Nash Equilibrium Approximation in Games of Incomplete Information

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Abstract

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The most commonly used solution concept in game theory is that of Nash Equilibrium.[1] However, except under fairly restrictive assumptions whose empirical validity often is questionable, many games cannot be solved analytically for NE solutions. As an alternative to NE Armantier, Florens and Richard (2000) (hereafter AFR) introduce the concept of Constrained Strategic Equilibrium (hereafter CSE). Essentially, they propose to restrict attention to appropriate subsets of strategies, typically indexed by an auxiliary parameter vector, and to search for an equilibrium solution within such subsets. The authors show that CSE offer a major computational advantage, and they provide a powerful algorithm based upon Monte Carlo simulations to determine the CSE numerically. The concept of CSE appeared to be relevant under two scenarios: the first one is directly related to the general notion of 'bounded rationality' and more specifically to the concept of Rules of Thumb ; in the second scenario, one would use the computational advantage of the CSE with the intent to approximate an analytically untractable NE solution. The objective of the present essay is to establish conditions under which a sequence of CSE approximates a NE, in the context of games of incomplete information. We also provide three criteria to document in practice whether the CSE is a good approximation of the NE and how far the CSE is from a NE.

It appears natural to approximate NEs since they often are analytically untractable and/or too complex to calculate. This suggestion finds additional heuristic support in the common observation that in those few cases where NE strategies can be computed, their 'smooth' graphs clearly suggest that it ought to be possible to approximate them by simpler functional forms, such as low degree polynomials, piecewise linear and/or exponential functions. See e.g. some of the graphs of NE strategies found in Marshall et al. (1994) or AFR (2000).

Several approaches may be considered to approximate numerically the NE. One can attempt to find the closest solution to the first order conditions (hereafter FOC) within a constraint set. For instance, in an auction problem Bajari (1996) propose to solve the system of differential equation resulting from the FOC by finite elements techniques. However this approach requires to explicit the FOC in the extensive form game which might next to impossible when the conditional individual expected utility functions are not differentiable or when actions are not continuous. Alternatively, one could approximate the NE by a CSE. This approach offers the advantages that it relies on "primitive" elements of the game (utility, distribution and strategy functions), it has a game theoretic interpretation

[1] Depending on the information available to players we shall consider Nash Equilibrium or Bayesian Nash Equilibrium. Hereafter both concepts in pure statregy are denoted NE.

in finite distance and finally, CSE can easily be computed with a flexible algorithm.

1. The general Model of incomplete Information

There are N players each of which is endowed with a privately known 'type' or 'signal' $\xi_i \in \Xi_i$ with $\Xi_i \subset \Re^p$ and $\Xi = \prod_{i=1}^{N} \Xi_i$. The types $\xi = (\xi_1, ..., \xi_N)$ are drawn from a joint distribution with cumulative distribution function (hereafter c.d.f.) $F(\xi)$ and density $f(\xi)$ (known to the players but not to the observer). Let $F_i(\xi_i)$ denote the marginal c.d.f. of ξ_i and $f_i(\xi_i)$ the corresponding density. This general framework includes as special cases of interest:

1. i.i.d. types,
$$f(\xi) = \prod_{i=1}^{N} f_1(\xi_i) \quad ; \tag{1.1}$$
2. exchangeable (affiliated) types,
$$f(\xi) = \int_{S_0} \prod_{i=1}^{N} f_1(\xi_i \mid \xi_0) \cdot f_0(\xi_0)\, d\xi_0 \quad , \tag{1.2}$$
where $\xi_0 \in \Xi_0$ denotes a 'linkage' random variable (e.g. the unknown value of an item being sold through a 'Common Value' auction) drawn from a distribution with p.d.f $f_0(.)$;
3. asymmetric independently distributed types,
$$f(\xi) = \prod_{i=1}^{N} f_i(\xi_i) \quad . \tag{1.3}$$

Let us denote $X_i \in \Re^q$ the set of possible actions x_i that player i can take. Players are endowed with individual Von Neuman-Morgenstern utility functions $U_i(x, \xi)$ where $x = (x_1, ..., x_N)$.

Consider Θ_i the set of all measurable functions $\varphi_i : \Xi_i \rightarrow X_i$ such that $U_i(\varphi_i(\xi_i), \xi)^2$ is integrable with respect to $F(.)$. Θ_i can be interpreted as the

[2] For the ease of exposition we adopt the usual notation: $\xi = (\xi_i, \xi_{-i}) = (\xi_1, ..., \xi_N)$ and $\varphi(\xi) = (\varphi_i(\xi_i), \varphi_{-i}(\xi_{-i})) = (\varphi_1(\xi_1), ..., \varphi_N(\xi_N))$.

set of all possible strategies transforming unobserved signals into actions,

$$\Xi_i \quad \rightarrow \quad X_i \tag{1.4}$$

$$\xi_i \quad \rightarrow \quad x_i = \varphi_i\left(\xi_i\right), \qquad i : 1 \rightarrow N \quad . \tag{1.5}$$

An important characteristic of games of incomplete information is that $\varphi_i\left(.\right)$ typically depends upon $F\left(.\right)$. As we shall see in section 2, we can often reduce the search for a NE to a more amenable set $H_i \subseteq \Theta_i$ consisting in the admissible strategies only. In the remainder we assume that $\varphi_i\left(.\right) \in H_i$. The number of players N (depending upon the situation, the decision to participate may be endogenous or exogenous), the joint distribution F, the utility functions $\left\{U_i\right\}_{i=1,...N}$ and the sets of admissible strategies $\left\{H_i\right\}_{i=1,...N}$ are common knowledge to all players. Symmetry assumes that the joint distribution F is exchangeable (i.e. F is invariant under a permutations of players), $\left(U_i, H_i, \Xi_i\right) = \left(U_j, H_j, \Xi_i\right)$ and the equilibrium strategies (subject to existence) are such that $\varphi_i = \varphi_j, \forall i \neq j$. Note that exchangeability reduces to the equality of marginal distributions when types are univariate or independent.

The strategic form of the game is based upon the set of individual expected utility functions

$$\widetilde{U}_i\left(\varphi\right) = E_\xi\left[U_i\left(\varphi\left(\xi\right), \xi\right)\right] \quad . \tag{1.6}$$

The extensive form of the game is based upon the set of conditional individual expected utility functions

$$\widehat{U}_i\left(\varphi; \xi_i\right) = E_{\xi_{-i}|\xi_i}\left[U_i\left(\varphi_i\left(\xi_i\right), \varphi_{-i}\left(\xi_{-i}\right); \xi_i, \xi_{-i}\right)\right] \quad . \tag{1.7}$$

2. Unconstrained NE solutions

Subject to existence, a Bayesian Nash Equilibrium in pure strategy in the set $H = \prod_{i=1}^{N} H_i$ is defined as a strategy profile $\varphi^{NE} = \left(\varphi_1^{NE}, ..., \varphi_N^{NE}\right)$ with $\varphi^{NE} \in \Phi_{NE}$[3] of mutually best responses strategies. We provide two definitions of the NE depending upon the form of the game.

Definition 2.1. *Nash Equilibrium in extensive form. A strategy profile* $\varphi^{NE} = \left(\varphi_1^{NE}, ..., \varphi_N^{NE}\right)$ *is a NE in the extensive form game* $\left(\varphi^{NE} \in \Phi_{NE}\right)$ *if and only if*

$$\widehat{U}_i\left(\varphi_i^{NE}\left(\xi_i\right), \varphi_{-i}^{NE}; \xi_i\right) \geq \widehat{U}_i\left(x_i, \varphi_{-i}^{NE}; \xi_i\right),$$

[3] Note that we do not assume that the NE is unique.

$$\forall x_i \in X_i,\ \forall \xi_i \in \Xi_i \text{ and } \forall i : 1 \to N \quad . \tag{2.1}$$

Definition 2.2. *Nash Equilibrium in strategic form. A strategy profile* $\widetilde{\varphi}^{NE} = \left(\widetilde{\varphi}_1^{NE}, ..., \widetilde{\varphi}_N^{NE}\right)$ *is a NE in the strategic form game* $\left(\widetilde{\varphi}^{NE} \in \widetilde{\Phi}_{NE}\right)$ *if and only if*

$$\widetilde{U}_i\left(\widetilde{\varphi}_i^{NE}, \widetilde{\varphi}_{-i}^{NE}\right) \geq \widetilde{U}_i\left(\varphi_i, \widetilde{\varphi}_{-i}^{NE}\right),$$

$$\forall \varphi_i \in H_i \text{ and } \forall i : 1 \to N \quad .$$

Following the original definition of Harsanyi (1967) the NE is typically defined in extensive form. The equivalence between the two equilibrium concept is widely known in Game theory and Decision theory (e.g. Schlaifer (1959)). Nevertheless, we feel it is important to provide a precise statement of this result since the CSE do not verify the equivalence.

Proposition 2.3. $\Phi^{NE} = \widetilde{\Phi}^{NE}$ *under the following assumption*

A1: $\forall \varphi_{-i} \in H_{-i}$ and $\forall \psi_i \in \Theta_i$ there exists $\varphi_i \in H_i$ such that

$$\widehat{U}_i\left(\varphi_i\left(\xi_i\right), \varphi_{-i}; \xi_i\right) \geq \widehat{U}_i\left(\psi_i\left(\xi_i\right), \varphi_{-i}; \xi_i\right),$$

$$\forall \xi_i \in \Xi_i \text{ and } \forall i : 1 \to N \quad . \tag{2.2}$$

Proof: TO BE COMPLETED

that if players i opponent restrict their strategy space to H_{-i}, then any strategy in $\Theta_i - H_i$ is dominated by a strategy in

A1 implies that a strategy φ_i is admissible in the set H_i if it is not strictly dominated for all $\varphi_{-i} \in H_{-i}$. In the remainder A1 is assumed to hold and we will consider either the extensive or the strategic form of the game to derive the NE solution.

Note that no general theorem ensures the existence of a NE solution in a game of incomplete information with continuous types and actions. In practice, the problems of existence and uniqueness are solved by the direct determination of an analytical equilibrium solution. This solution obtains from the following optimization and fixed point problems,

$$\varphi_i^{NE}\left(\xi_i\right) \in \underset{x_i \in X_i}{ArgMax}\ \widehat{U}_i\left(x_i, \varphi_{-i}^{NE}; \xi_i\right),\ \forall \xi_i \in \Xi_i \text{ and } \forall i : 1 \to N \quad . \tag{2.3}$$

The corresponding First Order Conditions (FOC) often are reformulated as

$$\frac{d}{dx_i}\widehat{U}_i\left(x_i,\varphi_{-i}^{NE};\xi_i\right)|_{x_i=\varphi_i^{NE}(\xi_i)}=0 \quad \forall\xi_i\in\Xi_i \text{ and } \forall i:1\rightarrow N, \tag{2.4}$$

which typically produce a set of equations (differential equations in the case of auctions) leading to the solution (provided that $\frac{d^2}{dx_i^2}\widehat{U}_i\left(x_i,\varphi_{-i}^{NE};\xi_i\right)|_{x_i=\varphi_i^{NE}(\xi_i)}\leq 0$ $\forall i:1\rightarrow N$). Note that this approach requires $\widehat{U}_i(.)$ to be twice continuously differentiable in x_i. If we define the operator

$$A_i[\varphi](\xi_i)=\frac{d}{dx_i}\widehat{U}_i\left(x_i,\varphi_{-i};\xi_i\right)_{|_{x_i=\varphi_i(\xi_i)}}, \qquad i:1\rightarrow N \quad ,$$

then a NE verifies $A_i\left[\varphi^{NE}\right](\xi_i)=0$ $\forall\xi_i\in\Xi_i$ and $\forall i:1\rightarrow N$.

Except under fairly restrictive assumptions (such as symmetry, risk neutrality, ...) it is often impossible to find an analytical solution to (2.3). Numerical methods may be applied when the FOC (2.4) have an explicit expression (e.g. Bajari 1996). Howevert in many complex games the FOC cannot be derived and we cannot applied usual numerical techniques.

3. Constrained Strategic Equilibrium

Constrained sets of strategies are implicitly defined here as subsets $H_i^{(k)}\subset H_i$. The definition of CSE now parallels that of a NE in strategic form, except that strategies are now restricted to $H_i^{(k)}$:

Definition 3.1. *A CSE in the set of strategies* $H^{(k)}=\prod_{i=1}^{N}H_i^{(k)}$ *is a strategic implementation of the game* $\varphi_{CSE}^{(k)}=\left(\varphi_{1,CSE}^{(k)},...,\varphi_{N,CSE}^{(k)}\right)$ *with* $\varphi_{CSE}^{(k)}\in\Phi_{CSE}^{(k)}$, *whereby the* $\varphi_{i,CSE}^{(k)}$*'s are mutually best responses in the strategic form game*

$$\widetilde{U}_i\left(\varphi_{i,CSE}^{(k)},\varphi_{-i,CSE}^{(k)}\right)\geq\widetilde{U}_i\left(\varphi_i^{(k)},\varphi_{-i,CSE}^{(k)}\right) \quad ,$$

$$\forall\varphi_i^{(k)}\in H_i^{(k)},\forall i:1\rightarrow N \quad .$$

AFR (2000) show that there exists a CSE in $H^{(k)}$ under the following assumptions, A2 $H_i^{(k)}$ is compact and convex $\forall i = 1, ..., N$,

A3 the function $\widetilde{U}_i\left(\varphi_i, \varphi_{-i}\right)$ is continuous in φ, $\forall i : 1 \to N$, $\forall \varphi \in H$,

A4 the function $\widetilde{U}_i\left(\varphi_i, \varphi_{-i}\right)$ is quasi concave in φ_i, $\forall i : 1 \to N$, $\forall \varphi_i \in H_i$.

AFR (2000) provide also primitive conditions on the utility function $U_i\left(\varphi\left(\xi\right), \xi\right)$ so that A3 and A4 are verified. In the remainder we assume that assumptions A2 to A4 are verified.

Provided that $\widetilde{U}_i\left(\varphi_i, \varphi_{-i}\right)$ is twice continuously differentiable in φ_i, the CSE can be defined as a fixed point of the constrained best response correspondence

$$\varphi_{i,CSE}^{(k)} \in \underset{\varphi_i^{(k)} \in H_i^{(k)}}{ArgMax}\ \widetilde{U}_i\left(\varphi_i^{(k)}, \varphi_{-i,CSE}^{(k)}\right) \quad \forall i : 1 \to N \quad . \tag{3.1}$$

The determination of this fixed point is greatly simplified with a parametrization of the strategies in $H_i^{(k)}$ by a vector of $\alpha_i^{(k)} \in \Re^k$. Such parametrization is always possible since $H_i^{(k)}$ is compact. This approach provides a major computational advantage since it requires to optimize over a finite set of parameters rather than an infinite set of functions as it is the case with NE (see AFR (2000) for numerical considerations).

The optimization problem in (3.1) leads to the FOC

$$\frac{\partial \widetilde{U}_i\left(\varphi_{i,CSE}^{(k)}, \varphi_{-i,CSE}^{(k)}\right)}{\partial \alpha_{i,t}^{(k)}} = E\left[\frac{\partial \varphi_{i,CSE}^{(k)}}{\partial \alpha_{i,t}^{(k)}}(\xi_i) A_i\left[\varphi_{i,CSE}^{(k)}\right](\xi_i)\right] = 0, \qquad i : 1 \to N, t : 1 \to k \quad , \tag{3.2}$$

If we decompose $\varphi_i^{(k)} \in H_i^{(k)}$ in

$$\varphi_i^{(k)} = \sum_{t=1}^{k} \alpha_{i,t}^{(k)} \Psi_t^{(k)}, \qquad i : 1 \to N \quad , \tag{3.3}$$

where $\left\{\Psi_t^{(k)}\right\}_{t=1,\dots,k}$ is a basis in $H_i^{(k)}$ then (3.2) can be written

$$< \Psi_t^{(k)}, A_i\left[\varphi_{i,CSE}^{(k)}\right] >= 0, \qquad i : 1 \to N, t : 1 \to k \quad , \tag{3.4}$$

where $< ., . >$ is the inner product defined with respect to $F(.)$. The operator $A_i\left[\varphi^{(k)}\right]$ is then orthogonal to every elements of the basis of $H_i^{(k)}$. As k gets larger the CSE has to verify more orthogonality conditions which intuitively suggests that $\lim_{k \to \infty} A_i\left[\varphi_{i,CSE}^{(k)}\right] = 0$. In other words the CSE should verify at the limit the FOC of a NE. This observations motivates the next section where we provides explicit conditions under which a sequence of CSE is an approximation of the NE.

4. General Approximation Theorem

In this section we assume that there exists a topology T such that $\underset{k\geq 1}{\cup} H^{(k)}$ is dense in H with respect to T and $H^{(k)} \subset H^{(k+1)}$ $\forall k \in N^*$. The first proposition show that a limit point of a sequence of CSE is a NE.

Proposition 4.1. *If the sequence* $\left\{\varphi_{CSE}^{(k)}\right\}_{k=1\to\infty}$ *where* $\varphi_{CSE}^{(k)} \in \Phi_{CSE}^{(k)}$ *has an accumulation point* $M \in H$ *with respect to* T *then there exists a NE in strategic form in* H *and* $M \in \Phi_{NE}$*. Besides, if* H *verifies A1 then* M *is a NE in extensive form in* H*.*

Proof:

Consider any strategy $\varphi \in H$. Since $\underset{k\geq 1}{U} H^{(k)}$ is dense in H, then there exists $\left\{\varphi^{(k)}\right\}_{k=1\to\infty}$ $\left(\varphi^{(k)} \in H^{(k)}\right)$ such that $\varphi^{(k)} \overset{k\to\infty}{\to} \varphi$. Then, since $\varphi_{CSE}^{(k)} \in \Phi_{CSE}^{(k)}$ we have $\forall i : 1 \to N$ and $\forall k : 1 \to \infty$

$$\widetilde{U}_i\left(\varphi_{i,CSE}^{(k)}, \varphi_{-i,CSE}^{(k)}\right) \geq \widetilde{U}_i\left(\varphi_i^{(k)}, \varphi_{-i,CSE}^{(k)}\right) \ . \tag{4.1}$$

If the sequence $\left\{\varphi_{CSE}^{(k)}\right\}_{k=1\to\infty}$ has an accumulation point $M = (M_i, M_{-i})$ such that $M \in H$ then there exists $\left\{k^{(m)}\right\}_{m=1\to\infty}$ such that $\left(\varphi_{i,CSE}^{\left(k^{(m)}\right)}, \varphi_{-i,CSE}^{\left(k^{(m)}\right)}\right) \overset{m\to\infty}{\to}$ (M_i, M_{-i}) and we still have, $\forall i : 1 \to N$ and $\forall k : 1 \to \infty$

$$\widetilde{U}_i\left(\varphi_{i,CSE}^{\left(k^{(m)}\right)}, \varphi_{-i,CSE}^{\left(k^{(m)}\right)}\right) \geq \widetilde{U}_i\left(\varphi_i^{\left(k^{(m)}\right)}, \varphi_{-i,CSE}^{\left(k^{(m)}\right)}\right) \ . \tag{4.2}$$

Finally, since $\widetilde{U}_i(.)$ is continuous in φ $\forall i : 1 \to N$, we can write the previous equation as m tends toward infinity,

$$\widetilde{U}_i(M_i, M_{-i}) \geq \widetilde{U}_i(\varphi_i, M_{-i}) \ \ \forall i : 1 \to N \tag{4.3}$$

Therefore $M \in \Phi_{NE}$.■

Proposition 4.2. *if* H *is compact with respect to the topology* T *then there exists a NE in* H *and any sequence* $\left\{\varphi_{CSE}^{(k)}\right\}_{k=1\to\infty}$ *has a subsequence that converges towards a NE.*

Proof:

If H is compact any sequence $\left\{\varphi^l\right\}_{l=1\to\infty} \in \prod_{l=1}^{\infty} H$, has at least an accumulation point in H. Since $\left\{\varphi_{CSE}^{(k)}\right\}_{k=1\to\infty} \in \prod_{k=1}^{\infty} H$ it has an accumulation point $M \in H$. Then, from the previous proposition M is a NE.■

The compactness of the strategy space is typically assumed in game of complete information to insure the existence of a NE. In games of incomplete information the exact structure of H is more delicate to determine and it may not be compact. Typically, H is a bounded subset of an infinite dimensional Banach space and therefore it cannot be compact with respect to the strong topology defined by the norm of the space (c.f. Brezis (19??)). However, H can be compact with respect to the weak$*$ topology. An example of such cases is provided in proposition 4.3.

Definition 4.3. *H_i is the set of uniformly bounded variation on $[a_i, b_i]$ if $\forall \varphi_i \in H_i$ there exists $W_i > 0$ such that $W_{a_i}^{b_i}(\varphi_i) \le W_i$ where*

$$W_{a_i}^{b_i}(\varphi_i) = \sup_{a_i=\xi_1<....<\xi_{T+1}=b_i} \sum_{t=1}^{T} \left|\varphi_i(\xi_t) - \varphi_i\left(\xi_{t+1}\right)\right| \tag{4.4}$$

is called the variation of the φ_i.

Consider the following assumptions

A5 $\Xi_i = [a_i, b_i]\ \forall i : 1 \to N$,

A6 $X_i \in \Re\ \forall i : 1 \to N$,

A7 H_i is the set of functions bounded at a_i $(|\varphi_i(a_i)| \le \overline{a_i}\ \forall \varphi_i \in H_i)$ and of uniformly bounded variation on $[a_i, b_i]$, $(\forall i : 1 \to N)$.

Proposition 4.4. *Under assumptions A1 to A7 there exists a NE in H and any sequence $\left\{\varphi_{CSE}^{(k)}\right\}_{k=1\to\infty}$ has a subsequence that converges towards a NE.*

Proof:

It suffices to show that the set H defined in proposition 4.3 is weakly compact. Consider $\varphi_i \in H_i$. It can be shown (e.g. Lojasciewics ???) that $|\varphi_i(\xi_i)| \le \overline{a_i} + W_i$, $\forall \xi_i \in [a_i, b_i]$. Consider $\overline{W}_i = \sup(\overline{a_i}, W_i)$ and a sequence in H_i $\left\{\varphi_i^l\right\}_{l=1\to\infty}$, then $\left|\varphi_i^l(\xi_i)\right| \le 2\overline{W}_i$. By Jordan's canonical decomposition $\varphi_i^l = \overline{\varphi}_i^l - \underline{\varphi}_i^l$ where $\overline{\varphi}_i^l$ and

$\underline{\varphi}_i^l$ are increasing and jointly bounded over $[a,b]$:

$$\left|\overline{\varphi}_i^l\right| = \frac{1}{2}\left|W_{a_i}^{b_i}\left(\varphi_i^l\right)+\varphi_i^l\left(\xi_i\right)\right| \leq \frac{3}{2}\overline{W}_i \tag{4.5}$$

$$\left|\underline{\varphi}_i^l\right| = \frac{1}{2}\left|W_{a_i}^{b_i}\left(\varphi_i^l\right)-\varphi_i^l\left(\xi_i\right)\right| \leq \frac{3}{2}\overline{W}_i \tag{4.6}$$

Helly's first theorem guarantees that any sequence of increasing and bounded functions over $[a_i, b_i]$ has a subsequence that converges at all continuity points over $[a_i, b_i]$. Now let us apply twice Helly's first theorem, first to the sequence $\left\{\underline{\varphi}_i^l\right\}_{l=1\to\infty}$ to obtain a convergent subsequence $\left\{\underline{\varphi}_i^{\beta(l)}\right\}_{\beta(l)=1\to\infty}$ and then to the sequence $\left\{\overline{\varphi}_i^{\beta(l)}\right\}_{\beta(l)=1\to\infty}$. This way we obtain two convergent subsequences:

$$\underline{\varphi}_i^{\alpha(l)}\left(\xi_i\right) \overset{l\to\infty}{\to} \underline{\varphi}_i\left(\xi_i\right) \text{ and } \overline{\varphi}_i^{\alpha(l)}\left(\xi_i\right) \overset{l\to\infty}{\to} \overline{\varphi}_i\left(\xi_i\right) \quad \forall \xi_i \in [a,b] \quad .$$

Hence,

$$\varphi_i^{\alpha(l)}\left(\xi_i\right) \overset{l\to\infty}{\to} \varphi_i\left(\xi_i\right) = \overline{\varphi}_i\left(\xi_i\right) - \underline{\varphi}_i\left(\xi_i\right) \quad \forall \xi_i \in [a,b] \quad .$$

Therefore, any sequence $\left\{\varphi_i^l\right\}_{l=1\to\infty}$ in H_i has a subsequence that converges weakly to a function φ_i of bounded variation.

Now, let us show that the variation of φ_i is smaller or equal to W_i. Consider any subdivision $a_i = \xi_1 < .... < \xi_{T+1} = b_i$ we know that

$$\sum_{t=1}^{T}\left|\varphi_i^l\left(\xi_t\right)-\varphi_i^l\left(\xi_{t+1}\right)\right| \leq W_i \ \forall l = 1 \to \infty \tag{4.7}$$

Now, take the limit as $l = 1 \to \infty$,

$$\sum_{t=1}^{T}\left|\varphi_i\left(\xi_t\right)-\varphi_i\left(\xi_{t+1}\right)\right| \leq W_i \tag{4.8}$$

Hence, taking the supremum of the left hand side, $W_{a_i}^{b_i}\left(\varphi_i\right) \leq W_i$.

Every sequence $\left\{\varphi_i^l\right\}_{l=1\to\infty}$ in H_i has a subsequence that weakly converges in H_i $\forall i : 1 \to N$. Therefore H is compact with respect to the weak topology.■

The assumption in proposition 4.3 is less restrictive than the compactness of the strategy space. Indeed, functions of bounded variation include most well defined bounded functions such as the continuous monotonic functions on $[a,b]$, the bounded functions with a countable number of discontinuity points, and the differentiable bounded function with derivatives changing signs a countable number of time.

Since we consider weak convergence, we can obtain a sequence $\{H^{(k)}\}_{k=1\to\infty}$ such that $\underset{k\geq 1}{\cup} H^{(k)}$ is dense in H, by usual approximation techniques (e.g. projection on the first elements of a given basis such as polynomials or piecewise linear functions). Note that if H includes discontinuous functions of bounded variation the sequence of CSE may not converge at the discontinuity points.

We show in proposition 4.3 that the CSEs converge toward the NE in strategic form in H. Using proposition 2.3 it is then sufficient to verify that H verifies A1, to obtain the convergence toward the NE in extensive form. In this case A! is equivalent to the following properties. First, the best response to a bounded variation function is a bounded variation function; second, the best response to a function of variation smaller than W is of variation smaller than W. Such conditions are verified in particular in the Independent Private Value model with support $[a, b]$ where H can be chosen as the set of non-decreasing functions bounded by the bisector. More generally, it has been shown (see Athey 1998) that in many games of incomplete information the utility function verifies the single crossing property which ensures that H is of bounded variation.

Finally, if the NE is not of bounded variation then two question arise: first, can we expect real agents to determine a NE which is a rather "wild" function? Second, will any numerical techniques be able to approximate this NE?

5. Criteria of convergence

In many games the set of admissible strategies is not clearly defined and it is not always possible to apply theoretic restrictions to eliminate strictly dominated strategies. In these cases it might be impossible to verify the conditions of an approximation theorem. Moreover, even when the approximation theorem can be applied, it would be useful to know for any given constraint set $H^{(k)}$ how distant the CSE $\varphi_{CSE}^{(k)}$ is from the NE solution. In this section a sequence $\left\{\varphi_{CSE}^{(k)}\right\}_{k=1\to\infty}$ is said to approximate a NE φ^{NE} if $\lim_{k=1\to\infty} \varphi_{CSE}^{(k)} = \varphi^{NE}$. We consider a sequence of constraint set $\{H^{(k)}\}_{k=1\to\infty}$ such that $\underset{k\geq 1}{\cup} H^{(k)}$ is dense in H where H is a set of admissible strategy verifying A1. We provide three criteria to document in practice whether the CSE is a good approximation of the NE and how far the

CSE is from a NE.

5.1. Convergence of the CSE sequence

Consider the criteria

$$C_1(k,t) = \left\| \varphi_{CSE}^{(k)} - \varphi_{CSE}^{(k+t)} \right\|.$$

where $\|.\|$ is a norm defined over H. Proposition 3.1 states that when a sequence of CSE converges it converges toward a NE. In other words, the sequence $\left\{ \varphi_{CSE}^{(k)} \right\}_{k=1\to\infty}$ is an approximation of the NE if and only if

$$\lim_{k\to\infty, t\to\infty} C_1(k,t) = 0 \quad .$$

In practice, we want to monitor the convergence of a CSE sequence by verifying that the criteria $C_1(k,1)$ converges toward 0. However, there is no result regarding the rate of convergence. Therefore, even when the criteria $C_1(k,1)$ is close to 0 this does not provide any explicit information about the quality of the approximation.

5.2. The best response to a CSE

Let us denote $\Phi_{BR}(\varphi) = \prod_{i=1}^{N} \Phi_{BR,i}(\varphi)$ where $\Phi_{BR,i}$ is player i best response correspondence defined as

$$\begin{aligned} \Phi_{BR,i} \quad &: \quad H \to H_i \\ \varphi \quad &\to \quad \Phi_{BR,i}(\varphi) = \left\{ \varphi_{BR,i} \in H_i / \widetilde{U}_i\left(\varphi_{BR,i}, \varphi_{-i}\right) \geq \widetilde{U}_i\left(\varphi_i, \varphi_{-i}\right) \right\} \end{aligned} \tag{5.1}$$

Note that $\Phi_{BR}(\varphi)$ is a subset of F. Let us assume that this best response correspondence is upper semicontinuous.

Consider a set $H^{(k)}$ and a constraint strategy profile $\varphi_{CSE}^{(k)} \in \Phi_{CSE}^{(k)}$. $\Phi_{BR,i}\left(\varphi_{CSE}^{(k)}\right)$ represents the set of best response strategies of player i in H_i when his opponents play the CSE strategy $\varphi_{CSE,-i}^{(k)}$ in $H^{(k)}$. Then, we can measure the distance between the CSE and its best response which leads to a second criteria:

$$C_2(k) = \left\| \varphi_{CSE}^{(k)} - \Phi_{BR}\left(\varphi_{CSE}^{(k)}\right) \right\| \quad .$$

Proposition 5.1. *Every sequence of CSE* $\left\{\varphi_{CSE}^{(k)}\right\}_{k=1\to\infty}$ *that approximates a NE verifies*

$$\lim_{k\to\infty} C_2(k) = 0 \quad .$$

Proof:

The proof is trivial: If $\varphi_{CSE}^{(k)}$ is an approximation of a NE equilibrium then

$$\lim_{k=1\to\infty} \varphi_{CSE}^{(k)} = \varphi^{NE} \quad .$$

This NE verifies, $\varphi^{NE} \in \Phi_{BR}\left(\varphi^{NE}\right)$. From the upper semicontinuity of $\Phi_{BR}(.)$ we have

$$\lim_{k=1\to\infty} \varphi_{CSE}^{(k)} \in \lim_{k=1\to\infty} \Phi_{BR}\left(\varphi_{CSE}^{(k)}\right) \quad .$$

which implies

$$\lim_{k=1\to\infty} C_2(k) = 0$$

■

One can monitor the quality of the approximation by looking at the distance between the CSE and its best responses in H. When $C_2(k)$ is sufficiently close to 0 then the CSE is a good approximation of the NE.

Note that $\varphi_i^{BR(k)}$ the best responses in H to $\varphi_{CSE,i}^{(k)}$ is tremendously easier to calculate than the actual NE. Indeed, $\varphi^{BR(k)}$ is determined by N independent maximization problems, while the NE requires to solve N maximizations combined with a system of N differential equations associated with the fixed point problem. Besides, the strategies of player i opponents are known and fixed to $\varphi_{CSE,i}^{(k)}$ which eliminates the uncertainty about other players actions. This, typically reduces considerably the dimension of the integral in the derivation of the (conditional) expected utility.

Finally, it might not be possible in some games to determined explicitly the function $\varphi_i^{BR(k)}$. In this case the criteria $C_2(k)$ can be approximated by $\widehat{C}_2^L(k)$ defined as

$$\widehat{C}_2^L(k) = \sum_{i=1}^{N}\sum_{l=1}^{L}\left|\varphi_i^{BR(k)}\left(\xi_i^l\right) - \varphi_{CSE,i}^{(k)}\left(\xi_i^l\right)\right| .$$

where $\varphi_i^{BR(k)}\left(\xi_i^l\right)$ is the best response to $\varphi_{CSE,-i}^{(k)}$ when player i receives the private signal ξ_i^l. The points ξ^l $(\forall l : 1 \to L)$ are determined exogenously and we suggest to use some fractiles of the private signal distribution $f(.)$.

5.3. The CSE as NE of a neighboring game

Consider the function $O(.)$

$$\begin{array}{ccc} \digamma & \rightarrow & H \\ f & \rightarrow & O(f) = \varphi_f^{NE} \end{array} ,$$

where $\digamma$ is a set of distributions and φ_f^{NE} is the NE of the game where private signals are drawn from f. Note that the element of $\digamma$ are assumed to be defined almost everywhere. For the ease of exposition assume that $O(.)$ is an homeomorphism[4] with an invert function $O^{-1}(\varphi) = f_\varphi(.)$ where $f_\varphi(.)$ is a distribution such that if the private signals where drawn from $f_\varphi(.)$ then φ would be a NE. This assumption should be interpreted as : neighboring distributions define neighboring games and they should have neighboring NE solution. Conversely, neighboring strategies should be the NE solution of neighboring games. Note that this assumption is necessary to anyone conducting empirical work and it is verified in common games such as the Independent Private values auction.

Then, a strategy φ is assumed to be admissible ($\varphi \in H$) if there exists a distribution $f_\varphi(.)$ in $\digamma$ for which φ is a NE.

Consider now a game where private signals are drawn from a given distribution $f(.)$. Consider also a set $H^{(k)}$ and a constraint strategy profile $\varphi_{CSE}^{(k)} \in \Phi_{CSE}^{(k)}$. $f_{\varphi_{CSE}^{(k)}}(.) = O^{-1}\left(\varphi_{CSE}^{(k)}\right)$ is then such that

$$\widehat{U}_i\left(\varphi_{CSE,i}^{(k)}(\xi_i), \varphi_{CSE,-i}^{(k)}; \xi_i / f_{\varphi_{CSE}^{(k)}}\right) \geq \widehat{U}_i\left(\psi_i(\xi_i), \varphi_{CSE,-i}^{(k)}; \xi_i / f_{\varphi_{CSE}^{(k)}}\right), \quad \forall \psi_i \in H_i \ \forall \xi_i \in \Xi_i \quad , \tag{5.2}$$

or equivalently in the strategic form game,

$$\widetilde{U}_i\left(\varphi_{CSE,i}^{(k)}, \varphi_{CSE,-i}^{(k)} / f_{\varphi_{CSE}^{(k)}}\right) \geq \widetilde{U}_i\left(\psi_i, \varphi_{CSE,-i}^{(k)} / f_{\varphi_{CSE}^{(k)}}\right) \quad \forall \psi_i \in H_i \quad . \tag{5.3}$$

where the conditional and unconditional expected utility are calculated with the density $f_{\varphi_{CSE}^{(k)}}(.)$.

[4]Note that the result should generalized to the more realistic case where $O(.)$ associates a subset of strategies to a subset of distributions.

This leads to a different measure of the distance between the CSE and an the NE,

$$C_3(k) = \left\| f - f_{\varphi^{(k)}_{CSE}} \right\| .$$

Proposition 5.2. *The sequence of CSE* $\left\{ \varphi^{(k)}_{CSE} \right\}_{k=1\to\infty}$ *approximates a NE if and only if*

$$\lim_{k\to\infty} C_3(k) = 0$$

Proof: The proof essentially relies on the continuity of $O(.)$ and its inverse. Consider a sequence of CSE $\left\{ \varphi^{(k)}_{CSE} \right\}_{k=1\to\infty}$ such that

$$\lim_{k\to\infty} \varphi^{(k)}_{CSE} = \varphi^{NE} \qquad .$$

Then, since $O^{-1}(.)$ is continuous

$$\lim_{k\to\infty} O^{-1}\left(\varphi^{(k)}_{CSE} \right) = O^{-1}\left(\varphi^{NE} \right) \qquad ,$$

$$\Rightarrow \lim_{k\to\infty} f_{\varphi^{(k)}_{CSE}} = f \qquad ,$$

$$\Rightarrow \lim_{k\to\infty} C_3(k) = 0 \qquad .$$

Conversely,

$$\lim_{k\to\infty} C_3(k) = 0$$

implies

$$\lim_{k\to\infty} O\left(\varphi^{(k)}_{CSE} \right) = O\left(\varphi^{NE} \right) \qquad ,$$

which by continuity of $O(.)$ reduces to

$$\lim_{k\to\infty} \varphi^{(k)}_{CSE} = \varphi^{NE} \qquad .$$

■

If a sequence of CSE $\left\{ \varphi^{(k)}_{CSE} \right\}_{k=1\to\infty}$ verifies proposition 5.2 then $\varphi^{(k)}_{CSE}$ can be interpreted as a NE in a slightly perturbed game where private signals are drawn from a distribution neighboring the original f. In other words, the CSEs are NE of games that become closer to the original game as k increases.

One can monitor the accuracy of the approximation by verifying that $C_3(k)$ is close enough to zero. In practice the determination of $f_{\varphi^{(k)}_{CSE}}$ requires to apply usual econometric techniques. For instance one can utilize, when available, the FOC associated with the determination of the NE as a moment condition.

$$A_i[\varphi](\xi_i) = 0 \quad \forall \xi_i \in \Xi_i \text{ and } \forall i : 1 \rightarrow N. \tag{5.4}$$

Provided identification we can apply the Method of Simulated Moment and estimate $f_{\varphi^{(k)}_{CSE}}$ by

$$\widehat{f}_{\varphi^{(k)}_{CSE}} = Arg \min_{f \in F} \sum_{l=1}^{L} \sum_{i=1}^{N} [A_i[\varphi](\xi_i)]^2 \quad \forall i : 1 \rightarrow N. \tag{5.5}$$

where $\widehat{U}_i\left(x_i, \varphi^{(k)}_{CSE,-i}; \xi^l_i / f\right)$ is the conditional expected utility calculated with the distribution $f(.)$ and ξ^l $(\forall l : 1 \rightarrow L)$ are private signals simulated from the distribution $f(.)$.

In most applications we assume that the private signals distribution belongs to a parametric family of distributions indexed by $\theta \in \Re^p$. In such cases, we have to estimate only the parameter $\widehat{\theta}$:

$$\widehat{\theta} = Arg \min_{\theta \in \Re^p} \sum_{l=1}^{L} \sum_{i=1}^{N} \left[\frac{d}{dx_i} \widehat{U}_i\left(x_i, \varphi^{NE}_{-i}; \xi^l_i / \theta\right) |_{x_i = \varphi^{NE}_i(\xi^l_i)} \right]^2 \quad \forall i : 1 \rightarrow N. \tag{5.6}$$

where ξ^l $(\forall l : 1 \rightarrow L)$ are simulated from $f(./\theta)$.

To conclude this section let us remind the reader that in empirical applications, which is our primary interest, the distribution of private signals is not known and needs to be estimated. In other words the game is not perfectly defined and the actual NE strategy will vary slightly depending upon the estimation of the distribution. In this context is seems reasonable to consider the concept of CSE that can be interpreted as NE of a game with a slightly different distribution.

6. Conclusion

[To be completed]

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