

Majority Voting in Multidimensional Policy Spaces :

Kramer-Shepsle versus Stackelberg¹

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Abstract

We study majority voting over a bidimensional policy space when the voters' type space is either uni- or bidimensional. We study two voting procedures widely used in the literature. The Stackelberg (ST) procedure assumes that votes are taken one dimension at a time according to an exogenously specified sequence. The Kramer-Shepsle (KS) procedure also assumes that votes are taken separately on each dimension, but not in a sequential way. A vector of policies is a Kramer-Shepsle equilibrium if each component coincides with the majority choice on this dimension given the other components of the vector. We study the existence and uniqueness of the ST and KS equilibria, and we compare them, looking e.g. at the impact of the ordering of votes for ST and identifying circumstances under which ST and KS equilibria coincide. In the process, we state explicitly the assumptions on the utility function that are needed for these equilibria to be well behaved. We especially stress the importance of single crossing conditions, and we identify two variants of these assumptions: a marginal version that is imposed on all policy dimensions separately, and a joint version whose definition involves both policy dimensions.

Keywords: Unidimensional and bidimensional type space, single crossing, one-sided separability

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1 Introduction

It is well known that majority voting suffers from what Bernheim and Slavov (2009) call the “curse of multidimensionality”: when the policy space is sufficiently rich, there is no policy option that gathers a majority of votes when faced with all other possible options –i.e., there is no Condorcet winner (see e.g. Plott 1967, Davis, DeGroot and Hinich 1972, McKelvey, Ordeshook and Ungar 1980, Banks, Duggan and Le Breton 2006 and Banks and Austen-Smith 1999).

In all rigorous formal versions of this result the respective roles of the properties of the types’ distribution function and of utility functions are not always clearly disentangled. For instance, in the spatial model of politics (where preferences are Euclidean), the symmetry of preferences is imposed and the focus is exclusively on the distribution of voters’ types. The first objective of this paper is to fill this small gap and to offer a pedestrian analysis of the existence of a Condorcet winner with a unidimensional type space.

Faced with this “curse of multidimensionality”, the applied political economy literature has followed various directions, including the obvious one of restricting the policy space to be unidimensional. Several lines of attack recognizing the issue of multidimensionality have consisted in a detailed game theoretical description of the collective decision making process and a subsequent analysis of its equilibrium outcomes. One example of such an approach is the analysis of sequential bargaining by Baron and Ferejohn (1989) and Banks and Duggan (2000). Another, more recent, example consists in the analysis of an electoral competition game with nonseparable preferences, where candidates differ in their exogenous characteristics (Krasa and Polborn (2009)) or strengths on certain policy dimensions (Krasa and Polborn (2010)) and where they are uncertain about voters’ preferences. In this paper, we adopt a bidimensional policy space and we focus on two widely used approaches having in common that votes never take place simultaneously on all dimensions.

The first approach assumes that citizens vote sequentially on each dimension. An exogenous ordering of the dimensions is considered and, at each voting stage, the outcomes of the preceding

votes are known to the voters. For instance, when there are two dimensions, a first majority vote is organized over one of the policy dimensions and is followed by a second majority vote over the other dimension. We call Stackelberg (ST) equilibria the policies that can be supported at equilibrium for a particular ordering of the dimensions. This sequential resolution has been used by many authors in political economy models (see e.g. Alesina, Baqir and Easterly 1999, Alesina, Baqir and Hoxby 2004, Cremer, De Donder and Gahvari 2004, Cremer *et al.* 2007, De Donder, Le Breton and Peluso 2009, Etro 2006, Gregorini 2009, Haimanko, Le Breton and Weber 2005).

The second approach assumes instead that there is no sequential ordering of the votes, but that they are taken separately on each dimension. Under the presumption that all dimensions except one have been settled, citizens cast their vote over the residual dimension. A solution is consistent if the vector of policies obtained through that procedure is self-supporting in a Nash-like manner. This idea has been independently developed by Kramer (1972) and Shepsle (1979) and hereafter we will call Kramer-Shepsle's equilibria (KS) the policy vectors meeting this consistency condition. More precisely, a vector is a Kramer-Shepsle's equilibrium if, for any dimension, the corresponding component in the vector coincides with the majority choice on this dimension given the other components of the policy vector. Shepsle considers the case where the collective decision processes may differ across dimensions and demonstrates existence under quite general conditions. He also illustrates through examples that the set of KS equilibria may display peculiar features. To the best of our knowledge, the only other theoretical contributions are two unpublished papers by Banks and Duggan (2004) and Duggan (2001) who examined the existence issue from a general perspective. This concept has also been studied by the applied political economy literature, e.g. by De Donder and Hindriks (1998), Diba and Feldman (1984), Nechyba (1997), Sadanand and Williamson (1991).

In this paper, we provide an analysis of the KS and ST equilibria in a general framework with a bidimensional policy space. We study their existence, uniqueness and we compare them, looking e.g. at the impact of the ordering of votes for ST and identifying circumstances under

which ST and KS equilibria coincide. In the process, we state explicitly the assumptions on the utility function that are needed for these equilibria to be well-behaved. We especially stress the importance of single-crossing conditions, and we identify two variants of these assumptions: a “marginal” version that is imposed on all policy dimensions separately, and a “joint” version whose definition involves both policy dimensions. We perform this analysis first with a unidimensional type space, and then with a bidimensional type space.

Our results run as follows. Starting with a unidimensional type space, we illustrate the “curse of multidimensionality” (of the policy space): when we assume that the utility function satisfies both marginal and joint single-crossing, there is generically no Condorcet winner and, perhaps more surprisingly, in most cases and for any policy proposal, it is possible to find a direction that is favored by almost all voters. We then study the KS and ST equilibria in this setting. We show that under marginal single-crossing, the KS solution(s) coincide with the set of componentwise ideal point(s) of the median type. Under strict concavity of the utility function, this implies that there exists a unique KS solution which is the unique ideal point of the median type. Assuming in addition strategic complementarity between policy dimensions results in the reduced utility function in the first stage of voting (given the anticipated choice in the second stage of voting) being single-crossing, so that the KS equilibrium coincides with the ST equilibrium. Although single-crossing and single-peakedness are two logically independent properties, we provide conditions on the derivatives of the direct utility function that ensure that a majority of the electorate has single-peaked reduced utility functions.

We next study a specific environment that has received a lot of attention in different literatures (e.g. on nation formation) and which does not satisfy the marginal single-crossing property. In this environment, voters differing in their preference for the type of a public good have to choose both its type and its quantity. While the majority-chosen public good’s type does not depend on its quantity, the opposite relationship is not true, a situation we dub one-sided separability. The literature has focused on the ST equilibrium where voters choose first the public good’s quantity. We show that this equilibrium corresponds to the KS equilibrium,

but that the ST equilibrium with the opposite sequence of votes (which, to the best of our knowledge, has not been studied previously) is more complex, with the identity of the second-stage decisive voter being affected by the first-stage voting decision. We provide a thorough analysis of how first-stage voting is impacted in that case (i.e., how voters bias their first-stage voting choices when anticipating the impact on the second-stage decisive voter's identity).

We then move to a bidimensional type space. There is little we can say at this level of generality about the existence or characteristics of the Stackelberg equilibria, and their relationship with the KS equilibria. We thus content ourselves with providing an example with a discrete number of types differing both in the location and in the shape of their indifference curves and where i) there are multiple KS equilibria, ii) not all KS equilibria correspond to ST equilibria (whatever the ordering of the votes) and iii) some KS equilibria do not correspond to any voter's most-preferred policy.

The paper is organized as follows: Section 2 presents the one-dimensional type general framework. Its first subsection analyzes simultaneous voting, the second subsection studies and compares Kramer-Shepsle and Stackelberg equilibria, while the third subsection is devoted to the analysis of a specific environment studied e.g. in the nation formation literature. Section 3 focuses on the case with two-dimensional types while section 4 concludes. Most proofs are relegated to Appendices.

2 One-Dimensional Types

Throughout the paper, we consider a population of voters who have to select a public policy in a two-dimensional policy space. A policy choice is therefore a vector $(x, y) \in Z$, where the set of feasible policy choices Z is assumed to be a convex, compact and rectangular subset $X \times Y$ of \mathfrak{R}^2 .¹ In this section, we assume that each voter is described by a one-dimensional type

¹This assumption implies that X and Y are compact intervals of the real line for $i = 1, 2$. The rectangularity assumption implies that the choice over one dimension does not have any implication on the feasible choices over the other dimension. A more general case is the subject of Banks and Duggan (2004).

$\theta \in \mathfrak{R}$. The statistical distribution of types is given by a continuous cumulative distribution function F whose support is the interval $[\underline{\theta}, \bar{\theta}]$ of \mathfrak{R} , with f denoting the corresponding density. The utility of a citizen of type θ for policy (x, y) is denoted by $U(\theta, x, y)$ that is assumed to be twice continuously differentiable and such that: $\frac{\partial^2 U(\theta, x, y)}{\partial x^2} < 0$ and $\frac{\partial^2 U(\theta, x, y)}{\partial y^2} < 0$.² Further, we will assume that for all $\theta \in [\underline{\theta}, \bar{\theta}]$, for all $y \in Y$ (respectively $x \in X$), the maximum of $U(\theta, \cdot, y)$ (respectively $U(\theta, x, \cdot)$) is attained in the interior of X (respectively Y).³ The following examples illustrate the broad spectrum of applications covered by this framework.⁴

Example 1 (Absolute Intensity of the Preference for Public Goods)

Let $Z = [0, \bar{x}] \times [0, \bar{y}]$, $\underline{\theta} > 0$ and $U(\theta, x, y) = \theta P(x, y) - (x + y)$ where P is twice continuously differentiable, increasing and such that $\frac{\partial^2 P(x, y)}{\partial x^2} < 0$, $\frac{\partial^2 P(x, y)}{\partial y^2} < 0$, $\frac{\partial P(0, y)}{\partial y} = \frac{\partial P(x, 0)}{\partial x} = \infty$, $\bar{\theta} \frac{\partial P(\bar{x}, y)}{\partial y} < 1$, $\bar{\theta} \frac{\partial P(x, \bar{y})}{\partial x} < 1$ for all $(x, y) \in Z$. In this setting, x and y denote the quantities of two different pure public goods produced under constant returns to scale and financed through per capita taxation. The parameter θ reflects the intensity of the preference for the bundle (x, y) of public goods (aggregated through P) with respect to the private numeraire.

Example 2 (Spatial Politics with Differentiated weights)

Let $Z = [\underline{\theta}, \bar{\theta}]^2$ and $U(\theta, x, y) = -\phi(\theta)(x - \theta)^2 - \psi(\theta)(y - \theta)^2$ where ϕ and ψ are two positive continuously differentiable functions. In this general framework, the parameter θ plays two roles. On one hand, it describes the favorite policy bundle of a citizen regardless of the specific features of ϕ and ψ . On the other hand, it also determines through these functions the respective weights placed by a citizen on the two dimensions. In the particular case where $\phi(\theta) = \psi(\theta) = 1$ for all θ in $[\underline{\theta}, \bar{\theta}]$, we obtain the spatial model of politics with the extra assumption that the support of the distribution is one dimensional (precisely here the diagonal).

Example 3 (Local Jurisdictions, Nation Formation and “One and a Half Dimen-

²These sign conditions imply that, for all θ , $U(\theta, x, y)$ is strictly concave in x for all y and strictly concave in y for all x .

³Most of our analysis extends to settings with corner solutions at the cost of some additional notation which has been avoided here.

⁴The working paper version of this paper (downloadable at http://idei.fr/doc/wp/2010/pdd_kramer_0110.pdf) contains several other examples.

sional” Preferences)

Let $Z = [0, \bar{x}] \times [0, 1]$, $\theta \in [0, 1]$ and $U(\theta, x, y) = v(x)\Psi(y - \theta) - x$ where v is such that $v(x) > 0$, $v'(x) > 0$, $v''(x) < 0$ and $v'(0) = \infty$, $v'(\bar{x})\Psi(0) < 1$, $\Psi(d) > 0$, Ψ is increasing to the left of 0, decreasing to the right of 0 and such that $\Psi''(d) < 0$. In this setting, x denotes the quantity of a pure public good while y now denotes a horizontal characteristic of this public good. This policy environment has been analyzed by many authors, including Alesina, Baqir and Easterly (1999), Alesina, Baqir and Hoxby (2004), Perroni and Scharf (2001) in the analysis of local jurisdictions, and Etro (2006) and Gregorini (2009) in the exploration of models of nation formation. It is also reminiscent of the voting environment of Groseclose (2007) where the horizontal dimension denotes ideology while the other dimension represents valence (defined as an advantage that a candidate has due to a non-policy factor, such as incumbency or charisma). All voters have the same preference on the valence dimension (hence the term “one-and-a-half dimensional” coined by Groseclose, 2007).

We first study the simultaneous voting game over the two dimensions before turning to sequential voting and the Kramer-Shepsle solution.

2.1 Simultaneous Voting

We now show that in the context of simultaneous voting over a bidimensional policy space with unidimensional voters’ types, the fact that utility functions satisfy reasonable “single-crossing” conditions does not guarantee the existence of a Condorcet winner. On the contrary, for any policy proposal, it is always possible to propose an alternative policy that is favored by almost all voters. For most of the paper, we assume the following monotonicity property:

Assumption 1 (Marginal Single-Crossing) *We assume that*

$$\frac{\partial^2 U(\theta, x, y)}{\partial \theta \partial x} > 0 \text{ and } \frac{\partial^2 U(\theta, x, y)}{\partial \theta \partial y} > 0$$

for all $(x, y) \in Z$ and $\theta \in \mathfrak{R}$.

Assumption 1 simply states that the marginal utility of both dimensions increases monotonically with the type of the agent. This monotonicity assumption implies that the classical single-crossing condition (which states that “leftist voters tend to favor left policies more than voters who are rightist in political preferences” (Myerson, 1996, p.23)) is satisfied on each dimension separately, hence the term of marginal single-crossing assumption.

It is easy to see that Assumption 1 is satisfied in Example 1. As for Example 2, we obtain $\frac{\partial^2 U(\theta, x, y)}{\partial \theta \partial x} = 2\phi(\theta) - 2\phi'(\theta)(x - \theta)$. The first term is always positive while the second term can take negative values. It is enough to bound the second term. Assumption 1 holds as soon as $\phi'(\theta)$ is not too large. If we denote by m the minimum of $\phi(\theta)$ over $[\underline{\theta}, \bar{\theta}]$, then it will hold whenever $|\phi'(\theta)| < \frac{m}{\theta - \underline{\theta}}$. The same analysis applies to $\frac{\partial^2 U(\theta, x, y)}{\partial \theta \partial y}$.

Assumption 1 does not hold for Example 3. We obtain $\frac{\partial^2 U(\theta, x, y)}{\partial \theta \partial x} = -v'(x)\Psi'(y - \theta)$ and $\frac{\partial^2 U(\theta, x, y)}{\partial \theta \partial y} = -\Psi''(y - \theta)$. The second-order derivative is always positive but the sign of the first-order derivative depends upon the position of y with respect to θ : its sign is positive if and only if $y > \theta$. This example is thus not covered by the results of this section and, given its importance in the literature, is analyzed separately in section 2.3.

We now introduce this definition.

Definition 1 *Assume people vote over the set Ω . We call $\omega \in \Omega$ a majority (voting) equilibrium, also called a Condorcet winner, if there is no $\omega' \neq \omega$ with $\omega' \in \Omega$ that is strictly preferred by more than one half of the voters to ω .*

We denote by $x(y, \theta)$ (respectively, $y(x, \theta)$) individual θ 's most-preferred value of x (resp., of y) for any given y (resp., given x). The following lemma (proved in Appendix 1) shows that the strict concavity of the utility function guarantees both the existence and unicity of a majority winner when voting over x for any given y (resp., over y for any x).⁵ Moreover, if Assumption 1 holds, Lemma 1 shows that the most-preferred value of x (respectively, of y) is increasing in θ , for any given y (resp., given x), and that the individual with the (unique) median type θ_{med} is decisive in both choices if they are taken separately.

⁵The elementary proof consists in showing that the CDF of marginal peaks has no flat sections.

Lemma 1 Let $U(\theta, x, y)$ be twice continuously differentiable and such that: $\frac{\partial^2 U(\theta, x, y)}{\partial x^2} < 0$ and $\frac{\partial^2 U(\theta, x, y)}{\partial y^2} < 0$. Then:

i) For all y (respectively x), there exists a unique (one-dimensional) Condorcet winner, which we denote by $x_m(y)$ (resp., $y_m(x)$).

ii) Under Assumption 1, the most-preferred value of x (respectively, of y) is increasing in θ , for any given y (resp., given x):

$$\frac{\partial x(y, \theta)}{\partial \theta} > 0 \text{ and } \frac{\partial y(x, \theta)}{\partial \theta} > 0.$$

Moreover, $x_m(y)$ (resp., $y_m(x)$) corresponds to the value of x (resp., of y) that is most-preferred by the individual with the median type, θ_{med} :

$$\begin{aligned} x_m(y) &= x(y, \theta_{med}) \quad \forall y \in \mathfrak{R}, \\ y_m(x) &= y(x, \theta_{med}) \quad \forall x \in \mathfrak{R}. \end{aligned}$$

Observe that we have imposed the strict concavity of the utility function separately with respect x and to y , but not with respect to (x, y) . We then introduce the following definition.

Definition 2 We call (x^*, y^*) a componentwise ideal point of an individual with type θ if $\underset{x \in X}{\text{ArgMax}} U(\theta, x, y^*) = x^*$ and $\underset{y \in Y}{\text{ArgMax}} U(\theta, x^*, y) = y^*$.

The following proposition shows that, if a Condorcet winner exists when voting simultaneously over the two dimensions, then it must be a componentwise ideal point of an individual with the median type, θ_{med} .⁶

Proposition 1 Consider the bidimensional majority voting setting with a unidimensional type space where Assumption 1 is satisfied. Then, the majority equilibrium (x^*, y^*) under simultaneous voting over both dimensions, if it exists, must be a componentwise ideal point of the median type voter θ_{med} :

$$x^* = x(y^*, \theta_{med}) \text{ and } y^* = y(x^*, \theta_{med}).$$

⁶An ideal point of an individual of type θ -i.e., a choice (x^*, y^*) such that $(x^*, y^*) = \underset{(x, y) \in X \times Y}{\text{ArgMax}} U(\theta, x, y)$, is of course a componentwise ideal point for such an individual. But the converse is not true, as we show in an example available upon request.

We now investigate under what conditions a vector (x^s, y^s) (called the status quo hereafter) is preferred by a majority of voters to any local deviation. We establish the conditions under which an individual votes in favor of a motion moving away from the status quo in a (arbitrary) direction $d = (d_x, d_y) \in \mathfrak{R}^2$. The change in the utility of a voter of type θ induced by d is⁷

$$\varphi(\theta) \equiv \frac{\partial U(\theta, (x^s, y^s))}{\partial x} d_x + \frac{\partial U(\theta, (x^s, y^s))}{\partial y} d_y.$$

The population of voters who favor a move from the status quo in the direction d is composed of all the types for which $\varphi(\theta) > 0$. A local Condorcet winner is defined as a policy pair (x^s, y^s) for which there exists an $\varepsilon > 0$ such that for any vector $(d_x, d_y) \in \mathfrak{R}^2$ belonging to the unitary circle, the mass of citizens who strictly prefer $(x^s + \varepsilon d_x, y^s + \varepsilon d_y)$ to (x^s, y^s) is less than or at most equal to $\frac{1}{2}$. We introduce the function

$$\Phi(d) = \int_{\{\theta \in [\underline{\theta}, \bar{\theta}]: \varphi(\theta) > 0\}} dF,$$

which measures the proportion of voters favoring a deviation in direction d from the status quo (x^s, y^s) . We show in Appendix 2 that if (x^s, y^s) is a local Condorcet winner, then $\Phi(d) \leq \frac{1}{2}$ for all d . This implies that, to check that a policy pair is a local Condorcet winner, it is sufficient to look at the sign of the function φ .

Observe that $\varphi(\theta_{med}) = 0$ since (x^*, y^*) is a componentwise ideal point of individual θ_{med} . From $\varphi'(\theta) \equiv \frac{\partial^2 U(\theta, x^*, y^*)}{\partial \theta \partial x} d_x + \frac{\partial^2 U(\theta, x^*, y^*)}{\partial \theta \partial y} d_y$, using Assumption 1, we obtain that $\varphi'(\theta) > 0$ if $d_x > 0$ and $d_y > 0$, which means that all individuals with $\theta > \theta_{med}$ are in favor of directions d with positive deviations from the status quo. By definition, this interval of types represents one half of the polity, so that $\Phi(d) = 1/2$. Similarly, $\varphi'(\theta) < 0$ if $d_x < 0$ and $d_y < 0$, so that all individuals with $\theta < \theta_{med}$ (and only them) favor the direction d , and $\Phi(d) = 1/2$. In words, if the deviation considered either increases or decreases both dimensions, then the individuals favoring this deviation are to be found only on one side of the median and are thus not numerous enough to defeat the status quo.

⁷The function φ also depends on d and on the status quo, but we simplify the notation by writing $\varphi(\theta)$. It is a first-order approximation of the change in utility - see Appendix 2 for the full statement.

We now turn to deviations with both a positive and a negative component. Individuals with $\theta > \theta_{med}$ benefit from the positive component of the deviation but suffer from the negative component, and vice versa for the individuals with $\theta < \theta_{med}$. The set of voters who favor such a deviation may then be disjoint and could comprise both people above and below θ_{med} . We now characterize this set and study whether it represents more than one half of the electorate.

Consider without loss of generality the case where d is such that $d_x > 0$ and $d_y < 0$. Recall that voters who favor a direction d are such that $\varphi(\theta) > 0$. Denoting by

$$MRS(\theta) = \frac{\frac{\partial U(\theta, x^*, y^*)}{\partial x}}{\frac{\partial U(\theta, x^*, y^*)}{\partial y}}$$

the (absolute value of)⁸ the marginal rate of substitution between x and y at (x^*, y^*) for individual θ , we obtain that voters who favor the direction d are such that $\theta > \theta_{med}$ together with $MRS(\theta) > -d_y/d_x$ (i.e., those for whom the utility gain from a larger value of x is larger than the utility loss from the lower value of y), or such that $\theta < \theta_{med}$ together with $MRS(\theta) < -d_y/d_x$ (i.e., those for whom the utility gain from a smaller value of y is larger than the utility loss from the larger value of x). The identification of the coalition of citizens $\Theta(d_x, d_y)$ supporting the deviation is illustrated on Figure 1 below, where we represent the MRS measured at (x^*, y^*) as a function of θ . It is important to note that this coalition need not be connected.

Insert Figure 1 about here

The construction itself shows that the circumstances for having (x^*, y^*) undefeated are very exceptional. Indeed, given the choice of $-d_y/d_x$, if the set $\{\theta \in [\underline{\theta}, \bar{\theta}] : MRS(\theta) = -d_y/d_x\}$ has measure 0 for F , then it must be the case that the coalition $\Theta(d_x, d_y)$ and its complement $[\underline{\theta}, \bar{\theta}] \setminus \Theta(d_x, d_y)$ have both a measure equal to $\frac{1}{2}$ with respect to F for the policy (x^*, y^*) to be

⁸Note that, under Assumption 1, the marginal rate of substitution at (x^*, y^*) is well defined for all $\theta \neq \theta_{med}$. Further, it is negative for all individuals since $\partial U(\theta, x^*, y^*)/\partial x > 0$ and $\partial U(\theta, x^*, y^*)/\partial y > 0$ for all $\theta > \theta_{med}$ while $\partial U(\theta, x^*, y^*)/\partial x < 0$ and $\partial U(\theta, x^*, y^*)/\partial y < 0$ for all $\theta < \theta_{med}$. Slightly abusing notation, we denote by $MRS(\theta_{med})$ the limit, as θ tends towards θ_{med} , of $MRS(\theta)$. From Assumption 1 and l'Hôpital's rule, it is easy to see that this limit exists.

a local Condorcet winner. This may happen for some specific value of $-d_y/d_x$ but then a small perturbation of d_y/d_x is likely to destroy this property.⁹

We then impose further structure on the problem in the hope of finding circumstances under which a local Condorcet winner exists. An interesting benchmark, often used in the political economy literature, is the case where the utility function exhibits the single-crossing or Spence-Mirrlees's condition (Gans and Smart 1996, Greenberg and Weber 1986, Rothstein 1990)—i.e., where the marginal rate of substitution is monotone¹⁰ in θ :

Assumption 2 (Local Joint Single-Crossing) *Let (x^*, y^*) be a componentwise ideal point of the median type. We say that U satisfies the property of local joint single-crossing with respect to (x^*, y^*) if*

$$\frac{\partial U(\theta, x^*, y^*)/\partial x}{\partial U(\theta, x^*, y^*)/\partial y} \text{ is strictly increasing in } \theta$$

for all $\theta \in \mathfrak{R}$.

We then obtain the following result.

Proposition 2 *In the bidimensional majority voting setting with a unidimensional type space, let (x^*, y^*) be a componentwise ideal point of a voter with median type. Under Assumptions 1 (marginal single-crossing) and 2 (local joint single-crossing), then :*

- a) *The policy bundle (x^*, y^*) is defeated at the majority by almost every deviation d such that $d_x d_y < 0$.*
- b) *Moreover, there exists a deviation $\tilde{d} = (\tilde{d}_x, \tilde{d}_y)$ with $\tilde{d}_x \tilde{d}_y < 0$ that is preferred by all voters (except θ_{med}) to (x^*, y^*) .*

In order to prove Proposition 2, we use Figure 2, where we make use of Assumption 2. The first panel depicts the case where $MRS(\theta_{med}) < -d_y/d_x$. In that case, all individuals below

⁹This reasoning does not hold when $MRS(\theta)$ is constant since, for any given directional deviation, the society is always divided equally.

¹⁰The subsequent analysis would carry through to the case where the MRS is monotone decreasing in type. Note that there is no logical connection between the two single-crossing conditions that we study (Assumptions 1 and 2) in the sense that neither implies nor precludes the other.

θ_{med} prefer the deviation. This is also the case for individuals with $\theta > \theta_{med}$ who are such that $MRS(\theta) > -d_y/d_x$. A strict majority favors d if this second group is not empty, which is the case provided that $MRS(\bar{\theta}) > -d_y/d_x$ —i.e., that d_y is not too large or d_x not too small (in absolute values). Figure 2(b) illustrates the case where $MRS(\theta_{med}) > -d_y/d_x$. In that case, all people with $\theta > \theta_{med}$ favor the deviation, together with individuals with $\theta < \theta_{med}$ for which $MRS(\theta) < -d_y/d_x$. As soon as this second group is not empty (which is the case if $MRS(\underline{\theta}) < -d_y/d_x$ —i.e., that d_y is not too small or d_x not too large, in absolute values), a strict majority of voters favor the deviation. This proves Proposition 2 (a).

The third panel of Figure 2 shows that, if the deviation \tilde{d} is such that $MRS(\theta_{med}) = -d_y/d_x$, all voters (except of course θ_{med}) favor this deviation, proving part b) of Proposition 2.

Insert Figure 2 about here

While the reader may not be surprised by part a) of Proposition 2, part b) is more surprising, since in that case there is a unanimity against the median voter's most-preferred policy, even under marginal and joint single-crossing conditions.

In Example 2, additional assumptions on $\phi(\theta)$ and $\psi(\theta)$ are necessary to ensure that U is strictly concave and satisfies the property of local joint single-crossing with respect to the unique ideal point (x^*, y^*) of the median type, since

$$\frac{\partial U(\theta, x^*, y^*)/\partial x}{\partial U(\theta, x^*, y^*)/\partial y} = \frac{\phi(\theta) \theta_{med} - \theta}{\psi(\theta) \theta_{med} - \theta} = \frac{\phi(\theta)}{\psi(\theta)}.$$

For instance, in the case where $[\underline{\theta}, \bar{\theta}] = [0, 1]$ and $\psi(\theta) = 1$, the property will be satisfied whenever $\phi'(\theta) > 0$. The following slight variant of Example 2 provides another illustration. Let $U(\theta, x, y) = -(x - \cos \theta)^2 - (y - \sin \theta)^2$ with $\theta \in [\frac{3\pi}{2}, 2\pi]$. We obtain that

$$\frac{\partial U(\theta, x, y)}{\partial x} = -2(x - \cos \theta) \text{ and } \frac{\partial U(\theta, x, y)}{\partial y} = -2(y - \sin \theta).$$

Since $\frac{\partial^2 U(\theta, x, y)}{\partial x \partial \theta} = -2 \sin \theta > 0$ and $\frac{\partial^2 U(\theta, x, y)}{\partial y \partial \theta} = 2 \cos \theta > 0$, Assumption 1 is satisfied. If F is uniform on $[\frac{3\pi}{2}, 2\pi]$, we have $\theta_{med} = \frac{7\pi}{4}$ and $(x^*, y^*) = (\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2})$. Moreover,

$$MRS(\theta) = MRS(\theta, x^*, y^*) = \frac{\cos \frac{7\pi}{4} - \cos \theta}{\sin \frac{7\pi}{4} - \sin \theta}. \quad (1)$$

We then deduce that $MRS'(\theta) = \frac{\sin \theta}{-\sin \theta - \frac{1}{2}\sqrt{2}} + (\cos \theta) \frac{\frac{1}{2}\sqrt{2} - \cos \theta}{(-\sin \theta - \frac{1}{2}\sqrt{2})^2}$. A careful analysis shows that $MRS'(\theta) < 0$ over the relevant range of values of θ i.e. up to a sign reversal, U satisfies the property of local joint single-crossing with respect to the unique ideal point (x^*, y^*) of the median type in that variant of Example 2. Any motion from (x^*, y^*) in the direction $(-1, 1)$ will be supported by almost all agents.¹¹ Indeed, from (1), using L'Hospital's rule we obtain

$$MRS\left(\frac{7\pi}{4}\right) = \frac{g'\left(\frac{7\pi}{4}\right)}{h'\left(\frac{7\pi}{4}\right)},$$

where $g(\theta) \equiv \cos \frac{7\pi}{4} - \cos \theta$ and $h(\theta) = \sin \frac{7\pi}{4} - \sin \theta$. Since $g'\left(\frac{7\pi}{4}\right) = -\sin \frac{7\pi}{4} = \frac{1}{2}\sqrt{2}$ and $h'\left(\frac{7\pi}{4}\right) = -\cos \frac{7\pi}{4} = -\frac{1}{2}\sqrt{2}$, we obtain that $MRS\left(\frac{7\pi}{4}\right) = -1$. If we want to work back in terms of normalized gradients, we get the vector $(-1, 1)$ as we need to multiply both $g'\left(\frac{7\pi}{4}\right)$ and $h'\left(\frac{7\pi}{4}\right)$ by -1 .

The take home message of this section is then that, except in very peculiar circumstances such as a perfectly symmetrical utility function, there is little hope of finding a Condorcet winner when voting simultaneously over the two dimensions, even when the type space is unidimensional and single-crossing conditions are satisfied.

We now move to the other equilibrium concepts studied in this paper, those proposed by Kramer and Shepsle, and by Stackelberg.

2.2 Kramer-Shepsle and Stackelberg equilibria

Let us examine first the Kramer-Shepsle equilibria. We first prove existence of such equilibria for a class of problems much larger than the class of problems considered in the previous section. Let \mathcal{U} be the class of utility functions U defined on Z such that $U_y^1 = U(., y)$ (respectively $U_x^2 = U(x, .)$) is strictly concave on X (respectively Y) for all y (respectively for all x). A profile is a mapping \mathbf{U} from $[\underline{\theta}, \bar{\theta}]$ into \mathcal{U} . We denote by R^1 (respectively R^2) the set of weak orders on X (respectively Y) induced by strictly concave utility functions on X (respectively Y). Given \mathbf{U} and $(x, y) \in Z$, we denote by $M^1(y)$ (respectively $M^2(x)$) the set of Condorcet

¹¹We thank the associate editor for pointing this out.

winners on the first dimension (respectively on the second dimension) when the choice on the second dimension (respectively the first dimension) is y (respectively x).¹²

Definition 3 *Given a profile \mathbf{U} , a Kramer-Shepsle (or KS) equilibrium for \mathbf{U} is a policy vector (x^{KS}, y^{KS}) such that*

$$x^{KS} \in M^1(y^{KS}) \text{ and } y^{KS} \in M^2(x^{KS}).$$

We prove in Appendix 3 that KS equilibria always exist for any profile \mathbf{U} satisfying the above assumptions.

Under Assumption 1, the correspondence M is a function: $M^1(y) = x_m(y) = x(y, \theta_{med})$ and $M^2(x) = y_m(x) = y(x, \theta_{med})$. This implies that (x, y) is a KS equilibrium iff (x, y) is a componentwise ideal point of a median type described by the following first order conditions:

$$\frac{\partial U(\theta_{med}, x^{KS}, y^{KS})}{\partial x} = 0 \text{ and } \frac{\partial U(\theta_{med}, x^{KS}, y^{KS})}{\partial y} = 0. \quad (2)$$

We have thus proved the following.

Proposition 3 *In the bidimensional majority voting setting with a unidimensional type space, under Assumption 1, any KS equilibrium (x^{KS}, y^{KS}) coincides with a componentwise ideal point of the θ_{med} type voter.*

Let us now move to the set of Stackelberg (or ST) equilibria that arise when there is a sequence of two votes. We assume (without loss of generality at this stage) that individuals are first called to vote over x and then, after having observed the voting outcome of this first round, that they vote over y . We solve for these ST equilibria and compare them with both the KS equilibria and with the ST equilibria under the opposite sequence (where voters choose first y and then x).

Solving backward, we know from Lemma 1 (i) that, for any outcome x in the first stage, there exists a unique majority equilibrium $y_m(x)$ in the second stage. From Lemma 1 (ii), we

¹²Under Assumption 1, these sets are singletons but in general it is not necessarily the case. However it is straightforward that both of them are non empty intervals.

know that under Assumption 1, $y_m(x)$ is the most-preferred value of y of the median type θ_{med} . In any case, this implies that, in the first stage, the reduced utility of a citizen of type θ for x is equal to

$$V(\theta, x) = U(\theta, x, y_m(x)).$$

Definition 4 *A Stackelberg (or ST) equilibrium when voters choose first x and then y is a policy vector (x^{ST}, y^{ST}) such that*

$$\int_{\{\theta \in [\underline{\theta}, \bar{\theta}]: V(\theta, x) - V(\theta, x^{ST}) > 0\}} f(\theta) d\theta \leq \frac{1}{2} \text{ for all } x \in \mathbb{R}$$

and $y^{ST} = y_m(x^{ST})$.

Of course, the first part of the definition of ST is not easy to test in general. Under the presumption that the function $y_m(x)$ is differentiable, the marginal first-stage utility of a citizen of type θ is given by

$$\frac{\partial V(\theta, x)}{\partial x} = \frac{\partial U(\theta, x, y_m(x))}{\partial x} + \frac{\partial U(\theta, x, y_m(x))}{\partial y} \frac{dy_m(x)}{dx} = 0. \quad (3)$$

The first term of (3) describes the direct effect of varying x on the individual's utility, while the second term describes the indirect effect through variations in the second-stage voting outcome.

We will make an extensive use of the following assumption:

Assumption 3 (Strategic complementarity) *We assume that the two policy dimensions are strategic complements:*

$$\frac{\partial^2 U(\theta, x, y)}{\partial x \partial y} \geq 0. \quad (4)$$

From this assumption, we deduce the following proposition.

Proposition 4 *If the function $U(\theta, x, y)$ is strictly concave in (x, y) for all θ and satisfies Assumptions 1 (marginal single-crossing) and 3 (strategic complementarity), we have (i) $dy_m(x)/dx \geq 0$, (ii) $dx_m(y)/dy \geq 0$ and (iii) $\frac{\partial^2 V(\theta, x)}{\partial x \partial \theta} > 0$. Moreover, there exists a unique ST equilibrium which coincides with the (unique) KS equilibrium.*

Proof. Under Assumption 1, $y_m(x)$ is the unique solution of the equation $\frac{\partial U(\theta_{med}, x, y)}{\partial y} = 0$. From our assumption that $\frac{\partial^2 U(\theta_{med}, x, y_m(x))}{\partial y^2} < 0$, we deduce from the implicit function theorem that y_m is differentiable, with

$$\frac{dy_m(x)}{dx} = -\frac{\frac{\partial^2 U(\theta_{med}, x, y_m(x))}{\partial x \partial y}}{\frac{\partial^2 U(\theta_{med}, x, y_m(x))}{\partial y^2}} \geq 0 \quad (5)$$

from Assumption 3. The proof of (ii) is obtained similarly.

From differentiating (3) and Assumptions 1 and 3, we deduce that $\frac{\partial^2 V(\theta, x)}{\partial x \partial \theta} > 0$. This implies that V is (strictly) single-crossing. Therefore, from Gans and Smart (1996) and Rothstein (1990), we deduce that V admits a majority equilibrium. We offer a direct proof by showing that this majority equilibrium coincides with $x^F(\theta_{med})$. To do so consider θ_{med} and $x^F(\theta_{med})$ the (unique) global peak of $V(\theta_{med}, x)$.¹³ For any θ , we define

$$\Psi(\theta, x) \equiv U(\theta, x^F(\theta_{med}), y_m(x^F(\theta_{med}))) - U(\theta, x, y_m(x)).$$

For any $x < x^F(\theta_{med})$, we can write

$$\Psi(\theta, x) = \int_x^{x^F(\theta_{med})} \frac{\partial V(\theta, t)}{\partial x} dt > 0.$$

Assuming that $\theta > \theta_{med}$, we further obtain

$$\Psi(\theta, x) - \Psi(\theta_{med}, x) = \int_{\theta_{med}}^{\theta} \int_x^{x^F(\theta_{med})} \frac{\partial^2 V(\hat{\theta}, t)}{\partial x \partial \theta} dt d\hat{\theta} > 0.$$

We deduce that $\Psi(\theta, x) > 0$ and therefore that a strict majority of voters prefer $x^F(\theta_{med})$ to x . A similar argument holds for any $x > x^F(\theta_{med})$. Hence, $x^F(\theta_{med})$ cannot be defeated by a majority, since any majority against $x^F(\theta_{med})$ would have to consist in part of agents $\theta > \theta_{med}$, a contradiction. ■

The intuition behind Proposition 4 is pretty simple. Assumptions 1 and 3 together ensure that the reduced utility V is single-crossing. The median voter then anticipates in the first stage

¹³If U is strictly concave, then for any θ , $U(\theta, x, y)$ has a unique (global) peak. By definition of $y_m(x)$, we deduce that $x^F(\theta_{med})$ is the unique (global) peak of $V(\theta_{med}, x)$.

that he will remain decisive in the second stage as well. In his first-stage choice of x , he then ignores (by an envelope theorem argument) the indirect effect of x on his utility, and chooses the optimal value of x given the value of y that will result in the second stage. The resulting policy bundle $(x^F(\theta_{med}), y_m(x^F(\theta_{med})))$ constitutes the unique Stackelberg equilibrium.

In Proposition 4, we have not assumed that V is single-peaked. If we had, denoting by $x^F(\theta)$ the peak of $V(\theta, x)$, we would deduce from the argument used in the last part of the proof of Proposition 4 that x^F is strictly increasing.

It is well known that single-crossing and single-peakedness are two logically independent properties. The following proposition states that under some additional assumptions on U , at least a majority of the electorate has single-peaked indirect utility functions.

Proposition 5 *In the bidimensional majority voting setting with a unidimensional type space, if the function $U(\theta, x, y)$ is strictly concave in (x, y) for all θ and satisfies Assumptions 1 (marginal single-crossing) and 3 (strategic complementarity) and if $\frac{\partial^3 U(\theta_{med}, x, y)}{\partial x^2 \partial y} < 0$, $\frac{\partial^3 U(\theta_{med}, x, y_m(x))}{\partial x \partial y^2} < 0$ and $\frac{\partial^3 U(\theta_{med}, x, y_m(x))}{\partial y^3} < 0$ then $V(\theta, x)$ is single-peaked in x for all $\theta \geq \theta_{med}$.*

These conditions on preferences involve the third derivatives of U . Note that these assumptions are satisfied in Example 1 if, for instance $P(x, y) = \psi(x)\psi(y)$ with $\psi > 0$, $\psi' > 0$, $\psi'' < 0$, $\psi''' < 0$ and with $\psi''\psi/(\psi')^2 < -1$ to ensure that U is concave in (x, y) .¹⁴

Propositions 4 and 5 assume that both dimensions are strategic complements. If dimensions are not strategic complements, then the argument used to prove that V is single-crossing does not hold and the issue of existence of a ST equilibrium arises. Moreover, even if we assume that V is single-peaked, the most-preferred first-stage value of x need not be monotone in θ anymore. In that case, it is necessary to consider the decreasing rearrangement \tilde{x} of x (as in the proof of part (i) of Lemma 1). Then the median outcome x_{med} is the solution to the equation

$$F(\theta : \tilde{x}(\theta) \leq x_{med}) = \frac{1}{2},$$

¹⁴Note that, although the assumptions $\Psi' > 0$, $\Psi'' < 0$ and $\Psi''' < 0$ can not be simultaneously met on the whole real line, the domain over which they are satisfied may be arbitrarily large. A polynomial example is available upon request from the authors.

and in general $x_{med} \neq x(\theta_{med})$.

To go beyond these general cases, we need to put more structure on the utility function. In the next section, we focus on a family of utility functions that has been studied at length, for instance in the nation formation literature.

2.3 One-sided Separability

In this subsection, we focus on the environment described in Example 3, which has received a great deal of attention in different fields. This setting is characterized by both a horizontal and a vertical dimension. As already pointed out, Assumption 1 (marginal single-crossing) is not satisfied so that this setting calls for a separate tailored treatment.

Let us assume that $\theta \in [0, 1]$ and

$$U(\theta, x, y) = v(x)\Psi(y - \theta) - x \text{ where } x \in \mathfrak{R}_+ \text{ and } y \in [0, 1]. \quad (6)$$

We assume that v is increasing and strictly concave, and such that $v'(0) = \infty$ and $v'(x)\Psi(0) < 1$ for x large enough and that Ψ is a function with values in \mathfrak{R}_{++} , symmetrical with respect to 0 and increasing to the left of 0.¹⁵ We also assume that the function Ψ is differentiable everywhere, so that $\Psi'(0) = 0$.¹⁶ This general form describes the situation of a public policy program with a vertical dimension x (the quantity or quality level of a public good) and a horizontal dimension y (a characteristic of the public good, such as its color, location,...). The type θ of a voter represents her most-preferred public good variant y among all feasible options: any departure from this ideal choice decreases her utility for any value of x . Also, for any fixed type of public good y , each voter derives a gross benefit from this public good consumption which increases with x . We assume that the unit cost of production of the public good is one, that there is a mass one of consumers, and that public provision is financed with a lump sum

¹⁵Therefore, it is decreasing to the right of 0. Alesina, Baqir and Easterly (1999), Etro (2006) and Gregorini (2009) consider the specific case where $\Psi(y, \theta) = \lambda - |\theta - y|$ where λ is a parameter larger than 1.

¹⁶This differentiability assumption is not necessary for our arguments but allows to significantly simplify some proofs.

tax. We thus have to subtract x from the gross utility to obtain the net utility of the public good. Note that we consider here a setting slightly more general than the one described as Example 3. The function $U(\cdot)$ is assumed to be strictly concave in x but not necessarily in y , as we make no concavity assumption on the function $\Psi(\cdot)$.

We start by looking at the ST procedure where citizens vote first over x and then over y . This is the sequence the jurisdiction and nation formation literature have focused on. Note first that the majority choice over y does not depend upon x , while the converse is not true, as an individual's willingness to pay for the public good depends on its location. We dub this property *one-sided separability*. Whatever the value of x , the majority choice over y , which we denote by y_{med} , is given by

$$y_m(x) = y(x, \theta_{med}) = y_{med} = \theta_{med}.$$

Given y_{med} , the reduced utility function takes the form

$$V(\theta, x) = v(x)\Psi(y_{med} - \theta) - x.$$

Given our assumptions on v and Ψ , V is a concave function of x with a peak at $x(\theta)$ where $x(\theta)$ is the unique solution x of the equation

$$v'(x)\Psi(y_{med} - \theta) = 1, \tag{7}$$

which is the familiar rule equating the marginal utility from the public good to its marginal taxation cost for individual θ . It is clear that this peak decreases continuously as θ moves away from θ_{med} , both to the left and to the right of θ_{med} . As the function $V(\cdot)$ is concave in x , we can apply the median voter theorem and assert that there exists a majority equilibrium value of x , which corresponds to the median most-preferred value of x when $y = y_{med}$. As should be obvious from (7), this decisive individual is *not* the individual with the median location θ_{med} , since this individual is the one with the largest willingness to pay for the public good, but rather the individual with the median distance to the median (i.e., the median value of $|y_{med} - \theta|$, since the function $\Psi(\cdot)$ is symmetrical around zero). We explain in Appendix 5 how to solve for the median optimal value of x , which we denote by x_{med} .

From the above arguments, we deduce that (x_{med}, y_{med}) is the unique ST equilibrium when voting first over x and then over y . It is also clear that this policy pair is the unique KS equilibrium as well, since $y_m(x) = y_{med}$ whatever the value of x . We thus have the following Proposition.

Proposition 6 *Given the utility function (6), the policy (x_{med}, y_{med}) is the (unique) Kramer-Shepsle equilibrium and it coincides with the Stackelberg equilibrium when people vote first over x and then over y .*

We now study the Stackelberg equilibrium when we reverse the vote sequence. Given an arbitrary value of y from the first vote, consider the second stage of the game—i.e., the vote over x . Since the utility function (6) is concave in x , we can apply the median voter theorem to learn that the majority-chosen x is the median most-preferred value of x given y . The most-preferred value of x of individual θ given y is

$$x(y, \theta) = (v')^{-1} \left(\frac{1}{\Psi(y - \theta)} \right),$$

which is symmetrical in θ around y , and decreasing as θ moves away from y . Assume without loss of generality that $y \leq y_{med}$. Two cases can materialize. In the first one, the decisive voters are the individuals located at a distance δ from y (to the left or to the right) and such that

$$F(y + \delta) - F(y - \delta) = \frac{1}{2}, \tag{8}$$

i.e., such that exactly 50% of the polity is located at a distance at most equal to δ from y (and thus prefer a larger value of x than $x(y, y \pm \delta)$). Note that equation (8) has a solution provided that y is such that $F(2y) \geq 1/2$. In words, the majority-chosen value of y must not be too far from the median (too small if we start with $y \leq y_{med}$ as assumed here, or too large if we had rather started with $y \geq y_{med}$). If y is far enough from y_{med} , then the decisive voter is the one with the median location, θ_{med} , with all the voters with $\theta < \theta_{med}$ preferring a larger (resp., lower) value of x than $x(y, \theta_{med})$ if $y \leq y_{med}$ (resp., if $y > y_{med}$) and all voters with $\theta > \theta_{med}$ preferring a lower (resp. larger) value of x if $y \leq y_{med}$ (resp., if $y > y_{med}$).

This shows that the identity of the decisive voter(s) in the second stage changes continuously with the choice made in the first stage. In terms of policy, this implies that

$$x_m(y) = \begin{cases} (v')^{-1} \left(\frac{1}{\Psi(y-\theta_{med})} \right) & \text{if } y \leq y^*, \\ (v')^{-1} \left(\frac{1}{\Psi(\delta(y))} \right) & \text{if } y^* \leq y \leq y^{**}, \\ (v')^{-1} \left(\frac{1}{\Psi(y-\theta_{med})} \right) & \text{if } y \geq y^{**}, \end{cases}$$

where (with an abuse of notation) y^* is the unique solution to the equation $F(2y) = 1/2$, y^{**} is the unique solution to the equation $F(2y - 1) = 1/2$ and $\delta(y)$ is given by (8).

Figure 3 depicts the case where F is uniform. Panel (a) shows that $\delta(y)$ is defined only when y is at most distant of $1/4$ from the median value of y , and is constant when it exists. If y is lower than $1/4$ or larger than $3/4$, the decisive voter in the choice of x is $1/2$, as shown in panel (b). For intermediate values of y , there are actually two types of decisive voters (panel b), both distant of $1/4$ from y (panel (a)). Panel (c) shows the majority-chosen value of x for any given y , $x_m(y)$: it first increases with y (since the decisive voter remains the same, while his distance from the chosen y decreases), then it is constant with y (even though the identity of the decisive voters changes with y , they all remain at the same distance from the chosen y), and finally decreases with y (as the distance between the decisive voter, located at $1/2$, and y increases).

Insert Figure 3 about here

The previous analysis shows that in the second stage, the decisive voter type changes continuously with the choice made in the first stage. Moving backward to the first stage voting over y , we assume that the indirect utility function of a citizen of type θ , which is given by $W(\theta, y) = v(x_m(y))\Psi(y - \theta) - x_m(y)$, is single-peaked in y for all θ . Proposition 7 (proved in Appendix 6) shows that individuals have no incentive to vote for $y = \theta$ in the first stage. Strategic considerations related to the second-stage choice of x drive them to vote for a value of x that differs in a systematic way from θ .

Proposition 7 *Given the utility function (6), if $W(\theta, y)$ is single-peaked in y for all θ , voting first over y and then over x , in the first stage:*

Voters of type $\theta < y^$ (with $F(2y^*) = 1/2$) always vote for a value of y larger than their peak θ ;*

*Voters of type $\theta > y^{**}$ (where $F(2y^{**} - 1) = 1/2$) always vote for a value of y smaller than their peak θ .*

Voters of type $y^ \leq \theta \leq y^{**}$ always vote for a value of y larger (resp., smaller) than their peak θ if $\delta(\theta)$ decreases (resp., increases) with θ . The sign of the derivative of $\delta(\theta)$ with respect to θ only depends upon the distribution function F .*

The intuition runs as follows. Individuals know that, if they obtain their “naive” most-preferred location $y = \theta$ in the first-stage, the majority-chosen public good level x will be much lower than their most-favored level, because they will be the ones with the largest willingness to pay for the public good. A small departure from $y = \theta$ then has a second-order direct cost (because, although less appealing, the location remains close to their first-best choice) but a first-order gain, provided that this departure leads to a larger amount of public good in the second stage. Voters whose peak is to the left of y^* anticipate that a first-stage choice close to their peak will result in the median voter θ_{med} being decisive in the second stage. A value of y slightly larger than θ will then induce a larger second-stage value of x , as it increases the willingness to pay for the public good of the θ_{med} individual (since it decreases the distance between the first-stage location choice and his most-preferred location). A similar reasoning explains why individuals located to the right of y^{**} always prefer a value of y that is smaller than their first-best choice θ . Individuals with intermediate preferences ($y^* \leq \theta \leq y^{**}$) anticipate that voters located at a distance $\delta(y)$ from y will be decisive in the second stage. They then bias their first-stage choice in order to decrease this distance, so that the decisive voter increases his most-preferred public good amount. We show in Appendix 6 that the distance $\delta(y)$ is a function of the distribution function F only.

Finally, while restrictive, the assumption that $W(\theta, y)$ is single-peaked in y for all θ is not

vacuous, as we show in an example in Appendix 6.

From Proposition 7, we gather that the first-stage, most-preferred values of x need not be monotone in θ (once strategic considerations are taken into account), so that the individual with the median type θ_{med} need not be the decisive voter. A more precise assessment of the identity of the first-stage median voter would necessitate the introduction of functional forms for the utility function Ψ and for the distribution function F . Observe that, in the special case where F is uniform as in the illustration above, the distance $\delta(y)$ is a constant (see Figure 3) so that individuals located between $1/4$ and $3/4$ have no incentive to distort their first-period choice and vote for $y = \theta$. The decisive individual in the first stage is then θ_{med} , and the first-stage choice of location is one half. In that special case, the KS equilibrium is also the ST equilibrium for both voting sequences.

3 Two-Dimensional Types

In this section, we move to the situation where the type of a voter is two-dimensional. The statistical distribution of types $\theta = (\theta_1, \theta_2)$ among the voters is now described by a continuous (i.e. absolutely continuous with respect to the Lebesgue measure on \mathfrak{R}^2) cumulative distribution function F whose support is (a subset of) \mathfrak{R}^2 ; we denote by f the corresponding density. The utility of a citizen of type θ for policy (x, y) remains denoted by $U(\theta, x, y)$, which is assumed to be twice continuously differentiable and concave in (x, y) .

We skip the analysis of the simultaneous voting setting as, in contrast to the one-dimensional case, it is very similar to what is done in theoretical political science (Banks and Austen-Smith (1999), Ordeshook (1986)). Instead, we focus on the analysis of the sets of Kramer-Shepsle and Stackelberg equilibria. A new phenomenon appears. In contrast to the one-dimensional type setting where the Kramer-Shepsle equilibrium was unique as soon as the voters' utility functions were strictly concave in both variables (see Proposition 3), in the two-dimensional type setting there may exist several KS equilibria.¹⁷

¹⁷The working paper version of this article shows that there always exists at least one KS equilibrium when

There is little we can say at this level of generality about the existence or characteristics of the Stackelberg equilibria, and their relationship with the KS equilibria. In this section, we content ourselves with providing an example (using the spatial model with quadratic preferences most often studied in the formal political science literature, such as in Banks and Austen-Smith (1999) or Ordeshook (1986)) illustrating that i) we may have multiple KS equilibria, ii) KS equilibria need not be Stackelberg equilibria and iii) KS equilibria need not correspond to any voter's most-preferred policy.

In this example, voters are heterogeneous with respect to both the location of their most-preferred policy and the shape of their indifference curves (i.e., the direction and intensity of the correlation between the two policy dimensions). We consider the case depicted in Figure 4 below, where 5 voters are identified by their ideal policies, located at the points a, b, c, d, e , respectively.

Insert Figure 4 about here

We retain for voters a, d and e the simplest configuration of circular level curves around their ideal points.¹⁸ The indifference curves of individuals b and c are instead represented by two ellipses centered around their ideal points, for which we choose different shapes.¹⁹ We depict in Figure 4a the lines $y(\theta, x)$ (obviously, $y(\theta, x)$ is a horizontal line through point θ for voters $\theta = a, d, e$), as well as $y_m(x)$ in bold. We proceed similarly in Figure 4b, showing the lines $x(\theta, y)$ together with $x_m(y)$. We report both $x_m(y)$ and $y_m(x)$ on Figure 5, and we obtain 3 KS equilibria: points c and d , but also a third point k^* that does not correspond to any voter's most-preferred location!

preferences satisfy the marginal single-crossing condition for both dimensions of types. It also shows that policy variables x and y are strategic complements when preferences satisfy both the marginal and joint single-crossing conditions.

¹⁸We do not represent level curves for these voters to avoid cluttering the figure further. See the working paper version of this article for an analytical description of the preferences in matrix terms.

¹⁹The equations of the ellipses represented in the figure are $(x - 4)^2 - 2(x - 4)(y - 3) + 2(y - 3)^2 = 1$ for the small ellipse around b , ($= 15$ for the bigger one) and $4(x - 5.4)^2 - 3(x - 5.4)(y - 4) + (y - 4)^2 = 9$ for the level curve of individual c .

Insert Figure 5 about here

As for Stackelberg equilibria, we observe numerically that both $V(\theta, x) = U(\theta, x, y_m(x))$ and $W(\theta, y, x_m(y))$ are single-peaked for all five voters. Moreover, location c constitutes the unique Stackelberg equilibrium, whatever the sequence of votes.

4 Conclusions

Majority voting over a multidimensional policy space leads in general to negative results, requiring very stringent conditions for the existence of an equilibrium outcome when voting simultaneously over all dimensions. Such results have induced political economy scholars to introduce specific and restrictive assumptions on individual preferences, on the distribution of individuals' types across the population and on the voting rule, often based on a sequential scheme.

Our paper takes one step back: it assumes utility functions and a distribution of types as general as possible, and it focuses on two specific alternatives to simultaneous majority voting. Our analysis of Kramer-Shepsle and Stackelberg equilibria leads to promising results. We show that it is possible to conclude about the existence of these equilibria starting from simple single-crossing conditions widely used in the literature. Under the same weak assumptions, we compare the characteristics of the solutions issued by the two voting procedures under exam, emphasizing the relevance of the median type preferred policy. We also study the uniqueness of equilibrium solutions, showing that multiple Kramer-Shepsle equilibria become plausible when the domain of individual preferences is richer. While developing our analysis in a general setting, we also study thoroughly an environment modelled in the political economy literature exploring issues such as the quantity and the location of public goods in modern democracies, the connection with the size of the nations and the stability of national borders to secession threats.

Both additional theoretical advances and further applications could enrich and complete our main findings. Along the first line, it would be interesting to study a model where the set of alternatives consists in a finite hypercube and where voter preferences are orderings. Along the second research line, we recommend a systematic comparison of KS and ST equilibria in the main models studied in the applied political economy literature, in the spirit of De Donder, Le Breton and Peluso (2009).

5 References

Alesina, A., Baqir, R. and W. Easterly. (1999) "Public Goods and Ethnic Divisions", *Quarterly Journal of Economics*, 114, 1243-1284.

Alesina, A., Baqir, R. and C. Hoxby. (2004) "Political Jurisdictions in Heterogeneous Communities", *Journal of Political Economy*, 112, 348-396.

Banks, J. and D. Austen-Smith (1999), *Positive Political Theory : Collective Preference*, Vol 1, Michigan University Press, Ann-Arbor.

Banks, J. and J; Duggan, J. (2000) "A Bargaining model of Collective Choice", *American Political Science Review*, 94, 73-88.

Banks, J. and J.Duggan (2004) "Existence of Nash Equilibria on Convex Sets", Wallis Working Papers WP20, University of Rochester, Wallis Institute of Political Economy.

Banks, J., Duggan, J. and M. Le Breton (2006) "Social Choice and Electoral Competition in the General Spatial Model", *Journal of Economic Theory*, 126, 194-234.

Baron, D. and J. Ferejohn (1989) "Bargaining in Legislatures", *American Political Science Review*, 83, 1181-1206.

Bernheim, B.D. and S.N. Slavov (2009) "A Solution Concept for Majority Rule in Dynamic Settings", *Review of Economic Studies* 76, 33-62.

Billingsley, P. (1999) *Convergence of Probability Measures*, Wiley, New York.

Cremer H., De Donder P. and F. Gahvari (2004), "Taxes, Budgetary Rule and Majority Voting", *Public Choice*, 119: 335-358.

Cremer H., De Donder, P., Maldonado, D. and P. Pestieau (2007) "Voting over the Size and Type of Social Security when some Individuals are Myopic", *Journal of Public Economics*, 91-10: 2041-2061.

Davis, O.A., DeGroot, M.H. and M.J. Hinich (1972) "Social Preference Orderings and Majority Rule", *Econometrica*, 40, 147-157.

De Donder P. and J. Hindriks (1998) "The Political Economy of Targeting", *Public Choice*, 95 (1-2), 177-200.

De Donder, P., Le Breton, M. and E. Peluso (2009) "Sequential Majority Choice Over a Two-Dimensional Policy Choice", CEPR Working Paper 7223.

Diba, B. and Feldman, A.M. (1984). "Utility functions for public outputs and majority Voting". *Journal of Public Economics*, 25, 235-243.

Duggan, J. (2001) "Non-Cooperative Games Among Groups", Wallis Working Papers WP21, University of Rochester, Wallis Institute of Political Economy.

Etro, F. (2006) "Political Geography", *Public Choice*, 16, 517-535.

Gans, J. and M. Smart (1996) "Majority Voting with Single-Crossing Preferences", *Journal of Public Economics*, 59, 219-238.

Greenberg, J. and S. Weber (1986) "Strong Tiebout Equilibrium under Restricted Preference Domain", *Journal of Economic Theory*, 38, 101-117.

Gregorini, F. (2009), "Political geography and income inequalities", Working Paper 152, Dipartimento di Economia Politica, Università di Milano - Bicocca.

Groseclose, T. (2007) "Black's Median Voter Theorem in 'One and a half' Dimensions", *Social Choice and Welfare*, 28, 321-335.

Haimanko, O., Le Breton, M. and S. Weber. (2005) "Transfers in a Polarized Country : Bridging the Gap between Efficiency and Stability", *Journal of Public Economics*, 89, 1277-1303.

Kramer, G.H. (1972) "Sophisticated Voting over Multidimensional Choice Spaces", *Journal of Mathematical Sociology*, 2, 165-180.

Krasa, S. and Polborn, M. (2009). "Political competition between differentiated candidates", CESIFO working paper 2560.

Krasa, S. and Polborn, M. (2010). "Competition between specialized candidates", *American Political Science Review*, 104, 745-765.

McKelvey, R., Ordeshook, P. and P. Ungar. (1980) "Conditions for Voting Equilibria in Continuous Voter Distributions", *Siam Journal of Applied Mathematics*, 39, 161-168.

Myerson, R. B. (1996) "Fundamentals of social choice theory." Discussion Paper 1162, Center for Mathematical Studies in Economics and Management Science, Northwestern University.

Nechyba, T.(1997) "Local Property and State Income Taxes: The Role of Interjurisdictional Competition and Collusion", *The Journal of Political Economy*, 105, 351-384.

Ordeshook, P. (1986) *Game Theory and Political Theory*, Cambridge University Press, Cambridge.

Perroni, C. and K. Scharf (2001) "Tiebout with Politics: Capital Tax Competition and Constitutional Choices", *Review of Economic Studies*, 68, 133-154.

Persson T. and G. Tabellini (2000) *Political Economics. Explaining Economic Policy*, MIT Press. Cambridge.

Plott, C. (1967) "A Notion of Equilibrium and its Possibility under Majority Rule", *American Economic Review*, 57, 787-806.

Rothstein, P. (1990) "Order Restricted Preferences and Majority Rule", *Social Choice and Welfare*, 7, 331-342.

Sadanand, A. B. and J. M. Williamson (1991). "Equilibrium in a Stock Market Economy with Shareholder Voting" *International Economic Review*, 32, 1-35.

Shepsle, K.A. (1979) "Institutional Arrangements and Equilibrium in Multidimensional Voting Models", *American Journal of Political Science*, 23, 27-59.

Tullock, G. (1967) "The General Irrelevance of the General Impossibility Theorem", *Quarterly Journal of Economics*, 81, 256-270.

Appendix 1: Proof of Lemma 1

i) Since the utility U is strictly concave with respect to x , for any given value of y (resp. x), the payoff of a citizen of type θ is maximized for a choice $x(y, \theta)$ (resp. $y(x, \theta)$) such that

$$\frac{\partial U(\theta, x(y, \theta), y)}{\partial x} = 0 \text{ (resp. } \frac{\partial U(\theta, x, y(x, \theta))}{\partial y} = 0).$$

From the implicit function theorem, we deduce

$$\frac{\partial x(y, \theta)}{\partial \theta} = -\frac{\frac{\partial^2 U(\theta, x(y, \theta), y)}{\partial \theta \partial x}}{\frac{\partial^2 U(\theta, x(y, \theta), y)}{\partial x^2}} \text{ and } \frac{\partial y(x, \theta)}{\partial \theta} = -\frac{\frac{\partial^2 U(\theta, x, y(x, \theta))}{\partial \theta \partial y}}{\frac{\partial^2 U(\theta, x, y(x, \theta))}{\partial y^2}}.$$

Take any y in Y and let $x_y(\cdot) : [\underline{\theta}, \bar{\theta}] \rightarrow \mathbb{R}$ be defined by $x_y(\theta) = x(y, \theta)$ and G_y be the cumulative distribution function defined on the interval X as follows:

$$G_y(x) = F(\{\theta \in [\underline{\theta}, \bar{\theta}] : x_y(\theta) \leq x\}).$$

We claim that there exists a unique value of x such that:²⁰

$$G_y(x) \geq \frac{1}{2} \text{ and } G_y^-(x) \leq \frac{1}{2}.$$

Indeed, suppose on the contrary that there exist two values $x_1 < x_2$ satisfying the above inequalities. Since $\frac{1}{2} \geq G_y^-(x_2) \geq G_y(x_1) \geq \frac{1}{2}$, we deduce that

$$G_y(x_1) = \frac{1}{2}$$

Since $\frac{1}{2} \geq G_y(x_2) \geq G_y(x_1)$, we obtain that $G_y(x_2) = \frac{1}{2}$. This implies that the cumulative function G_y is constant with the value $\frac{1}{2}$ on the interval $[x_1, x_2]$. We now show that this is not possible. Consider the sets

$$A_1 \equiv \{\theta \in [\underline{\theta}, \bar{\theta}] : x_y(\theta) \leq x_1\} \text{ and } A_2 \equiv \{\theta \in [\underline{\theta}, \bar{\theta}] : x_y(\theta) \leq x_2\}.$$

Let θ_1 and θ_2 be such that $x_y(\theta_1) = x_1$ and $x_y(\theta_2) = x_2$. Since x_y is continuous, we deduce from the intermediate value theorem that such values of θ exist. Suppose without loss of

²⁰For any increasing function G and any real number x , $G^-(x) = \lim_{y \rightarrow x, y < x} G(y)$ denotes the left limit of G at x .

generality that $\theta_1 < \theta_2$. From the intermediate value theorem again, for any value of x^* in $]x_1, x_2[$, there exists $\theta^* \in]\theta_1, \theta_2[$ such that $x_y(\theta^*) = x^*$. Let $\epsilon > 0$. From the continuity of x_y , there exists a small interval $[\theta^* - \delta, \theta^* + \delta] \subsetneq]\theta_1, \theta_2[$ with $\delta > 0$ and $x_y(\theta) \in (x^* - \epsilon, x^* + \epsilon)$ for all $\theta \in [\theta^* - \delta, \theta^* + \delta]$. Since F has full support, we deduce that $F([\theta^* - \delta, \theta^* + \delta]) > 0$ and therefore $G_y((x^* - \epsilon, x^* + \epsilon)) > 0$. This implies that $G_y(x_2) \geq G_y(x_1) + G_y((x^* - \epsilon, x^* + \epsilon)) > \frac{1}{2}$. But this contradicts the earlier claim.

Let x^* be the unique solution x of the inequalities $G_y(x) \geq \frac{1}{2}$ and $G_y^-(x) \leq \frac{1}{2}$. To conclude, it remains to show that x^* is the unique Condorcet winner. It is clearly a Condorcet winner as for any $x < x^*$, by strict concavity of $U(\theta, \cdot, y)$ all types in the set $\{\theta \in [\underline{\theta}, \bar{\theta}] : x_y(\theta) \geq x^*\}$ strictly prefer x^* to x . A similar argument holds for any $x > x^*$.

We now show that there is no other Condorcet winner. Suppose that there is another one, say x^{**} . Without loss of generality, assume that $x^{**} < x^*$. Then, from the definition of x^* , $G_y(x^{**}) < \frac{1}{2}$. Since G_y is right continuous, there exists n large enough such that $G_y(x^{**} + \frac{1}{n}) < \frac{1}{2}$. This implies $F(\{\theta \in [\underline{\theta}, \bar{\theta}] : x_y(\theta) \geq x^{**} + \frac{1}{n}\}) > \frac{1}{2}$. By concavity of $U(\theta, \cdot, y)$ all types in the set $\{\theta \in [\underline{\theta}, \bar{\theta}] : x_y(\theta) \geq x^{**} + \frac{1}{n}\}$ strictly prefer $x^{**} + \frac{1}{n}$ to x^{**} . This contradicts our assumption that x^{**} is a Condorcet winner.

ii) From Assumption 1, we deduce from above that:

$$\frac{\partial x(y, \theta)}{\partial \theta} > 0 \text{ and } \frac{\partial y(x, \theta)}{\partial \theta} > 0.$$

From these monotonicity properties, we obtain immediately that $x_m(y) = x(y, \theta_{med})$ and $y_m(x) = y(x, \theta_{med})$.

Appendix 2: If (x^s, y^s) is a local Condorcet winner, then $\Phi(d) \leq \frac{1}{2}$ for all d

First note that

$$\begin{aligned} & U(\theta, (x^s + \varepsilon d_x, y^s + \varepsilon d_y)) - U(\theta, (x^s, y^s)) \\ &= \varepsilon \varphi(\theta) + \varepsilon^2 \left[\frac{1}{2} \frac{\partial^2 U(\theta, (x^s, y^s))}{\partial x^2} (d_x)^2 + \frac{1}{2} \frac{\partial^2 U(\theta, (x^s, y^s))}{\partial y^2} (d_y)^2 \right. \\ & \quad \left. + \frac{\partial^2 U(\theta, (x^s, y^s))}{\partial x \partial y} (d_x d_y) \right] + M(\theta) O(\varepsilon^2), \end{aligned}$$

where $M(\theta)$ is a constant depending upon θ . Assume that (x^s, y^s) is a local Condorcet winner, but that there exists d , such that $\Phi(d) > \frac{1}{2}$. Since the measure F is regular,²¹ for any $\rho > 0$ there exists a compact subset $\Omega_\rho \subseteq \{\theta \in [\underline{\theta}, \bar{\theta}] : \varphi(\theta) > 0\}$ such that $F(\{\theta \in [\underline{\theta}, \bar{\theta}] : \varphi(\theta) > 0\} \setminus \Omega_\rho) < \rho$. Select ρ such that $\int_{\Omega_\rho} dF > \frac{1}{2}$ and let $C = \text{Sup}_{\theta \in \Omega_\rho} \frac{1}{2} \frac{\partial^2 U(\theta, (x^s, y^s))}{\partial x^2} + \frac{1}{2} \frac{\partial^2 U(\theta, (x^s, y^s))}{\partial y^2} + \frac{\partial^2 U(\theta, (x^s, y^s))}{\partial x \partial y} + M(\theta)$ and $c = \text{Inf}_{\theta \in \Omega_\rho} \varphi(\theta)$. For all ε such that $\varepsilon < \frac{c}{C}$, we deduce that $U(\theta, (x^s + \varepsilon d_x, y^s + \varepsilon d_y)) - U(\theta, (x^s, y^s)) > 0$ in contradiction to our assumption that (x^s, y^s) is a local Condorcet winner.

Appendix 3: Existence of KS equilibria

To prove this claim, we have to prove that the set valued mapping M from Z into Z such that $M(x, y) = M^1(y) \times M^2(x)$ has a fixed point. We claim that the correspondence is upper hemi continuous. Indeed, let $x_n \rightarrow x$ and $y_n \rightarrow y$ when $n \rightarrow \infty$. By continuity, we deduce that from all $\theta \in [\underline{\theta}, \bar{\theta}]$, $U(\theta, x, y_n) \rightarrow U(\theta, x, y)$ for all x and $U(\theta, x_n, y) \rightarrow U(\theta, x, y)$ for all y . Let $R_n^1(\theta)$ and $R_n^2(\theta)$ be the marginal weak orders induced on X and Y by $U(\theta, \cdot, y_n)$ and $U(\theta, x_n, \cdot)$. From what precedes, for all $\theta \in [\underline{\theta}, \bar{\theta}]$, $R_n^1(\theta)$ converges to $R^1(\theta)$ and $R_n^2(\theta)$ converges to $R^2(\theta)$ for the topology of closed convergence where $R^1(\theta)$ and $R^2(\theta)$ are the marginal weak orders induced on X and Y by $U(\theta, \cdot, y)$ and $U(\theta, x, \cdot)$. This almost sure convergence with respect to θ implies that the marginal mapping $U_{y_n}^1$ (respectively $U_{x_n}^2$) converges to the marginal mapping U_y^1 (respectively U_x^2) for the Δ metric defined in Banks, Duggan and Le Breton (2006). The upper hemicontinuity of M^1 and M^2 follow from proposition 25 in Banks, Duggan and Le Breton. (2006)²². Existence of a KS equilibrium follows from Kakutani 's theorem.

²¹Heuristically, a regular measure on a topological space is a measure for which every measurable set is "approximately open" and "approximately closed". Any Borel probability measure on any metric space is a regular measure. Therefore, all interesting probability measures are regular. We refer the reader to Billingsley (1999) for a concise definition.

²²When there is a finite number of individuals, existence follows from Duggan (2001) or Shepsle (1979). The extra effort needed here arises from the fact that we consider a continuous distribution of voters. A direct proof avoiding the appeal to proposition 25 in Banks, Duggan and Le Breton (2006) could be provided but would require extra cumbersome notations.

Appendix 4: Proof of Proposition 5

Note first that if $\theta \geq \theta_{med}$, then the ideal point $(x(\theta), y(\theta))$ of $U(\theta, x, y)$ is such that $x(\theta) \geq x(\theta_{med})$ and $y(\theta) \geq y(x(\theta), \theta_{med})$. Since $U(\theta, x, y)$ is strictly concave, then, if the function y_m is concave, for all $\theta \geq \theta_{med}$, we have that

$$V(\theta, x) = \text{Max } U(\theta, x, y) \text{ subject to the constraint } y \leq y_m(x)$$

is single-peaked with respect to x . The concavity of the y_m function guarantees that the indirect utility of all $\theta \geq \theta_{med}$ is single-peaked, because the location of their utility peak compared to y_m ensures that they maximize their concave utility on a convex set.

We now prove that if U satisfies the properties assumed in the proposition, then y_m is indeed concave. From the differentiation of (5), we obtain

$$\begin{aligned} \frac{d^2 y_m(x)}{dx^2} = & - \frac{\left[\frac{\partial^3 U(\theta_{med}, x, y_m(x))}{\partial x^2 \partial y} + \frac{\partial^3 U(\theta_{med}, x, y_m(x))}{\partial x \partial y^2} \frac{dy_m(x)}{dx} \right] \frac{\partial^2 U(\theta_{med}, x, y_m(x))}{\partial y^2}}{\left(\frac{\partial^2 U(\theta_{med}, x, y_m(x))}{\partial y^2} \right)^2} \\ & + \frac{\left[\frac{\partial^3 U(\theta_{med}, x, y_m(x))}{\partial x \partial y^2} + \frac{\partial^3 U(\theta_{med}, x, y_m(x))}{\partial y^3} \frac{dy_m(x)}{dx} \right] \frac{\partial^2 U(\theta_{med}, x, y_m(x))}{\partial x \partial y}}{\left(\frac{\partial^2 U(\theta_{med}, x, y_m(x))}{\partial y^2} \right)^2} \end{aligned}$$

Since $\frac{\partial^2 U(\theta_{med}, x, y_m(x))}{\partial y^2} < 0$, $\frac{dy_m(x)}{dx} > 0$ and $\frac{\partial^2 U(\theta_{med}, x, y_m(x))}{\partial x \partial y} > 0$, if $\frac{\partial^3 U(\theta_{med}, x, y)}{\partial x^2 \partial y} < 0$, $\frac{\partial^3 U(\theta_{med}, x, y_m(x))}{\partial x \partial y^2} < 0$ and $\frac{\partial^3 U(\theta_{med}, x, y_m(x))}{\partial y^3} < 0$ then $\frac{d^2 y_m(x)}{dx^2} < 0$.

Appendix 5: Majority choice of x in section 2.3

The most-preferred value of x decreases from $\bar{x} \equiv v'^{-1}\left(\frac{1}{\Psi(0)}\right)$ to $\underline{x} \equiv v'^{-1}\left(\frac{1}{\text{Min}(\Psi(y_{med}), \Psi(y_{med}-1))}\right)$ as θ moves away from θ_{med} . Without loss of generality, suppose that $\Psi(y_{med}) \leq \Psi(y_{med}-1)$. The proportion $B(x)$ of voters with an ideal peak below the fixed level x is given by :

$$B(x) = \begin{cases} F\left(y_{med} - \Psi^{-1}\left(\frac{1}{v'(x)}\right)\right) & \text{if } x \leq x^*, \\ F\left(y_{med} - \Psi^{-1}\left(\frac{1}{v'(x)}\right)\right) + \left[1 - F\left(y_{med} + \Psi^{-1}\left(\frac{1}{v'(x)}\right)\right)\right] & \text{if } x \geq x^*, \end{cases}$$

where x^* is the unique solution to the equation

$$v'(x) = \frac{1}{\Psi(y_{med}-1)}.$$

When F is symmetric, $y_{med} = \frac{1}{2}$, $x^* = \underline{x}$ and B is a cumulative distribution function on $[\underline{x}, \bar{x}]$ defined as follows:

$$B(x) = 2F\left(\frac{1}{2} - \Psi^{-1}\left(\frac{1}{v'(x)}\right)\right).$$

Then, the majority choice x_{med} is the unique solution x to the equation :

$$F\left(\frac{1}{2} - \Psi^{-1}\left(\frac{1}{v'(x)}\right)\right) = \frac{1}{4}.$$

For instance, when F is uniform, x_{med} is the peak of a voter located at a distance from the median equal to $\frac{1}{4}$.

Appendix 6: Proof of Proposition 7

The first-order condition for y of an individual θ is given by

$$v'(x_m(y))\Psi(y - \theta)\frac{dx_m(y)}{dy} + v(x_m(y))\Psi'(y - \theta) - \frac{dx_m(y)}{dy} = 0.$$

Our objective is to assess under what circumstances the value of y that maximizes $W(\theta, y)$ differs from θ (which is the “true peak” of the utility function—i.e., the value of y that maximizes $U(\theta, x, y)$ for any given value of x). To this effect, we evaluate the derivative of $W(\theta, y)$ at $y = \theta$ to obtain

$$\frac{\partial W(\theta, y)}{\partial y} \Big|_{y=\theta} = \frac{dx_m(\theta)}{dy} [v'(x_m(\theta))\Psi(0) - 1] + v(x_m(\theta))\Psi'(0).$$

The function $x_m(y)$ is characterized by the equality $v'(x_m(y))\Psi(d) - 1 = 0$, where $d = \delta(\theta)$ if $y^* \leq \theta \leq y^{**}$ and $d = \theta - \theta_{med}$ if $\theta < y^*$ or $\theta > y^{**}$. Therefore, the above derivative is equal to

$$\begin{aligned} \frac{\partial W(\theta, y)}{\partial y} \Big|_{y=\theta} &= v'(x_m(\theta)) [\Psi(0) - \Psi(d)] \frac{dx_m(\theta)}{dy} + v(x_m(\theta))\Psi'(0) \\ &= v'(x_m(\theta)) [\Psi(0) - \Psi(d)] \frac{dx_m(\theta)}{dy}, \end{aligned}$$

as $\Psi'(0) = 0$. Since $\Psi(0) - \Psi(d) > 0$, the sign of the derivative at $y = \theta$ is the same as the sign of $dx_m(\theta)/dy$. If $\theta < y^*$, $x_m(\theta) = x(\theta, \theta_{med})$ so that $dx_m(\theta)/dy > 0$. If $\theta > y^{**}$, $x_m(\theta) = x(\theta, \theta_{med})$ so that $dx_m(\theta)/dy > 0$. If $y^* \leq \theta \leq y^{**}$, $x_m(\theta) = x(\theta, \theta \pm \delta(\theta))$ and we obtain that

$$\frac{dx_m(y)}{dy} = -\frac{v'(x_m(y))\Psi'(\delta(y))}{v''(x_m(y))\Psi(\delta(y))} \frac{d\delta(y)}{dy}.$$

Since $\Psi'(\delta) < 0$, the sign of $dx_m(y)/dy$ is the opposite of the sign of $d\delta(y)/dy$. From the definition of $\delta(y)$ and the implicit function theorem, we obtain that

$$\frac{d\delta(y)}{dy} = \frac{f(y - \delta(y)) - f(y + \delta(y))}{f(y + \delta(y)) + f(y - \delta(y))},$$

so that the sign of $d\delta(y)/dy$ depends exclusively upon the shape of the density function f .

We now provide an example where $W(\theta, y)$ is single-peaked in y for all θ . Consider for instance the case where F is uniform, $v(x) = 2\sqrt{x}$ and $\Psi(y - \theta) = K - (y - \theta)^2$ where K is a sufficiently large positive constant. We obtain that

$$x_m(y) = \begin{cases} (K - \frac{1}{4} - y^2 + y)^2 & \text{if } y \leq \frac{1}{4} \text{ or } y \geq \frac{3}{4}, \\ (K - \frac{1}{16})^2 & \text{if } \frac{1}{4} \leq y \leq \frac{3}{4}, \end{cases}$$

and therefore that

$$W(\theta, y) = \begin{cases} 2(K - \frac{1}{4} - y^2 + y)(K - (y - \theta)^2) - (K - \frac{1}{4} - y^2 + y)^2 & \text{if } y \leq \frac{1}{4} \text{ or } y \geq \frac{3}{4}, \\ 2(K - \frac{1}{16})(K - (y - \theta)^2) - (K - \frac{1}{16})^2 & \text{if } \frac{1}{4} \leq y \leq \frac{3}{4}. \end{cases}$$

This indirect utility function is single-peaked for all $\theta \in [0, 1]$. The graph of $W(\theta, y)$ for several values of θ when $K = 1$ is represented in Figure 6.

Insert Fig. 6 about here

Figure 1: Voters favoring direction $d = (d_x, d_y)$

General case

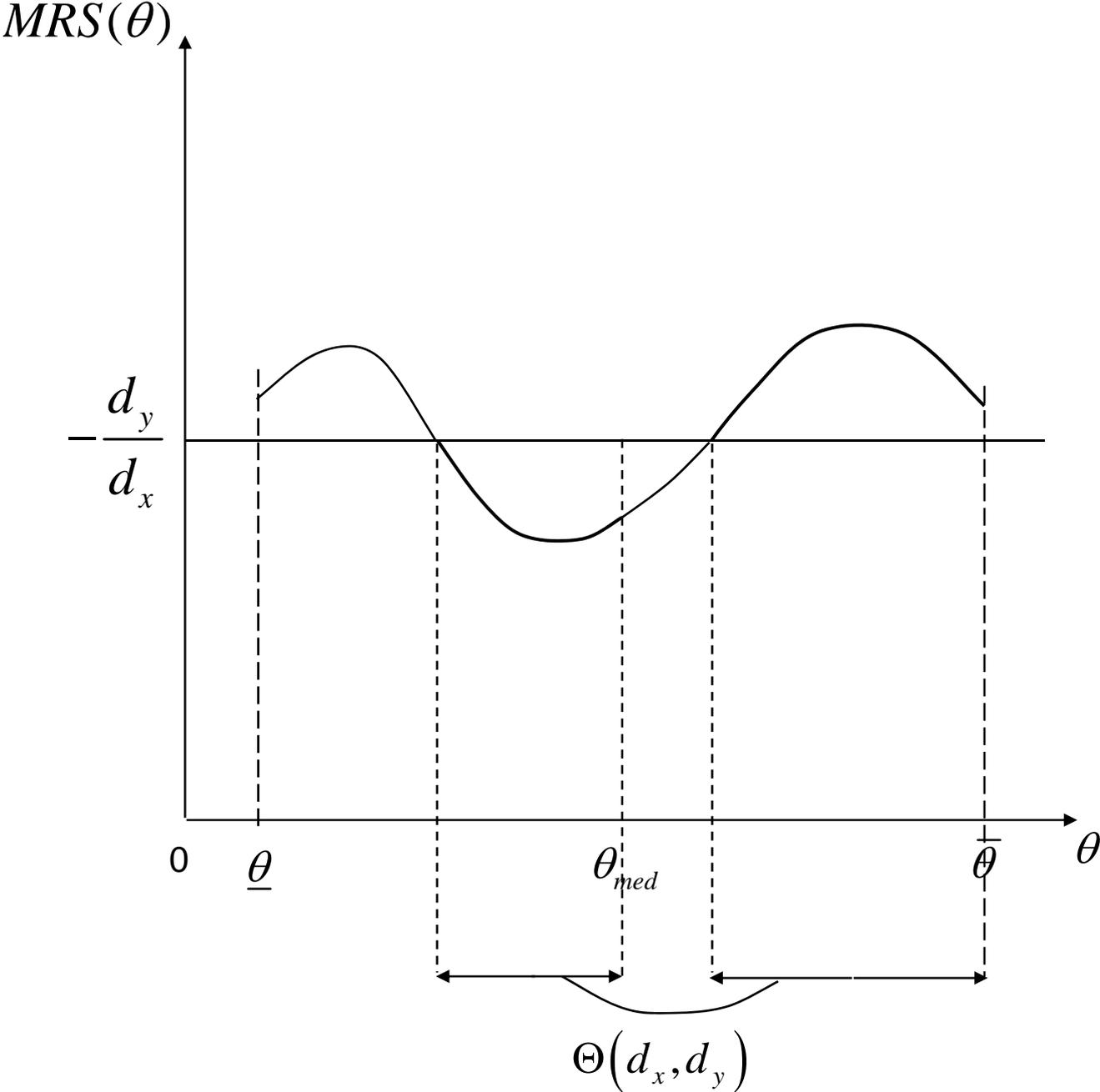


Figure 2 : Voters favoring direction $d=(d_x,d_y)$

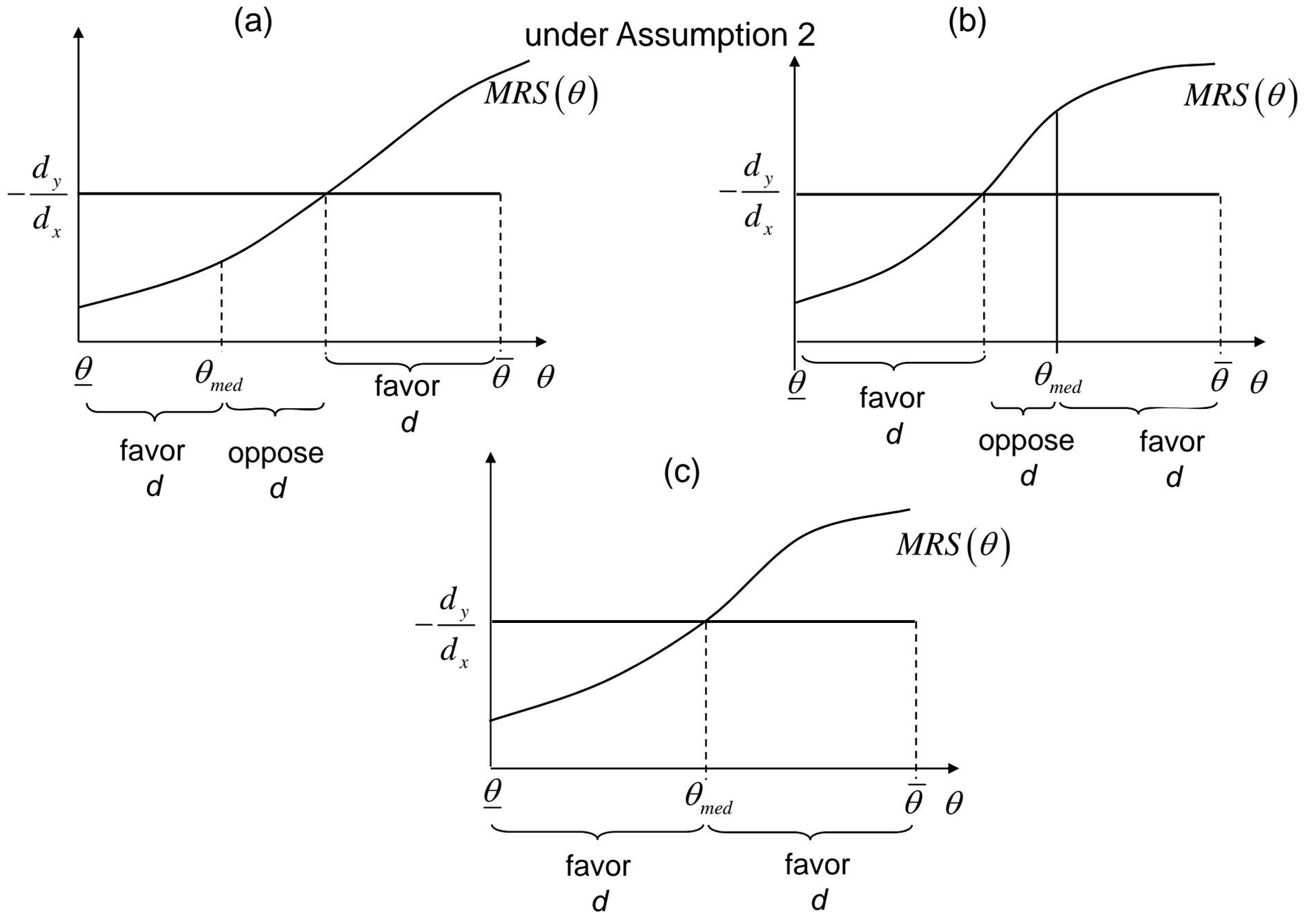
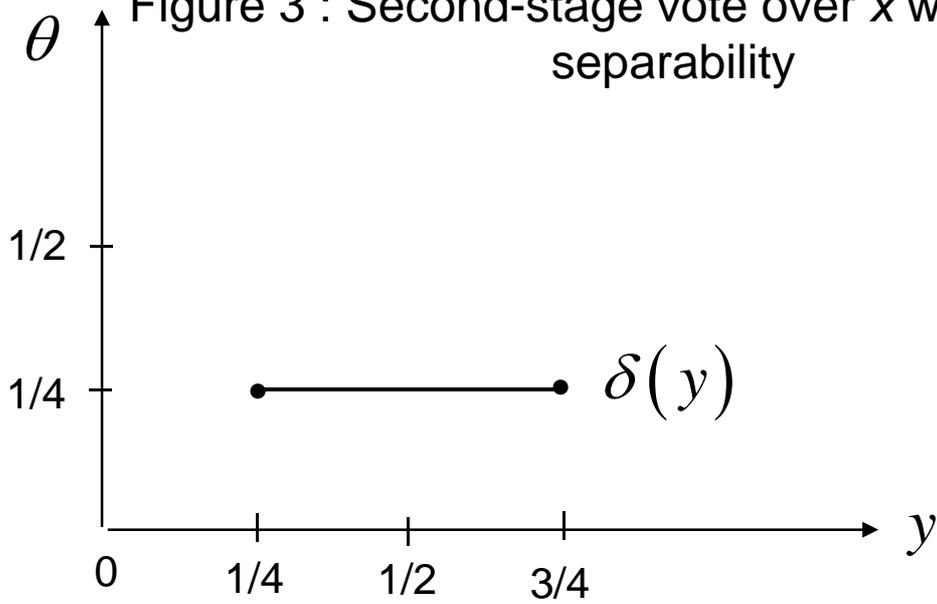
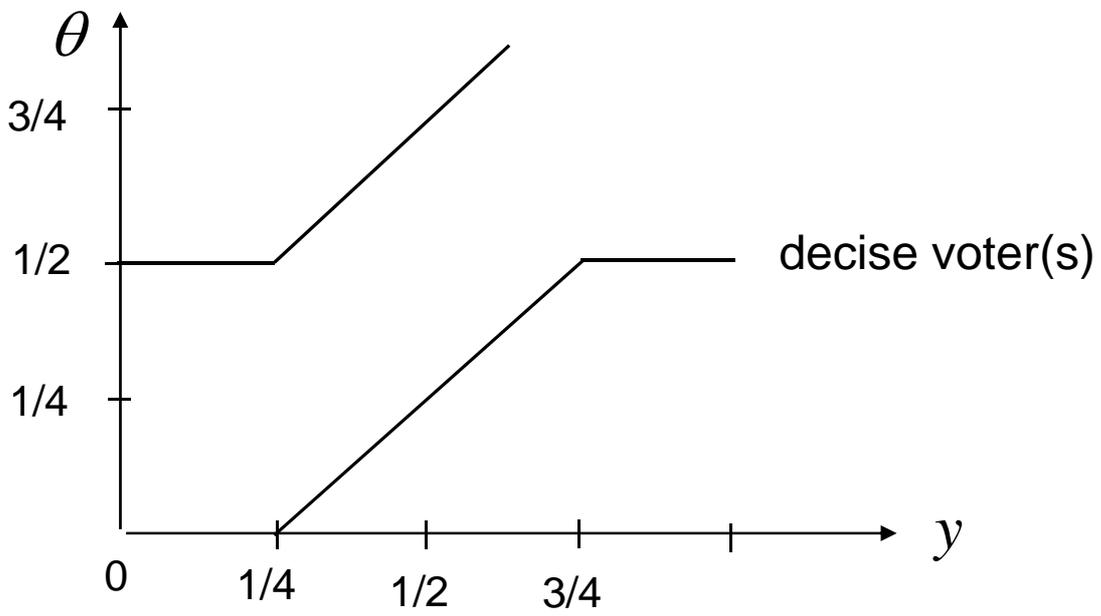


Figure 3 : Second-stage vote over x with one-sided separability

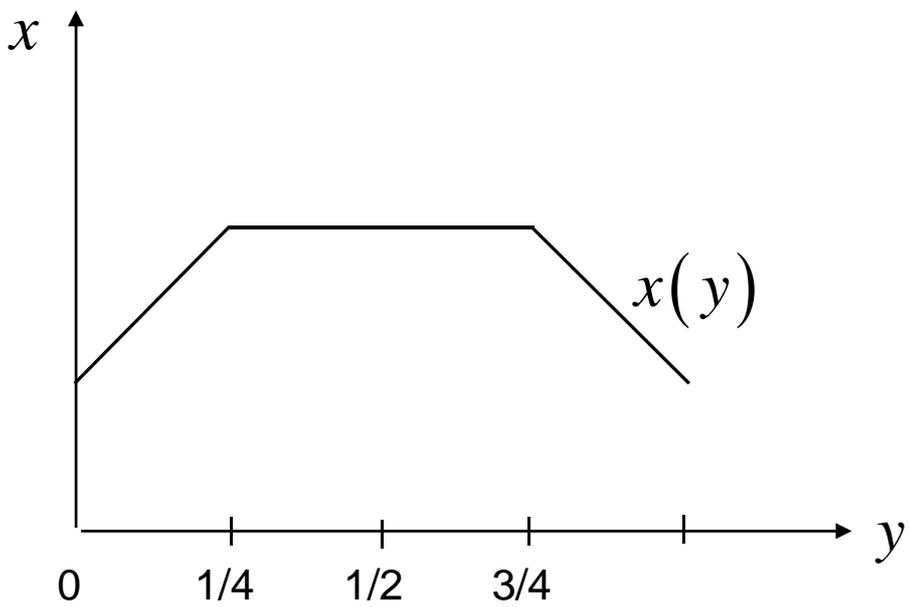
(a)

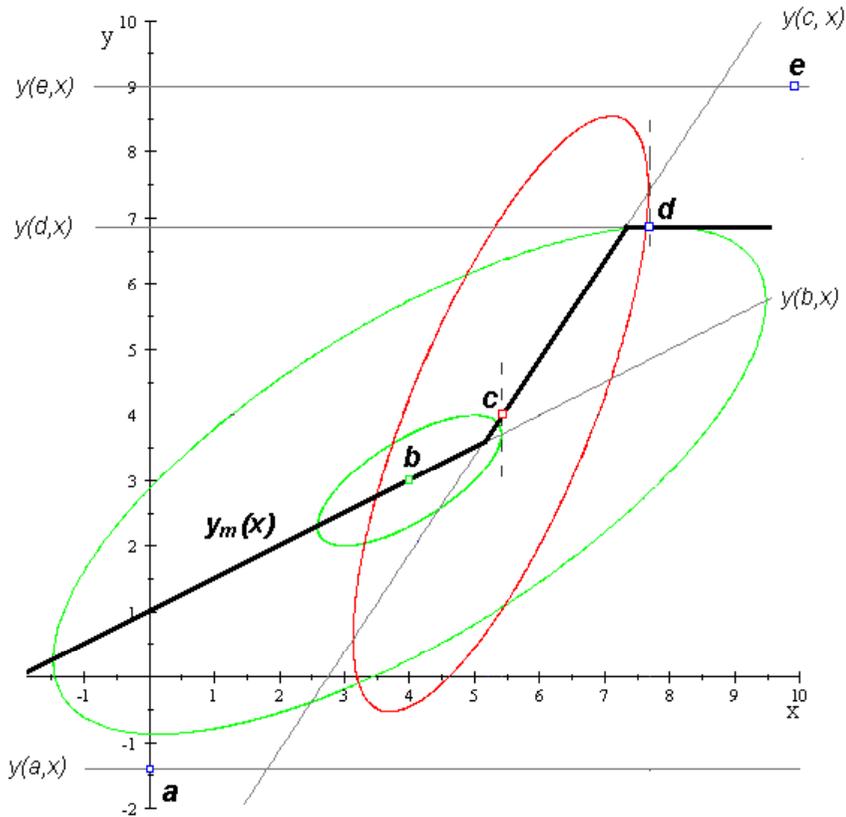


(b)

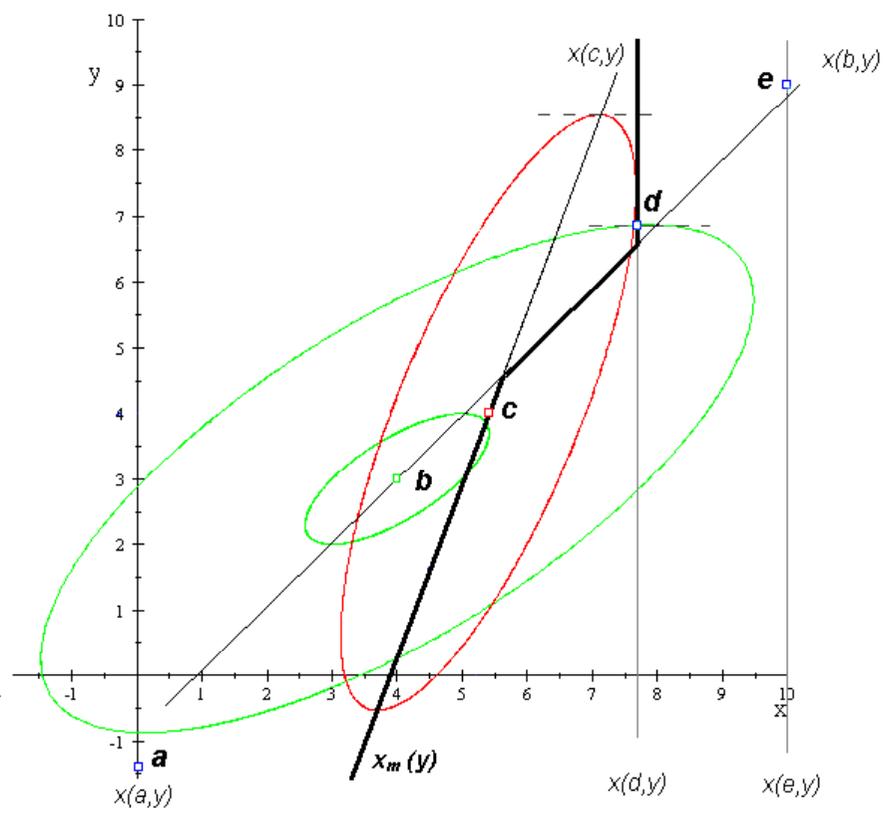


(c)





Panel 1



Panel 2

Figure 4: An example with quadratic preferences

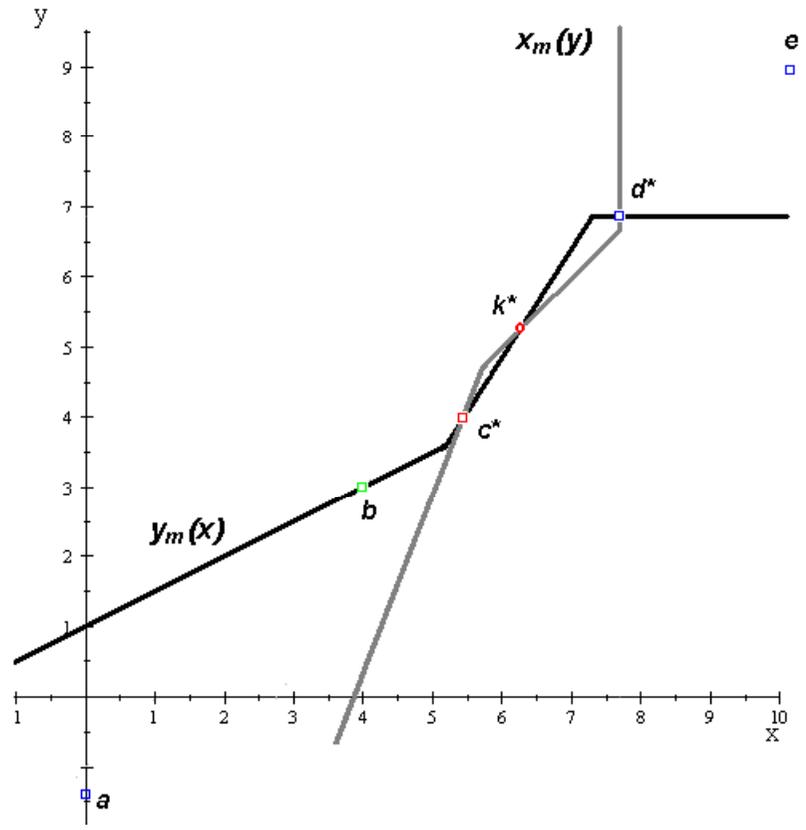


Figure 5: KS equilibria with quadratic example

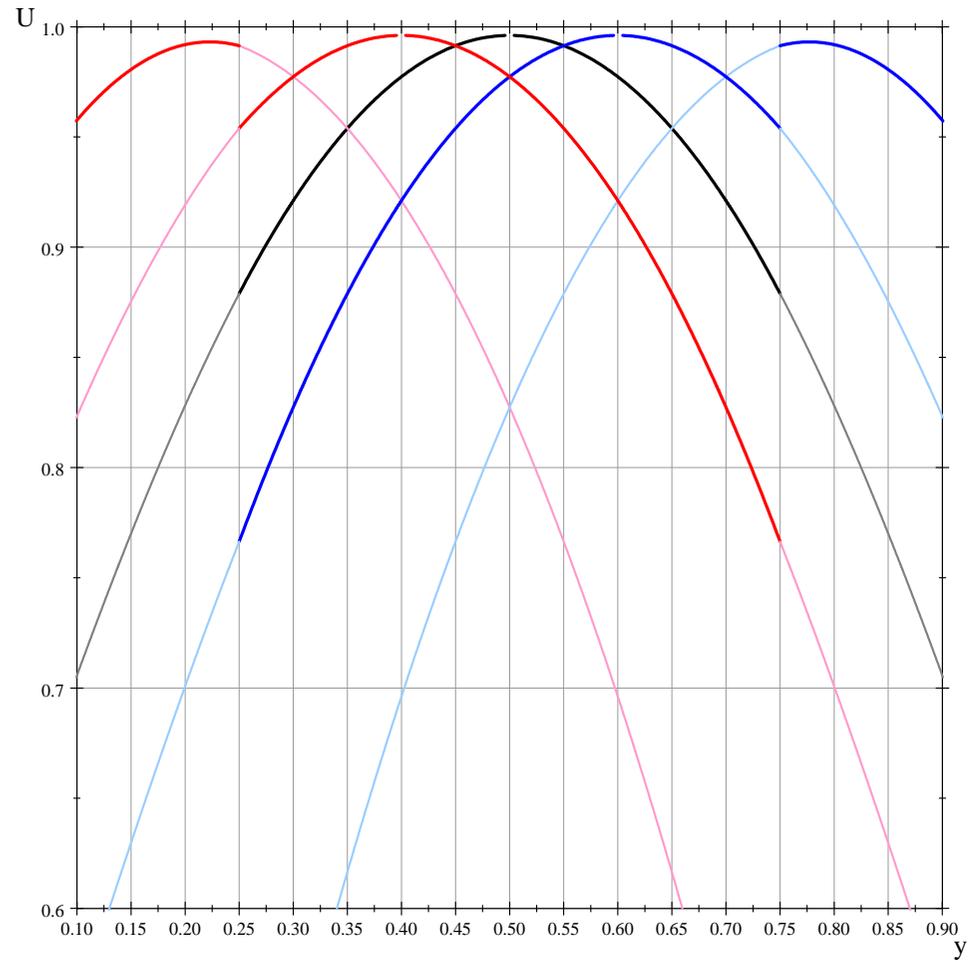


Figure 6: $W(\theta, y)$ for $\theta = \{0.2, 0.4, 0.5, 0.6, 0.8\}$ when $K = 1$