation to the actual NE solution.

It is quite obvious that, in the context of the French aerospace industry at least, CSE solutions not only provide excellent approximations to an existing NE solution but, in addition, are computationally more tractable as well as easier to interpret (whence their potential usefulness as Rules of Thumb strategies).

6.2. Asymmetry and collusion

CSE offers the key advantage that it can be computed in situations where NE solutions are analytically intractable. Important examples of such situations are auctions where bidders are asymmetric and/or where (subgroups of) bidders collude. There are very few such cases for which there currently exist operational numerical algorithms to compute NE solutions. See, for example, Marshall et al.

(N)	2	3	4	5	6	7
$d_1^*(p; N)$	1.266	1.088	0.941	0.823	0.728	0.652
$d_2^*(p; N)$	0.544	0.608	0.661	0.704	0.738	0.765
$d_3^*(p; N)$	0.0410	0.0352	0.0304	0.0267	0.0236	0.0211

Table 6.5: Coefficients of the CSE solution (k=2)

Parameter	a_C	b_C	a_Q	b_Q	γ
Estimated Value	2.181	0.950	5.266	5.260	0.210
Standard deviation	0.114	0.121	0.137	0.150	0.433

Table 6.6: Parameter estimates with CSE solutions (k=4)

Variable	C	Q	Profit Margin
Expectation	0.902	0.497	0.491
Standard deviation	0.438	0.147	0.197

Table 6.7: Moments with CSE solutions (k=4)

N	$d_1^*(\hat{\theta}; N)$	$d_2^*(\hat{\theta}; N)$	$d_3^*(\hat{\theta}; N)$	$d_4^*(\hat{\theta}; N)$	$d_5^*(\hat{\theta}; N)$
2	1.380	0.0297	0.470	-0.107	0.0083
3	1.201	0.0564	0.509	-0.119	0.0093
4	1.059	0.0721	0.539	-0.128	0.0100
5	0.950	0.0824	0.560	-0.137	0.0105
6	0.865	0.0924	0.571	-0.137	0.0107
7	0.798	0.1050	0.574	-0.138	0.0107

Table 6.8: Coefficients of the CSE solutions (k=4)

(1994) for numerical solutions to the case of first price IPV auctions with two subgroups of bidders who are symmetric within groups but asymmetric across. In the present section we provide two examples for which, as far as we know, the NE solutions cannot (presently) be computed and for which, nevertheless, our MC algorithm produces CSE solutions with no difficulties. For the sake of illustration, we rely upon the institutional framework of the French aerospace industry as described earlier but limit ourselves to computing CSE solutions for artificial choices of parameter values.[14]

6.2.1. Example 1: Coalition

We consider here the case where a single player faces a coalition of $N - 1$ players. We assume that the coalition is represented at the main auction by a sole bidder who submits a bid corresponding to the type with highest score within the coalition.[15] All players draw their (private) type from the bivariate distribution characterized by assumptions A.6 to A.8. The parameter vector μ is set equal to $\mu_0 = (2.0 \quad 1.0 \quad 5.0 \quad 5.0 \quad 0.3)$ implying expected cost 0.886 with standard deviation 0.463 and expected quality 0.5 with standard deviation 0.15. The quality threshold Q_0 is set equal to 0.4. The bid function of player 1 and that of the coalition sole bidder are both assumed to be of the form given in equation (6.10). The CSE coefficients $d_i^*(\mu_0; N)$ for each of the two bidders are found in Tables 6.9 and 10 together with expected profits and probabilities of winning[16] (the latter figures are per capita under the implicit assumption that the coalition allocation mechanism is symmetric). Symmetric expected profit is also included for reference. The corresponding bid functions are illustrated in figure 4. In addition to demonstrating the feasibility of a CSE solution, the numbers in Tables 6.9 and 6.10 confirm earlier findings by Marshall et al. (1994) in a simpler model: the very existence of a coalition benefits the outsider significantly more than the insiders. This finding, to be confirmed by a larger scale study, raises obvious questions as to the viability of coalitions in the procurement environment discussed here. A

[14] We have no particular reasons to suspect major asymmetries and/or collusive behavior among participants to the procurement analyzed earlier and, moreover, it would be vain to attempt estimating a more complex model than the one we already estimated in view of the small sample size.

[15] We do not attempt to model here an 'incentive compatible' collusive mechanism, neither do we address the fundamental issue of the viability of a non-inclusive coalition. A complete study of the problem goes beyond the objectives of our paper but the results presented below raise obvious questions as to the viability of the coalition under consideration.

[16] The numbers in the first column ($N = 2$) of tables 9 and 10 ought to be the same. The fact that the actual figures are very close to one another illustrates the excellent numerical accuracy of our simulation algorithm.

full scale study of this important issue goes beyond the objectives of the present paper but belongs to our current research agenda.

N	2	3	4	5	6	7
$d_1^*(\mu_0; N)$	1.981	1.541	1.413	1.317	1.259	1.213
$d_2^*(\mu_0; N)$	0.366	0.440	0.548	0.579	0.597	0.612
$d_3^*(\mu_0; N)$	0.0507	0.0407	0.0362	0.0337	0.0323	0.0311
Expected profit	0.360	0.263	0.222	0.199	0.186	0.176
Winning probability	0.502	0.414	0.372	0.350	0.336	0.326
Symmetric Expected profit	0.361	0.206	0.140	0.104	0.080	0.067

Table 6.9: CSE solution and outcome for player 1 (coalition)

N	2	3	4	5	6	7
$d_1^*(\mu_0; N)$	1.979	1.803	1.742	1.716	1.704	1.689
$d_2^*(\mu_0; N)$	0.369	0.423	0.443	0.451	0.454	0.459
$d_3^*(\mu_0; N)$	0.0507	0.0462	0.0446	0.0440	0.0436	0.0433
Expected profit	0.361	0.211	0.157	0.127	0.106	0.091
Winning probability	0.498	0.586	0.627	0.651	0.665	0.673
Symmetric Expected profit	0.361	0.206	0.140	0.104	0.080	0.067

Table 6.10: CSE solution and per capita outcome for coalition of N-1 players

6.2.2. Example 2: Asymmetry

In this example player 1 draws her type from a different distribution from that of players 2 to N. Actually we assume that only the marginal distribution for cost differ among the two groups. The mean and standard deviation of cost for player 1 are 1.0 and 0.2, respectively. Players 2 to N draw cost from a more favorable distribution with mean 0.8 and standard deviation 0.2. The quality threshold stays fixed at 0.4. Here again both bid functions are assumed to be of the form given in equation (6.10). The CSE coefficients $d_i^*(\mu_0; N)$ for both classes of players are found in Tables 6.11 and 6.12 together with (per capita) expected profit and probability of winning. The corresponding bid functions are illustrated in figure 5 (the symmetric bid function correspond to the case where all bidders are similar to player 1). We note that a relatively modest increase in expected cost (one standard deviation) significantly penalizes player 1. Note also that player 1 tries to (partially) make up for an unfavorable cost distribution

N	$d_1^*(\mu_0; N)$	$d_2^*(\mu_0; N)$	$d_3^*(\mu_0; N)$	expected profit	winning probability
2	1.540	0.507	0.0394	0.223	0.462
3	0.934	0.701	0.0239	0.091	0.267
4	0.684	0.781	0.0175	0.047	0.172
5	0.563	0.820	0.0144	0.028	0.120
6	0.486	0.845	0.0124	0.019	0.091
7	0.439	0.859	0.0113	0.014	0.071

Table 6.11: CSE solution and outcome for player 1

N	$d_1^*(\mu_0; N)$	$d_2^*(\mu_0; N)$	$d_3^*(\mu_0; N)$	expected profit	winning probability
2	1.744	0.442	0.0447	0.324	0.538
3	1.125	0.640	0.0288	0.163	0.369
4	0.844	0.730	0.0216	0.098	0.278
5	0.694	0.777	0.0178	0.066	0.221
6	0.604	0.806	0.0155	0.049	0.181
7	0.539	0.827	0.0138	0.038	0.158

Table 6.12: CSE solution and outcome for player j (j=2,...N)

by decreasing the cost coefficient and increasing the quality coefficient in her bid function.

7. Conclusion

First and foremost, the MC simulation algorithm we have developed for computing CSE solutions and embedding these calculations within a general estimation algorithm appears to work extremely well. It outperforms standard NE calculations in those few cases where NE solutions might be available and, more importantly, provides operational solutions when NE solutions are analytically and numerically intractable. Non trivial examples of such cases were analyzed in our paper. The combination of our CSE algorithm with the general inference principle developed by Florens et al. (1997) produces an operational integrated methodology applicable to a broad range of empirical game theoretic models, offering a number of exciting avenues for research. In conclusion of our paper, we briefly discuss a few such issues that belong to our immediate research agenda.

(1) As we are facing a broad range of alternative strategic and econometric specifications, we critically need to develop specification tests for empirical game theoretic models. Moreover, in an approximation framework specification

tests are also essential to evaluate how closely the CSE have converged toward the NE. We brie‡y evoked in the paper the possibility to overidentify the model to serve as the basis of a speci…cation test.

(2) We are currently working on a formal approximation theorem of NE solutions by CSE's. Preliminary convergence results have been obtained for speci…c classes of game, such as auctions or games for which the strategies are of bounded variation. In addition, we also intend to develop mathematical guidelines for selecting appropriate classes of restricted strategies.

(3) The CSE algorithm we developed provides us with an operational procedure for investigating the empirical relevance of recently developed theoretical concepts of 'bounded rationality'. Therefore, we intend to extend the use of our algorithm to produce real-life economic applications of bounded rationality, of which very few currently exist. In particular, it ought to be possible to explicitly model 'complexity' to validate functionally simple CSE solutions.

(4) Finally, we do intend to apply the concept of CSE to broader classes of games than that discussed here. Obvious areas of interest are repeated games, learning or mixed strategies equilibriums, among others. We also propose to apply our algorithm to other applied micro or Industrial Organization economic problems, such as principal-agent, non linear pricing, adverse selection or moral hazard, all problems for which NE solutions can at best be obtained under strong simplifying assumptions, whose empirical validity often is highly questionable.

8. Appendix 1: Continuity of $\widetilde{U}_i(\varphi;\theta)$ in φ.

Consider $U_i(x;\xi)$ an Hölder continuous function in $x \in \mathbb{R}^{p'}$ ($\forall \xi \in \Omega$ and Ω compact) with an exponent $\alpha \geq 1$ independent of ξ, and a constant $k(\xi)$. Then, $\forall \{\varphi^1, \varphi^2\} \in H^2$ and $\forall \xi \in \Omega$ we have

$$\left| U_i\left(\varphi^1(\xi;\theta);\xi\right) - U_i\left(\varphi^2(\xi;\theta);\xi\right)\right| \leq k(\xi)\left[d\left(\varphi^1(\xi;\theta),\varphi^2(\xi;\theta)\right)\right]^{\alpha}; \quad \text{(A.1.1)}$$

where $d(.)$ is an usual metric on $\mathbb{R}^{p'}$. Consider two conjugate numbers q_1 and q_2 ($q_1 > 0$, $q_2 > 0$ and $1/q_1 + 1/q_2 = 1$, or $q_1 = 1$ and $q_2 = \infty$). We assume that $k \in L_F^{q_1}\left(\int_\Omega |k(\xi)|^{q_1} F(\partial\xi) < \infty\right)$ and $H \subset L_F^{q_2}$ (in particular, when $q_2 = \infty$ strategies are bounded). Then, the function $\widetilde{U}_i(\varphi;\theta) = E_{\xi|\theta}[U_i(\varphi(\xi;\theta);\xi)]$ is continuous in φ. Indeed, $\forall \{\varphi^1, \varphi^2\} \in H^2$

$$\left|\widetilde{U}_i\left(\varphi^1;\theta\right) - \widetilde{U}_i\left(\varphi^2;\theta\right)\right| = \left| E_{\xi|\theta}\left[U_i\left(\varphi^1(\xi;\theta);\xi\right)\right] - E_{\xi|\theta}\left[U_i\left(\varphi^2(\xi;\theta);\xi\right)\right]\right|$$

$$\leq E_{\xi|\theta}\left[\left|U_i\left(\varphi^1(\xi;\theta);\xi\right) - U_i\left(\varphi^2(\xi;\theta);\xi\right)\right|\right] \leq E_{\xi|\theta}\left[|k(\xi)|\left[d\left(\varphi^1(\xi;\theta),\varphi^2(\xi;\theta)\right)\right]^{\alpha}\right]$$

$$\leq \left(\int |k(\xi)|^{q_1} F(\partial\xi)\right)^{1/q_1}\left(\int \left[d\left(\varphi^1(\xi;\theta),\varphi^2(\xi;\theta)\right)\right]^{\alpha q_2} F(\partial\xi)\right)^{1/q_2} \quad \text{(A.1.2)}$$

The last inequality follows from the Hölder inequality. Therefore, $\widetilde{U}_i(\varphi;\theta)$ is continuous in φ with respect to the $L_F^{q_2}$ metric.

9. Appendix 2: Quasi concavity of $\widetilde{U}_i\left(\varphi_i;\varphi_{-i};\theta\right)$ in φ_i.

If $U_i\left(\varphi_i(\xi;\theta);\varphi_{-i}(\xi;\theta);\xi\right)$ is concave in φ_i then, $\forall \{\varphi_i^1;\varphi_i^2\} \in H_i^2$, $\forall \xi \in \Omega$ and for all $0 < \lambda < 1$;

$$U_i\left(\lambda\varphi_i^1(\xi;\theta) + (1-\lambda)\varphi_i^2(\xi;\theta);\varphi_{-i}(\xi;\theta);\xi\right) \geq$$

$$\lambda U_i\left(\varphi_i^1(\xi;\theta);\varphi_{-i}(\xi;\theta);\xi\right) + (1-\lambda)U_i\left(\varphi_i^2(\xi;\theta);\varphi_{-i}(\xi;\theta);\xi\right). \quad \text{(A.2.1)}$$

Let us multiply both side of the inequality by $f(\xi)$ and integrate over Ω; we have

$$\int_\Omega U_i\left(\lambda\varphi_i^1(\xi;\theta) + (1-\lambda)\varphi_i^2(\xi;\theta);\varphi_{-i}(\xi;\theta);\xi\right) f(\xi)\,\partial\xi \geq$$

$$\lambda\int_\Omega U_i\left(\varphi_i^1(\xi;\theta);\varphi_{-i}(\xi;\theta);\xi\right) f(\xi)\,\partial\xi + (1-\lambda)\int_\Omega U_i\left(\varphi_i^2(\xi;\theta);\varphi_{-i}(\xi;\theta);\xi\right) f(\xi)\,\partial\xi$$

(A.2.2)

or, equivalently,

$$V_i\left(\lambda\varphi_i^1+(1-\lambda)\varphi_i^2;\varphi_{-i}\right)\geq\lambda V_i\left(\varphi_i^1;\varphi_{-i}\right)+(1-\lambda)V_i\left(\varphi_i^2;\varphi_{-i}\right)\qquad\text{(A.2.3)}$$

and $V_i\left(\varphi_i;\varphi_{-i};\theta\right)$ is concave in φ_i.

Similarly, if the conditional expectation $E_{\xi_{-i(s)}|\xi_{(s)};\theta}\left[U_i\left(\varphi_{(s)}\left(\xi_{(s)};\theta\right);\varphi_{-i(s)}\left(\xi_{-i(s)};\theta\right);\xi_{(s)};\xi_{-i(s)}\right)\right]$ where $\xi_{(s)}=\left(\xi_1;\ldots;\xi_{(s)}\right)$ is concave in φ_i then $\forall\left(\varphi_i^1;\varphi_i^2\right)\in H_i^2;\ \forall\xi_{(s)}\in\Xi_{(s)}$ and for all $0<\lambda<1$

$$\int_{\Xi_{-i(s)}}U_i\left(\lambda\varphi_i^1(\xi;\theta)+(1-\lambda)\varphi_i^2(\xi;\theta);\varphi_{-i}(\xi;\theta);\xi\right)f_{-i(s)/\xi_{(s)}}\left(\xi_{-i(s)}\right)\partial\xi_{-i(s)}\geq$$

$$\lambda\int_{\Xi_{-i(s)}}U_i\left(\varphi_i^1(\xi;\theta);\varphi_{-i}(\xi;\theta);\xi\right)f_{-i(s)/\xi_{(s)}}\left(\xi_{-i(s)}\right)\partial\xi_{-i(s)}+$$

$$(1-\lambda)\int_{\Xi_{-i(s)}}U_i\left(\varphi_i^2(\xi;\theta);\varphi_{-i}(\xi;\theta);\xi\right)f_{-i(s)/\xi_{(s)}}\left(\xi_{-i(s)}\right)\partial\xi_{-i(s)}\qquad\text{(A.2.4)}$$

where $f_{-i(s)/\xi_{(s)}}(.)$ is the marginal p.d.f of $\xi_{-i(s)}$ conditionally on $\xi_{(s)}$. If we multiply both side of the inequality by $f_{(s)}\left(\xi_{(s)}\right)$ and integrate over $\Xi_{(s)}$ we obtain the same results as in and $V_i\left(\varphi_i;\varphi_{-i};\theta\right)$ is concave in φ_i.

10. Appendix 3: Derivation of formula (6.1)-(6.3)

We proceed under the working assumption that the bid function is of the form

$$p=q.\omega\left(\frac{c}{q}\right);\qquad\text{(A.3.1)}$$

where ω is monotone increasing (actually, we can run our proof without that condition at the cost of added complications, and verify in the end that condition (A.1) obtains). Let $h=\omega^{-1}$. The boundary condition in assumption (6) is rewritten as $\omega\left(\overline{C}/Q_0\right)=\overline{C}/Q_0$. Let $Y=Q/P$ denote the (random) quality-price ratio. The derivation of the (symmetric) bid function for a qualified firm proceeds in several steps.

(1) Elementary probability calculations produce the following results

$$\Pr\left(Y\leq\frac{1}{a}\,|\,Q=q\right)=1-F_{C|Q}\left(qh(a)\,|\,q\right);\qquad\text{(A.3.2)}$$

for $a \geq \omega(0)$ (which is the only case relevant for the equilibrium condition derived below), whence

$$\Pr\left(Y \le \frac{1}{a} \,\Big|\, Q \geq Q_0\right) = 1 - \frac{B(a;\omega;Q_0)}{1 - F_Q(Q_0)} \quad ; \tag{A.3.3}$$

with

$$B(a;\omega;Q_0) = \int_{Q_0}^{1} f_Q(u) \cdot F_{C|Q}(uh(a) \mid u)\, du \quad . \tag{A.3.4}$$

(2) Let $Y_{(n;Q_0)}$ denote the highest quality-price ratio among $n = N - 1$ qualified rival firms, conditionally on there being at least one. We have

$$\Pr\left(Y_{(n;Q_0)} \le \frac{1}{a}\right) = \sum_{k=1} P_k \Pr\left(Y \le \frac{1}{a} \,\Big|\, Q \geq Q_0\right)^k \quad ; \tag{A.3.5}$$

where P_k, which denotes the probability that k firms qualify conditionally on there being at least one, is given by

$$P_k = \frac{\binom{n}{k} \cdot [1 - F_Q(Q_0)]^k \cdot [F_Q(Q_0)]^{n-k}}{1 - [F_Q(Q_0)]^n}; \qquad k : 1 \to n \,. \tag{A.3.6}$$

Hence

$$\Pr\left(Y_{(n;Q_0)} \le \frac{1}{a}\right) = \frac{D(a;\omega;Q_0)}{1 - [F_Q(Q_0)]^n} \quad ; \tag{A.3.7}$$

with

$$D(a;\omega;Q_0) = [1 - B(a;\omega;Q_0)]^n - [F_Q(Q_0)]^n \quad . \tag{A.3.8}$$

(3) The payoff of a firm with type $(c;q)$ which faces $Y_{(n;Q_0)}$ and bids $q \cdot \Omega(x)$ is given by

$$U\left(x; Y_{(n;Q_0)}; c; q\right) = q\left(\omega(x) - \frac{c}{q}\right); \qquad \text{if} \quad \frac{1}{\omega(x)} \geq Y_{(n;Q_0)} \quad ; \tag{A.3.9}$$

and equals zero, otherwise. Its expected profit is

$$U^*(x;c;q) = \frac{q}{1 - [F_Q(Q_0)]^n} \cdot R\left(x; \frac{c}{q}\right) \quad ; \tag{A.3.10}$$

with

$$R\left(x; \frac{c}{q}\right) = \left(\omega(x) - \frac{c}{q}\right) \cdot D(\omega(x);\omega;Q_0) \quad . \tag{A.3.11}$$

Since $\omega(x) \geq \omega(0)$, we also have

$$B(\omega(x); \omega; Q_0) = \int_{Q_0}^{\infty} f_Q(u) \cdot F_{C|Q}(ux \mid u)\, du \quad ; \tag{A.3.12}$$

which actually does not depend upon ω. Whence

$$R\left(x; \frac{c}{q}\right) = \left(\omega(x) - \frac{c}{q}\right) \cdot \Delta(x; Q_0) \quad ; \tag{A.3.13}$$

where $\Delta(x; Q_0)$ has been defined in equation (2.16). The equilibrium condition is given by

$$\frac{\partial}{\partial x} R\left(x; \frac{c}{q}\right) = 0; \qquad \text{at} \quad x = \frac{c}{q} \quad ; \tag{A.3.14}$$

or, equivalently,

$$\omega'(x) \cdot \Delta(x; Q_0) + [\omega(x) - x] \cdot \Delta'(x; Q_0) = 0 \quad : \tag{A.3.15}$$

The solution of this differential equation is given by

$$\omega(x) = x + \frac{K - \int_0^x \Delta(u; Q_0)\, du}{\Delta(x; Q_0)} \quad : \tag{A.3.16}$$

The boundary condition implies that

$$K = \int_0^{\bar{C}/Q_0} \Delta(u; Q_0)\, du \quad ; \tag{A.3.17}$$

which completes the proof. It is fairly straightforward to demonstrate that condition (A.15) implies that the bid function $\pi(c; q)$ is monotone in both arguments.

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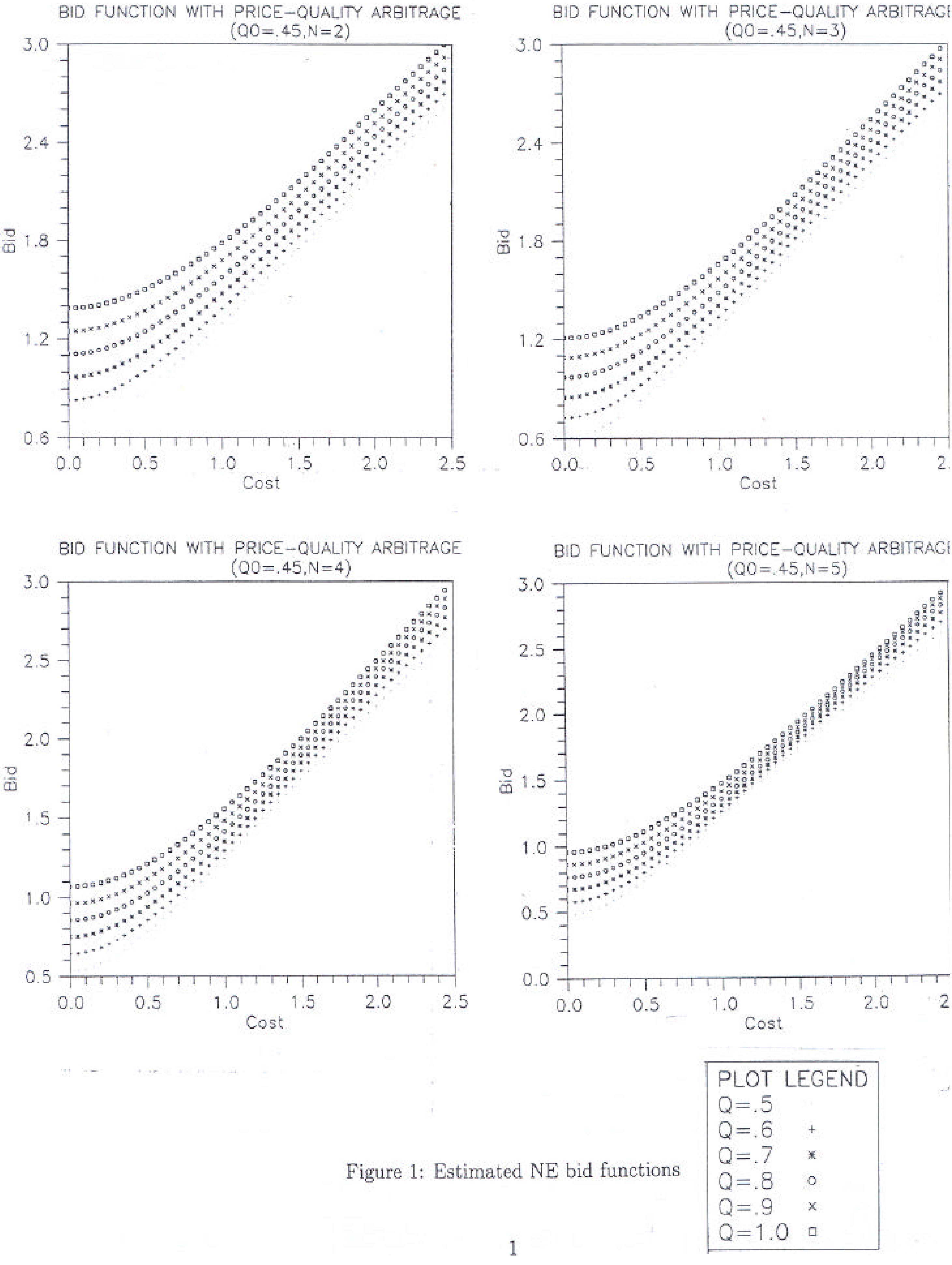
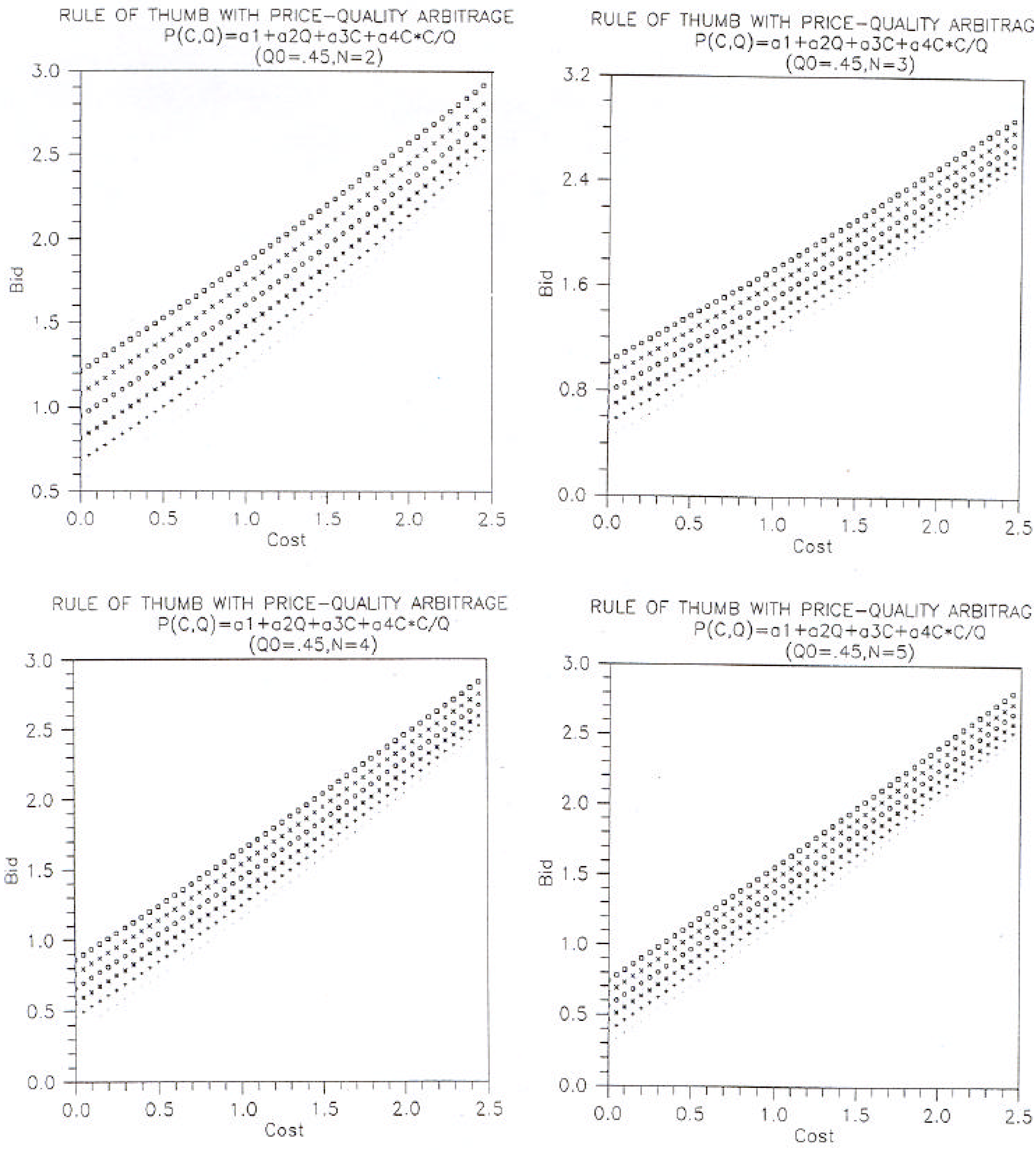


Figure 1: Estimated NE bid functions



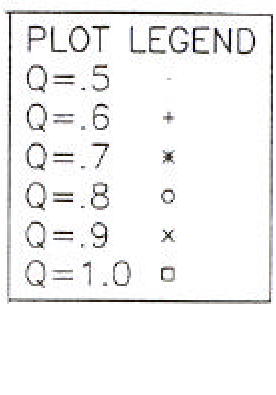
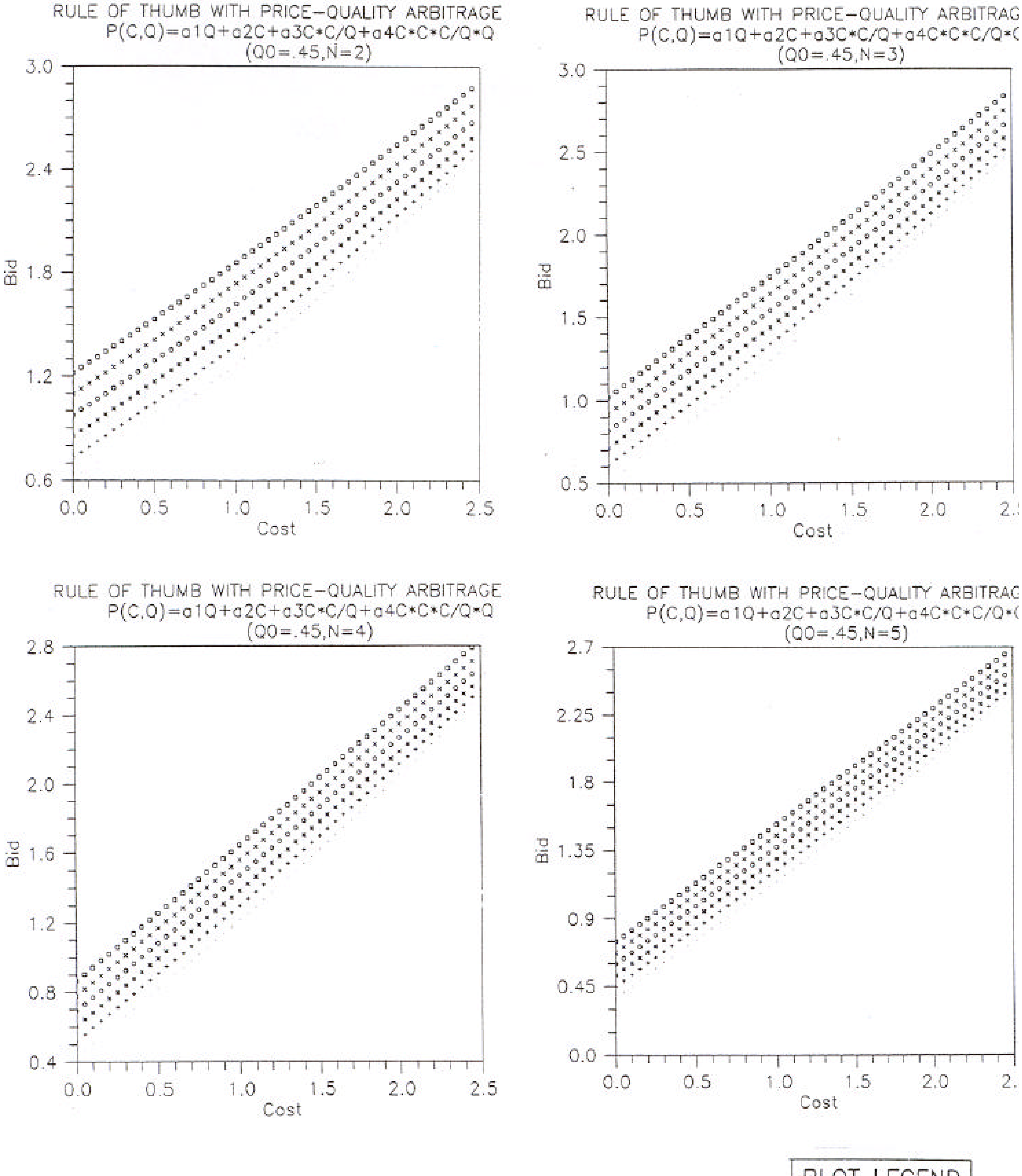


Figure 2: Estimated CSE bid Functions (L=2)



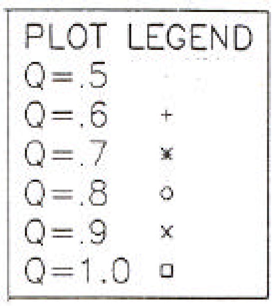


Figure 3: Estimated CSE bid functions(L=4)

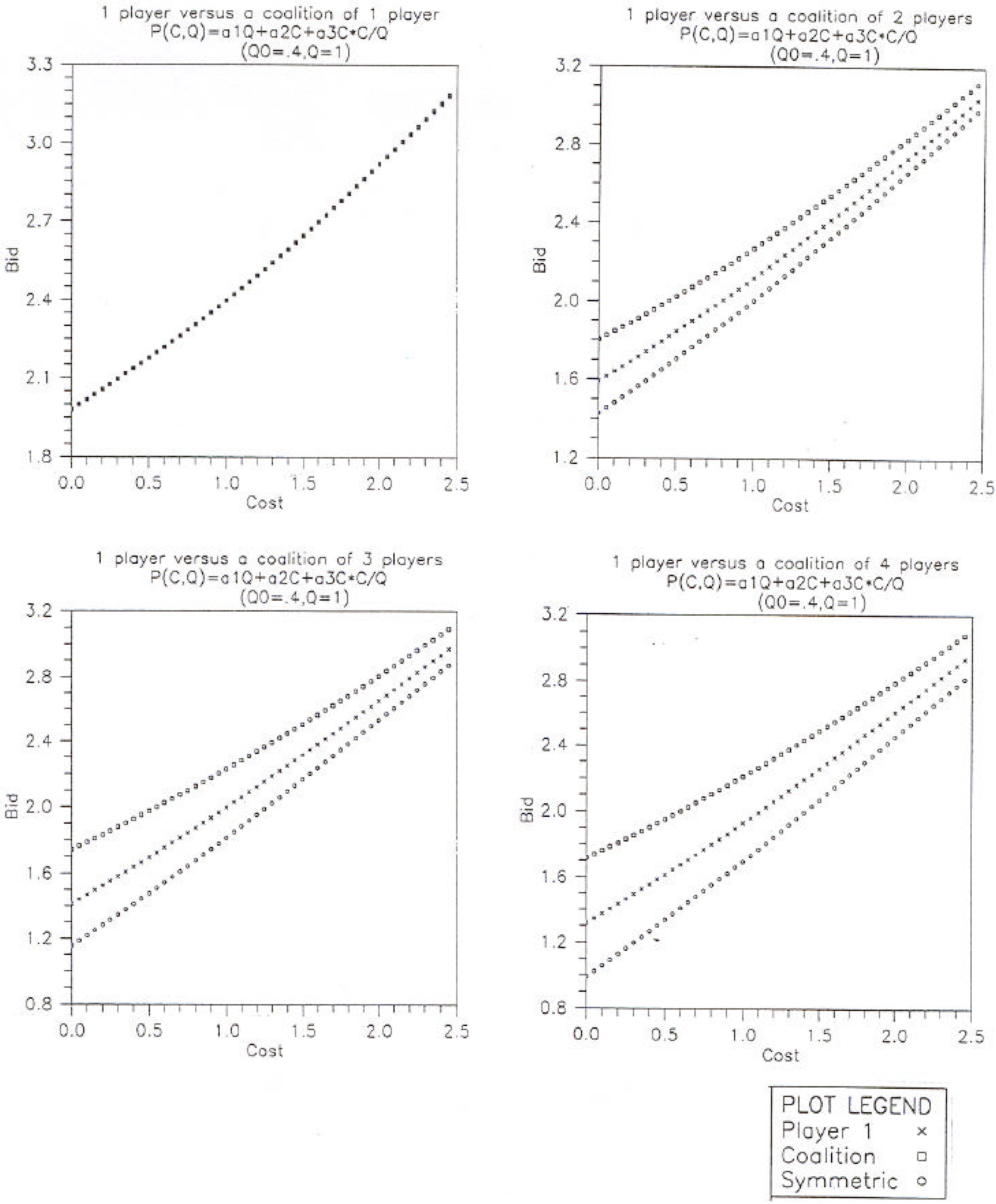


Figure 4: Simulation of Rules of Thumb with Price-Quality Arbitrage

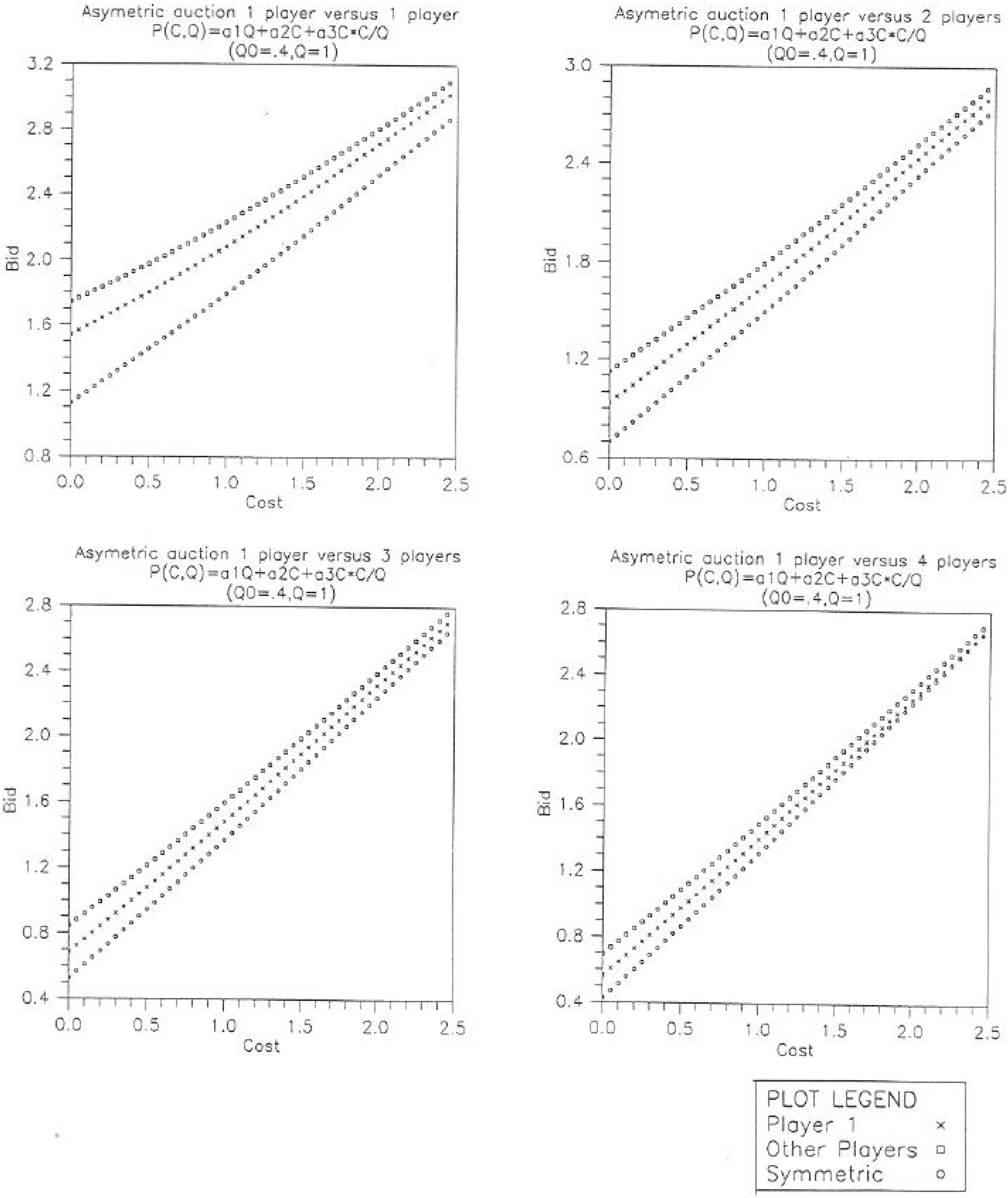


Figure 5: Simulation of Rules of Thumb with Price-Quality Arbitrage