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Simulating Equilibrium in Multi-Product Postal Markets Following De-regulation and Liberalization

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1. Introduction

In this paper we describe the design and construction of a simulator of liberalized multi-product postal markets. The simulator uses a novel approach to explore equilibrium in these markets under various schemes for relaxing postal price regulation.

A market is "liberalized" when an incumbent supplier faces potential competitors who will enter the market if it is profitable and will remain out of the market if it is not. Liberalized markets can arise in many contexts but are most often associated with price de-regulation. The general idea is that the threat of entry will impose sufficient discipline on an incumbent's prices to make the pre-existing price regulation at-least-partly redundant. An example of partial liberalization of U.S. postal markets occurred in the mid-1970s when UPS, FedEx and others were permitted to offer expedited mail and parcel delivery services in direct competition with the U.S Postal Service (USPS). More recently, successive postal directives of the European Union (EU) have imposed wholesale liberalizations on the domestic postal markets of its member countries. International mail markets provide another example. These markets are almost always naturally liberalized at the collection end because most national postal authorities are committed by international agreements to collect mail for delivery in other countries but usually find that it is impractical to exclude entrants (see Pearsall 2016).

Economists typically analyze postal markets under assumptions that limit the entrant/competitors (ECs) to pure strategies consisting of single combinations of products. However, our simulations show that when an EC's entry decisions are endogenous the resulting pattern of entry is likely to be a stochastic mix of product combinations. In these cases the postal operator's (PO's) prices limit and equalize the profit that an EC can derive on entry from employing two or more product combinations. The EC's entries and exits then converge on a stochastic mix of these equally profitable combinations. In equilibrium the resultant entry frequencies leave the PO with no incentive at the margin to adjust its prices to improve the expected value of its objective. We have found nothing describing such an equilibrium within the familiar economics literature.

Our simulator generates the price choices of a PO who is always present in the market, the price choices of ECs when they choose to enter, and the patterns of product combinations chosen by the ECs. The simulator does this by finding a Nash equilibrium for a non-zero-sum, non-cooperative, multi-person game. The ECs react to the PO's pricing choices by choosing a combination of products to offer in the market and prices for the products that maximize their profit. The PO observes the frequencies of the ECs' product choices and sets its own prices to maximize the expected value of its economic objective (which may differ from profit) subject to whatever constraints have been imposed by the regulator. The Nash equilibrium consists of a pair of strategies that are optimal against each other.

The game is solved using the method of fictitious play (aka Brown's method). The simulator alternately derives the strategy for each player that is optimal given the most recent strategy employed by its opponent. The process converges iteratively on prices and frequencies that describe a Nash equilibrium. In effect, the simulator solves the game by replicating the operation of the liberalized market over time.

The simulator is designed to enable us to simulate postal markets over a range of market conditions, regulatory constraints and PO objectives. In its current configuration the simulator will accept demand models with up to six matched postal services each for the PO and the ECs. The demand functions for both the PO and ECs are approximated as linear functions and are calibrated by specifying elasticity matrices, volume levels, market shares and diversion rates when the PO and ECs offer all of their services at matched prices. For simplicity, when there is more than one EC, the ECs' are assumed to be identical to each other so they divide equally the demand that would arise if there were only one EC. The cost functions of the PO and ECs are approximated as linear functions and are calibrated to costs input for the volumes used to calibrate the demand functions. There may be up to sixty-four different product combinations for an EC. Demand and cost functions for each combination are derived under assumptions that respect the requirements of neo-classical demand and cost theory.

The simulator allows us to explore the operation and equilibrium of postal markets under a variety of more-or-less restrictive regulatory regimes. Users may designate a reserved area for the PO in the form of services for which the ECs are not permitted to offer directly competitive products. Conversely, the user may designate services that the ECs will always offer even at a loss. Price floors and price caps may be individually specified for each of the PO's products. In addition a global price cap may be imposed upon the collection of prices chosen by the PO. The weights for the global price index are chosen by the user.

Our simulator allows the user to select the objective quantity that the PO maximizes. The basic choices are: profit, welfare, cost and revenue. The latter three may be maximized subject to a floor on the PO's profit. The simulator can be run using weighted combinations of the objectives. It is also possible to reweight the individual costs of the PO's products to reflect asymmetric preferences. For example, the PO may prefer to incur labor costs rather than costs for materials and capital.

Finally, at each step of the method of fictitious play, the PO has a variety of technical options for inferring the frequencies of the ECs' product combination selections from those made at previous steps. For example, the PO may employ frequencies calculated as simple or weighted averages of the EC's past selections.

In the following section we describe the characteristics of liberalized markets that make it reasonable to describe these markets as non-zero-sum, non-cooperative, multi-person games. Section 3 provides a semi-formal description of the game. In Section 4 we outline the method of fictitious play for finding the Nash equilibrium for these games. The remaining sections of the paper describe in some detail how the simulator is constructed and provide an example of its intended application. In Section 5 we propose a method for deriving a linear demand model from existing estimates of demand elasticities and assumed parameters. In Section 6 the method is extended to the derivation of linear cost models for the PO and EC. At each step of the simulator the EC chooses a product combination and prices that maximize its profit. This is described in Section 7. Similarly, at each step the PO chooses prices to maximizes its objective subject to various linear constraints. Necessary conditions for a solution to this problem are derived in Section 8. In Section 9 we describe how the components of the simulator are assembled and solved for the strategies used by the PO and ECs. In Section 10 we present the results of an illustrative simulation of liberalized US postal markets. The paper concludes briefly in Section 11 with our plans for the simulator's future use.

2. Design of the Simulator

The concepts underlying our simulator are applicable when a PO remains in an open market and acts as the price leader. This is the common condition of postal markets following an opening to potential entrants. Typically, the pre-existing price regulation is relaxed but not entirely eliminated and administrative rules or competition laws delay the PO's responses to the entry and pricing decisions of an unregulated EC. Entry and exit by an EC normally require substantial lead times also. However, an EC's prices are not subject to regulation and can be changed much more rapidly than either the POs prices or the EC's selections of product offerings. Therefore, the EC is usually in a position to observe the PO's prices before it must set its own.

The role of price leader following market opening is not an advantage to an incumbent PO. In fact it may be a considerable disadvantage since it prevents the PO from engaging in predatory pricing - price discrimination based upon the product combinations and prices selected by the ECs on entry. If a PO can discriminate it will not set its prices without knowing if ECs are present. Instead, it will act as a Cournot oligopolist and employ a different set of prices for each combination of products that it may encounter from ECs. Price leadership by the PO is an essential condition for liberalized markets as we have defined them.

In the immediate aftermath of a postal liberalization postal markets will be finely tuned to the service offerings of the incumbent PO. ECs attempting to distinguish their service offerings from those of the PO will find that the high ground is already occupied. For example, an EC's best opportunity to enter the US market for Standard mail may be to deliver printed advertising to every address within broadly-defined areas. Unaddressed mail can be delivered at very low cost because it needs almost no processing but the service is of lower value because a mailer mostly loses the ability to precisely target recipients. At present USPS is prevented by the U.S. Congress from accepting such mail. Standard mail is all addressed so an EC delivering advertising mail indiscriminately within local areas would be offering a bulk mail service that is inferior to USPS Standard mail.

Most of the analysis underlying our simulator is an extension of models and concepts that apply to a simple case where the PO offers a single mail service at a single price and a single EC that also offers only a single service at a single price.[1] The service offered by the EC may not be identical to that offered by the PO so the two services are not necessarily perfect substitutes. This type of market has properties that define a non-cooperative non-zero-sum two-person game with a Nash equilibrium. The two players of the game are the PO and the EC offering the substitute service. The PO's pure strategies are the different prices that it may set. The EC has two pure strategies: to be either *in* or *out* of the market. Both the PO and the EC may be regarded as choosing strategies from compact and convex sets. The PO's price must be non-negative and is upper-bounded by a price that is sufficiently high to drive the demand for the PO's services to zero even if the EC is not present. The EC's set of strategies may be extended to include probabilistic mixes of its two pure strategies. The payoffs to the two players are determined by their objective functions which are differentiable and strictly concave over their respective sets of feasible strategies.

Since each player chooses a strategy from a compact and convex set, a Nash equilibrium consisting of a pair of strategies that are optimal against each other exists. Moreover, because of the strict concavity of the PO's objective, the equilibrium price is unique. Equilibrium takes one of three forms, monopoly, duopoly with price leadership, or stochastic entry with limit-pricing. In a monopoly the EC finds that it is unprofitable to be *in* even when the PO sets a monopoly price. Consequently, the EC is always out and the market becomes a PO monopoly. In duopoly with price leadership the EC is always *in*. The equilibrium for stochastic entry with limit-pricing is a pair of strategies consisting of a price, which leaves the EC indifferent between entering or not entering, and a mixed strategy indexed by a probability of entry for the EC.

3. A Liberalized Multi-Product Postal Market

When there are multiple services offered by the PO its prices are denoted by the price vector P_I. On entering the postal market the EC selects a combination of products from a set of possible services that would normally be differentiated somewhat from those offered by the PO. Let $t \in T$ designate a combination of services for the EC contained

[1] Single-product models describing markets along these lines may be found in Pearsall and Trozzo (2008), Pearsall (2011) and Pearsall (2016).

within a set T of all possible combinations that the EC might offer, including the option of not offering any services. T

The EC's prices for the services in the combination t are the elements of a price vector P_E^t. If the EC chooses to offer the services designated by t, it takes the PO's price vector, P_I, as given and sets prices on the services, P_E^t , to maximize its profit function, $f_E^t(P_I, P_E^t)$. The EC's profit functions are assumed to be strictly concave for any P_I that the PO might choose and for any $t \in T$ except for the special case of non-entry for which there is no price vector and a zero profit. A profit function $f_E^t(P_I, P_E^t)$ always reaches a maximum at a single vector of prices. Therefore, for any product combination t we may treat the EC as setting prices for the services offered according to a reaction function $P_E^t(P_I) = ArgMax_{P_E^t}\{f_E(P_I, P_E^t)\}$. In effect, the EC's prices are determined by a post-optimization for every possible product combination with the PO's prices known in advance to the EC.

The PO's pure strategies for a single play of the game are real price vectors P_I chosen within a compact and convex region S of real number space R^n. The dimension of R^n is the number of the PO's services. We would usually expect P_I to be non-negative and bounded from above, but the PO's prices may be subject to a variety of additional regulatory constraints. For example, individual floors on prices may be imposed to avoid cross-subsidies; individual caps may be imposed to control the PO's exercise of market power; the level of the PO's prices may be regulated with a global price cap; and the PO's profit may be constrained by a break-even condition or a limit on the PO's rate of return.

The EC's pure strategies for a single play of the game are indexed to the elements of the set T of possible product combinations. The game is extended to include mixed strategies for the EC by assigning probabilities to the EC's possible product combinations. Let μ_t be the probability assigned to the product combination t. A mixed strategy consists of an assignment of a probability μ_t to each of the elements of T so that $0 \leq \mu_t \leq 1 \; \forall \; t \in T$ and $\sum_{t \in T} \mu_t = 1$.

The PO's price vector P_I is set before the PO knows the EC's specific choice of a product combination $t \in T$. Similarly, the EC chooses a product combination on entering (or decides not to enter) by choosing t without knowing the PO's prices. By including mixed strategies we extend the EC's choices to include random selections of product combinations according to the probabilities $\mu_t \; \forall \; t \in T$. The EC's set of all possible strategies becomes compact and convex when extended to include all of the EC's possible mixed strategies.

The EC's objective is to maximize its expected profit. Therefore, the EC's expected payoff is just $\sum_{t\in T} \mu_t \, f_E^t\big(P_I, P_E^t(P_I)\big)$. The PO's objective may also be to maximize expected profit. However, the PO's objective may differ from profit if it is subject to regulation. So we assume only that the PO's objective is to maximize a concave function of the price vector P_I when the EC follows its reaction function for each product combination $t \in T$. The payoff to the PO if the EC chooses the product combination t is the strictly concave function $f_I^t\big(P_I, P_E^t(P_I)\big)$. The PO's objective is to maximize the expected value of its payoffs, $\sum_{t\in T} \mu_t \, f_I^t\big(P_I, P_E^t(P_I)\big)$. Since the probabilities μ_t are all non-negative and at least one must be positive, the PO's objective is a strictly concave function of P_I.

A Nash equilibrium consists of a pair of feasible strategies for the two players which are simultaneously optimal against each other. The EC's mixed strategy of entry and exit using various service combinations solves the problem:

$$Max_{\mu_t}\left\{\sum_{t\in T} \mu_t \, f_E^t\big(P_I, P_E^t(P_I)\big) \middle| 0 \le \mu_t \le 1 \; \forall \, t \in T \text{ and } \sum_{t\in T} \mu_t = 1\right\}$$

given the prices chosen by the PO. Ordinarily, the EC simply sets $\mu_t = 1$ for the pure strategy that yields the largest profit $f_E^t\big(P_I, P_E^t(P_I)\big)$. However, it is necessary to formulate the EC's problem in a way that accommodates ties. If two or more product combinations yield the same maximum profit, then the EC's maximization problem is degenerate and probabilistic mixes of the equi-profitable combinations are all solutions to the EC's maximization problem.

The PO's strategy is a vector of prices for its own products that simultaneously solve the problem:

$$Max_{P_I}\left\{\sum_{t\in T} \mu_t \, f_I^t\big(P_I, P_E^t(P_I)\big) \middle| P_I \in S\right\}$$

given the probabilities that describe the EC's entries and exits. These problems each have concave objective functions and feasible strategies drawn from compact and convex vector spaces. Therefore, the maximization problems for the PO and EC jointly determine a Nash equilibrium. A Nash equilibrium is unique with respect to the price vector P_I because the PO's objective is strictly concave.

4. The Method of Fictitious Play

The traditional language and descriptions of game theory encourage us to view a game's equilibrium strategies as though they are deliberately chosen by the players. However, this is misleading as a way to view equilibrium in a liberalized market. It is better to view the equilibrium price vector, $P_{I,}$ and probabilities of entry, $\mu_t \; \forall \; t \in T$, as a matched pair describing the probabilistic outcome of repeated plays of the game.

The method of fictitious play mimics the operation over time of a liberalized market in which the EC and PO each maximize their own objectives using the information they might reasonably be expected to possess from knowing their own costs and observing the market. The players do not use information that they would be unlikely possess in an actual liberalized market. In particular, the PO does not anticipate the EC's selections of product combinations using information other than the EC's past selections. The information that the players rely on consists of the past record of the EC's selections of product combinations and the PO's most recent selection of prices. We also assume that the EC's reaction function $P_E^t(P_I)$ can be deduced by the PO for any of the product combinations $t \in T$.

The method proceeds iteratively in steps indexed by the superscript $0,1,..,i-1,i,i+1,...etc.$ At step *i* the EC knows the PO's price from the previous step, P_I^{i-1}, and chooses a product combination *t* to maximize its profit, i.e., it solves the problem $Max_t\left\{f_E^t\left(P_I^{i-1},P_E^t\left(P_I^{i-1}\right)\right) \mid t \in T\right\}$. Simultaneously, the PO uses an estimate of the frequencies $\mu_t^{i-1} \forall \; t \in T$ derived from the EC's past selections of product combinations. The estimate is regarded as predetermined as the PO maximizes its objective, i.e., it solves the problem $Max_{P_I}\left\{\sum_{t\in T}\mu_t^{i-1} \, f_I^t\left(P_I,P_E^t(P_I)\right) | P_I \in S\right\}$. Note that the PO uses the EC's reaction function for each *t*.

The players' solutions to the two problems at step *i* are used to construct the prices and frequencies for the next step. At step *i*+1the EC simply takes the price vector that solves the PO's maximization problem as P_I^i. The PO's re-estimation of the product frequencies is more complicated. Basically, the PO adds a data point consisting of the EC's choice *t* to the sample it uses to estimate $\mu_t^i \forall \; t \in T$. However, there are various ways that a PO might estimate the frequencies from the sample. The approach used in basic applications of the method of fictitious play is to compute the frequencies as sample proportions. Alternatively, the frequencies may be estimated after truncating and/or censoring the sample or after weighting the points in the sample. In fact the method of fictitious play can be made to work with virtually all of the usual statistical methods for estimating frequencies from a time series.

The method begins at an arbitrary starting point consisting of an assumed pair of feasible strategies P_I^0 and $\mu_t^0 \forall\, t \in T$. As the steps increase the strategies approach a probability limit which is the Nash equilibrium for the game. The method of fictitious play simulates the market behavior of the EC and the PO in a way that is wholly consistent with micro-economic theory. Neither the EC nor the PO needs to be aware of the fact that they are playing a game. The PO need not deliberately engage in limit-pricing and the EC need not intentionally randomize its choice of product combinations. The PO and the EC are merely responding to the entry and price signals they receive from the market. Nevertheless, limit-pricing by the PO and stochastic entry by the EC are often descriptive of their behavior over time.

5. The Demand Model

The simulator employs demand models that are linear in form:

$$\begin{bmatrix} Q_I \\ Q_E \end{bmatrix} = \begin{bmatrix} \alpha_I \\ \alpha_E \end{bmatrix} + \begin{bmatrix} A & B \\ B' & C \end{bmatrix} \begin{bmatrix} P_I \\ P_E \end{bmatrix}.$$

The component vectors and coefficient matrix of the model are partitioned to conform to the products and prices of the PO (subscript "I") and of the EC (subscript "E").

P_I, Q_I the price and quantity vectors for the PO's products.

P_E, Q_E the price and quantity vectors for the EC's products.

$\begin{bmatrix} \alpha_I \\ \alpha_E \end{bmatrix}$ the vector of equation intercepts.

$\begin{bmatrix} A & B \\ B' & C \end{bmatrix}$ the matrix of price derivatives.

The model is linear so the equation intercepts and price derivatives are all fixed values. We regard the income effects of postal price changes as negligible. Therefore, the entire matrix of price derivatives must be symmetric in accordance with neo-classical demand theory. In particular, the principal submatrices A and C are symmetric. It is also desirable but not strictly essential that the matrix of price derivatives be negative definite.

The simulator requires a linear demand model for each combination of postal products that the EC may offer. The most obvious distinction between these models is that the rows and columns corresponding to products that the EC chooses not to offer are entirely zeroed out. However, there are other differences between the matrices that we would also expect to see. For example, we would expect any of the PO's products to exhibit a larger elasticity of demand when the EC offers a directly competing service and a smaller elasticity when the EC does not.

It is unlikely that estimates of many of the required demand models will exist prior to a liberalization of postal markets. In the remainder of this section we offer a method for extracting the needed demand models in a consistent manner from a limited set of elasticity estimates and assumed parameter values. The method assumes that the EC's possible products roughly match those already offered by the PO. We have tested our method for models with 6 products each for the PO and the EC using various elasticity matrices taken from fits to USPS data of a branching AIDS model (Bzhilyanskaya *et al* 2015). The results demonstrate that our method is a practical way to generate useable demand models with very limited information.

The method takes two steps. In the first step an estimate of the incumbent PO's elasticity matrix is transformed so that when the price derivatives are calculated for centered values for prices and volumes the resultant derivative matrix is symmetric. In the second step the demand equations are extrapolated to include all of the EC's possible product combinations. These extrapolations require some additional information in the form of two sets of values that are stipulated for a base case in which the EC offers all of the same products at the same prices as the PO. The stipulated values are the market shares of the PO (and the EC) and the "diversion" rates at which the PO and EC exchange demand for comparable services at the margin. For example, suppose that an EC succeeds in increasing the demand for a product by lowering his price for the product. A diversion rate of 0.90 for the service means that 90 percent of the increase is the result of diverting volume from the PO's comparable product.

The formulas that are applied in the first step are:

$\hat{E}_{ij}, \sigma_{ij}^2$ an estimate of the elasticity of demand for the PO's product i with respect to the price of the PO's product j and the variance of the estimate.

P_i, Q_i the centered price and volume for the PO's product *i*.

$\hat{\beta}_{ij} = \frac{dQ_i}{dP_j} = \hat{E}_{ij}\frac{Q_i}{P_j}$ the centered derivative.

$var(\hat{\beta}_{ij}) = \hat{E}_{ij}^2\frac{Q_i^2}{P_i^2}$ the variance of the centered derivative.

$\beta_{ij} = \frac{var(\hat{\beta}_{ji})\hat{\beta}_{ij} + var(\hat{\beta}_{ij})\hat{\beta}_{ji}}{var(\hat{\beta}_{ij}) + var(\hat{\beta}_{ji})}$ for $i \neq j$ the centered and symmetric derivative.

$E_{ij} = \beta_{ij}\frac{P_j}{Q_i}$ for $i \neq j$ and $E_{ii} = \hat{E}_{ii}$.[2]

The elasticity estimates and assumed parameter values for a two-product example of our method are shown as red numbers enclosed in boxes in Table 1. Also shown in the table are the results of the first step.

Table 1: Two Products Example

Estimated Incumbent Elasticity Matrix

	Product 1	Product 2	Row Sum
Product 1	-0.4000	0.1000	-0.3000
Product 2	0.3000	-0.7000	-0.4000

dQ/dP Centered

	Product 1	Product 2
Product 1	-52,000	2,167
Product 2	3,000	-1,167

Elasticity Standard Deviations

	Product 1	Product 2
Product 1	0.1500	0.0500
Product 2	0.2000	0.1000

dQ/dP Centered Std. Dev.

	Product 1	Product 2
Product 1	19,500	1,083
Product 2	2,000	167

Centering Price and Volume

	Price	Volume
Product 1	0.5000	65,000
Product 2	3.0000	5,000

dQ/dP Centered and Symmetric

	Product 1	Product 2
Product 1	-52,000	2,356
Product 2	2,356	-1,167

	Market Shares		Diversion
	Incumbent	Entrant	Rate
Product 1	0.9000	0.1000	0.8000
Product 2	0.4000	0.6000	0.6000

Centered and Symmetric Elasticities

	Product 1	Product 2	Row Sum
Product 1	-0.4000	0.1087	-0.2913
Product 2	0.2356	-0.7000	-0.4644

The method begins with the estimated elasticity matrix and the corresponding matrix of standard deviations shown in the upper left-hand side of Table 1. In displaying matrices

[2] This equation preserves the diagonal elements of the initial matrix of elasticities. Alternatively, the row sums of the initial matrix may be preserved by setting $E_{ii} = \sum_j \hat{E}_{ij} - \sum_{j \neq i} E_{ij}\, E_{ii}$. It may also be necessary to further adjust the elasticity estimates so that the matrix of partial derivatives is negative definite.

of elasticities and derivatives we follow the convention of listing volumes vertically and prices horizontally. For instance, 0.1000 is the estimated cross-elasticity of demand of the PO's product 1 with respect to the price of its product 2.

The estimated elasticities are combined with the centering prices and volumes to obtain the centered derivatives and their standard deviations on the upper right-hand side of Table 1. The matrix of centered derivatives is asymmetric. The two off-diagonal derivatives are unequal at the centering price and volume. The standard deviations of these elements are used to resolve this inconsistency by combining these elements into a single most likely value. This value appears in the off diagonal locations of the centered and symmetric derivatives matrix. This is the matrix A of the linear demand model when the EC does not enter any postal markets at all.

Table 2: Demand Model Two Products Example

Elasticity Matrices

Case 0

In		PO-1	PO-2	EC-1	EC-2	Row Sum
1	PO-1	-0.400	0.109	0.000	0.000	-0.291
1	PO-2	0.236	-0.700	0.000	0.000	-0.464
0	EC-1	0.000	0.000	0.000	0.000	0.000
0	EC-2	0.000	0.000	0.000	0.000	0.000

Case 1

In		PO-1	PO-2	EC-1	EC-2	Row Sum
1	PO-1	-1.210	0.109	0.810	0.000	-0.291
1	PO-2	0.212	-0.700	0.024	0.000	-0.464
1	EC-1	7.289	0.109	-9.111	0.000	-1.713
0	EC-2	0.000	0.000	0.000	0.000	0.000

Case 2

In		PO-1	PO-2	EC-1	EC-2	Row Sum
1	PO-1	-0.400	0.043	0.000	0.065	-0.291
1	PO-2	0.236	-2.078	0.000	1.378	-0.464
0	EC-1	0.000	0.000	0.000	0.000	0.000
1	EC-2	0.236	0.919	0.000	-1.531	-0.377

Case 3

In		PO-1	PO-2	EC-1	EC-2	Row Sum
1	PO-1	-1.210	0.043	0.810	0.065	-0.291
1	PO-2	0.212	-2.078	0.024	1.378	-0.464
1	EC-1	7.289	0.043	-9.111	0.065	-1.713
1	EC-2	0.212	0.919	0.024	-1.531	-0.377

dQ/dP Derivative Matrices

Case 0

In		PO-1	PO-2	EC-1	EC-2	Intercept
1	PO-1	-52,000	2,356	0	0	83,933
1	PO-2	2,356	-1,167	0	0	67,322
0	EC-1	0	0	0	0	0
0	EC-2	0	0	0	0	0

Case 1

In		PO-1	PO-2	EC-1	EC-2	Intercept
1	PO-1	-141,556	2,120	94,756	0	75,540
1	PO-2	2,120	-1,167	236	0	67,322
1	EC-1	94,756	236	-118,444	0	17,638
0	EC-2	0	0	0	0	0

Case 2

In		PO-1	PO-2	EC-1	EC-2	Intercept
1	PO-1	-52,000	942	0	1,413	83,933
1	PO-2	942	-1,385	0	919	26,929
0	EC-1	0	0	0	0	0
1	EC-2	1,413	919	0	-1,531	4,131

Case 3

In		PO-1	PO-2	EC-1	EC-2	Intercept
1	PO-1	-141,556	848	94,756	1,272	75,540
1	PO-2	848	-1,385	94	919	26,929
1	EC-1	94,756	94	-118,444	141	17,638
1	EC-2	1,272	919	141	-1,531	4,131

The results of the second step in our method are displayed in Table 2. Here we have a linear demand model for each of the EC's four possible product combinations. These are denoted cases 0 to 3. The products that are offered by the PO and EC for each case are designated by a "1" in the left-hand column.

The elasticity matrices for the cases appear down the left-hand side of the table. The labeling of the rows and columns of the matrices identifies the supplier and the product. These elasticities were derived from the results of step 1 using the following formulas:

a_{ij}, b_{ij}, c_{ij}, b'_{ij} elements of the elasticity sub-matrices corresponding to *A, B, C, B'*.

$d_i = 0$ entrant does not offer product i, $d_i = 1$ entrant offers product *i.*

s_i PO market share of product *i;* $(1 - s_i)$ EC market share of product *i.*

r_i diversion rate for product *i.*

$$a_{ii} = \begin{cases} E_{ii} & \text{for } d_i = 0 \\ E_{ii}\left(1 + r_i(1 - s_i)/s_i\right)/(1 - r_i^2) & \text{for } d_i = 1 \end{cases}$$

$$a_{ij} = \begin{cases} E_{ij} & \text{for } i \neq j \text{ and } d_j = 0 \\ E_{ij} s_j & \text{for } i \neq j \text{ and } d_j = 1 \end{cases}$$

$$c_{ii} = \begin{cases} 0 & \text{for } d_i = 0 \\ E_{ii}\,(1 + s_i/(1 - s_i))/(1 - r_i^2) & \text{for } d_i = 1 \end{cases}$$

$$c_{ij} = \begin{cases} 0 & \text{for } i \neq j \text{ and } d_i = 0 \text{ or } d_j = 0 \\ E_{ij}\left(1 - s_j\right) & \text{for } i \neq j \text{ and } d_i = 1 \text{ and } d_j = 1 \end{cases}$$

$$b_{ii} = \begin{cases} 0 & \text{for } d_i = 0 \\ -c_{ii} r_i\,(1 - s_i)/s_i & \text{for } d_i = 1 \end{cases}$$

$$b_{ij} = \begin{cases} 0 & \text{for } i \neq j \text{ and } d_i = 0 \text{ or } d_j = 0 \\ E_{ij}\left(1 - s_j\right) & \text{for } i \neq j \text{ and } d_i = 1 \text{ and } d_j = 1 \end{cases}$$

$$b'_{ii} = \begin{cases} 0 & \text{for } d_i = 0 \\ b_{ii}\, s_i/(1 - s_i) & \text{for } d_i = 1 \end{cases}$$

$$b'_{ij} = \begin{cases} 0 & \text{for } i \neq j \text{ and } d_i = 0 \\ E_{ij} s_j & \text{for } i \neq j \text{ and } d_i = 1 \end{cases}$$

The coefficients of the demand models for the four cases are arranged down the right-hand side of Table 2. The formulas transform the centered elasticity matrix of Table 1 in a way that preserves several essential properties of the resultant elasticity and derivative matrices for the cases. First, the PO's elasticities are unchanged when the EC does not offer a directly competing product. Second, in all cases the row sums of the elasticity matrices for the PO are identical to those of the elasticity matrices in Table 1. Therefore, the PO always responds identically to proportionate changes in all postal prices, including the prices of any products offered by an EC. Third, the derivatives matrices are all symmetric as required by micro-economic theory. The elements of the derivatives matrices in Table 2 are calculated from their corresponding elasticities at the centering prices and volumes. Finally, if the original derivatives matrix is negative definite, it is very likely that all of the derivative matrices derived from it will also be negative definite.

6. The Cost Model

The simulator employs cost functions for the PO and EC that are also linear:

$$C_I = m_I' Q_I + F_I \text{ for the Incumbent PO and}$$

$$C_E = m_E' Q_E + F_E \text{ for the Potential Entrant EC}$$

m_I, m_E vectors of marginal costs for the PO and EC.

F_I, F_E cost equation intercepts for the PO and EC.

There is a single cost function for the PO since the PO always offers the same set of products. However, there may be different cost functions for the EC depending on the EC's product combinations. As before we treat each product combination as a distinct case and construct a cost function for the EC for each case. Also, we rely on initial cost estimates for the PO and assumed values for several parameters. Table 2 continues the two-product example of the previous section.

In the upper left-hand corner of Table 3 are shown the estimates of the PO's unit volume variable costs for the products under the assumption that the PO is alone in the market and produces the entire basis volume of each. These volumes are the same as the centering volumes in Table 1 but this correspondence is not required by our method. We assume that similar estimates of unit volume variable costs for the same basis volumes are also available for the EC.

Again, we proceed in two steps. First, marginal costs are derived for the basis case in which the PO and EC offer all possible services and charge identical prices. For convenience we have assumed that the basis prices and volumes are the same as those used to center the linear demand equations in Table 1. The demand model divides the basis volume for each product between the PO and the EC according to the assumed market shares exhibited in Table 1. This division is shown in Table 3. The marginal costs for the basis volumes are derived by adjusting the initial unit volume variable costs upwards to correct for the effects of economies of scope and scale lost with lower volumes. The adjustment roughly follows the same cost accounting arithmetic used by USPS and the PRC to attribute postal costs to various services.

$\overline{m}_I$, $\overline{m}_E$ vectors of the initial unit volume variable costs for the PO and EC.

D_I, D_E cost "drivers" for the PO and EC computed by converting the volumes of all products to into an equivalent volume of the first product using given unit equivalents.

ϵ_L, ϵ_E volume "variabilities" for the PO and EC. Volume variability is defined as the elasticity of variable cost with respect to the cost driver, *D*. US postal accounting methods treat volume variability as fixed.

The vectors m_I and m_E are derived by applying the formulas:

$$m_I = \overline{m}_I(D_I/(D_I + D_E))^{\epsilon_I - 1} \text{ and } m_E = \overline{m}_E(D_E/(D_I + D_E))^{\epsilon_E - 1}.$$

These marginal cost vectors appear in the PO's cost function and in the EC's cost function for all of the EC's product combinations.

Table 3: Cost Model Two Products Example

	Initial Unit V V Costs		Basis Marginal Costs		Specific Fixed Cost	
	Incumbent	Entrant	Incumbent	Entrant	Incumbent	Entrant
Product 1	0.200	0.250	0.237	0.344	500	0
Product 2	2.500	1.750	2.961	2.408	100	200

	Basis Price	Basis Volume	Basis Volumes Incumbent	Basis Volumes Entrant		Prod. 1 Unit Equivalent
Product 1	0.500	65,000	58,500	6,500	Product 1	1.000
Product 2	3.000	5,000	2,000	3,000	Product 2	12.500

Volume Variability	
Incumbent	Entrant
0.600	0.700

Basis Case	Incumbent	Entrant
Volume Variable Cost	19,781	9,460
Infra Vol. Var. Cost	13,187	4,054

Basis Equiv. Volume	
Incumbent	Entrant
83,500	44,000

Inc. Cost Intercept
15,787

	Incumbent	Entrant
Revenue	35,250	12,250
Variable Cost	32,968	13,514
Specific Fixed Cost	600	200
Institutional Fixed Cost	2,000	500
Profit	-318	-1,964
Basis Cost Intercept	15,787	4,754

	Ent. Basis Volumes						
Case	Product 1	Product 2	Vol. Var. Cost	Infra Vol. Var. Cost	Variable Cost	Specific Fixed Cost	Ent. Cost Intercept
0	0	0	0	0	0	0	0
1	6,500	0	2,236	958	3,194	0	1,458
2	0	3,000	7,224	3,096	10,320	200	3,796
3	6,500	3,000	9,460	4,054	13,514	200	4,754

The second step of our method is to calculate the intercepts for the cost equations. The intercepts position the cost equations so that they will each replicate the total cost of the PO or the EC when the EC selects a particular product combination. The intercepts F_I, F_E themselves are composed of several kinds of costs that are assumed to remain fixed when the simulator uses the linear cost model. These components are: infra volume variable cost (IVVC) which captures the accumulated effects of economies of scope and scale, specific fixed cost (SFC) which is the sum of those fixed costs that are incurred only when the PO and EC offer the associated products, and institutional fixed cost (IFC) which is always incurred unless the PO or EC chooses to exit all postal markets. The total cost of the PO or EC is the sum of the intercept $F_I$or F_E and the PO or EC's volume variable costs (VVC) $m_I'Q_I$ or $m_E'Q_E$.

Properly defined, the variable cost of the PO or IC is the sum of its VVC and IVVC, not VVC alone. Under the USPS and PRC accounting system IVVC and VVC can be related using the volume variabilityϵ: $\mathrm{VVC} = \epsilon(\mathrm{VVC} + \mathrm{IVVC})$. Therefore, IVVC can be derived from VVC using $\mathrm{IVVC} = ((1-\epsilon)/\epsilon)\mathrm{VVC}$. The components of SFC for the products and the IFC for the PO and EC must be supplied by users. Altogether the PO and EC's cost components, revenue and profit in the basis case of the two-product example are shown in Table 3. The basis cost intercept is the sum of its components IVVC, SFC and IFC.

This arithmetic is simply repeated for each of the EC's product combination as shown in the last four lines of Table 3. The values of F_E for each of the EC's cost functions corresponding to the four cases are on the right-hand side. Notice that the bottom line just duplicates the basis case for the EC. The PO's cost function does not change from case to case so F_I is the incumbent basis case intercept.

7. The Entrant's Problem

The PO is the price leader in a liberalized market. Therefore, the EC sets its price vector for whatever product combination it has chosen knowing the price vector P_I that has been selected by the PO. The EC will do this by setting $P_E$to maximize its profit for the chosen product combination.

The simulator constructs the EC's profit function from the components of the linear demand and cost models that apply to the EC. These components are:

$Q_E = \alpha_E + B'P_I + CP_E$ the EC's demand functions, and

$C_E = m_E' Q_E + F_E$ the EC's cost function.

The EC's profit function is:

$$\pi_E = P_E' Q_E - C_E = (P_E - m_E)'[\alpha_E + B' P_I + C P_E].$$

We may delete the rows and columns of vectors and matrices when they apply to products that the EC does not offer. Then the EC's profit function π_E is a strictly concave function of P_E with a unique maximum at the point where the profit function's price gradient is zero. This first-order condition is:

$$\nabla \pi_E = [\alpha_E + B' P_I + C P_E] + C(P_E - m_E) = 0.$$

When we solve the condition for P_E we get the vector function:

$$P_E(P_I) = m_E/2 - C^{-1}[\alpha_E + B'^{P_I}]/2$$

which is the EC's reaction function. Both the profit function and the reaction function are dependent on the EC's choice of a product combination so there is a π_E^t and $P_E^t(P_I^i)$ for every $t \in T$.

The sub-matrix *C* must be non-singular to form the reaction function. When α_E, B and C are derived as shown in Section 5 they have zeros in the rows and columns for products that the EC does not offer (those for which $d_i = 0$). We avoid the singularity by deleting these rows and columns. This is inconvenient for computations because it requires doing the arithmetic with arrays of all different sizes. An easy solution is to insert a "1" on the diagonal elements of the matrix *C* wherever the element corresponds to an omitted product. Similarly, a "0" can be inserted in m_E for each omitted product. With these changes the matrix *C* becomes non-singular, the reaction function installs a price of zero for the products not offered by the EC, and the EC's profit is unaffected.

Recall that at each step of the method of fictitious play the EC chooses a product combination, *t*, by solving $Max_t \left\{ f_E^t \left(P_I^{i-1}, P_E^t\left(P_I^{i-1}\right)\right) \mid t \in T \right\}$ given the PO's price vector P_I^{i-1} from the previous step. The simulator makes this selection by calculating P_E and π_E for every possible $t \in T$. The calculation for each *t* employs the demand and cost model coefficients for the corresponding case. The prototype simulator has six products each for the PO and EC so there can be as many as 64 cases to be compared. The EC's product selection is the feasible product combination $t \in T$ yielding the highest profit π_E.

Not all of the 64 possible product combinations may be eligible for selection. The simulator permits users to designate a reserved area for the PO consisting of products that only the PO may offer. The EC is prohibited from offering products of its own that compete directly with those in the reserved area. Conversely, the user may designate a predetermined area of products that it is assumed the EC will always offer. These products are present in all of the product combinations chosen by the EC even if the EC must suffer a loss to produce them. Both a reserved area and a predetermined area restrict the size of the set of feasible product combinations T. The reserved area eliminates combinations that include the EC's directly competing products. The predetermined area eliminates product combinations that do not include the EC's products that are predetermined.

8. The Incumbent's Problem

The PO's objective may be either to maximize its profit or to maximize some other quantity such as welfare, sales or the command of resources. All of these alternatives have been proposed as objectives for enterprises under regulation. Profit, of course, is the objective assumed by neo-classical micro-economic theory. It is the objective that implicitly underlies most normative theorizing about postal price regulation. "Welfare" is defined operationally as the sum of consumers' surplus on the PO's products plus the PO's profit. It is the objective used by Baumol and Bradford (1970) to obtain Boiteux/Ramsey prices for an efficient price-regulated monopoly. "Sales" is equated to the PO's revenue. Sales maximization has been proposed by Baumol (1959) as closer to a typical enterprise's actual objective than profit maximization. Finally, "command of resources" can be equated to the PO's cost. Maximizing its command of resources would be the objective of a PO acting as a public bureaucracy according to a theory of bureaucratic behavior due to Niskanen (1971). All of these objectives, except for profit, are normally accompanied by a floor on the PO's profit level. However, the floor does not necessarily have to be an economic profit of zero.

The simulator allows a user to choose among these objectives singly or to employ a weighted combination of them. It is also possible to specify unit volume variable costs that differ from marginal costs for the cost maximization objective. This feature enables the simulator to approximate other objective measures that may play a part in determining the

PO's market behavior. For example, a government- run PO might be obliged to assist politically by stimulating employment. An employment bias in the PO's objective can be introduced by adjusting the marginal costs to magnify their labor component. The adjusted marginal costs then become the unit volume variable costs that are used when the PO maximizes cost.

As the price leader, the PO sets its prices without knowing precisely which product combination the EC will select in response. Therefore, all of the product combinations that the EC might choose are relevant for the PO's selection of a price vector P_I. For any single feasible EC product combination $t \in T$ the PO must take into account the following:[3]

$Q_I = \alpha_I + AP_I + BP_E$ the PO's demand functions, and

$P_E(P_I) = m_E/2 - C^{-1}\,[\alpha_E + B'P_I]/2$ the EC's reaction function.

These may be combined to obtain demand functions for the PO that embed the EC's reaction function, are linear functions of only P_I, and depend on *t*:

$Q_I = \gamma + EP_I$ with $\gamma = \alpha_I + Bm_E/2 - BC^{-1}\alpha_E/2$ and $E = A - BCB'/2$.

We also make use of the PO's cost function which does not depend on *t:*

$C_I = m_I'Q_I + F_I$ the PO's cost function.

Singly, the alternative objective quantities of the PO may be written as functions of just the PO's price vector:

$\pi_I = (P_I - m_I)'[\gamma + EP_I] - F_I$ profit.

$W_I = -[\gamma + EP_I]'E^{-1}[\gamma + EP_I] + \pi_I$ welfare from the PO's products.

$C_I = \overline{m}_I'[\gamma + EP_I] + F_I$ cost with the adjusted marginal costs $\overline{m}_I$.

$R_I = P_I{}'[\gamma + EP_I]$ revenue.

The simulator's generalized objective for the PO is a weighted average using the user's selections for profit, welfare, cost and revenue. Three of the PO's objectives may be accompanied by a profit floor $\pi_I \geq \overline{\pi}_I$. The profit floor is accounted for in the PO's generalized objective by adding another term for profit calculated using a multiplier. The multiplier $\rho \geq 0$ is an add-on weight for profit that is adjusted (if necessary) until $\pi_I = \overline{\pi}_I$ when a profit floor is in effect. If there is no floor or the floor is ineffective, $\rho = 0$. The

[3] Notation such as a superscript "t" designating the product combination to which the demand function and reaction function apply has been omitted.

PO's generalized objective is $\omega_0 W_I + \omega_I \pi_i + \omega_2 C_I + \omega_3 R_I$ with the objective weights including ρ normalized so that they sum to one.

At each step of the method of fictitious play the PO chooses its price by solving $Max_{P_I}\{\sum_{t\in T} \mu_t\, f_I^t\big(P_I, P_E^t(P_I)\big)|P_I \in S\}$ given the PO's estimate of the frequencies, $\mu_t \forall\ t \in T$, with which the EC uses its product combinations.[4] We may now provide specific forms for the elements of this problem. The objective is to maximize $\sum_{t\in T} \mu_t[\omega_0 W_I + \omega_I \pi_i + \omega_2 C_I + \omega_3 R_I]$ subject to a set of linear inequalities that define the compact and convex set S of feasible price vectors. The simulator is designed to accept two kinds of constraints. First, the PO may be required to observe a global price cap. This imposes a single constraint $v'P_I \leq g,\ v$ is a vector of volume weights for the calculation of a global price index which is capped at the value g. Second, the postal regulator may impose caps and floors individually on the prices of the PO's products as in the US. The US price caps are intended to prevent USPS from freely exercising its market power in those markets where it is considered dominant. The floors are imposed to prevent cross subsidies. The simulator allows a user to explore different regulatory environments by mixing and matching a global cap with various individual caps and floors, and by varying the vector of volume weights for a global cap.

The simulator is designed so that price floors for all of the PO's prices are assumed by default. Users must indicate when a price cap applies to a product in which case the corresponding floor is over-ridden.[5] This restriction prevents price caps and floors from ever working at cross purposes. The constraints for the caps are $P_I \leq \bar{P}_c$ and for the floors are $P_I \geq \bar{P}_f$ where $\bar{P}_c$ and $\bar{P}_f$ are vectors of the cap and floor levels. However, the user chooses, product-by-product, whether the cap or the floor constraint is applicable.

Altogether, the specific problem that is solved for the PO at each step is:

$$Max_{P_I}\{\textstyle\sum_{t\in T} \mu_t\, [\omega_0 W_I + \omega_I \pi_i + \omega_2 C_I + \omega_3 R_I]|v'P_I \leq g, P_I \leq \bar{P}_c, P_I \geq \bar{P}_f\}.$$

The problem is solved using the Lagrange multiplier method. We form a Lagrangian from the problem's objective and constraints using multipliers for the constraints. The Lagrangian for the PO's maximization problem is:

[4] The superscript "i" designating the iteration for the frequencies has been omitted.

[5] Users may install zeros for the price floors to negate them without also capping them.

$L_I = \sum_{t\in T}\mu_t\,[\omega_0 W_I + \omega_I \pi_i + \omega_2 C_I + \omega_3 R_I] + \theta_g[v' P_I - g] + \theta_c'[P_I - \bar{P}_c] +$
$\theta_f'\left[P_I - \bar{P}_f\right]$ with $\theta_g \le 0$, $\theta_c \le 0$ and $\theta_f \le 0$.

$\theta_g$is a scalar multiplier; θ_c and $\theta_f$are vectors of multipliers. The Lagrangian is formed so that price caps have non-positive multipliers and price floors have non-negative multipliers. These signs conform to the expected effects on the maximized objective of increases in the caps and floors.

The price gradient of the Lagrangian is:

$$\nabla L_I = \sum_{t\in T}\mu_t\,[\omega_0 \nabla W_I + \omega_I \nabla\pi_i + \omega_2 \nabla C_I + \omega_3 \nabla R_I] + \theta_g v + \theta_c + \theta_f$$

The gradient of the Lagrangian has components corresponding to the price gradients of the PO's alternative objective quantities. These components are:

$\nabla\pi_I = EP_I + \gamma + Em_I$ profit.

$\nabla W_I = 2EP_I + \gamma - Em_i$ welfare from the PO's products.

$\nabla C_I = E\bar{m}_I$ cost with adjusted marginal costs.

$\nabla R_I = 2EP_I + \gamma$ revenue.

The price vector P_I is the same for all of the EC's product combinations. Therefore, we may substitute the weighted averages $\bar{E} = \sum_{t\in T}\mu_t E$ and $\bar{\gamma} = \sum_{t\in T}\mu_t\gamma$ wherever the weighted averages appear in the Lagrangian gradient to get:

$\nabla L_I = \omega_0[\bar{E}P_I - \bar{E}m_I] + \omega_1[2\bar{E}P_I + \bar{\gamma} - \bar{E}m_I] + \omega_2\bar{E}\bar{m}_I + \omega_3[2\bar{E}P_I + \bar{\gamma}] + \theta_g v +$
$\theta_c + \theta_f$

We may further simplify ∇L_I by rearranging and collecting terms. Let $k = \omega_0 + 2\omega_1 + 2\omega_3$ and $K = (\omega_0 + \omega_1 + \omega_2)\bar{E}m_I - \omega_2\bar{E}(\bar{m}_I - m_I) - (\omega_1 + \omega_3)\bar{\gamma}$. At any step of the method of fictitious play k and K are fixed. The Lagrangian gradient becomes the linear functional $\nabla L_I = k\bar{E}P_I + \theta_g v + \theta_c + \theta_f - K$ with variables P_I, θ_g, θ_c and θ_f.

Necessary conditions for a solution to the PO's maximization problem are that the Lagrangian gradient equal zero, that the constraint terms in the PO's Lagrangian also all equal zero, and that the Lagrange multipliers have all the correct signs. These conditions may be simplified somewhat by exploiting the fact that the price caps and floors are not simultaneously effective. We may combine the multiplier vectors for these constraints as $\theta = \theta_c + \theta_f$. And we may combine the matching constraint terms in the Lagrangian as

$\theta'[P_I - \bar{P}]$ where $\bar{P}$ is a vector composed of the chosen elements of $\bar{P}_c$ and $\bar{P}_f$. The necessary conditions are:

$$k\bar{E}P_I + \theta_g v + \theta - K = 0,$$

$$\theta_g[v'P_I - g] = 0, \theta'[P_I - \bar{P}] = 0,$$

$$\theta_g \leq 0, \theta \begin{cases} \leq \text{ for constraint elements corresponding to price caps} \\ \geq 0 \text{ for constraint elements corresponding to price floors} \end{cases}$$

9. The Method of Solution

The necessary conditions are solved at each step of the method of fictitious play to generate the PO's price vector P_I used in the next step. The method of solution is to use the necessary conditions to form a linear equation system under the assumption that a specific combinations of the price constraints are binding. Such a system is formed and solved for every possible combination of binding price constraints. The solutions that do not violate the constraints or the sign conditions on the Lagrange multipliers are then compared with respect to their objective values.[6] The price vector P_I is taken from the solution with the highest objective value.

The linear system is composed of the Lagrangian gradient set equal to zero and the equation for the global price index. The equation system takes two forms depending upon whether or not the global price cap is a binding constraint.

$$\begin{bmatrix} k\bar{E} & v \\ v' & 0 \end{bmatrix} \begin{bmatrix} P_I \\ \theta_g \end{bmatrix} = \begin{bmatrix} K \\ g \end{bmatrix} - \begin{bmatrix} \theta \\ 0 \end{bmatrix} \text{ the global price cap constraint is binding.}$$

$$\begin{bmatrix} k\bar{E} & 0 \\ v' & 1 \end{bmatrix} \begin{bmatrix} P_I \\ -g \end{bmatrix} = \begin{bmatrix} K \\ 0 \end{bmatrix} - \begin{bmatrix} \theta \\ 0 \end{bmatrix} \text{ the global price cap constraint is nonbinding.}$$

In the first equation system, P_I, θ_g and θ are variables and g is the predetermined value of the global price cap. In the second, g and θ_g switch roles with $\theta_g = 0$. The variables of the second system are P_I, g and θ. We may combine these two equation systems by using the switch $d = \begin{cases} 1 \text{ if the global price cap constraint is binding,} \\ 0 \text{ if the global price cap constraint is nonbinding} \end{cases}$

The combined linear system is:

[6] If there are no combinations for which both the prices and multipliers meet these conditions, then the comparison is conducted among the solutions that do not violate the constraints without regard for the signs of the multipliers.

$$\begin{bmatrix} k\bar{E} & dv \\ v' & (1-d) \end{bmatrix} \begin{bmatrix} P_I \\ d\theta_g - (1-d)g \end{bmatrix} = \begin{bmatrix} K \\ dg \end{bmatrix} - \begin{bmatrix} \theta \\ 0 \end{bmatrix}$$

We can solve this linear system if we know which of the conditions $P_I \leq \bar{P}_c$ and $P_I \geq \bar{P}_f$ are binding. Let $P_I = \begin{bmatrix} P_B \\ P_N \end{bmatrix}$ be a partition of the PO's price vector into components that correspond to the prices that are bounded, P_B, and prices that are not bounded , P_N. A price is bounded if it is equal in the solution to its cap or floor; it is not bounded if is not equal to its cap or floor. $\bar{P} = \begin{bmatrix} \bar{P}_B \\ \bar{P}_N \end{bmatrix}$ and $\theta = \begin{bmatrix} \theta_B \\ \theta_N \end{bmatrix}$are the conforming partitions of $\bar{P}$ and θ. We extend the partitioning to the equation system as follows:

$$\begin{bmatrix} k\bar{E}_{BB} & k\bar{E}_{BN} & dv_B \\ k\bar{E}_{NB} & k\bar{E}_{NN} & dv_N \\ v'_B & v'_N & (1-d) \end{bmatrix} \begin{bmatrix} P_B \\ P_N \\ d\theta_g - (1-d)g \end{bmatrix} = \begin{bmatrix} K_B \\ K_N \\ dg \end{bmatrix} - \begin{bmatrix} \theta_B \\ \theta_N \\ 0 \end{bmatrix}$$

Now, consider the necessary condition $\theta'[P_I - \bar{P}] = 0$. Using the partition notation this condition is $\theta'_B[P_B - \bar{P}_B] + \theta'_N[P_N - \bar{P}_N] = 0$. We have $P_B = \bar{P}_B$for prices that are bound, and we have $\theta_N = 0$ for prices that are not bound. These substitutions may be made in the partitioned equation system and the system rearranged to obtain a linear system that can be solved for the remaining variables $\theta_B, P_N$and θ_g(or g). The rearranged linear system is:

$$\begin{bmatrix} I & k\bar{E}_{BN} & dv_B \\ 0 & k\bar{E}_{NN} & dv_N \\ 0 & v'_N & (1-d) \end{bmatrix} \begin{bmatrix} \theta_B \\ P_N \\ d\theta_g - (1-d)g \end{bmatrix} = \begin{bmatrix} K_B \\ K_N \\ dg \end{bmatrix} - \begin{bmatrix} k\bar{E}_{BB} \\ k\bar{E}_{NB} \\ v'_B \end{bmatrix} \bar{P}_B$$

"I" denotes an identity matrix with the same dimensions as $\bar{E}_{BB}$. The coefficient matrix on the left-hand side is non-singular so the rearranged linear system has the straight forward solution:

$$\begin{bmatrix} \theta_B \\ P_N \\ d\theta_g - (1-d)g \end{bmatrix} = \begin{bmatrix} I & k\bar{E}_{BN} & dv_B \\ 0 & k\bar{E}_{NN} & dv_N \\ 0 & v'_N & (1-d) \end{bmatrix}^{-1} \left\{ \begin{bmatrix} K_B \\ K_N \\ dg \end{bmatrix} - \begin{bmatrix} k\bar{E}_{BB} \\ k\bar{E}_{NB} \\ v'_B \end{bmatrix} \bar{P}_B \right\}$$

The simulator constructs and solves a linear system for every possible division of the PO's prices into bounded and unbounded prices. It will solve these systems twice if a global price constraint is present; once with $d = 0$ and once with $d = 1$. For each solution it is determined whether or not the values of $P_I = \begin{bmatrix} \bar{P}_B \\ P_N \end{bmatrix}$ satisfiy the constraints $v'P_I \leq g, P_I \leq \bar{P}_c$ and $P_I \geq \bar{P}_f$. It is also determined if the calculated multipliers $\theta_g$and $\theta = \begin{bmatrix} \theta_B \\ 0 \end{bmatrix}$ have the correct signs $\theta_g \leq 0$, $\theta_c \leq 0$ and $\theta_f \leq 0$. If these tests are passed the solution is feasible. Objective values of the solutions are calculated from $\sum_{t \in T} \mu_t \, [\omega_0 W_I + \omega_I \pi_i + \omega_2 C_I + \omega_3 R_I]$. The incumbent PO's pricing problem is solved by the feasible solution with the highest objective value.

The method of solution includes an algorithm for determining the multiplier ρ when the PO's maximization problem includes the profit floor $\pi_I <= \bar{\pi}_I$. There can be three different outcomes from adding this constraint to the PO's problem. First, the constraint may be ineffective. Then, ρ is set at 0. and the PO's objective includes only a negligible add-on weight. Second, the constraint may make the PO's problem infeasible. When this happens ρ is set at 1000.0 and the PO's objective is effectively converted to maximizing the PO's profit. Third, we may have $\pi_I = \bar{\pi}_I$ for $0 < \rho < 1000$. In this case the value of the multiplier ρ is found by applying the algorithm.

The algorithm applies a version of the method of chords. The basic idea is to approximate the functional relationship between π_I and ρ, let us call it $\pi_I = f(\rho)$, and then use the inverse function to calculate $\rho = f^{-1}(\bar{\pi}_I)$. A parabolic section is used as the approximating function $\hat{f}$ and the function is fit at each iteration using the three points from past iterations that most closely straddle $\bar{\pi}_I$. $\hat{\rho} = \hat{f}^{-1}(\bar{\pi}_I)$ is then used to solve the PO's maximization problem. If the resulting profit $\hat{\pi}_I$ is close enough to $\bar{\pi}_I$, then the algorithm terminates. If not, then the data point $(\hat{\rho},\ \bar{\pi}_I)$ is used to improve the approximation $\hat{f}$ in the region of $\bar{\pi}_I$ and the process is repeated. The process converges monotonically (and rapidly!) on a solution because the points used to re-estimate $\hat{f}$ are successively closer to $\bar{\pi}_I$.

10. An Illustrative Simulation of Liberalized U.S. Postal Markets

Table 4 holds an example of a simulation of US postal markets after an assumed de-regulation and liberalization. The example was devised using demand elasticities and their standard errors for FY 2013 taken from an AIDS model recently fit to USPS data by Bzhilyanskaya *et al*. The volumes and prices used to center the demand model are USPS volumes and revenues per piece for FY 2015. Volumes, revenues, costs, etc. are shown in thousands (000); prices, etc. are in constant dollars. USPS unit costs are roughly equal to USPS volume variable costs per piece in FY 2015.

The input data and control settings have been devised to provide an example that illustrates the capabilities of the simulator, rather than to accurately portray the outcome of an actual liberalization and de-regulation of US postal markets. To calibrate the demand model we assumed that all USPS market shares are 0.5 and that all of the diversion rates are 0.9. For the cost model we simply assumed that an EC's cost function would be identical to the USPS cost function with fixed costs omitted.

The control values for the simulation are shown boxed in red in Table 4. To produce the example we assigned USPS a reserved area comprised of just First-Class mail. Entrants were assumed to always be present in the market for Competitive Expedited Package Services. Otherwise, the ECs were free to enter and exit the remaining four markets. Profit maximization was selected as the objective for USPS. Therefore, no profit floor was imposed.

The price of First-Class mail was capped at \$0.55. It was assumed that the prices of the remaining five classes would be de-regulated by eliminating any price caps that apply to these classes under current law. Therefore, the remaining regulatory price controls consist of default price floors for all of the classes except First-Class mail. There is no global price cap so the simulator just computes the price index using the given weights.

The simulator was run for 500 iterations but only the last 200 iterations were used to calculate the averages labeled "Simulated" in the table. Table 4 exhibits the prices, entry frequencies and other statistics for both the last iteration and for the simulation averages.

The simulated results for this case are characteristic of the results we have obtained with many tests of the simulator. The simulator has converged upon a solution that exhibits limit-pricing by USPS and stochastic selection of product combinations by a single potential entrant. The "Incumbent" prices shown in Table 4 confront the "Entrant" with six product combinations for which the EC takes the same loss of about -6,822,058. The identity of the services in each combination are shown at the bottom left-hand side of Table 4..

Table 4: A Simulation of Liberalized US Postal Markets

Simulation

Macro Controls (Hot Keys)			Previous	Incumbent Price Simulated	Last Iteration		Entrant Price (when Entered) Last Iteration	Simulated		Global Price Cap Weights	Cost Objective Unit V V Cost
Key	Effect	1Cls	0.5500	0.5500	0.5500	1Cls			1Cls	63,149,837	0.2047
\x	Execute from Start	PrOth	5.1069	5.1069	5.1069	PrOth	4.9877	4.9888	PrOth	6,945,272	4.2528
\c	Continue Execution	2Per	0.7003	0.6992	0.6997	2Per	0.6753	0.6697	2Per	5,838,175	0.3598
\l	One Iteration Loop	3Std	0.2136	0.2135	0.2135	3Std	0.1939	0.1936	3Std	80,090,273	0.1382
\p	Single Pass	4Pkg	3.7437	3.7427	3.7431	4Pkg	3.4363	3.4277	4Pkg	564,576	1.3602
		PclSR	2.1050	2.1061	2.1057	PclSR		1.8519	PclSR	7,875,043	1.4134

Statistical Frequency Models			Frequency of Entrant Product Offerings Freq. Model	Simulated	Case Frequency		Expected Volume Incumbent	Entrant		Last Iteration	Simulated
1	Simple Average (SA)								Inc. Expected Profit	-4,859,825	-4,861,865
2	SA w/o Run Up	1Cls	0.0000	0.0000	0.0000	1Cls	52,558,066	0	Incumbent Profit	-5,516,705	-4,619,967
3	SA w/ Run Up w/ Censoring	PrOth	1.0000	1.0000	1.0000	PrOth	3,080,847	4,421,514	Ent. Maximum Profit	-6,818,589	-6,822,058
4	SA w/o Run Up w/ Censoring	2Per	0.5083	0.5102	0.4876	2Per	3,140,382	1,243,318	Incumbent Welfare	37,142,733	36,788,670
5	Weighted Average (WA)	3Std	0.7490	0.7495	0.7214	3Std	39,438,631	53,489,212	Inc. Adjusted Cost	65,814,431	65,814,070
6	WA w/o Run Up	4Pkg	1.0000	1.0000	1.0000	4Pkg	213,890	300,492	Incumbent Revenue	60,954,606	60,952,205
7	WA w/ Run Up w/ Censoring	PclSR	0.3485	0.3481	0.3532	PclSR	2,324,487	1,554,878			
8	WA w/o Run Up w/ Censoring								Global Cap Index	0.6694	0.6693

Infeasibility Codes		Incumbent Objective Weights Objective	Weight	Selection		All Entrants	per Entrant	Number of Entrants Last Iteration	Simulated	Sample Size
0	Volumes all Non-negative	Welfare	0.0000	0	Revenue	38,467,059	38,467,059			200
1	Negative Entrant Volume	Profit	1.0000	1	V V Cost	29,895,332	29,895,332	1.0000	1.0000	Iteration
2	Negative Incumbent Volume	Adj'd Cost	0.0000	0	Non-V V Cost	15,390,316	15,390,316			500
3	Negative Ent. & Inc. Volumes	Revenue	0.0000	0	Profit	-6,818,589	-6,818,589	Incumbent Case	Entrant Case	End Simulation
10 to 13	Also Negative Entrant Price							1	30	500

Multipliers for Price Caps and Floors

Mail Catagories		Index	Multiplier	Price Floor	Price Cap		Reserved (1) Entered (-1)	Price Capped
			0	None	0.6500	Global		0
1Cls	First-Class Mail	1Cls	26,687,459	0.2047	0.5500	1Cls	1	1
PrOth	Competitive Expedited Package Services	PrOth	0	4.2528	5.6121	PrOth	-1	0
2Per	Periodicals	2Per	0	0.3598	0.2722	2Per	0	0
3Std	Standard Mail	3Std	0	0.1382	0.2211	3Std	0	0
4Pkg	Market Dominant Package Services	4Pkg	0	1.3602	1.4278	4Pkg	0	0
PclSR	Parcel Select and Return Services	PclSR	0	1.4134	1.9937	PclSR	0	0

Method of Chords for Profit Constraint

Old	Inc. Profit	Rho
Upper Limit	-5,493,755	0.993648
Start Rho	-4,859,825	0.992336
Lower Limit	-5,500,001	0.992335
New	Inc. Profit	Rho
Upper Limit	-4,859,825	0.000100
Finish Rho	-5,500,000	0.000000
Lower Limit	-5,500,001	0.992335

Incumbent Infeasible	Profit Constraint Multiplier Old Rho	New Rho	Profit Floor	no Floor	Status Unbounded	Frequency Model
0	0.000100	0.000000	-5,500,000	0	-1	4

Binomial Coefficients for New Objective

a	b	c
-3.38773E+15	6.72798E+15	-3.34041E+15

Case No.	Flags (Include=1, Exclude=0) 1Cls	PrOth	2Per	3Std	4Pkg	PclSR	Current Case	Ent. Case Lookup	Incumbent Profit	Entrants Profit	Entrant Case Frequency Freq. Model	Simulated	Infeasibility Code
22	0	1	1	0	1	0		22	-274,699	-6,826,766	0.1971	0.2388	0
26	0	1	0	1	1	0		26	-4,783,059	-6,819,500	0.2178	0.1990	0
30	0	1	1	1	1	0	*****	30	-5,516,705	-6,818,589	0.2365	0.2090	0
54	0	1	1	0	1	1		54	-2,913,880	-6,835,910	0.0539	0.0398	2
58	0	1	0	1	1	1		58	-7,425,724	-6,824,934	0.2739	0.3134	2
62	0	1	1	1	1	1		62	-8,149,172	-6,830,894	0.0207	0.0000	2

Social Welfare All Products/Producers

	Last Iteration	Simulated
Consumers' Surplus	73,352,341	73,320,588
Producers' Surplus	-12,335,293	-11,442,026
Social Welfare	61,017,048	61,878,562

	Last Iteration	Simulated
Entrants Revenue	38,467,059	36,791,185
Entrants Vol. Var. Cost	29,895,332	28,999,949
Ent. Intra V V & Fixed	15,390,316	14,613,295

The entrant alternates its choice among these six combinations at frequencies derived from the last 200 iterations of the simulation. In effect, the EC employs a mixed strategy of entry and exit. For the individual products these frequencies translate into simulated entry probabilities of 1.000 each for Competitive Expedited Package Services and Market Dominant Package Services, 0.508 for Periodicals, 0.749 for Standard mail and 0.349 for Parcel Select and Return services. The entrant does not enter the prohibited market for First-Class mail.

Together, the "Incumbent Prices" and the simulated "Frequency of Entrant Product Offerings" describe an equilibrium pair of strategies as defined by game theory. They are optimal against each other. Neither the PO nor the EC have an incentive to change. Together the strategies establish both players' prices and the EC's probabilities for offering the six combinations of postal services. Other product combinations are not used either because they violate the assumed regulatory restrictions or because they result in lower profits for the ECs.

The example exhibits the competitive effects of the assumed regulatory regime. The price cap on First-Class mail is effective and has a positive multiplier value. USPS encounters competitors that are always present in the markets for Competitive Expedited Package Services and Market Dominant Package Services and Periodicals; and infrequently in the remaining markets that are open to entrants.

USPS's assumed position as the price leader places it at a disadvantage in postal markets where it faces competition. This fact is mostly evident from the "Incumbent" and "Entrant" prices at the top of Table 4. USPS gets underpriced in every market where and when an entrant is present. For example, when an EC enters the market for Standard Regular mail its price per piece averages \$0.1936 while the PO's average price for the directly competing service is \$0.2135.

Profits, costs, revenues, profits, and incumbent welfare (defined as consumers' surplus on the PO's products plus the PO's profit) and the global cap index are shown down the right-hand side of Table 4. Welfare calculated for all products and all producers of postal services are shown on the lower right. Values are exhibited for the last iteration and for averages over the selected sample. These are included to help users of the simulator to judge convergence and to permit comparisons with the results of other simulations.

11. Conclusion

It is rarely possible to conduct scientific experiments with an actual economic system. Simulation offers a practical alternative by substituting a model intended to mimic the system. However, the rules for setting up controlled experiments, taking observations, analyzing results and reaching valid conclusions all remain about the same as they would be for an actual experiment. Our simulator mimics the behavior of POs and ECs in inter-related postal markets that are de-regulated and opened to entry in various ways. However, the simulator is purely an experimental apparatus. To reveal empirically the behavioral characteristics of postal markets the simulator must serve within an appropriately designed research plan.

The design we have in mind is a set of experiments intended to identify an improved regulatory regime for USPS. The simulator's dimensions match the number of major classes of US mail. It can use demand estimates that are available from existing econometric studies of US postal markets, cost estimates that can be extracted from USPS Cost and Revenue Analysis (CRA) reports and parameters that can be judged with fair accuracy. It can account for uncertainty regarding USPS' corporate economic objective by allowing us to simulate all of the basic possibilities. And it enables us to mix and match regulatory arrangements by selecting among a range of regulatory devices including reserved areas, individual price caps and floors, a global price cap and a constraint imposed on USPS' profits.

Our simulator incorporates a major technical innovation. We treat the product selections of potential EC's as endogenous. At every step of a simulation the EC's choose a product configuration to maximize profits in response to the PO's prices. In turn, the PO chooses prices to maximize its objective given its estimate of the frequencies at which the ECs chooses the various product configurations. Test runs of the simulator quickly reveal the signature characteristics of equilibrium in liberalized postal markets. These are limit pricing by the PO and the stochastic use of product configurations by the ECs.

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