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"Corporate Policies with Temporary and Permanent Shocks"

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Corporate Policies with Temporary and Permanent Shocks*

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Abstract

We develop a dynamic model of investment, cash holdings, financing, and risk management policies in which firms face financing frictions and are subject to permanent and temporary cash flow shocks. In this model, target cash holdings depend on the long-term prospects of the firm, implying that the payout policy of the firm, its financing policy, and its cash-flow sensitivity of cash display a more realistic behavior than in prior models with financing frictions. In addition, risk management policies are richer and depend on the nature of cash flow shocks and potential collateral constraints. Lastly, the timing of investment and the firms initial asset mix both reflect financing frictions and the joint effects of permanent and temporary shocks.

Keywords: Corporate policies; permanent vs. temporary shocks; financing frictions.

JEL Classification Numbers: G31, G32, G35.

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During the past two decades, dynamic corporate finance models have become part of the mainstream literature in financial economics, providing insights and quantitative guidance for investment, financing, cash management, or risk management decisions under uncertainty. Two popular cash flow environments have been used extensively in this literature. In one, shocks are of permanent nature and cash flows are governed by a geometric Brownian motion (i.e. their growth rate is normally distributed). This environment has been a cornerstone of dynamic capital structure models (see e.g. Leland (1998) or Strebulaev (2007)) and real-options models (see e.g. McDonald and Siegel (1986) or Morellec and Schürhoff (2011)). In the other, shocks are of temporary nature and short-term cash flows are modeled by the increments of an arithmetic Brownian motion (i.e. cash flows are normally distributed). This has proved useful in models of liquidity management (see e.g. Décamps, Mariotti, Rochet, and Villeneuve (2011) or Bolton, Chen, and Wang (2011)) and in models of dynamic agency (see e.g. DeMarzo and Sannikov (2006) or Biais, Mariotti, Plantin, and Rochet (2007)).[1]

Assuming that shocks are either of permanent or temporary nature has the effect of dramatically simplifying dynamic models. However, corporate cash flows cannot generally be fully described using solely temporary or permanent shocks. Many types of production, market, or macroeconomic shocks are of temporary nature and do not affect long-term prospects. But long-term cash flows also change over time due to various firm, industry, or macroeconomic shocks that are of permanent nature. In addition, focusing on one type of shocks produces implications that are sometimes inconsistent with the evidence. For example, in models based solely on permanent shocks, cash flows cannot be negative without having negative asset values, the volatilities of earnings and asset value growth rates are equal, and innovations in cash flows are perfectly correlated with those in asset values (see Gorbenko and Strebulaev (2010)). In liquidity management models based solely on temporary shocks, cash holdings are the only state variable for the firm's problem, equity issues always have the same size, and the cash-flow sensitivity of cash is either zero or one.

Our objective in this paper is therefore twofold. First, we seek to develop a dynamic model of investment, financing, cash holdings, and risk management decisions in which firms

[1] See Strebulaev and Whited (2012) for a recent survey of models based on permanent shocks. See Moreno-Bromberg and Rochet (2014) for a recent survey of liquidity models based on temporary shocks. See Biais, Mariotti, and Rochet (2013) for a recent survey of dynamic contracting models.

are exposed to both permanent and temporary cash flow shocks. Second, we want to use this model to shed light on existing empirical results and generate novel testable implications.

A prerequisite for our study is a model that captures in a simple fashion the joint effects of permanent and temporary shocks on firms' policy choices. In this paper, we base our analysis on a model of cash holdings and financing decisions with financing frictions in the spirit of Décamps, Mariotti, Rochet, and Villeneuve (2011), to which we add permanent shocks, an initial investment decision, and an analysis of risk management policies.

Specifically, we consider a firm with a valuable real investment opportunity. In order to undertake the investment project, the firm needs to raise costly outside funds. The firm has full flexibility in the timing of investment but the decision to invest is irreversible. The investment project, once completed, produces a stochastic stream of cash flows that depend on both permanent and temporary shocks. To account for the fundamentally different nature of these shocks, we model the firm cash flows in the following way. First, cash flows are subject to profitability shocks that are permanent in nature and governed by a geometric Brownian motion, as in standard real options and dynamic capital structure models. Second, for any given level of profitability, cash flows are also subject to short-term shocks that expose the firm to potential losses. These short-term cash flow shocks may be of temporary nature but they may also be correlated with permanent shocks, reflecting the level of cash flow persistence. In the model, the losses due to short-term shocks can be covered either using cash holdings or by raising funds at a cost in the capital markets. The firm may also hedge its exposure to permanent and temporary shocks by investing in financial derivatives or by changing its exposure to these shocks (asset substitution). When making investment, liquidity, financing, and hedging decisions, management maximizes shareholder value.

Using the model, we generate two sorts of implications. First, we show that a combination of temporary and permanent shocks can lead to policy choices that are in stark contrast with those in models based on a single source of risk. Second, our analysis demonstrates that temporary and permanent risks have different, often opposing, implications for corporate policies. Combining them produces implications that are consistent with a number of stylized facts and allows us to generate a rich set of testable predictions.

We highlight the main empirical implications. Compared to standard real options models in which firms have enough resources to fund investment and shocks are always permanent in nature, we find that the combination of financing frictions and temporary shocks significantly delays investment.[2] This delay is due to two separate effects. First, the cost of external finance increases the cost of investment, making the investment opportunity less attractive and leading to an increase in the profitability level required for investment. Second, the combination of temporary shocks and financing frictions reduces the value of the firm after investment, further delaying investment.

Importantly, this result is very different from those in prior studies, such as Boyle and Guthrie (2003), in which firms are solely exposed to permanent shocks and face financing frictions when seeking to invest in new projects. In such models, potential future financing constraints feed back in current policy choices by encouraging early investment. Our analysis therefore highlights another way by which financing constraints can distort investment behavior: The threat of future cash shortfalls increases future financing costs and reduces the value of the asset underlying the firm's growth option, thereby leading to late exercise of the investment opportunity. We also show that the effect can be quantitatively important. In our base case environment for example, investment is triggered for a profitability level that is 10% higher than in models without temporary shocks and financing frictions.

Another direct effect of temporary shocks and future financing frictions is that they alter the optimal asset mix of the firm. Notably, we show that as financing frictions or the volatility of temporary shocks increase, the firm decides to hold larger cash balances at the time of investment, so that its asset mix gets distorted towards safer assets.

After investment, the value of a constrained firm depends not only on the level of its cash reserves, as in prior dynamic models with financing frictions, but also on the value of the permanent shock (i.e. profitability). Notably, one interesting and unique feature of our model is that the ratio of cash holdings over profitability is the state variable of the firm's problem. This is largely consistent with the approach taken in the empirical literature, but it has not been clearly motivated by theory.[3] Given that the empirical literature uses a

[2] See for example the early papers of McDonald and Siegel (1986) and Dixit (1989) or the recent contributions of Carlson, Fisher, and Giammarino (2004), Lambrecht (2004), Manso (2008), Grenadier and Malenko (2010), Carlson, Fisher, and Giammarino (2010), or Grenadier and Malenko (2011). Dixit and Pindyck (1994) and Stokey (2009) provide excellent surveys of this literature.

[3] Opler, Pinkowitz, Stulz, and Williamson (1999) state about their measure of cash holdings: "We deflate

related proxy, it may not seem a very notable observation that cash holdings are scaled by profitability. However, the observation that "effective cash = cash/profitability" implies that more profitable firms hold more cash. That is, as the long-term prospects of the firm improve following positive permanent shocks, the firm becomes more valuable and finds it optimal to hoard more cash. By contrast, negative permanent shocks decrease firm value and, consequently, the optimal level of precautionary cash reserves.

We show in the paper that this relation between permanent shocks and target cash holdings has numerous implications. First, a standard result in corporate-liquidity models based solely on temporary shocks is that the cash-flow sensitivity of cash is either zero or one. In contrast, our model predicts that firms will demonstrate a non-trivial and realistic cash-flow sensitivity of cash, due to the effects of permanent shocks on target cash holdings. In our model, this sensitivity is measured by an explicit expression that depends on a number of firm, industry, or market characteristics. In particular, this sensitivity increases with financing frictions, consistent with the evidence in Almeida, Campello, and Weisbach (2004). Second, the relation between permanent shocks and target cash holdings implies that when firms access capital markets to raise funds, the size of equity issues is not constant as in prior models, but depend on the firm's profitability (i.e. on the value of the permanent shock). In particular, a unique prediction of our model is that more profitable firms access equity markets less often but raise more funds when accessing financial markets.

A third key implication of the relation between permanent shocks and target cash holdings concerns the effects of risk and uncertainty on cash holdings and firm value. Notably, we show in the paper that firm value increases in correlation between short-term and permanent shocks, that is, in the persistence of cash flow shocks. This is not immediately expected because two correlated shocks of temporary nature would allow for diversification if correlation decreased. So firm value would decrease in correlation between temporary shocks. Intuitively, the firm benefits from increased correlation between short-term and permanent shocks because it is then able to generate cash flows when they are needed to maintain scaled cash holdings. Another related implication is that an increase in the volatility of permanent cash flow shocks can increase firm value as long as permanent shocks are correlated with

liquid asset holdings by the book value of total assets, net of liquid assets, which we call net assets hereafter."

temporary shocks. (As in previous models, volatility in temporary shocks decreases firm value.) This effect arises despite the concavity of the value function and is due to the fact that volatility in permanent cash flow shocks can help managing liquidity when short-term shocks display persistence. Similar intuition applies to our predictions regarding optimal cash policies. Target cash holdings decrease with the persistence of cash flow shocks and can decrease with the volatility of permanent shocks if persistence is positive. Importantly, we also find that permanent shocks have large quantitative effects on firm value and optimal policies. Using conservative parameter values, the inclusion of permanent shocks in the model increases firm value by 19% and decreases target cash holdings by 12%.

Turning to risk management strategies, we show that derivatives usage should depend on whether the risk stems from temporary or permanent shocks. Specifically, if the firm's risk and futures prices are positively correlated, then hedging temporary shocks involves a short futures position while hedging permanent shocks may involve a long futures position. (And vice versa if the correlation is negative.) This means that hedging permanent shocks takes a position not contrary but aligned to exposure. In these cases, the firm prefers to increase cash flow volatility to increase cash flow correlation to permanent profitability shocks.

We also show that managing risk either by derivatives or by directly selecting the riskiness of assets (i.e. asset substitution) leads to the same outcome if the risk is due to temporary shocks. However, derivatives and asset substitution are not equivalent when managing the risk from permanent shocks. This is due to the fact that asset substitution does not generate immediate cash flows whereas derivatives do. This may not matter for an unconstrained firm, but it is a fundamental difference for a financially constrained firm. One prediction of the model is thus that a firm in distress would engage asset substitution with respect to permanent shocks but not in derivative hedging. Lastly, we show that when risk management is costly, constrained firms hedge less, consistent with the evidence in Rampini, Sufi, and Viswanathan (2014) that collateral constraints pay a major role in risk management. Again, these predictions are very different from those in models based on a single source of risk (see for example Bolton, Chen, and Wang (2011)).

As relevant as it is to analyze an integrated framework combining both temporary and permanent shocks, there are surprisingly only a few attempts in the literature addressing this problem. Gorbenko and Strebulaev (2010) consider a dynamic model without financing frictions, in which firm cash flows are subject to both permanent and temporary shocks. Their study focuses leverage choices. Our paper instead analyzes liquidity, refinancing, risk management, and investment policies. Another important difference between the two papers is that we model temporary shocks with a Brownian process instead of a Poisson process. Besides being more tractable, our modeling approach may also be more suitable for studying some of the corporate policies examined in the present paper, such as liquidity management or risk management. Grenadier and Malenko (2010) build a real options model in which firms are uncertain about the permanence of past shocks and have the option to learn before investing. In their model, there are no financing frictions and, as a result, no role for cash holdings and no need to optimize financing decisions.[4]

Our work is also directly related to the recent papers that incorporate financing frictions into dynamic models of corporate financial decisions. These include Bolton, Chen, and Wang (2011), Décamps, Mariotti, Rochet, and Villeneuve (2011), Gryglewicz (2011), Bolton, Chen, and Wang (2013), and Hugonnier, Malamud, and Morellec (2014). A key simplifying assumption in this literature is that cash flows are only subject to transitory shocks. That is, none of these papers has permanent shocks together with temporary shocks. As discussed above, this has the effect of producing empirical predictions that are sometimes inconsistent with the available evidence.

The paper is organized as follows. Section 1 describes the model. Section 2 solves for the value of a financially constrained firm and for the real option to invest in this firm. Section 3 derives the model's empirical implications. Section 4 examines risk management policies. Section 5 concludes. Technical developments are gathered in the Appendix.

1 Model

Throughout the paper, agents are risk neutral and discount cash flows at a constant rate $r > 0$. Time is continuous and uncertainty is modeled by a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$

[4]In a recent empirical study, Chang, Dasgupta, Wong, and Yao (2014) show that decomposing corporate cash flows into a transitory and a permanent component helps better understand how firms allocate cash flows and whether financial constraints matter in this allocation decision.

with the filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$, satisfying the usual conditions.

We consider a firm that owns an option to invest in a risky project. The firm has full flexibility in the timing of investment but the decision to invest is irreversible. The direct cost of investment is constant, denoted by $I > 0$. The project, once completed, produces a continuous stream of cash flows that are subject to both permanent and temporary shocks. Permanent shocks change the long-term prospects of the firm and influence cash flows permanently by affecting the productivity of the firm's assets. We denote the productivity of assets by $A = (A_t)_{t \geq 0}$ and assume that it is governed by a geometric Brownian motion:

$$dA_t = \mu A_t dt + \sigma_A A_t dW_t^P, \tag{1}$$

where μ and $\sigma_A > 0$ are constant parameters and $W^P = (W_t^P)_{t \geq 0}$ is a standard Brownian motion. In addition to these permanent shocks, cash flows are subject to short-term shocks that do not necessarily affect long-term prospects. Notably, we consider that operating cash flows dX_t after investment are proportional to A_t but uncertain and governed by:

$$dX_t = \alpha A_t dt + \sigma_X A_t dW_t^X, \tag{2}$$

where α and σ_X are strictly positive constants and $W^X = (W_t^X)_{t \geq 0}$ is a standard Brownian motion. W^X is allowed to be correlated with W^P with correlation coefficient ρ, in that

$$\mathbb{E}[dW_t^P dW_t^X] = \rho dt. \tag{3}$$

The dynamics of cash flows can then be rewritten as

$$dX_t = \alpha A_t dt + \sigma_X A_t (\rho dW_t^P + \sqrt{1 - \rho^2} dW_t^T), \tag{4}$$

where $W^T = (W_t^T)_{t \geq 0}$ is a Brownian motion independent from W^P. This decomposition implies that short-term cash flow shocks dW_t^X consist of temporary shocks dW_t^T and persistent shocks dW_t^P and that ρ is a measure of persistence of short-term cash flow shocks. In what follows we refer to σ_X as the volatility of short-term shocks or, when it does not cause confusion, as the volatility of temporary cash flow shocks.

The permanent nature of innovations in A implies that a unit increase or decrease in A increases or decreases the expected value of each future cash flow. To illustrate this property, it is useful to consider an environment in which the firm has a frictionless access to capital markets, as in e.g. Leland (1994) or McDonald and Siegel (1986). In this case, the value of the firm after investment V^{FB} is simply given by the present value of all future cash flows produced by the firm's assets. That is, we have

$$V^{FB}(a) = \mathbb{E}_a \left[\int_0^{\infty} e^{-rt} dX_t \right] = \frac{\alpha a}{r - \mu}. \tag{5}$$

Equation (5) shows that a shock that changes A_t via dW_t^P is permanent in the sense that a unit increase in A_t will increase all future expected levels of profitability by that unit (adjusted for the drift). A shock to W_t^T is temporary because, keeping everything else constant, it has no impact on future cash flows. That is, when cash flow shocks are not persistent, i.e. when $\rho = 0$, short-term cash flow shocks do not affect future level of cash flows. When cash flows shock are perfectly persistent, i.e. when $\rho = 1$, any cash flow shocks impact all future cash flows. Realistically, cash flow shocks are persistent but not perfectly and ρ is expected to take values between 0 and 1 for most firms.

The modeling of cash flows in equations (1) and (2) encompasses two popular frameworks as special cases. If $\mu = \sigma_A = 0$, we obtain the stationary framework of the models of liquidity management (see Décamps, Mariotti, Rochet, and Villeneuve (2011), Bolton, Chen, and Wang (2011), Hugonnier, Malamud, and Morellec (2014)) and dynamic agency (see DeMarzo and Sannikov (2006) or DeMarzo, Fishman, He, and Wang (2012)). As we show below, adding permanent shocks in these models gives rise to two sources of dynamic uncertainty that makes corporate policies intrinsically richer. If $\sigma_X = 0$, we obtain the model with time-varying profitability applied extensively in dynamic capital structure models (see Goldstein, Ju, and Leland (2001), Hackbarth, Miao, and Morellec (2006), Strebulaev (2007)) and real-options analysis (see Dixit and Pindyck (1994), Carlson, Fisher, and Giammarino (2006), Morellec and Schürhoff (2011)). Our model with temporary and permanent shocks differs from the latter in that earnings and asset volatilities differ and innovations in current cash flows are imperfectly correlated with those in asset values. As discussed in Gorbenko and Strebulaev (2010), all these features are consistent with empirical stylized facts.

In the absence of short-term shocks, the cash flows of an active firm are always positive since A is always positive. The short-term shock W^X exposes the firm to potential losses, that can be covered either using cash reserves or by raising outside funds. Specifically, we allow management to retain earnings inside the firm and denote by M_t the firm's cash holdings at any time $t > 0$. These cash reserves earn a constant interest rate $r - \lambda$ inside the firm, where $\lambda \in (0, r]$ is a cost of holding liquidity. We also allow the firm to increase its cash holdings or cover operating losses by raising funds in the capital markets.

When raising outside funds at time t, the firm has to pay a proportional cost $p > 1$ and a fixed cost ϕA_t so that if the firm raises some amount e_t from investors, it gets $e_t/p - \phi A_t$. As in Bolton, Chen, and Wang (2011), the fixed cost scales with firm size so that the firm does not grow out from the fixed cost. The net proceeds from equity issues are then stored in the cash reserve, whose dynamics evolve as:

$$dM_t = (r - \lambda)M_t dt + dX_t + \frac{dE_t}{p} - d\Phi_t - dL_t, \tag{6}$$

where L_t in the cumulative dividend paid to shareholders, E_t is the cumulative gross external financing raised from outside investors, and Φ_t is the cumulative fixed cost of financing.

Equation (6) is an accounting identity that indicates that cash reserves increase with the interest earned on cash holdings (first term on the right hand side), the firm's earnings (second term), and outside financing (third term), and decrease with financing costs (fourth term) and dividend payments (last term). In this equation, the cumulative gross financing raised from outside investors E_t and the cumulative fixed cost of financing Φ_t are formally defined as:[5]

$$\Phi_t = \sum_{n=1}^{\infty} \phi A_{\tau_n} 1_{\tau_n \leq t} \quad \text{and} \quad E_t = \sum_{n=1}^{\infty} e_n 1_{\tau_n \leq t},$$

for some increasing sequence of stopping times $(\tau_n)_{n=1}^{\infty}$ that represent the dates at which the firm raises funds from outside investors and some sequence of nonnegative random variables $(e_n)_{n=1}^{\infty}$ that represent the gross financing amounts.

[5]Technically, $((\tau_n)_{n\geq 1}, (e_n)_{n\geq 1}, L)$ belongs to the set $\mathcal{A}$ of admissible policies if and only if $(\tau_n)_{n\geq 1}$ is a non-decreasing sequence of $\mathbb{F}$-adapted stopping times, $(e_n)_{n\geq 1}$ is a sequence of nonnegative $(\mathcal{F}_{\tau_n})_{n\geq 1}$-adapted random variables, and L is a non-decreasing $\mathbb{F}$-adapted and right-continuous process with $L_0 \geq 0$.

The firm can abandon its assets at any time after investment by distributing all of its cash to shareholders. Alternatively, it can be liquidated if its cash buffer reaches zero following a series of negative shocks and raising outside funds to cover the shortfall is too costly. We consider that the liquidation value of assets represents a fraction $\omega < 1$ of their unconstrained value $V^{FB}(a)$ plus current cash holdings. The liquidation time is then defined by $\tau_0 \equiv \{t \geq 0 \,|\, M_t = 0\}$. If $\tau_0 = \infty$, the firm never chooses to liquidate.

Objective function. We solve the model backwards, starting with the value and optimal policies of an active firm. In a second stage, we derive the value-maximizing investment policy for the firm's growth option together with the value of the growth option.

The objective of shareholders after investment is choose the dividend, financing, and default policies that maximize firm value. (We also analyze risk management strategies in section 4.) There are two state variables for shareholders' optimization problem: Profitability A_t and the cash balance M_t. We can thus write shareholders' problem as:

$$V(a,m) = \sup_{(\tau_n)_{n\geq 1},(e_n)_{n\geq 1},L} \mathbb{E}_{a,m}\left[\int_0^{\tau_0} e^{-rt}(dL_t - dE_t) + e^{-r\tau_0}\frac{\omega\alpha A_{\tau_0}}{r-\mu}\right]. \tag{7}$$

The first term on the right hand side of equation (7) represents the present value of payments to incumbent shareholders until the liquidation time τ_0, net of the claim of new investors on future cash flows. The second term represents the firm's discounted liquidation value.

Consider next shareholders' investment decision. As discussed above, in the presence of short-term shocks and financing frictions, the firm will find it optimal to hold cash after investment. Thus, solving shareholders' investment problem entails finding both the optimal time to invest as well as the value-maximizing initial level of cash reserves m_0. Denote the value of the investment opportunity by $G(a)$. Shareholders' optimization problem before investment can then be formally written as:

$$G(a) = \sup_{\tau, m_0 \geq 0} \mathbb{E}_a\left[e^{-r\tau}\left(V(A_\tau, m_0) - p(I + m_0 + \phi A_\tau)\right)\right]. \tag{8}$$

It is important to note that the realizations of temporary shocks do not matter before investment since they have no impact on the profitability of investment. That is, persistent short-term shocks matter only in as much as they relate to the long-term productivity. Therefore, the only state variable in the investment problem is the productivity of the asset underlying the project, a. However, the parameters governing the short-term shocks, σ_X, α, and ρ, do influence the value of the investment opportunity and the optimal investment decision via their impact on the post-investment value of the firm.

2 Model solution

2.1 Value of an active firm

In this section, we base our analysis of shareholders' problem (7) on heuristic arguments. These arguments are formalized in the Appendix. To solve problem (7) and find the value of an active firm, we need to determine the financing, payout, and liquidation policies that maximize shareholder value after investment. Consider first financing and liquidation decisions. Because of the fixed cost of financing, it is natural to conjecture that it is optimal for shareholders to delay equity issues as much as possible. That is, if any issuance activity takes place, this must be when cash holdings drop down to zero, so as to avoid liquidation. At this point, the firm will either issue shares if the fixed cost of financing is not too high or it will liquidate. Consider next payout decisions. In the model, cash reserves allow the firm to reduce refinancing costs or the risk of inefficient liquidation. Because the benefit of an additional dollar retained in the firm is decreasing in the firm's cash reserves and keeping cash inside the firm entails an opportunity cost λ on any dollar saved, we conjecture that the optimal payout policy is characterized by a profitability-dependent target cash level $m^*(a)$, such that all earnings are retained when the firm's cash balance is below this level and all earnings are paid out when the cash balance is above this level.

To solve for firm value after investment, we first consider the region $(0, m^*(a))$ over which it is optimal to retain earnings. In this region, the firm does not deliver any cash flow to shareholders and equity value satisfies:

$$\begin{aligned} rV(a,m) &= \mu a V_a(a,m) + (\alpha a + (r-\lambda)m)\, V_m(a,m) \\ &+ \frac{1}{2}a^2 \left(\sigma_A^2 V_{aa}(a,m) + 2\rho\sigma_A\sigma_X V_{am}(a,m) + \sigma_X^2 V_{mm}(a,m)\right). \end{aligned} \tag{9}$$

where V_x denote the first-order derivative of the function V with respect to x and V_{xy} denotes the second-order partial derivative of V with respect to x and y. The left-hand side of this equation represents the required rate of return for investing in the firm's equity. The right-hand side is the expected change in equity value in the region where the firm retains earnings. The first two terms capture the effects of changes in profitability and cash savings on equity value. The last term captures the effects of volatility in cash flows and productivity. In our model with permanent and temporary shocks, changes in productivity affect not only the value of an active firm but also the value of cash reserves to shareholders in that $V_{am}(a, m) \neq 0$.

Equation (10) is solved subject to the following boundary conditions. First, when cash holdings exceed the dividend boundary function $m^*(a)$, the firm places no premium on internal funds and it is optimal to make a lump sum payment $m - m^*(a)$ to shareholders. As a result, we have

$$V(a, m) = V(a, m^*(a)) + m - m^*(a), \tag{10}$$

for all $m \geq m^*(a)$. Substracting $V(a, m^*(a))$ from both sides of this equation, dividing by $m - m^*(a)$, and taking the limit as m tends to $m^*(a)$ yields the condition

$$V_m(a, m^*(a)) = 1. \tag{11}$$

As V is assumed to be C^2 across the boundary function $m^*(a)$, condition (11) in turn implies the high-contact condition (see Dumas (1992)):

$$V_{mm}(a, m^*(a)) = 0, \tag{12}$$

that determines the location of the dividend boundary function.

When the fixed cost of external finance ϕ is not too large, the firm raises funds every time its cash buffer is depleted. In this case, the value-matching condition at zero is

$$V(a, 0) = V(a, \overline{m}(a)) - p(\overline{m}(a) + \phi a), \tag{13}$$

so that the value of the shareholders' claim when raising outside financing is equal to the continuation value (first term on the right-hand side) less issuance costs (second term). The value-maximizing issue size $\overline{m}(a)$ is then determined by the first-order condition:

$$V_m(a, \overline{m}(a)) = p, \tag{14}$$

which ensures that the marginal cost of outside funds is equal to the marginal benefits of cash holdings at the post-issuance level of cash reserves. As shown by this equation, the size of equity issues is not constant as in previous contributions, but depends on the firm's productivity. Lastly, when the fixed cost of financing makes an equity issue unattractive, liquidation is optimal at $m = 0$ and we have:

$$V(a, 0) = \frac{\omega \alpha a}{r - \mu}. \tag{15}$$

While there are two state variables for shareholders' optimization problem (10)-(15), this problem is homogeneous of degree one in a and m. We can thus write:

$$V(a, m) = aV(1, m/a) \equiv aF(c), \tag{16}$$

where $c \equiv \frac{m}{a}$ represents the scaled cash holdings of the firm and $F(c)$ is the scaled value function. Using this observation, the boundary conditions can be rewritten in terms of the scaled value function as a standard free boundary problem with only one state variable, the scaled cash holdings of the firm that evolve between the liquidation/refinancing trigger located at zero and the payout trigger c^*.

We can now follow the same steps as above to derive shareholders' modified (or scaled) optimization problem after investment. When scaled cash holdings are in $(0, c^*)$, it is optimal for shareholders to retain earnings and the scaled value function $F(c)$ satisfies:[6]

$$(r - \mu)F(c) = (\alpha + c(r - \lambda - \mu))F'(c) + \frac{1}{2}(\sigma_P^2 c^2 - 2\rho\sigma_A\sigma_X c + \sigma_X^2)F''(c). \tag{17}$$

[6]Using equation (16), we have that $V_m(a, m) = F'(c)$, $V_{mm}(a, m) = \frac{1}{a}F''(c)$, $V_a(a, m) = F - cF'(c)$, $V_{aa}(a, m) = \frac{c^2}{a}F''(c)$, and $V_{am}(a, m) = -\frac{c}{a}F''(c)$. Plugging these expressions in the partial differential equation (10) yields the ordinary differential equation (17) (with A as numéraire).

At the payout trigger c^*, $F(c)$ satisfies the value-matching and high-contact conditions

$$
\begin{aligned}
F'(c^*) &= 1, && (18)\\
F''(c^*) &= 0. && (19)
\end{aligned}
$$

Additionally, when the firm runs out of cash, shareholders can either refinance or liquidate assets. As a result, the scaled value function satisfies

$$F(0) = \max\left(\max_{c\in[-\phi,\infty)} \left(F(c) - p\left(c+\phi\right)\right); \frac{\omega\alpha}{r-\mu}\right). \tag{20}$$

When refinancing at zero is optimal, scaled cash holdings after refinancing $\overline{c}$ are given by the solution to the first-order condition:

$$F'(\overline{c}) = p. \tag{21}$$

Lastly, in the payout region $c > c^*$, the firm pays out any cash in excess of c^* and we have

$$F(c) = F(c^*) + c - c^*. \tag{22}$$

Before solving shareholders' problem, we can plug the value-matching and high-contact conditions (18)-(19) in equation (17) to determine the value of the firm at the target level of scaled cash holdings c^*. This shows that equity value satisfies

$$V(a, m^*(a)) = aF(c^*) = \frac{\alpha a}{r-\mu} + \left(1 - \frac{\lambda}{r-\mu}\right) m^*(a). \tag{23}$$

Together with equation (5), equation (23) implies that equity value in a constrained firm holding $m^*(a)$ units of cash is equal to the first best equity value minus the cost of holding liquidity, which is the product of the target level of cash holdings $m^*(a)$ and the present value of the unit cost of holding cash $\frac{\lambda}{r-\mu}$.

The following proposition summarizes these results and characterizes shareholders' optimal policies and value function after investment.

Proposition 1. *Consider a firm facing issuance costs of securities ($\phi > 1$, $p > 1$), costs of carrying cash ($0 < \lambda \leq r$), and short-term shock that are not perfectly persistent ($\rho < 1$). Then, the following holds:*

1. *The value of the firm, $V(m, a)$ solving problem (7), satisfies the relation $V(m, a) = aF(\frac{m}{a})$, where (F, c^*) is the unique solution to the system (17)-(22).*

2. *The function $F(c)$ is increasing and concave over $(0, \infty)$. $F'(c)$ is strictly greater than one for $c \in (0, c^*)$, where $c^* \equiv \inf\{c > 0 \,|\, F'(c) = 1\}$, and equal to one for $c \in [c^*, \infty)$.*

3. *If issuance costs are high, it is never optimal to issue new shares after investment, $F(0) = \frac{\omega\alpha}{r-\mu}$, and the firm is liquidated as soon as it runs out of cash.*

4. *If issuance costs are low, $F(0) = \max_{c \in [-\phi, \infty)} (F(c) - p\,(c + \phi)) > \frac{\omega\alpha}{r-\mu}$ and it is optimal to raise a dollar amount $e_n^* = p(\overline{c} + \phi)A_{\tau_n}$ from investors at each time τ_n at which the firm runs out of cash, where $\overline{c} \equiv (F')^{-1}(p)$.*

5. *When $m \in (0, m^*(a))$, the marginal value of cash is increasing in profitability. Any cash held in excess of the dividend boundary function $m^*(a) = c^* a$ is paid out to shareholders. Payments are made to shareholders at each time τ satisfying $M_\tau = c^* A_\tau$.*

Proposition 1 delivers several results. First, as in previous dynamic models with financing frictions (such as Bolton, Chen, and Wang (2011) or Décamps, Mariotti, Rochet, and Villeneuve (2011)), firm value is concave in cash reserves, which implies that shareholders behave in a risk-averse way. In particular, it is never optimal for shareholders to increase the risk of (scaled) cash reserves. Indeed, if the firm suffers from a series of shocks that deplete its cash reserves, it incurs some cost to raising external funds. In an effort to avoid these costs and preserve equity value, the firm behaves in a risk-averse fashion.

Second, Proposition 1 shows that when the cost of external funds is not too high, it is optimal for shareholders to refinance when the firm's cash reserves are depleted. In addition, the optimal issue size depends on the profitability of assets at the time τ_n of the equity issue and is given by $e_n^* = p(\overline{c} + \phi)A_{\tau_n}$. Thus, a unique feature of our model is that the size of equity issues is not constant. Rather, more profitable firms make larger equity issues.

Third, prior research has shown that the marginal value of cash should be decreasing in cash reserves and increasing in financing frictions (see e.g. Décamps, Mariotti, Rochet, and Villeneuve (2011)). Proposition 1 shows that the marginal value of cash should also be increasing in profitability in that $V_{am} > 0$. As we show below, this result has important consequences for the cash flow sensitivity of cash.

Lastly, Proposition 1 shows that cash reserves are optimally reflected down at $m^*(a) = c^*a$. When cash reserves exceed $m^*(a)$, the firm is fully capitalized and places no premium on internal funds, so that it is optimal to make a lump sum payment $m - m^*(a)$ to shareholders. As we show in section 3.1 below, the desired level of reserves results from the trade-off between the cost of raising funds and the cost of holding liquid reserves.

Two special cases of our model are worth emphasizing. First, when $\rho = 1$, the volatility of the scaled cash holdings vanishes at $\frac{\sigma_X}{\sigma_A}$. Second, when $\lambda = 0$ liquid reserves can be stored within the firm at no cost. In each of these cases, Proposition 1 is modified as follows.

Proposition 2. *The following holds:*

1. *If $\lambda = 0$ and $\rho = 1$, the following strategy is optimal:*
   - *If $m \geq \frac{\sigma_X}{\sigma_A}a$ shareholders receive a lump sum payment of $a(c - \frac{\sigma_X}{\sigma_A})$ at time 0 and a continuous payment $((r-\mu)\frac{\sigma_X}{\sigma_A} + \alpha)A_t$ per unit of time afterwards.*
   - *If $m \leq \frac{\sigma_X}{\sigma_A}a$ shareholders receive nothing up to the hitting time of $\frac{\sigma_X}{\sigma_A}$ where they receive a continuous payment $((r-\mu)\frac{\sigma_X}{\sigma_A} + \alpha)A_t$ per unit of time afterwards.*
2. *If $\lambda = 0$ and $\rho < 1$, then shareholders always agree to postpone dividend distributions and no optimal payout policy exists.*

2.2 Value of the option to invest

Consider next the option to invest in the project. Following the literature on investment decisions under uncertainty (see Dixit and Pindyck (1994)), it is natural to conjecture that the optimal investment strategy for shareholders should be to invest when the value of the active firm exceeds the cost of investment by a sufficiently large margin.

In models without financing frictions, this margin reflects the value of waiting and postponing investment until more information about asset productivity is available. In addition to this standard effect arising from the irreversibility of the investment decision, our model incorporates a second friction: Operating the asset may create temporary losses and financing these losses is costly. Our analysis thus generalizes the canonical real options model to the presence of financing frictions.

Because of financing frictions, shareholders' optimization problem before investment involves choosing both the timing of investment and the initial level of cash reserves. For any investment time τ, the optimal initial level of cash reserves m_0, if positive, must satisfy the first-order condition in problem (8). That is, we must have:

$$V_m(A_\tau, m_0) = p. \tag{24}$$

This is the same condition as the one used in equation (14) for optimal cash reserves after refinancing. Thus, the initial level of cash reserves, if positive, is given by $m_0 = \overline{c}a$.

Next, for any initial level of reserves, the investment policy takes a form of a barrier policy whereby the firm invests as soon as the asset productivity process reaches some endogenous upper barrier. We denote the optimal threshold/barrier by a^*. Investment is then undertaken the first time that A_t is at or above a^*.

Since the firm does not deliver any cash flow before investment, standard arguments imply that the value of the investment opportunity $G(a)$ satisfies for any $a \in (0, a^*)$:

$$rG(a) = \mu a G'(a) + \frac{1}{2}\sigma_A^2 a^2 G''(a). \tag{25}$$

At the investment threshold (i.e. when $a = a^*$), the value of the option to invest $G(a)$ must equal the value of an active firm minus the cost of acquiring the assets and the costs of raising the initial cash. This requirement, together with $m_0 = \overline{m}(a) = \overline{c}a$, yields the value-matching condition:

$$G(a^*) = a^* F(\overline{c}) - p(\overline{c}a^* + \phi a^*) - pI. \tag{26}$$

Optimality of a^* further requires that the slopes of the pre- and post-investment values are equal when $a = a^*$. That is, the value of the investment opportunity $G(a)$ satisfies the smooth-pasting condition:

$$G'(a^*) = F(\overline{c}) - p(\overline{c} + \phi). \tag{27}$$

Solving shareholders' optimization problem yields the following result.

Proposition 3. *The following holds:*

1. *If the costs of external finance are low, in that $F(0) > \omega\alpha/(r-\mu)$, then the value of the option to invest is given by*

$$G(a) = \begin{cases} \left(\frac{a}{a^*}\right)^{\xi} (a^* F(0) - pI), & \forall a \in (0, a^*), \\ aF(0) - pI, & \forall a \geq a^*, \end{cases} \tag{28}$$

where the value-maximizing investment threshold satisfies

$$a^* = \frac{\xi}{\xi - 1} \frac{pI}{F(0)}, \tag{29}$$

with

$$\xi = \frac{\sigma_A^2 - 2\mu}{2\sigma_A^2} + \sqrt{\left(\frac{\sigma_A^2 - 2\mu}{2\sigma_A^2}\right)^2 + \frac{2r}{\sigma_P^2}} > 1. \tag{30}$$

Investment is undertaken the first time that $A_t \geq a^*$ and the firm's cash reserves at the time of investment are given by $m_0 = \overline{c}a^*$.

2. *If the costs of external finance are high, in that $F(0) = \omega\alpha/(r-\mu) > p\phi$, then the value of the option to invest is given by*

$$G(a) = \begin{cases} \left(\frac{a}{a^*}\right)^{\xi} (a^*(F(0) - p\phi) - pI), & \forall a \in (0, a^*), \\ a(F(0) - p\phi) - pI, & \forall a \geq a^*, \end{cases} \tag{31}$$

where the value-maximizing investment threshold satisfies

$$a^* = \frac{\xi}{\xi - 1} \frac{pI}{F(0) - p\phi}, \tag{32}$$

and ξ is defined in equation (30). Investment is undertaken the first time that $A_t \geq a^*$. No cash is raised in addition to the investment cost I and it is optimal to liquidate right after investment.

3. *If the costs of external finance are very high, in that $F(0) = \omega\alpha/(r - \mu) \leq p\phi$, the firm never invests and the value of the option to invest satisfies*

$$G(a) = 0, \ \forall a > 0. \tag{33}$$

As in standard real options models, Proposition 3 shows that, the value of the option to invest is the product of two terms when issuance costs are low: The net present value of the project at the time of investment, given by $a^* F(0) - pI$, and the present value of \$1 to be obtained at the time of investment, given by $\left(\frac{a}{a^*}\right)^\xi$. When issuance costs are high, it is either optimal to liquidate right after investment or to refrain from investing altogether.

Focusing on the more interesting case in which the costs of external finance are low, one can note that when $p = 1$ and the firm cash flows are not subject to temporary shocks ($\sigma_X = 0$), the optimal investment threshold becomes

$$a^*_{FB} = \frac{\xi}{\xi - 1} \frac{I}{F^{FB}}, \tag{34}$$

where $F^{FB} = \frac{V^{FB}(a)}{a} = \frac{\alpha}{r - \mu}$. The same threshold obtains for an investor without financing frictions (i.e. when $p = 1$ and $\phi = 0$). This equation can also be written as $a^*_{FB} F^{FB} = \frac{\xi}{\xi - 1} I$, where the right-hand side of this equation is the adjusted cost of investment. This adjusted cost reflects the option value of waiting through the factor $\frac{\xi}{\xi - 1}$.

Equation (34) recovers the well-known investment threshold of real options models (see for example Dixit and Pindyck (1994)). Except for the two special cases of our model ($p = 1$ and $\sigma_X = 0$ or $p = 1$ and $\phi = 0$), we have that $F(0)$ is strictly lower than F^{FB}, so that the investment threshold of Proposition 3 is strictly higher than the standard real options threshold. $F(0)$ is lower than F^{FB} because the firm faces financing frictions and holding liquidity inside the firm is costly. Our model uncovers how these two frictions affect the optimal investment policy and the value of the investment opportunity.

3 Model analysis

3.1 Permanent shocks and the value of a constrained firm

3.1.1 Comparative statics

Do temporary and permanent shocks have *qualitatively* the same effects on firm value and optimal policies? To answer this question, we consider in this section the effects of the parameters driving the dynamics of temporary and permanent shocks on the value of a constrained firm $F(c)$ and on target cash holdings c^*.

The following lemma derives comparative statics with respect to an exogenous parameter $\theta \in \{\sigma_X, \sigma_A, \rho, \phi, p, \alpha, \mu\}$. To make the dependence of F and c^* on θ explicit, we write $F = F(., \theta)$ and $c^* = c^*(\theta)$. Focusing on the refinancing case (results for the liquidation case are reported in the Appendix), we have that:

Proposition 4. *The following holds:*

1. *Firm value satifies*

$$\frac{\partial F}{\partial p}(c,p) < 0,\ \frac{\partial F}{\partial \phi}(c,\phi) < 0,\ \frac{\partial F}{\partial \alpha}(c,\alpha) > 0,\ \frac{\partial F}{\partial \mu}(c,\mu) > 0,\ \text{and}\ \frac{\partial F}{\partial \rho}(c,\rho) > 0.$$

2. *Target cash reserves satisfy*

$$\frac{dc^*(p)}{dp} > 0,\ \frac{dc^*(\phi)}{d\phi} > 0,\ \frac{dc^*(\alpha)}{d\alpha} < 0,\ \frac{dc^*(\mu)}{d\mu} > 0,\ \text{and}\ \frac{dc^*(\rho)}{d\rho} < 0.$$

Several results follow from Proposition 4. First, and consistent with economic intuition, firm value decreases and the target level of liquid reserves increases with financing frictions (as captured by p and ϕ). Second, both the growth rate of profitability μ and the mean cash flow rate α increase firm value. The target level of cash reserves also increases with the growth rate of the permanent shock, as the firm becomes more valuable. Interestingly, however, target cash reserves decrease with the mean cash flow rate, as it becomes less likely that the firm will need to raise costly funds as its cash flows increase. Third, the effect of persistence of short-term shocks ρ on firm value is also unambiguously positive. It is not immediately expected that firm value increases in correlation ρ between short- and long-term shocks. Indeed, if the firm faced two shocks of temporary nature, the result would be opposite. Lower correlation of two temporary shocks would allow for diversification and firm value would decrease in correlation between temporary shocks. Our result shows that correlation between short-term and permanent shocks works differently.

To understand why firm value increases with the persistence of cash flows ρ, think about a firm hit by a positive permanent shock. Its expected profitability increases and, in order to maintain scaled cash holdings, the firm needs to increase (unscaled) cash holdings. If short-term shocks are correlated with permanent shocks (i.e. if there is persistence in cash flow shocks), in expectation cash flows temporally increase and the firm has the means to increase cash holdings. If short-term shocks are not correlated with permanent shocks, the firm may not be able to do so and its value will benefit less from the positive permanent shock. It is also interesting to observe that an increase in the persistence of cash flows decreases target cash holdings. The intuition for the negative effect of persistence is that with higher persistence the firm gets positive cash flows shocks when they are needed to maintain scaled cash holdings, so that target cash holdings can be lower.

The effects of volatility on firm value and cash holdings are more difficult to characterize. Applying Proposition 8 in the Appendix, we can measure the effect of the volatility of short-term shocks σ_X on the (scaled) value of an active firm. Keeping persistence ρ constant, σ_X is also a measure of the volatility of temporary shocks. Notably, we have that:

$$\frac{\partial F}{\partial \sigma_X}(c, \sigma_X) = \mathbb{E}_c \left[\int_0^{\tau_0} e^{-(r-\mu)t} \left(-\rho \sigma_A C_{t^-} + \sigma_X \right) \frac{\partial^2 F}{\partial c^2}(C_{t^-}, \sigma_X) dt \right]. \tag{35}$$

Given that the function $F(c)$ is concave, we have that $\frac{\partial F(c)}{\partial \sigma_X} < 0$ if $\rho \leq 0$. For $\rho \in (0, 1)$, the sign of $\frac{\partial F(c)}{\partial \sigma_X}$ is not immediately clear. However, numerical simulations suggest that the effect of increased volatility of short-term shocks on firm value is negative, consistently with previous literature (see e.g. Décamps, Mariotti, Rochet, and Villeneuve (2011)).[7]

Consider next the effect of the volatility of permanent shocks on firm value. Applying Proposition 8 in the Appendix, we have:

$$\frac{\partial F}{\partial \sigma_A}(c, \sigma_A) = \mathbb{E}_c \left[\int_0^{\tau_0} e^{-(r-\mu)t} \left(\sigma_A C_{t^-} - \rho \sigma_X \right) C_{t^-} \frac{\partial^2 F}{\partial c^2}(C_{t^-}, \sigma_A) dt \right]. \tag{36}$$

Clearly, this equation shows that $\frac{\partial F(c)}{\partial \sigma_A} < 0$ if $\rho \leq 0$. When $\rho \in (0,1)$, the effect of an increase in the volatility of permanent shocks on firm value is ambiguous. The reason is that firm value decreases in the volatility of the state variable c, and σ_A may either increase or decrease this volatility. Indeed, the instantaneous variance of c is given by $\sigma_A^2 c^2 - 2\rho\sigma_A\sigma_X c + \sigma_X^2$. Its derivative with respect to σ_A is $2\sigma_A c^2 - 2\rho\sigma_X c$. Hence, the volatility of permanent shocks may increase firm value for low c and low σ_A and decrease firm value for high c and high σ_A. The intuition for the positive effect is that volatility in permanent cash flow shocks can help the firm manage its liquidity needs when cash flow shocks are persistent.

Lastly, note that the target level of cash holdings satisfies (see the Appendix):

$$\frac{dc^*(\theta)}{d\theta} = -\frac{r-\mu}{\lambda} \left(\frac{\partial F}{\partial \theta}(c^*(\theta), \theta) + c^*(\theta) \frac{\partial [\frac{\lambda}{r-\mu}]}{\partial \theta} - \frac{\partial [\frac{\alpha}{r-\mu}]}{\partial \theta} \right) \tag{37}$$

It follows from the previous discussion on the effects of σ_X and σ_A on $F(c)$ that $\frac{\partial c^*}{\partial \sigma_X} > 0$ and $\frac{\partial c^*}{\partial \sigma_A} > 0$ if $\rho \leq 0$, $\frac{\partial c^*}{\partial \sigma_X} > 0$, and $\frac{\partial c^*}{\partial \sigma_A} \gtrless 0$ if $\rho \in (0,1)$. These results mirror the results obtained for firm value. It is again interesting to observe that an increase in the volatility of permanent shocks may decrease target cash holdings.

For completeness, Figure 1 plots target cash holdings c^* and the scaled issuance size $\overline{c}$ as functions of the volatility of short-term shocks σ_X, the volatility of permanent shocks σ_A, the persistence of cash flows ρ, the fixed and proportional financing costs ϕ and p, and the carry cost of cash λ. The parameter values used to produce these panels are reported in Table

[7] It is clear from equation (35) that $c^* \leq \frac{\sigma_X}{\rho\sigma_A}$ is a sufficient condition for the negative derivative w.r.t. σ_X. The inequality $c^* \leq \frac{\sigma_X}{\rho\sigma_A}$ always holds at and near our baseline parameter values, but it can be violated if the cost of carrying cash λ is very low. Despite extensive simulation, we have not been able to find any instance of a positive effect of σ_X on F.

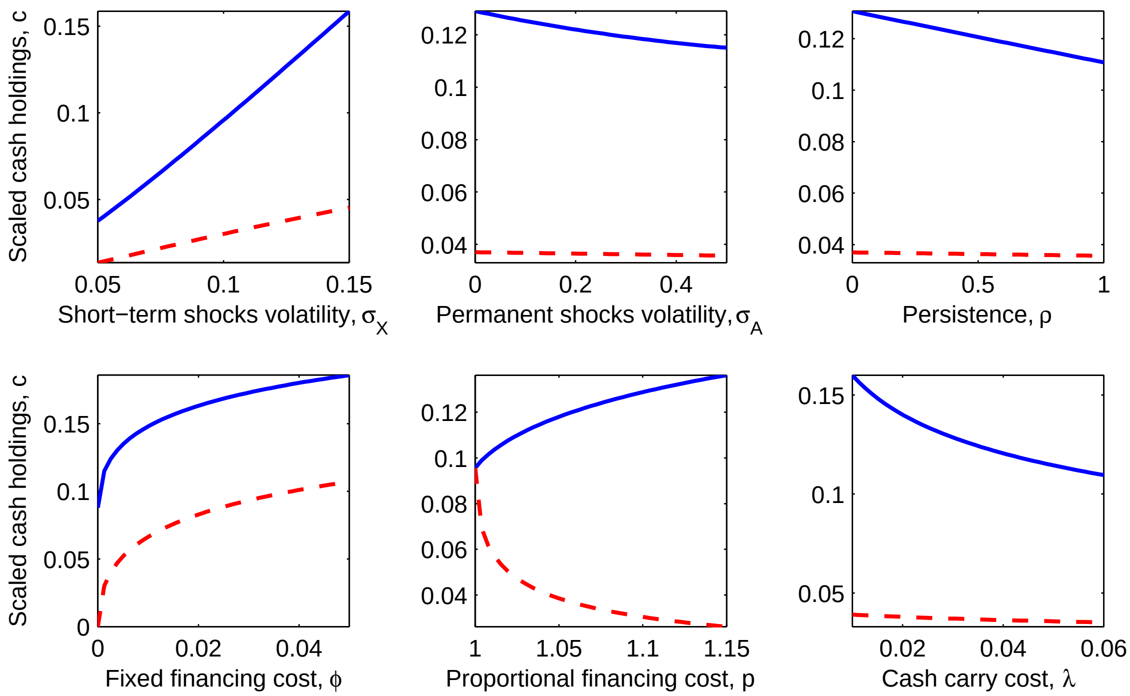

Figure 1: The effect of exogenous parameters on the target cash holdings c^* (solid curves) and on the scaled issuance size $\bar{c}$ (dashed curves) in the refinancing case. The constant parameter values are given in Table 1.

1 below. These panels confirm the above comparative statics results. They also show that the size of equity issues should increase with the fixed costs of external finance (since the benefit of issuing equity must exceed ϕ) and decrease with the proportional costs of external finance (since firm value is concave and $F'(\bar{c}) = p$). As in prior models, the effects of the other parameters on $\bar{c}$ mirror those of these parameters on target cash holdings.

3.1.2 How much do permanent shocks matter?

The previous section has shown that permanent shocks have qualitatively different effects on optimal policies than temporary shocks. The question we ask next is whether permanent shocks have non-trivial quantitative effects. To answer this question, we examine the predictions of the model for the firm's financing and cash holdings policies.

To do so, we select model parameters to match previous studies. Notably, following models with temporary shocks (e.g. Bolton, Chen, and Wang (2011, 2013)), we set the risk-free rate of return to $r = 6\%$, the mean cash flow rate to $\alpha = 0.18$, the diffusion coefficient on short-term shocks to $\sigma_X = 0.12$, and the carry cost of cash to $\lambda = 0.04$. We base the value of liquidation costs on the estimates of Glover (2014) and set $1 - \omega = 45\%$. Financing costs are set equal to $\phi = 0.002$ and $p = 1.06$. These values imply that when issuing equity, the firm pays a financing cost of 10.6%. The parameters of the permanent shocks are set equal to $\mu = 0.01$ and $\sigma_A = 0.25$ (consistent with the estimates of Morellec, Nikolov, and Schürhoff (2012)). Lastly, the persistence of short-term shocks is set to $\rho = 0.5$. The benchmark case parameter values are summarized in Table 1. We also examine the effects of varying these parameters on the firm's policy choices.

Figure 2 shows the effects of introducing time-varying profitability via persistent shocks in a dynamic model with financing frictions. To better understand the sources of changes, separate plots are shown in which we first introduce a positive drift only (Panel A with $\mu = 0.01$ and $\sigma_A = 0$), then a positive volatility only (Panel B with $\mu = 0$ and $\sigma_A = 0.25$), and finally in which we combine both drift and volatility effects (Panel C with $\mu = 0.01$ and $\sigma_A = 0.25$). Introducing a positive growth in cash flows is similar to introducing a capital stock that appreciates deterministically at the rate μ. As a result of this drift in cash flows, firm value is increased by 18% at the target level of cash reserves. However, target (scaled)

Variable	Symbol	Parameter	Symbol	Value
		Baseline Model		
Cash holdings	M	Growth rate of asset productivity	μ	0.01
Scaled cash holdings	C	Mean rate of cash flows	α	0.18
Asset productivity	A	Volatility of permanent shocks	σ_A	0.25
Cumulative cash flows	X	Volatility of short-term shocks	σ_X	0.12
Cumulative payout	L	Persistence of short-term shocks	ρ	0.5
Cumulative external financing	E	Riskfree rate	r	0.06
Cumulative fixed financing cost	Φ	Carry cost of liquidity	λ	0.04
Active firm value	V	Proportional financing cost	p	1.06
Scaled active firm value	F	Fixed financing cost	ϕ	0.002
Investment option value	G	Asset liquidation-value ratio	ω	0.55
Payout boundary	c^*	Investment cost	I	10
Financing target	$\overline{c}$			
Investment threshold	a^*			
		Risk Management		
Futures price	Y	Futures volatility	σ_Y	0.2
Futures position	h	Correlation between futures and firm permanent shocks	χ_P	0.7
Hedge ratio	g	Correlation between futures and firm temporary shocks	χ_T	0.7
		Margin-requirement ratio	π	10

Table 1: The model's main variables and parameters. The parameters are used in the benchmark case in numerical analyses.

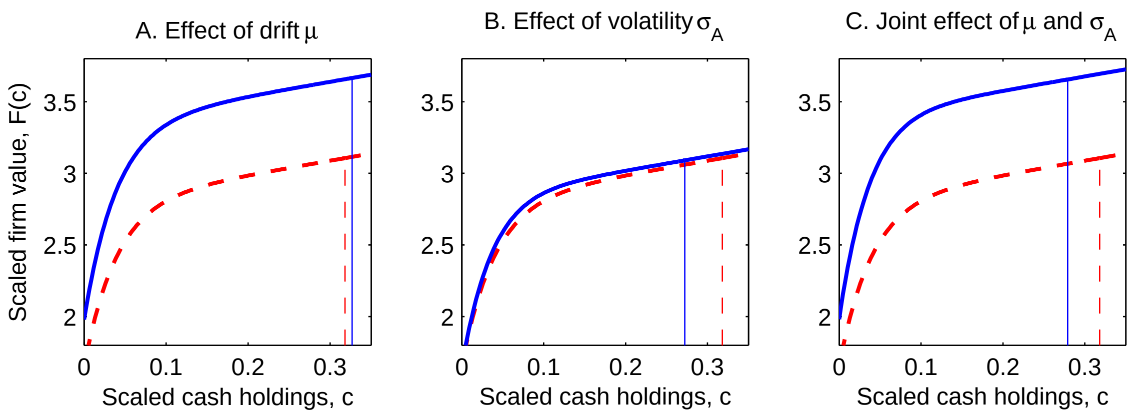

Figure 2: The effect of permanent shocks on firm value and target cash holdings in the liquidation case. The dashed curves represent the case with only temporary shocks ($\sigma_A = \mu = 0$) in all the panels. The solid curves are with permanent shocks, with $\mu = 0.01$ and $\sigma_A = 0$ in Panel A, $\mu = 0$ and $\sigma_A = 0.25$ in Panel B, and $\mu = 0.01$ and $\sigma_A = 0.25$ in Panel C. In all the cases, the vertical lines depict the target scaled cash holdings c^*. The other parameter values are given in Table 1.

cash holdings are mostly unaffected by the introduction of a permanent drift (an increase by less than 3%) as risk does not change.

By contrast, Figure 2 shows that adding volatility in A changes the target level of scaled cash holdings significantly without having a material effect on the value of the firm. In our base case parametrization for example, optimal cash holdings decrease by 14% since the volatility of scaled cash holdings is reduced by the introduction of volatility in A (in that we have $\sqrt{\sigma_A^2 c^2 - 2\rho\sigma_A\sigma_X c + \sigma_X^2} < \sigma_X$ over the relevant range). As shown by the figure, the joint effect of μ and σ_A is substantial on both firm value (an increase by 19% at the target) and target cash holdings (a decrease by 12%).

Figure 3 shows that similar results obtain in the refinancing case. Again the drift μ of permanent shocks affects mostly the value function and has little impact on optimal policies. The volatility σ_A of permanent shocks affects significantly optimal policies but not the value function.

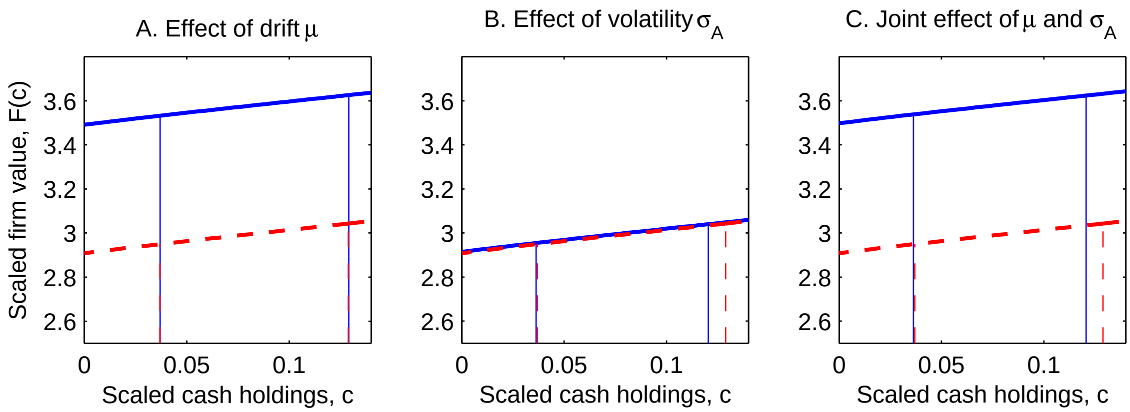

Figure 3: The effect of permanent shocks on firm value and target cash holdings in the refinancing case. The dashed curves represent the case with only temporary shocks ($\sigma_A = \mu = 0$) in all the panels. The solid curves are with permanent shocks, with $\mu = 0.01$ and $\sigma_A = 0$ in Panel A, $\mu = 0$ and $\sigma_A = 0.25$ in Panel B, and $\mu = 0.01$ and $\sigma_A = 0.25$ in Panel C. In all the cases, the vertical lines depict the scaled issuance size $\bar{c}$ and the target scaled cash holdings c^*. The other parameter values are given in Table 1.

3.2 Cash-flow sensitivity of cash

Corporate liquidity models featuring solely temporary shocks characterize optimal cash holdings and dividend policies using a constant target level of cash holdings (see e.g. Bolton, Chen, and Wang (2011), Décamps, Mariotti, Rochet, and Villeneuve (2011), or Hugonnier, Malamud, and Morellec (2014)). This generates the prediction that firms at the target distribute all positive cash flows or, equivalently, that cash holdings are insensitive to cash flows. As firms off the target retain all earnings, the predicted propensity to save from cash flows is either one or zero. Our model generates a more realistic firm behavior at the target cash level and provides an explicit measure of the cash-flow sensitivity of cash.

To illustrate this feature, suppose that the firm's cash holdings are at the target level, i.e. $M_t = c^* A_t$. As we show below, this is a most relevant assumption since the bulk of the probability mass of the stationary distribution of cash holdings is at the target level. Consider now what happens upon the realization of a cash flow shock dX_t. Profitability A_t changes in expectation by

$$\mathbb{E}[dA_t|dX_t] = \mu A_t dt + \sigma_A A_t \frac{\rho}{\sigma_X A_t}(dX_t - \alpha A_t dt). \tag{38}$$

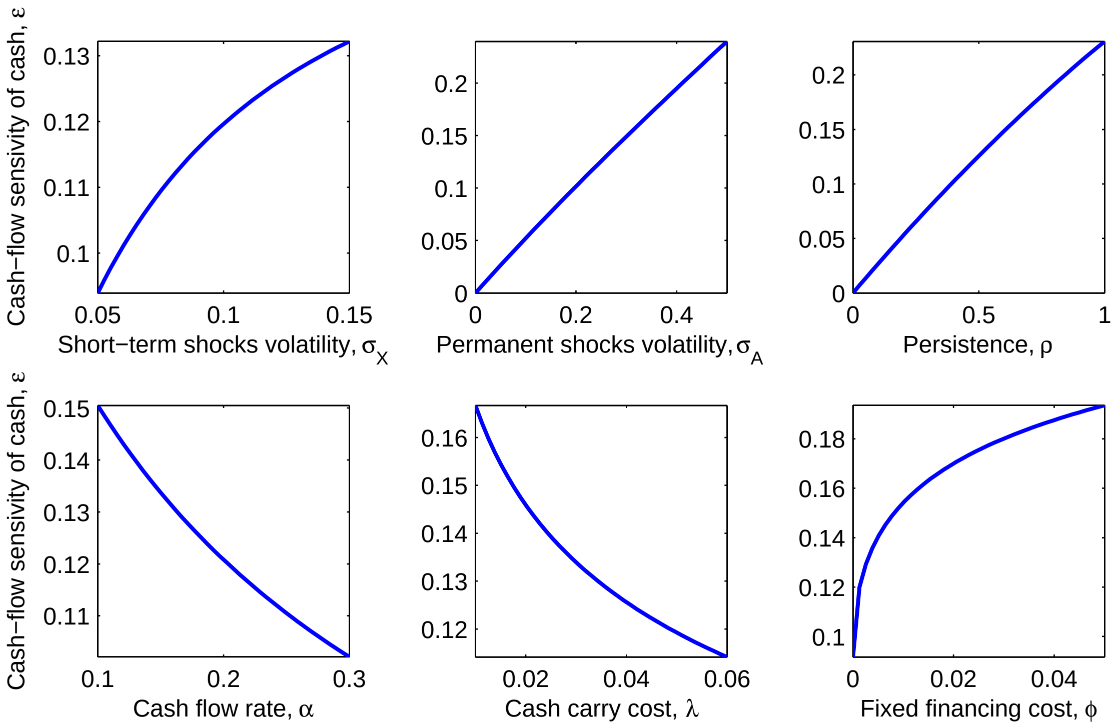

Figure 4: The effect of exogenous parameters on the cash-flow sensitivity of cash ϵ in the refinancing case. Input parameter values are given in Table 1.

Target cash holdings then change to $c^*(A_t+dA_t)$ and the increase or decrease in cash holdings conditional on $d\tilde{X}_t$ can be expressed as

$$\mathbb{E}[c^* dA_t | dX_t] = c^* \left(\mu - \frac{\alpha \rho \sigma_A}{\sigma_X} \right) A_t dt + \frac{\rho \sigma_A c^*}{\sigma_X} dX_t. \tag{39}$$

Excluding the deterministic part, we then have

$$\epsilon = \frac{\rho \sigma_A c^*}{\sigma_X} \tag{40}$$

which is a measure of the cash-flow sensitivity of cash.

It is clear that without permanent shocks (i.e. when $\sigma_A = 0$) or without persistence of temporary shocks (i.e. when $\rho = 0$), the cash-flow sensitivity of cash ϵ is zero. As shown by equation (40), the cash-flow sensitivity of cash ϵ in our model depends directly on the parameters of temporary and permanent shocks, ρ, σ_A, and σ_X, and indirectly on the other parameters of the model via the target level of cash holdings c^*. In particular, since c^* increases in the cost of refinancing, the cash-flow sensitivity of cash increases in financing frictions, consistent with the evidence in Almeida, Campello, and Weisbach (2004). Figure 4 presents the effects of various parameters of the model on our measure of the cash-flow sensitivity of cash in the refinancing case. As shown by the figure, the sensitivity increases in volatilities of both short-term and permanent shocks and in the persistence of cash flow shocks. The effect of σ_X on ϵ is due to the fact that an increase in the volatility of short-term shocks increases target cash holdings c^*. Lastly, note that the values of ϵ in Figure 4 are in the range reported in Almeida, Campello, and Weisbach (2004).

To support our claim that the bulk of the probability mass of the stationary distribution of cash holdings is at the target level, we next examine the stationary distribution of cash holdings implied by the model. This is done by simulating the model dynamics with the baseline parameter values in the refinancing case.[8] Figure 5 and Table 2 present the results and show that the stationary distribution of cash holdings is very skewed. The median level of cash reserves of 0.1009 is very close to the target level of cash reserves of 0.1206. The

[8]We can only compute the stationary distribution of cash holdings for the refinancing case since, in the liquidation case, the firm liquidates with probability 1.

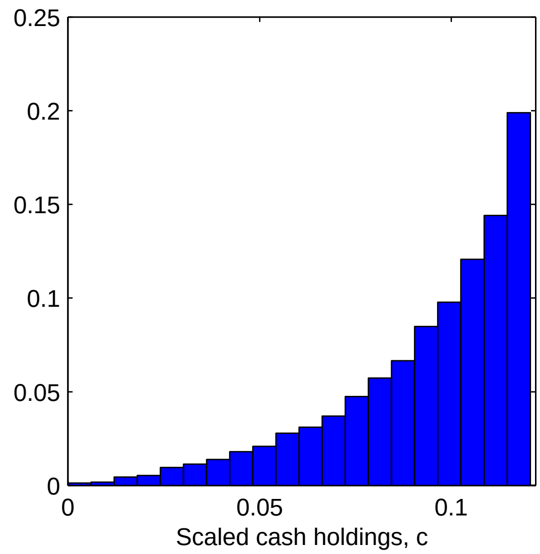

Figure 5: The stationary distribution of scaled cash holdings in the refinancing case. The parameter values are given in Table 1.

concentration of cash holdings close to the target level arises for two reasons. One is the outcome of the optimal policies that attempt at warding off costly financial distress. Second, and uniquely to our model, persistence in temporary shocks make safe firms even safer. This is related to the time varying volatility of scaled cash holdings. In our base case environment, this volatility decreases in c in the whole relevant domain. In particular, the volatility is the highest at low cash reserves and makes a firm in distress to quickly either recover with retained earnings or resolve to a new equity issuance. By contrast, a firm at the target cash level tends to stay there as the volatility of its scaled cash holdings is low. Panel B of Table 2 shows that the distribution of scaled cash reserves makes firms frequent and persistent dividend payers and infrequent equity issuers.

3.3 Investing in financially-constrained firms

A key result of Proposition 3 is that financing frictions reduce the value of an active firm and delay investment, in that the selected investment threshold for a constrained firm satisfies $a^* > a^*_{FB}$. The results in Proposition 3 are therefore very different from those in prior studies, such as Boyle and Guthrie (2003), in which firms face financing constraints when seeking to invest in new projects. In such models, potential future financing constraints (i.e. potential

Panel A: Stationary distribution				
	Mean	Median	5%	95%
Scaled cash holdings, c	0.094	0.101	0.042	0.12
Panel B: Simulated annual values				
Positive payouts in all quarters				53.5%
Issuing equity				16.1%

Table 2: The stationary distribution of scaled cash holdings c in the refinancing case. The parameter values are given in Table 1.

future reductions in the firm's financial resources) feed back in current policy choices and encourage early investment. Our analysis therefore highlights another way by which financing constraints can distort investment behavior: The threat of future cash shortfalls increases future financing costs and reduces the value of the asset underlying the firm's growth option, thereby leading to late exercise of the investment opportunity.

More generally, financing frictions have two separate effects on the timing of investment in our model. First, they increase the cost of investment, thereby delaying investment. Second, they reduce the value of an active firm (i.e. the value of the underlying asset), further delaying investment. Table 3 shows how these two effects vary with input parameter values. In our base case environment, Case 1 in the table, financing frictions increase the investment threshold by 9.1% and two thirds of the delay in investment is due to financing frictions at the time of investment. As shown by the table, a firm with more volatile cash flows ($\sigma_X = 0.15$) and higher costs of holding cash ($\lambda = 0.06$) optimally invests at a yet higher threshold relative the first-best with close to 50% of the delay coming from the post-investment financing frictions. A firm with a relatively low cash flow volatility and low costs of holding cash (Case 3) invests at a lower threshold but still much above the first-best threshold. In this case, the bulk of the delay is due to financing frictions at investment.

To provide a more complete picture, Figure 6 plots the selected investment thresholds for an unconstrained firm and for a constrained firm as functions of the volatilities of short-term and permanent shocks σ_X and σ_A, the persistence of cash flow shocks ρ, the proportional cost of outside funds p, and the carry cost of cash λ. As shown by Figure 6, the effect of future financing constraints on the firm's investment policy increases with the carry cost of

Parameters	Delay in investment due to financing constraints (as % of a^*_{FB})	% of the delay due to at-investment constraints
1. $\sigma_X = 0.12$, $\lambda = 0.04$	9.1%	67.3%
2. $\sigma_X = 0.15$, $\lambda = 0.06$	11.5%	53.6%
3. $\sigma_X = 0.09$, $\lambda = 0.02$	7.3%	83.3%

Table 3: Quantitative effects of financing constraints on the investment threshold and their decomposition. The constant parameter values are given in Table 1.

cash λ, the volatility of short-term shocks σ_X, and financing costs p, and decreases with liquid reserves c and the persistence of cash flows ρ. Except for p, these effects in our model are driven by the cost of financing frictions after investment and follow from the effects of these parameters on the value of an active firm.

Our model also has implications for the relation between investment and uncertainty. Notably, we have shown that an increase in the volatility of short-term shocks raises the risk of future funding shortfalls, thereby reducing the value of an active firm and investment incentives. Therefore, our model predicts that in most economic environments, increasing σ_X will decrease investment rates. Another determinant of risk in our model is the persistence of cash flow shocks. Since an increase in the persistence of cash flows unambiguously increases the value of a constrained firm, another novel prediction of our model is that increasing ρ should increase investment rates. As shown by Figure 6, the effect is quantitatively small.

A key difference between our model and traditional real options models is that firms face financing frictions and are exposed to short-term cash flow shocks. As discussed in section 3.1.1, financing frictions and short-term shocks lead the firm to value inside equity and to hold cash balances at the time of investment as a precautionary motive. At the same time, however, financing frictions and the uncertainty associated with short-term shocks lead the firm to delay investment and, thus, to an increase in the value of productive assets at the time of investment. Figure 7 shows that the first effect is more important quantitatively in our base case environment, so that comparative statics for the asset mix of the firm mirror those for target cash holdings at the time of investment.

Lastly, an interesting feature of our real options model is that the value function starts as $G(a)$ before investment, a function of only a that is convex in a, and changes to $V(a, m) =$

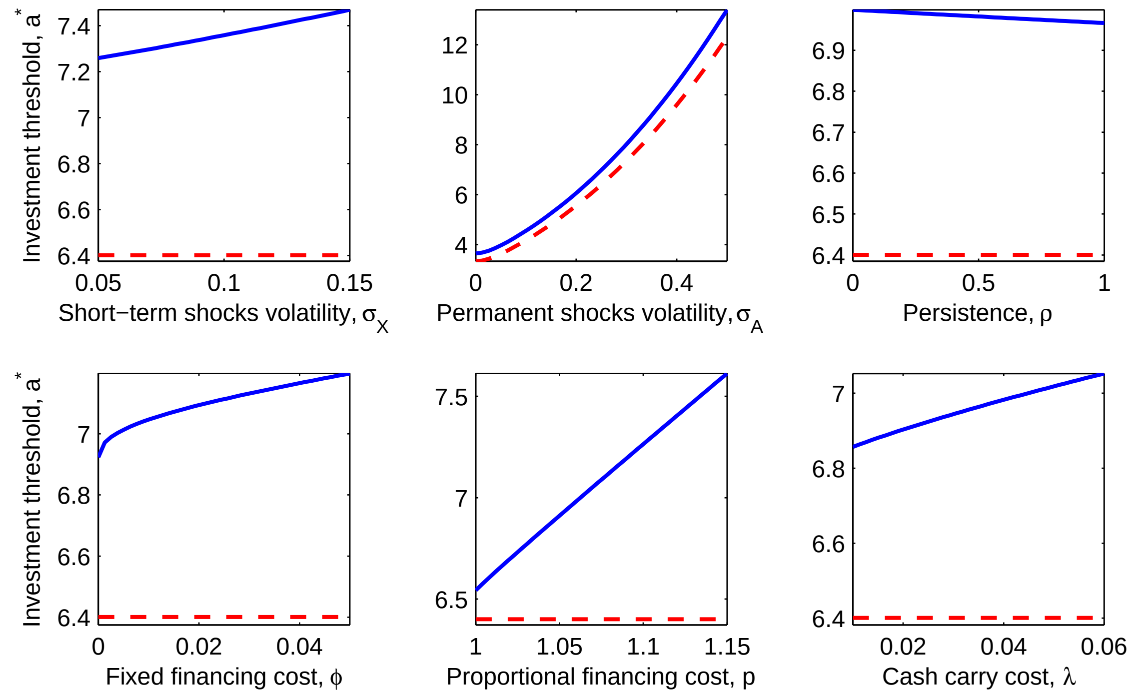

Figure 6: The effect of exogenous parameters on the optimal investment threshold a^* in the refinancing case (solid curves) and in the first best (dashed curves). The constant parameter values are given in Table 1.

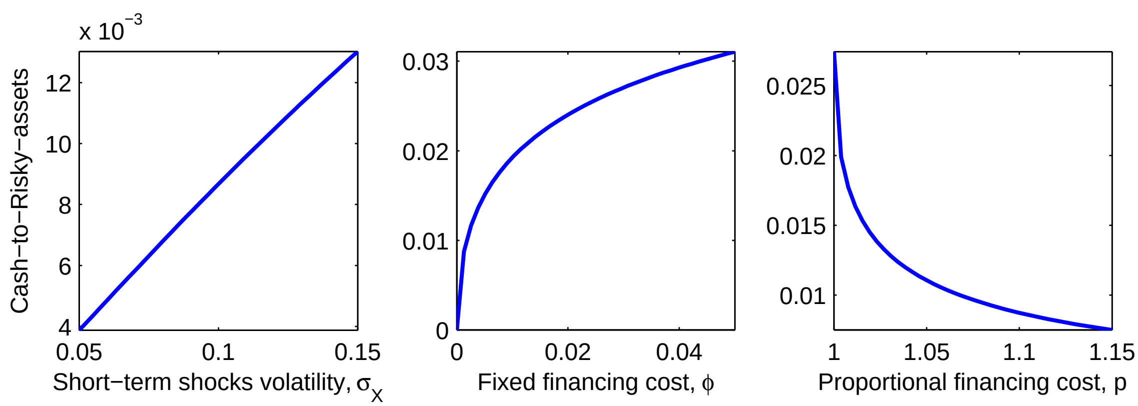

Figure 7: The effect of financing frictions and volatility of temporary shocks on the asset mix of the firm at the time of investment. Input parameter values are given in Table 1.

$aF(m/a)$ after investment, a function of a and m that is concave in a. One potential implication of this property is that the firm's strategy with respect to asset risk (and exposure to shocks) would be different before and after investment. That is, before investment the firm, if it had a choice, would select assets/technologies with high risk. After investment, the firm would like to mitigate risk using the strategies described below.

4 Risk management

In this section, we analyze risk management in the presence of temporary and permanent shocks to determine whether the management of these two sources of risk is substantially different. To investigate this issue, we assume that the firm manages its risk exposure using derivatives such as futures contracts (as in Bolton, Chen, and Wang (2011) and Hugonnier, Malamud, and Morellec (2014)). We consider futures contracts with price Y_t that is governed by the stochastic differential equation:

$$dY_t = \sigma_Y Y_t dZ_t, \tag{41}$$

where σ_Y is a positive constant and $Z = (Z_t)_{t \geq 0}$ is a standard Brownian motion.

We denote by h_t the firm's position in the futures contracts (measured in dollar). As discussed below, this hedging position may be constrained by requirements of maintaining a margin account. The dynamics of cash reserves with futures hedging are then given by:

$$dM_t = (r - \lambda)M_t dt + \alpha A_t dt + \sigma_X A_t dW_t^X - dU_t + dL_t + h_t \sigma_Y dZ_t. \tag{42}$$

As shown by this equation, one important aspect of hedging with derivatives contracts is that it produces additional short-term cash flows ($h_t \sigma_Y dZ_t$). Asset substitution does not have this feature. As a result, and as argued below, cash holdings and financing constraints will then be important in determining whether firms manage their risks by using derivatives contracts or by changing asset exposure to permanent and temporary shocks.

4.1 Costless risk management

We start our analysis by considering an environment in which hedging is costless (or unconstrained) in that there are no requirements of maintaining a margin account. Suppose first that the firm manages only temporary shocks using futures contracts (by the firm's choice or because only futures correlated with temporary shocks are available). Let χ_T denote the correlation between Z_t and W_t^T (Z_t and W_t^P are uncorrelated here). Since firm value is concave in cash reserves M, we expect that the firm will completely eliminate its temporary risk exposure via dynamic hedging.

Using the same steps as above, it is immediate to show that the value of an active firm satisfies in the earnings retention region:

$$\begin{aligned} rV(a,m) &= \mu a V_a(a,m) + (\alpha a + (r-\lambda)m)V_m(a,m)) \\ &+\frac{1}{2}a^2\left(\sigma_A^2 V_{aa}(a,m) + 2\rho\sigma_A\sigma_X V_{am}(a,m) + \sigma_X^2 V_{mm}(a,m)\right) \\ &+\max_h \frac{1}{2}\left\{h^2\sigma_Y^2 V_{mm}(a,m) + 2\chi_T\sqrt{1-\rho^2}h\sigma_Y\sigma_X a V_{mm}(a,m)\right\}. \end{aligned} \tag{43}$$

The value-maximizing hedging policy is determined by solving the first-order condition with respect to h (the second-order condition holds since $V_{mm} < 0$). This yields:

$$h_T^* = -\frac{\chi_T\sqrt{1-\rho^2}\sigma_X}{\sigma_Y}a. \tag{44}$$

Substituting (44) into equation (42) reveals that the optimal hedge of temporary shocks removes all the correlated risk so that the volatility of firm cash flows decreases by $\chi_T\sqrt{1-\rho^2}\sigma_X$.

Suppose next that the firm manages only its exposure to permanent shocks. Let χ_P denote the correlation with between Z_t and W_t^P (Z_t and W_t^T are uncorrelated here). In this case, the value of the firm satisfies in the earnings retention region:

$$\begin{aligned} rV(a,m) &= \mu a V_a(a,m) + (\alpha a + (r-\lambda)m)V_m(a,m)) \\ &+\frac{1}{2}a^2\left(\sigma_A^2 V_{aa}(a,m) + 2\rho\sigma_A\sigma_X V_{am}(a,m) + \sigma_X^2 V_{mm}(a,m)\right) \\ &+\max_h \frac{1}{2}\left\{h^2\sigma_Y^2 V_{mm}(a,m) + 2\chi_P\rho h\sigma_Y\sigma_X a V_{mm}(a,m)\right. \\ &\left.+2\chi_P h\sigma_Y\sigma_A a V_{am}(a,m)\right\}, \end{aligned} \tag{45}$$

and the first-order condition yields:

$$h_P^* = -\frac{\chi_P \rho \sigma_X a V_{mm}(a,m) + \chi_P \sigma_A a V_{am}(a,m)}{\sigma_Y V_{mm}(a,m)} = -\frac{\chi_P \rho \sigma_X}{\sigma_Y} a + \frac{\chi_P \sigma_A}{\sigma_Y} m, \tag{46}$$

where the second equality follows from the fact that $V_{mm}(a,m) = \frac{1}{a} F''(c)$ and $V_{am}(a,m) = -\frac{c}{a} F''(c)$. Substituting the expression for h_P^* in equation (42) shows that optimal hedging of permanent shocks adds two terms the dynamics of cash reserves. The first one, $-\chi_P \rho \sigma_X A_t dZ_t$, serves to remove the correlated risk from firm cash flows. The second one, $\chi_P \sigma_A M_t dZ_t$, is specific to hedging of permanent shocks and has a double impact. First, it increases the volatility of cash flows. Second, it simultaneously increases the persistence of cash flow shocks.

Comparing the optimal hedging positions h_T^* and h_P^*, we find that hedging policies with respect to temporary and permanent shocks are markedly different. First, the signs of h_T^* and h_P^* can be opposite. This implies that if futures returns are positively correlated with the firm's risk (both $\chi_T > 0$ and $\chi_P > 0$), then the firm always takes a short position in futures to hedge temporary shocks but takes a combination of short and long positions to manage exposure to permanent shocks.[9] The former behavior is expected and known (see for example Bolton, Chen, and Wang (2011)) but the latter seems striking.

Despite the fact that the scaled value function is concave, risk management of permanent shocks with derivatives may imply taking a position that is not contrary but *aligned* with the exposure. To understand this result, note that the positive sign in h_P^* stems from the positive sign of $V_{am} = -\frac{c}{a} F''$ as opposed to the negative signs of V_{aa} and V_{mm}. This means that the marginal value of cash increases in profitability. As mentioned earlier, the hedge increases both cash flow volatility and persistence and the firm benefits from persistence in cash flows, i.e. from generating liquidity when long-term prospects improve.[10]

The second difference between h_T^* and h_P^* is in the dependence on profitability and liquidity. To analyze and compare hedging positions, we need to scale these positions by the firm's exposure. The firm in our model hedges cash flows with expected profitability A_t so this

[9] The long position dominates, in particular, if cash flow persistence ρ is low and if cash m is large compared to profitability a.

[10] It would be misleading to call the firm's risk management policy to permanent shocks as "speculation," since taking a position that is not contrary to the exposure actually reduces risk.

denominator of a hedge ratio seems to follow the usual practice in risk management literature (see e.g. Tufano (1996)). Let $g_t = h_t/A_t$ denote the hedge ratio at time t. Accordingly, optimal hedge ratios are given by

$$g_T^* = -\frac{\chi_T\sqrt{1-\rho^2}\sigma_X}{\sigma_Y} \text{ and } g_P^* = -\frac{\chi_P \rho \sigma_X}{\sigma_Y} + \frac{\chi_P \sigma_A}{\sigma_Y} c. \tag{47}$$

These equations show that the hedge ratio with respect to temporary shocks is constant while the hedge ratio with respect to permanent shocks is linear in scaled cash holdings c. This last observation implies that if profitability is large compared to cash holdings (so that c is low), then profitability shocks on their own are sufficient to generate the required cash flows and the firm needs less cash flows from positions in derivatives. Finally, note that if the futures price is correlated with both W and B, the hedging position and the hedge ratio are respectively given by $h^* = h_T^* + h_P^*$ and $g^* = g_T^* + g_P^*$.

According to present accounting standards (SFAS), hedges need to be accounted differently depending on their nature. Two main types are cash flow hedging and fair value hedging (see e.g. Disatnik, Duchin, and Schmidt (2014)). Cash flow hedging relates to hedging of shocks that affect the firm's cash flows streams. Fair value hedging relates to hedging against shocks to the value of its assets and liabilities, irrespective of the realized cash flow stream associated with these assets. These two can be distinguished from accounting data of US industrial firms.[11] There is a clear mapping from our hedging of temporary shocks to cash flow hedging and from our hedging of permanent shocks to fair value hedging. Our hedging of permanent shocks is essentially hedging the value of the firm's assets. This makes our distinction between these two forms of hedging relevant and testable.

4.2 Costly risk management

Suppose now that hedging positions are not unbounded and are instead constrained by requirements of maintaining a margin account. Specifically, assume that the firm's net futures position cannot exceed the amount on the margin account by more than a factor π. Assuming that the margin account earns the same interest as the common cash account,

[11] Decomposing earnings and cash flows into temporary and permanent shocks is a common practice in the accounting literature (see e.g. Kothari (2001) or Dechow, Ge, and Schrand (2010)).

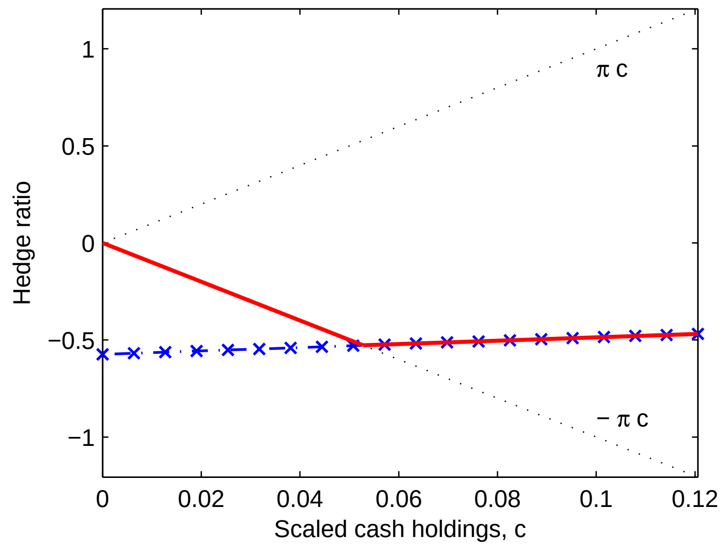

Figure 8: Optimal hedge ratios g^*. The dotted lines represent the margin-account constrains, the x-marked line represents the unconstrained hedge ratio, and the thick curve depicts the constrained hedge ratio. The parameter values are given in Table 1.

all cash holdings can be moved to the margin account if needed, and so the margin-account constraint is equivalent to limiting the futures position to a π multiple of cash holdings, or $|h_t| \leq \pi M_t$. In terms of the hedge ratio, the constraint can then be written as $|g_t| \leq \pi C_t$.

Figure 8 plots the hedge ratio under margin requirements. Input parameter values for the figure are set as follows: $\sigma_Y = 0.2$, $\chi_T = 0.7$, $\chi_P = 0.7$, and $\pi = 10$. The values of χ_T and χ_P imply that the futures price is positively correlated with both temporary and permanent shocks and that the unconstrained hedge ratio is sum of a negative constant (g_T^*) and of an increasing function of c (g_P^*). The pattern of costly risk management is such that constrained firms (i.e. firms with low c and also with low value) hedge less due to difficulties with meeting margin requirements. This is consistent with the evidence in Rampini, Sufi, and Viswanathan (2014) that collateral constraints play a major role in risk management. As long as risk management involves also permanent shocks (as is the case in Figure 8), firms with large cash reserves, that are no longer constrained by margin requirements, decrease (the absolute value of) hedging as c increases (or as firm value increases). If risk management involves only temporary shocks, then firms with large cash reserves have constant hedge ratios.

4.3 Hedging using derivatives versus asset substitution

An alternative to risk management using derivatives is to change the firm's assets to achieve a different risk exposure. Notably, the firm may employ assets or processes that have lower or higher temporary or permanent risks. This is a version of asset substitution. An important difference between asset substitution and hedging with derivatives is that the former does not generate cash flows. Whether a risk management strategy generates cash flows or not is not important in models with unconstrained financing (like Leland (1998)), but this is relevant in a model with financing frictions like ours (see also Mello and Parsons (2000)).

Suppose that the firm can manage costlessly its asset risk via unconstrained selection of volatilities of short-term or permanent shocks, σ_X and σ_A. Consider first short-term shocks. Using equation (17), it is immediate to see that the first order derivative of (scaled) firm value with respect to σ_X is always negative, so that the optimal policy is to set $\sigma_X = 0$. The same outcome would obtain if the firm could select its exposure to temporary shocks. This shows that the outcome of derivative hedging and asset risk management are the same: The firm aims at removing all exposure to short-term and temporary shocks and the two methods are equivalent.

Consider next permanent shocks. Using equation (17), we have that the first order derivative of firm value with respect to σ_A, given by $(\sigma_A c - \rho\sigma_X)cF''(c)$, is always negative if $\rho \leq 0$. In these instances, it is optimal to set $\sigma_A = 0$. If instead $\rho > 0$, the optimal exposure σ_A to the permanent shock W^P satisfies:

$$\sigma_A = \frac{\rho\sigma_X}{c}. \tag{48}$$

Plugging the expression for σ_A in the volatility of scaled cash holdings, we get a resulting volatility given by $\sigma_X\sqrt{1-\rho^2}$. Two observations are in order. First, the value-maximizing firm is willing to maintain a positive volatility of permanent shocks. In essence, this happens because volatility of scaled cash holding c is not the lowest at $\sigma_A = 0$ but when σ_A is at a right proportion to σ_X, ρ, and c (such that (48) holds). Second, the optimal volatility of permanent shocks is large when c is small. A high σ_A contributes to the volatility of c positively and directly by changing the volatility of permanent shocks, via $\sigma_A^2 c^2$, and indirectly via the covariance term, $2\rho\sigma_A\sigma_X c$. If c is low, the direct volatility effect, being quadratic in c, is dwarfed by the covariance term. By selecting a high exposure to permanent shocks σ_A, the firm can benefit from the increased covariance with little cost of increased variance.

Managing permanent risk with either derivatives or asset substitution boils down to balancing the effect of risk management on the volatility and persistence of cash flows. Typically, risk management of either type would increase beneficial persistence at the cost of an increased volatility. The difference between derivatives and asset risk management is that the former manipulates short-term cash flow volatility and the latter affects long-term asset-profitability volatility. This implies that the two strategies have different incentives with varying c. For example, derivative hedging looses some of its potential when a firm enters distress, i.e. when c is low. A firm with little cash, cannot afford to generate cash flow shocks to benefit from persistence, as this would put it at risk of running out of cash quickly. By contrast, and as discussed above, a distressed firm would have strong incentives to engage in asset substitution to increase σ_A.

5 Conclusion

We develop a dynamic model of investment, cash holdings, financing, and risk management policies in which firms face financing frictions and are subject to both permanent and temporary cash flow shocks. Using this model, we show that combining permanent and temporary shocks helps explain corporate behavior and produces predictions that are in line with the available evidence. Notably, while in corporate-liquidity models based solely on temporary shocks the cash-flow sensitivity of cash is either zero or one, our model predicts that firms will demonstrate a non-trivial and realistic cash-flow sensitivity of cash, due to the effects of permanent shocks on target cash holdings. In addition, we show that when firms access capital markets to raise funds, the size of equity issues is not constant as in prior models, but depend on the firm's profitability.

We also investigate in the paper how the timing of investment and the initial asset mix of the firm reflect financing frictions and the joint effects of permanent and temporary shocks. We find that that as financing frictions or the volatility of temporary shocks increase, the firm decides to hold larger cash balances at the time of investment, so that its asset mix gets distorted towards safer assets. Finally, we find that financing frictions and temporary shocks delay investment and have large effects on the timing of investment.

Lastly, we show that in the presence of permanent shocks risk management policies are richer and depend on the nature of the cash flow shocks and potential collateral constraints. Notably, we show that if the firm's risk and futures prices are positively correlated, then hedging temporary shocks involves a short futures position while hedging permanent shocks may require a long futures position. (And vice versa if the correlation is negative.) We also show that managing risk either by derivatives or by directly selecting the riskiness of assets (i.e. asset substitution) leads to the same outcome if the risk is due to temporary shocks. However, derivatives and asset substitution are not equivalent when managing the risk from permanent shocks. Finally, we show that when risk management is costly, constrained firms hedge less, consistent with the evidence in Rampini, Sufi, and Viswanathan (2014). Again, these predictions are very different from those in models based on a single source of risk.

Appendix

A. Proof of Proposition 1

The proof goes through three steps. Step 1 shows that problem (7) can be re-written as a one-dimensional control problem. Step two solves the variational system (17), (20), (22). Step 3 shows that the solution to (17), (20), (22) coincides with the solution of the one-dimensional control problem and derives the optimal dividend and issuance policies. To avoid confusion, throughout the proof, V^* and F^* denote the value functions of control problems while V and F denote the solution to variational systems.

Step 1. Let $\tilde{\mathbb{P}}$ be the probability defined by

$$\left(\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}\right)_{|\mathcal{F}_t} = Z_t \equiv \exp\{-\frac{1}{2}\sigma_A^2 t + \sigma_A W_t^P\}, \quad \forall\, t \geq 0, \tag{49}$$

on $(\Omega, \mathcal{F})$. By Girsanov's Theorem, $(\tilde{W}_t^P, W_t^T)_{t\geq 0}$ with $\tilde{W}_t^P = -\sigma_A t + W_t^P$, is a bi-dimensional Brownian motion under the probability $\tilde{\mathbb{P}}$. We have:

Proposition 5. *The value function V^* of problem (7) satisfies*

$$V^*(a,m) = aF^*\left(\frac{m}{a}\right), \tag{50}$$

The function F^* is defined on $[0,\infty)$ by

$$F^*(c) = \sup_{((\tau_n)_{n\geq 1},(e_n)_{n\geq 1},L)\in\mathcal{A}} f(c;(\tau_n)_{n\geq 1},(e_n)_{n\geq 1},L). \tag{51}$$

with

$$f(c;(\tau_n)_{n\geq 1},(e_n)_{n\geq 1},L) = \mathbb{E}_c^{\tilde{\mathbb{P}}}\left[\int_0^{\tau_0} e^{-(r-\mu)t}(d\tilde{L}_t - d\tilde{E}_t) + e^{-(r-\mu)\tau_0}\frac{\omega\alpha}{r-\mu}\right], \tag{52}$$

and

$$\begin{aligned} C_0 = c, \quad dC_t &= (\alpha + C_t(r-\lambda-\mu))\,dt + \sqrt{\sigma_A^2 C_t^2 - 2\rho\sigma_X\sigma_A C_t + \sigma_X^2}\,dW_t^C \\ &\quad + \frac{d\tilde{E}_t}{p} - d\tilde{\Phi}_t - d\tilde{L}_t, \end{aligned} \tag{53}$$

where $W^C = (W_t^C)_{t\geq 0}$ Brownian motion under $\tilde{\mathbb{P}}$,

$$\tau_0 = \inf\{t \geq 0 \,|\, C_t = 0\}. \tag{54}$$

and

$$\tilde{\Phi}_t = \sum_{n\geq 1} \phi 1\!\!1_{\{\tau_n \leq t\}}, \tag{55}$$

$$\tilde{E}_t = \sum_{n\geq 1} \tilde{e}_n 1\!\!1_{\{\tau_n < t\}} \text{ with } \tilde{e}_n = e_n A_{\tau_n}, \tag{56}$$

$$\tilde{L}_t = \int_0^t \frac{1}{A_s}\, dL_s, \tag{57}$$

Proof of Proposition 5. Applying the Ito's formula to $\left(e^{-r(t\wedge\tau_0)} M_{t\wedge\tau_0}\right)_{t\geq 0}$ and letting t go to ∞ yields

$$\mathbb{E}\left[\int_0^{\tau_0} e^{-rt}(dL_t - dE_t)\right] = m + \mathbb{E}\left[\int_0^{\tau_0} e^{-rt}(-\lambda M_t + \alpha A_t)dt\right] - \mathbb{E}\left[\int_0^{\tau_0} e^{-rt}(\frac{p-1}{p} dE_t + d\Phi_t)\right]$$

which we re-write under the form

$$\begin{aligned}\frac{1}{a}\mathbb{E}\left[\int_0^{\tau_0} e^{-rt}(dL_t - dE_t)\right] &= \frac{m}{a} + \mathbb{E}\left[\int_0^{\tau_0} e^{-(r-\mu)t} Z_t(-\lambda\frac{M_t}{A_t} + \alpha)dt\right] \\ &\quad - \mathbb{E}\left[\int_0^{\tau_0} e^{-(r-\mu)t} Z_t(\frac{p-1}{p}\frac{dE_t}{A_t} + \frac{d\Phi_t}{A_t})\right].\end{aligned}$$

The change of probability measure (49) yields

$$\begin{aligned}\frac{1}{a}\mathbb{E}\left[\int_0^{\tau_0} e^{-rt}(dL_t - dE_t)\right] &= \frac{m}{a} + \mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_0^{\tau_0} e^{-(r-\mu)t}(-\lambda\frac{M_t}{A_t} + \alpha)dt\right] \\ &\quad - \mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_0^{\tau_0} e^{-(r-\mu)t}(\frac{p-1}{p}\frac{dE_t}{A_t} + \frac{d\Phi_t}{A_t})\right].\end{aligned} \tag{58}$$

Then, applying Ito's formula to $(\frac{M_t}{A_t})_{t\geq 0}$ yields

$$\begin{aligned}\frac{M_0}{A_0} = \frac{m}{a}, \quad d\left(\frac{M_t}{A_t}\right) &= \left(\alpha + \frac{M_t}{A_t}(r-\lambda-\mu)\right)dt + \left(\sigma_X\rho - \frac{M_t}{A_t}\sigma_A\right)d\tilde{W}_t^P \\ &\quad + \sigma_X\sqrt{1-\rho^2}dW_t^T + \frac{1}{A_t}\left(\frac{dE_t}{p} - d\Phi_t - dL_t\right),\end{aligned}$$

or equivalently,

$$\begin{aligned}\frac{M_0}{A_0} = \frac{m}{a}, \quad d\left(\frac{M_t}{A_t}\right) &= \left(\alpha + \frac{M_t}{A_t}(r-\lambda-\mu)\right)dt + \sqrt{\sigma_A^2(\frac{M_t}{A_t})^2 - 2\rho\sigma_X\sigma_A\frac{M_t}{A_t} + \sigma_X^2}\, dW_t^C \\ &\quad + \frac{1}{A_t}\left(\frac{dE_t}{p} - dL_t\right) - d\tilde{\Phi}_t,\end{aligned}$$

where $(W_t^C)_{t\geq 0}$ is a Brownian motion under $\tilde{\mathbb{P}}$. Applying Ito's formula to $\left(e^{-r(t\wedge\tau_0)}\frac{M_{t\wedge\tau_0}}{A_{t\wedge\tau_0}}\right)_{t\geq 0}$, letting t go to ∞, and rearranging terms, we get

$$\begin{aligned}\mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_0^{\tau_0} e^{-(r-\mu)t}\frac{1}{A_t}(dL_t - dE_t)\right] &= \frac{m}{a} + \mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_0^{\tau_0} e^{-(r-\mu)t}(-\lambda\frac{M_t}{A_t} + \alpha)dt\right] \\ &\quad -\mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_0^{\tau_0} e^{-(r-\mu)t}(\frac{p-1}{p}\frac{dE_t}{A_t} + d\tilde{\Phi}_t)\right].\end{aligned}$$

Noting that $\mathbb{E}[e^{-r\tau_0}\frac{\omega\alpha}{r-\mu}A_{\tau_0}] = a\mathbb{E}^{\tilde{\mathbb{P}}}[\frac{\omega\alpha}{r-\mu}e^{-(r-\mu)\tau_0}]$, we deduce then from (58)

$$\mathbb{E}\left[\int_0^{\tau_0} e^{-rt}(dL_t - dE_t) + e^{-r\tau_0}\frac{\omega\alpha A_{\tau_0}}{r-\mu}\right] = a\mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_0^{\tau_0} e^{-(r-\mu)t}\frac{1}{A_t}(dL_t - dE_t) + e^{-(r-\mu)\tau_0}\frac{\omega\alpha}{r-\mu}\right].$$

To conclude the proof, note that problem

$$\sup_{((\tau_n)_{n\geq 1},(e_n)_{n\geq 1},L)\in\mathcal{A}} \mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_0^{\tau_0} e^{-(r-\mu)t}\frac{1}{A_t}(dL_t - dE_t) + e^{-(r-\mu)\tau_0}\frac{\omega\alpha}{r-\mu}\right]$$

where the admissible policies $(\tau_n)_{n\geq 1}, (e_n)_{n\geq 1}, L$ are related by

$$\begin{aligned}C_0 = c, \quad dC_t &= (\alpha + C_t(r-\lambda-\mu))\,dt + \sqrt{\sigma_A^2 C_t^2 - 2\rho\sigma_X\sigma_A C_t + \sigma_X^2}\,dW_t^C \\ &\quad + \frac{1}{A_t}\left(\frac{d\tilde{E}_t}{p} - d\tilde{L}_t\right) - d\Phi_t,\end{aligned}$$

together with (56), (57) is equivalent to problem (51)-(57). □

The two next steps solve problem (51). To this end, we solve first the variational system (17), (20), (22) (step 2). Then, we show that its solution coincides with the solution of problem (51) (step 3).

Step 2 The following holds.

Proposition 6. *There exists a unique solution (F, c^*) to the variational system (17), (20), (22) that is concave and twice continuously differentiable over $(0,\infty)$.*

The proof mimics the proof of Proposition A1 in DMRV (2011). The arguments must be slightly adapted because, in the ordinary differential equation (17), the drift $(\alpha + c(r-\lambda-\mu))$ can take negative values and $\Sigma(c) \equiv \sigma_A^2 c^2 - 2\rho\sigma_X\sigma_A c + \sigma_X^2$ is non-constant. For completeness, we develop below the main steps of the proof with a particular focus on the arguments that require a slight adaptation. We refer to DMRV (2011) for more details.

Proof of Proposition 6: We start by considering the family of ordinary differential equations parametrized by $c_1 > 0$,

$$-(r-\mu)F(c) + (\alpha + c(r-\lambda-\mu))F'(c) + \frac{1}{2}(\sigma_A^2 c^2 - 2\rho\sigma_X\sigma_A c + \sigma_X^2)F''(y) = 0,$$
$$0 < c < c_1; \quad (59)$$
$$F'(c_1) = 1; \quad (60)$$
$$F''(c_1) = 0. \quad (61)$$

Because $\rho \in [-1,1)$, $\Sigma(c) \equiv \sigma_A^2 c^2 - 2\rho\sigma_X\sigma_A c + \sigma_X^2 > 0$ and (59)-(61) admits a unique solution F_{c_1} over $[0, c_1]$ for any $c_1 > 0$. The next lemma establishes monotonicity and concavity property of F_{c_1}.

Lemma 1. *The following holds*

(i) If $0 < \lambda \leq r - \mu$ then, for any $c_1 > 0$, $F'_{c_1} > 1$ and $F''_{c_1} < 0$ over $[0, c_1)$.

(ii) If $\lambda > r - \mu$ then, for any $0 < c_1 < \frac{\alpha}{\lambda+\mu-r}$, $F'_{c_1} > 1$ and $F''_{c_1} < 0$ over $[0, c_1)$.

Proof of Lemma 1: Differentiating (59) yields $\frac{1}{2}\Sigma(c_1)F'''_{c_1}(c_1) - \lambda F'_{c_1}(c_1) = 0$, which implies $F'''_{c_1}(c_1) > 0$ because $\lambda > 0$. Since $F''_{c_1}(c_1) = 0$ and $F'_{c_1}(c_1) = 1$, it follows that $F''_{c_1} < 0$ and thus $F'_{c_1} > 1$ over some interval $(c_1 - \varepsilon, c_1)$ where $\varepsilon > 0$. Now, suppose by way of contradiction that $F'_{c_1}(c) \leq 1$ for some $c \in [0, c_1 - \varepsilon]$, and let $\tilde{c} = \sup\{c \in [0, c_1 - \varepsilon] \mid F'_{c_1}(c) \leq 1\} < c_1$. Then, $F'_{c_1}(\tilde{c}) = 1$ and $F'_{c_1} > 1$ over $(\tilde{c}, c_1)$, so that $F_{c_1}(c_1) - F_{c_1}(c) > c_1 - c$ for all $c \in (\tilde{c}, c_1)$. Since $F_{c_1}(c_1) = \frac{\alpha}{r-\mu} + \frac{r-\lambda-\mu}{r-\mu}c_1$, this implies that for any such c,

$$\begin{aligned}
F''_{c_1}(c) &= \frac{2}{\Sigma(c)}\left\{(r-\mu)F_{c_1}(c) - (\alpha + c(r-\lambda-\mu))F'_{c_1}(c)\right\} \\
&< \frac{2}{\Sigma(c)}\{(r-\mu)(c - c_1 + F_{c_1}(c_1)) - (\alpha + (r-\lambda-\mu)c)\} \qquad (62) \\
&= \frac{2}{\Sigma(c)}\lambda(c - c_1) \\
&< 0. \qquad (63)
\end{aligned}$$

To get (62), remark that, by assumption, in each case (i) and (ii), we have $\alpha + (r-\lambda-\mu)c > 0$ for any $c \in (\tilde{c}, c_1)$. To conclude, note that (63) contradicts the fact that $F'_{c_1}(\tilde{c}) = F'_{c_1}(c_1) = 1$. Therefore $F'_{c_1} > 1$ over $[0, c_1)$, from which it follows that $F''_{c_1} < 0$ over $[0, c_1)$. $\square$

If there exists a solution F to (17), (20), (22) that is twice continuously differentiable over $(0, \infty)$, then, by construction, F must coincide over $[0, c_1]$ with some F_{c_1}, for an appropraite choice of c_1. This choice is dictated by the boundary condition (20) that F must satisfy at zero. The next lemma studies the behavior of F_{c_1} and F'_{c_1} at zero as c_1 varies.

Lemma 2. *In each of the two cases of Lemma 1, $F_{c_1}(0)$ is a strictly decreasing and concave function of c_1, whereas $F'_{c_1}(0)$ is a strictly increasing and convex function of c_1.*

Proof of Lemma 2: consider H_0 and H_1 the solutions to ODE

$$-(r-\mu)H(c)+(\alpha+c(r-\lambda-\mu))H'(c)+\frac{1}{2}(\sigma_A^2c^2-2\rho\sigma_X\sigma_Ac+\sigma_X^2)H''(c)=0$$

over $[0,\infty)$ characterized by the initial conditions $H_0(0)=1, H_0'(0)=0, H_1(0)=0$, and $H_1'(0)=1$. H_0' and H_1' are strictly positive over $(0,\infty)$. The Wronskian $W_{H_0H_1}\equiv H_0H_1'-H_1H_0'$ of H_0 and H_1 satisfies $W_{H_0H_1}(0)=1$ and

$$W_{H_0H_1}'(c)=-\frac{2}{\Sigma^2(c)}(\alpha+c(r-\lambda-\mu))W_{H_0H_1},$$

so that $W_{H_0H_1}>0$ which implies that for each $c_1>0$, $F_{c_1}=F_{c_1}(0)H_0+F_{c_1}'(0)H_1$ over $[0,c_1]$. Using the boundary condition $F_{c_1}(c_1)=\frac{\alpha+c_1(r-\lambda-\mu)}{r-\mu}$ and $F_{c_1}'(c_1)=1$, we obtain that

$$\begin{aligned}\frac{dF_{c_1}(0)}{dc_1} &= -\frac{1}{W_{H_0H_1}(c_1)}\frac{\lambda}{r-\mu}H_1'(c_1) < 0,\\ \frac{d^2F_{c_1}(0)}{d^2c_1} &= -\frac{1}{W_{H_0H_1}(c_1)}\frac{2\lambda}{\Sigma(c_1)}H_1(c_1) < 0\end{aligned}$$

and,

$$\begin{aligned}\frac{dF_{c_1}'(0)}{dc_1} &= \frac{1}{W_{H_0H_1}(c_1)}\frac{\lambda}{r-\mu}H_0'(c_1) > 0,\\ \frac{d^2F_{c_1}'(0)}{d^2c_1} &= \frac{1}{W_{H_0H_1}(c_1)}\frac{2\lambda}{\Sigma(c_1)}H_0(c_1) > 0.\end{aligned}$$

□

Since $\lim_{c_1\downarrow 0}F_{c_1}(0)=\frac{\alpha}{r-\mu}>\frac{\omega\alpha}{r-\mu}$ and $\lim_{c_1\downarrow 0}F_{c_1}'(0)=1<p$, it follows from Lemma 2 that there exists a unique $\hat{c}_1>0$ such that $F_{\hat{c}_1}(0)=\frac{\omega\alpha}{r-\mu}$, and that there exists a unique $\tilde{c}_1>0$ such that $F_{\tilde{c}_1}'(0)=p$. Note that:

$\hat{c}_1$ satisfies $\hat{c}_1<\frac{\alpha}{\lambda}(1-\omega)$. Indeed, the concavity property implies $F_{c_1}(0)<F_{c_1}(c_1)-c_1$. A computation yields $F_{c_1}(c_1)-c_1\le\frac{\omega\alpha}{r-\mu}$ iff $c_1\ge\frac{\alpha}{\lambda}(1-\omega)$, (in the case $\lambda>r-\mu$, we have $\frac{\alpha}{\lambda}(1-\omega)<\frac{\alpha}{r-\lambda-\mu}$, and thus the assumption of assertion (ii) of lemma 1 is satisfied).

$\hat{c}_1>\tilde{c}_1$ if and only if $F_{\hat{c}_1}'(0)>p$. Furthermore, Lemma 1 along with the fact that $F_{c_1}'(c_1)=1$ implies that if $c_1\ge\tilde{c}_1$, there exists a unique $c_p(c_1)\in[0,c_1)$ such that $F_{c_1}'(c_p(c_1))=p$. This corresponds to the unique maximum over $[0,\infty)$ in case (i) of Lemma 1, (resp. over $[0,c_1)$ in case (ii) of Lemma 1) of the function $c\mapsto F_{c_1}(c)-p(c+\phi)$. Observe that, by construction, $c_p(\tilde{c}_1)=0$. This leads to the two cases:

1. Issuance costs are high, that is

$$\max_{[-\phi,\infty)} (F_{\hat{c}_1}(y) - p(c+\phi)) = \frac{\omega\alpha}{r-\mu}. \tag{64}$$

This is the case if $\hat{c}_1 \leq \tilde{c}_1$, or, equivalently, $F'_{\hat{c}_1}(0) \leq p$, or if $\hat{c}_1 > \tilde{c}_1$ but $F_{\hat{c}_1}(c_p(\hat{c}_1)) - p(c_p(\hat{c}_1) + \phi) \leq 0$. Define then the function F by

$$F(c) = \begin{cases} \frac{\omega\alpha}{r-\mu} & c < 0, \\ \\ F_{\hat{c}_1}(c) & c \geq 0. \end{cases}$$

Note that, by construction, $F(0) = \frac{\omega\alpha}{r-\mu}$. Furthermore, condition (64) implies that the function $c \mapsto F(c) - p(c+\phi)$ reaches its maximum over $[-\phi, \infty)$ at $-\phi$. Letting $c^* = \hat{c}_1$, it is then easy to check that (F, c^*) solves the variational system (17), (20), (22).

2. Issuance cost are low, that is

$$\max_{[-\phi,\infty)} (F_{\hat{c}_1}(y) - p(c+\phi)) > \frac{\omega\alpha}{r-\mu}. \tag{65}$$

This is the case if $\hat{c}_1 > \tilde{c}_1$, or, equivalently, $F'_{\hat{c}_1}(0) > p$, and $F_{\hat{c}_1}(c_p(\hat{c}_1)) - \phi(c_p(\hat{c}_1) + \phi) > 0$. One has the following lemma which corresponds to Lemma A.3 in DMRV (2011) .

Lemma 3. *If (65) holds, there exists a unique $c'_1 \in (\tilde{c}_1, \hat{c}_1)$ such that*

$$F_{c'_1}(0) = F_{c'_1}(c_p(c'_1)) - p(c_p(c'_1) + \phi).$$

Then, define the function F by

$$F(c) = \begin{cases} \frac{\omega\alpha}{r-\mu} & c < 0, \\ \\ F_{c'_1}(c) & c \geq 0. \end{cases}$$

Lemma 1 along with $c'_1 < \hat{c}_1$ implies that $F(0) > \frac{\omega\alpha}{r-\mu}$. Furthermore, as $c'_1 > \tilde{c}_1$, the function $c \mapsto F(c) - p(c+\phi)$ reaches its maximum over $[-\phi, \infty)$ at $\bar{c} \equiv c_p(c'_1)$. Letting $c^* = c'_1$, it is easy to check that (F, c^*) solves the variational system (17), (20), (22).

The remaining of the proof of Proposition 6 coincides with the proof of Proposition A1 in DMRV (2011). □

Step 3 We now show that the functions F^* and F coincide. The next Lemma states that F is an upper bound for F^*

Lemma 4. *For any admissible policy $((\tau_n)_{n\geq1}, (e_n)_{n\geq1}, L)$, the solution F to (17), (20), (22) satisfies*

$$F(c) \geq f(c; (\tau_n)_{n\geq1}, (e_n)_{n\geq1}, L); \quad c > 0$$

The proof of Lemma 4 is standard and follows from Lemma A4 in DMRV (2011). To prove that $F = F^*$, it thus remains to construct an admissible policy, the value of which coincides with the function F. To this end, we consider the scaled cash reserve process C^* defined as the solution to the Skorokhod problem

$$\begin{aligned} C_t^* &= m + \int_0^t (\alpha + C_s^*(r - \lambda - \mu))\, ds + \sqrt{\sigma_A^2 C_s^{*2} - 2\rho\sigma_X\sigma_A C_s^* + \sigma_X^2}\, dW_s^C \\ &\quad + \sum_{n\geq 1} \bar{c} 1_{\{\tau_n^* \leq t\}} - L_t^*, && (66) \\ C_t^* &\leq c^*, && (67) \\ L_t^* &= \int_0^t 1_{\{C_s^* = c^*\}}\, dL_s^*, && (68) \end{aligned}$$

where the sequence of stopping times $(\tau_n^*)_{n\geq 1}$ is recursively defined by

$$\tau_0^* \equiv 0, \quad \tau_n^* \equiv \inf\{t > \tau_{n-1}^* \,|\, C_{t^-}^* = 0 \text{ and } C_t^* = \bar{c} > 0\}; \quad n \geq 1, \tag{69}$$

with $\inf \emptyset \equiv \infty$ by convention. Standard results on the Skorokhod problem imply that there exists a unique solution (C^*, L^*) to (66)-(69). Condition (68) requires that cumulative scaled dividends increase only when the scaled cash reserves reach the boundary c^*, whereas (66)–(67) express that this causes the scaled cash reserves to be reflected back at c^*. Two cases can arise. If (64) holds, then $\bar{c} = 0$ and the project is liquidated as soon as C^* drops down to zero, so that $\tau_0^* = \inf\{t \geq 0 \,|\, C_{t^-}^* = 0\} < \infty$, $\tilde{\mathbb{P}}$–almost surely. If (65) holds, then $\bar{c} = c_p(c^*) > 0$, and the process C^* discontinuously jumps to $\bar{c}$ each time it drops down to zero, so that $\tau_0^* = \infty$, $\tilde{\mathbb{P}}$–almost surely. This corresponds to a situation in which, for any $n \geq 1$, $e^* = F^*(\bar{c}) - F^*(0) = p(\bar{c} + \phi)$. Drawing again on DMRV (2011), we obtain

Proposition 7. *The value function F^* for problem (51) coincides with the function F solution to (17), (20), (22) that is twice continuously differentiable over $(0, \infty)$. The optimal issuance and dividend policies are given by $((\tau_n^*)_{n\geq 1}, (e_n^*)_{n\geq 1}, L^*)$, where*

$$\tau_n^* = \infty, \quad i_n^* = 0; \quad n \geq 1$$

if condition (64) holds, and

$$\tau_n^* = \inf\{t > \tau_{n-1} \,|\, C_{t^-}^* = 0\}, \quad e_n^* = p(\bar{c} + \phi); \quad n \geq 1$$

if condition (65) holds.

Finally, Proposition 7 together with Proposition 5 leads to Proposition 1.

B. Proof of Proposition 2

We only prove point 1 as the other points are straightforward to establish.

1. When $m \geq \frac{\sigma_X}{\sigma_A} a$, the cash reserves associated to the strategy under consideration satisfy

$C_t = \frac{\sigma_X}{\sigma_A}$ for all t. Therefore,

$$V(m,a) \geq m - a\frac{\sigma_X}{\sigma_A} + \mathbb{E}\left[\int_0^{+\infty} e^{-rt}((r-\mu)\frac{\sigma_X}{\sigma_A} + \alpha)A_t\,dt\right] = m + \frac{\alpha a}{r-\mu}.$$

The reverse inequality always holds because $m + \frac{\alpha a}{r-\mu}$ is the first best value.

2. When $m \leq \frac{\sigma_X}{\sigma_A}a$, the proof is more involved. We consider two cases.

High issuance costs: The cash reserves evolve as

$$dM_t = rM_t dt + A_t(\alpha dt + \sigma_X dW_t^T) - dL_t.$$

Let us introduce the stopping times

$$\tau = \inf\{t \geq 0,\ M_t = \frac{\sigma_X}{\sigma_A}A_t\}$$

and,

$$\tau_0 = \inf\{t \geq 0,\ M_t = 0\}.$$

Obviously, the law of these stopping times depend on the dividend policy L. When necessary, we will emphazise this dependence by noting $\tau(L)$ and $\tau_0(L)$.
Integration by parts yields

$$d(e^{-rt}M_t) = e^{-rt}A_t(\alpha dt + \sigma_X dW_t^T) - e^{-rt}dL_t.$$

Therefore,

$$\mathbb{E}\left[e^{-r(\tau\wedge\tau_0)}M_{\tau\wedge\tau_0}\right] = m + \mathbb{E}\left[\int_0^{\tau\wedge\tau_0} e^{-rt}A_t\alpha\,dt\right] - \mathbb{E}\left[\int_0^{\tau\wedge\tau_0} e^{-rt}\,dL_t\right].$$

or equivalently, because $M_{\tau_0} = 0$,

$$\mathbb{E}\left[\int_0^{\tau\wedge\tau_0} e^{-rt}\,dL_t\right] + \mathbb{E}\left[e^{-r\tau}M_\tau \mathbb{1}_{\tau\leq\tau_0}\right] = m + \mathbb{E}\left[\int_0^{\tau\wedge\tau_0} e^{-rt}A_t\alpha\,dt\right].$$

Using the dynamic programming principle, we have

$$V(m,a) = \sup_{L\in\mathcal{A}} \mathbb{E}\left[\int_0^{\tau\wedge\tau_0} e^{-rt}\,dL_t + e^{-r\tau}\left(M_\tau + \frac{\alpha A_\tau}{r-\mu}\right)\mathbb{1}_{\tau\leq\tau_0} + e^{-r\tau_0}\frac{\omega\alpha A_\tau}{r-\mu}\mathbb{1}_{\tau\geq\tau_0}\right].$$

Combining the last two equations gives

$$V(m,a) = m + \sup_{L\in\mathcal{A}} \mathbb{E}\left[\int_0^{\tau\wedge\tau_0} e^{-rt}A_t\alpha\,dt + e^{-r\tau}\frac{\alpha A_\tau}{r-\mu}\mathbb{1}_{\tau\leq\tau_0} + e^{-r\tau_0}\frac{\omega\alpha A_\tau}{r-\mu}\mathbb{1}_{\tau\geq\tau_0}\right].$$

The strong Markov property yields

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^{\tau\wedge\tau_0} e^{-rt}A_t\alpha\,dt + e^{-r\tau}\frac{\alpha A_\tau}{r-\mu}\mathbb{1}_{\tau\le\tau_0} + e^{-r\tau_0}\frac{\omega\alpha A_\tau}{r-\mu}\mathbb{1}_{\tau\ge\tau_0}\right]\\
= \;& \frac{\alpha a}{r-\mu} - (1-\omega)\mathbb{E}[e^{-r\tau_0}\frac{\alpha A_{\tau_0}}{r-\mu}\mathbb{1}_{\tau\ge\tau_0}].
\end{aligned}
$$

Observe that $\mathbb{E}[e^{-r\tau_0}\frac{\alpha A_{\tau_0}}{r-\mu}\mathbb{1}_{\tau\ge\tau_0}] = a\mathbb{E}^{\tilde{\mathbb{P}}}[e^{-(r-\mu)\tau_0}\frac{\alpha}{r-\mu}\mathbb{1}_{\tau\ge\tau_0}]$ where $\tilde{\mathbb{P}}$ is defined in (49). Now, as a functional of L, $a\mathbb{E}^{\tilde{\mathbb{P}}}[e^{-(r-\mu)\tau_0}\frac{\alpha}{r-\mu}\mathbb{1}_{\tau\ge\tau_0}]$ is clearly increasing and therefore the supremum defining V is reached by $L=0$.

Low issuance costs: We assume that equity issuance is affordable. The cash reserves evolves as

$$dM_t = rM_tdt + A_t(\alpha dt + \sigma_X dW_t^T) - dL_t + \frac{dE_t}{p} - d\Phi_t.$$

The Dynamic Programming principle yields

$$
\begin{aligned}
V(m,a) \;=\;& \sup_{((\tau_n)_{n\ge1},(e_n)_{n\ge1},L)\in\mathcal{A}} \mathbb{E}\left[\int_0^{\tau\wedge\tau_0} e^{-rs}\,(dL_t - dE_t) + e^{-r\tau}\left(M_\tau + \frac{\alpha A_\tau}{r-\mu}\right)\mathbb{1}_{\tau\le\tau_0}\right.\\
&+ \left. e^{-r\tau_0}V(0,A_{\tau_0})\mathbb{1}_{\tau\ge\tau_0}\right].
\end{aligned}
$$

Proceeding as in the case “high issuance costs”, we obtain

$$
\begin{aligned}
V(m,a) \;=\;& m + \sup_{((\tau_n)_{n\ge1},(e_n)_{n\ge1},L)\in\mathcal{A}} \left(\frac{\alpha a}{r-\mu} - \frac{p-1}{p}\mathbb{E}\left[\int_0^{\tau\wedge\tau_0} e^{-rt}\,dE_t\right] - \mathbb{E}\left[\int_0^{\tau\wedge\tau_0} e^{-rt}\,d\Phi_t\right]\right.\\
&+ \left. a\left(F(0) - \frac{\alpha}{r-\mu}\right)\mathbb{E}^{\tilde{\mathbb{P}}}[e^{-(r-\mu)\tau_0(L)}\mathbb{1}_{\tau(L)\ge\tau_0(L)}]\right).
\end{aligned}
$$

Because $p\ge1$, it is optimal not to issue equity before zero and, because $F(0) < \frac{\alpha}{r-\mu}$, it is optimal not to pay dividend before reaching the level $\frac{\sigma_X}{\sigma_A}$.

C. Comparative statics

To make the dependence of F, $\overline{c}$, and c^* on θ explicit, we write $F = F(.,\theta)$, $\overline{c} = \overline{c}(\theta)$, and $c^* = c^*(\theta)$.[12] We have the following result:

Proposition 8. *Let θ be one of the deep parameters of the model.*

[12] We maintain the notation c^* when there is no ambiguity.

1. If issuance costs are high (liquidation case), then firm value satisfies

$$\frac{\partial F}{\partial \theta}(c,\theta) = \mathbb{E}_c\left[\int_0^{\tau_0} e^{-(r-\mu)t}\left(-\frac{\partial[r-\mu]}{\partial\theta}F(C_t^*,\theta)+\frac{\partial[\alpha+(r-\lambda-\mu)C_t^*]}{\partial\theta}\frac{\partial F}{\partial c}(C_t^*,\theta)\right.\right.$$
$$\left.\left.+\frac{1}{2}\frac{\partial[\sigma_A^2C_t^{*2}-2\rho\sigma_A\sigma_X C_t^*+\sigma_X^2]}{\partial\theta}\frac{\partial^2F}{\partial c^2}(C_t^*,\theta)\right)dt+e^{-(r-\mu)\tau}\frac{\partial[\omega\alpha/(r-\mu)]}{\partial\theta}\right].$$

2. If issuance costs are low (refinancing case), then firm value satisfies

$$\frac{\partial F}{\partial \theta}(c,\theta) = \mathbb{E}_c\left[\int_0^{\infty} e^{-(r-\mu)t}\left(-\frac{\partial[r-\mu]}{\partial\theta}F(C_{t^-}^*)+\frac{\partial[\alpha+(r-\lambda-\mu)C_{t^-}^*]}{\partial\theta}\frac{\partial F}{\partial c}(C_{t^-}^*,\theta)\right.\right.$$
$$\left.+\frac{1}{2}\frac{\partial[\sigma_A^2C_{t^-}^{*2}-2\rho\sigma_A\sigma_X C_{t^-}^*+\sigma_X^2]}{\partial\theta}\frac{\partial^2F}{\partial c^2}(C_{t^-}^*,\theta)\right)dt$$
$$\left.-\left(\frac{\partial F}{\partial\theta}(\overline{c}(\theta),\theta)-\frac{\partial F}{\partial\theta}(0,\theta)\right)\sum_{n\geq 1}e^{-r\tau_n^*}\right].$$

3. In both the liquidation and refinancing cases, the target level of cash holdings satisfies

$$\frac{dc^*(\theta)}{d\theta} = -\frac{r-\mu}{\lambda}\left(\frac{\partial F}{\partial\theta}(c^*(\theta),\theta)+c^*(\theta)\frac{\partial[\frac{\lambda}{r-\mu}]}{\partial\theta}-\frac{\partial[\frac{\alpha}{r-\mu}]}{\partial\theta}\right) \tag{70}$$

Using Proposition 8, we can measure the effects of the model parameters on the (scaled) value of an active firm and the target level of liquid reserves.

Proof of Proposition 8: We prove the case the case 2 (refinancing case). The proof of case 1 is similar. Applying the Ito's lemma, we get

$$\begin{aligned}e^{-(r-\mu)T}\frac{\partial F}{\partial\theta}(C_T^*,\theta) &= \frac{\partial F}{\partial\theta}(c,\theta)+\int_0^T e^{-(r-\mu)t}\left[-(r-\mu)\frac{\partial F}{\partial\theta}(C_{t^-}^*,\theta)+\mathcal{L}\frac{\partial F}{\partial\theta}(C_{t^-}^*,\theta)\right]dt\\ &+\int_0^T e^{-(r-\mu)t}\frac{\partial^2F}{\partial c\partial\theta}(C_{t^-}^*,\theta)((\sigma_X\rho-C_{t^-}^*\sigma_A)d\tilde{W}_t^P+\sigma_X\sqrt{1-\rho^2}dW_t^T)\\ &-\int_0^T e^{-(r-\mu)t}\frac{\partial^2F}{\partial c\partial\theta}(C_{t^-}^*,\theta)dL_t^*\\ &+\sum_{t\in[0,T]}e^{-(r-\mu)t}\left(\frac{\partial F}{\partial\theta}(C_t^*,\theta)-\frac{\partial F}{\partial\theta}(C_{t^-}^*,\theta)\right)\end{aligned} \tag{71}$$

for all $T\geq 0$ and where the operator $\mathcal{L}$ is defined by

$$\mathcal{L}u(c) = (\alpha+c(r-\lambda-\mu))u'(c)+\frac{1}{2}(\sigma_P^2c^2-2\rho\sigma_A\sigma_X c+\sigma_X^2)u''(c).$$

Let us consider each term of the right hand side of (71). We deduce from (17) that the first term of the right hand side of (71) satisfies

$$\begin{aligned} & -(r-\mu)\frac{\partial F}{\partial\theta}(C^*_{t^-},\theta)+\mathcal{L}\frac{\partial F}{\partial\theta}(C^*_{t^-},\theta) \\ = & -(r-\mu)\frac{\partial F}{\partial\theta}(C^*_{t^-},\theta)+(\alpha+C^*_{t^-}(r-\lambda-\mu))\frac{\partial^2 F}{\partial\theta\partial c}(C^*_{t^-},\theta) \\ & +\frac{1}{2}(\sigma_A^2 C^{*2}_{t^-}-2\rho\sigma_A\sigma_T C^*_{t^-}+\sigma_X^2)\frac{\partial^3 F}{\partial\theta\partial c^2}(C^*_{t^-},\theta) \\ = & \frac{\partial[r-\mu]}{\partial\theta}F(C^*_{t^-},\theta)-\frac{\partial[\alpha+C^*_{t^-}(r-\lambda-\mu)]}{\partial\theta}\frac{\partial F}{\partial c}(C^*_{t^-},\theta) \\ & -\frac{1}{2}\frac{\partial[\sigma_A^2 C^{*2}_{t^-}-2\rho\sigma_A\sigma_X C^*_{t^-}+\sigma_X^2]}{\partial\theta}\frac{\partial^2 F}{\partial c^2}(C^*_{t^-},\theta). \end{aligned}$$

Because, $\frac{\partial^2 F}{\partial c\partial\theta}(.,\theta)$ is bounded over $(0,c^*(\theta)]$, the third term of the right hand side of (71) is a square integrable martingale. The fourth term of the right hand side of (71) is identically zero. Indeed, differentiating $\frac{\partial F}{\partial c}(c^*(\theta),\theta)=1$ with respect to θ and using the fact that $\frac{\partial F^2}{\partial c^2}(c^*(\theta),\theta)=0$ yields $\frac{\partial F^2}{\partial c\partial\theta}(c^*(\theta),\theta)=0$, which, together with (68) implies the result. Lastly, because C^* has paths that are continuous except at the dates $(\tau^*_n)_{n\geq 0}$ at which new shares are issued, one has

$$\sum_{t\in[0,T]} e^{-(r-\mu)t}\left(\frac{\partial F}{\partial\theta}(C^*_t,\theta)-\frac{\partial F}{\partial\theta}(C^*_{t^-},\theta)\right) = \left(\frac{\partial F}{\partial\theta}(\overline{c}(\theta),\theta)-\frac{\partial F}{\partial\theta}(0,\theta)\right)\sum_{n\geq 1} e^{-r\tau^*_n}\mathbb{1}_{\tau^*_n\leq T}.$$

Taking expectations in (71) yields

$$\begin{aligned} \frac{\partial F}{\partial\theta}(c,\theta) = & \ \mathbb{E}_c[\int_0^T e^{-(r-\mu)t}(-\frac{\partial[r-\mu]}{\partial\theta}F(C^*_{t^-},\theta)+\frac{\partial[\alpha+C^*_{t^-}(r-\lambda-\mu)]}{\partial\theta}\frac{\partial F}{\partial c}(C^*_{t^-},\theta) \\ & +\frac{1}{2}\frac{\partial[\sigma_A^2 C^{*2}_{t^-}-2\rho\sigma_A\sigma_X C^*_{t^-}+\sigma_X^2]}{\partial\theta}\frac{\partial^2 F}{\partial c^2}(C^*_{t^-},\theta))\,dt] \\ & -\left(\frac{\partial F}{\partial\theta}(\overline{c}(\theta),\theta)-\frac{\partial F}{\partial\theta}(0,\theta)\right)\sum_{n\geq 1} e^{-r\tau^*_n}+\mathbb{E}\left[e^{-(r-\mu)T}\frac{\partial F}{\partial\theta}(C^*_T,\theta)\right]. \end{aligned}$$

To conclude, we show that $\lim_{T\longrightarrow\infty}\mathbb{E}\left[e^{-(r-\mu)T}\frac{\partial F}{\partial\theta}(C^*_T,\theta)\right]=0$. Because, $\frac{\partial^2 F}{\partial c\partial\theta}(.,\theta)$ is bounded over $(0,c^*(\theta)]$, we have

$$e^{-(r-\mu)T}\frac{\partial F}{\partial\theta}(C^*_T,\theta)\leq e^{-(r-\mu)T}K(1+C^*_T)\leq e^{-(r-\mu)T}K(1+c^*(\theta))$$

for all T, where K is a positive constant, and the third inequality follows from the fact that $C^*_T\leq c^*(\theta)$ $\mathbb{P}$ almost surely, thus the result.

Differentiating equation (22) of the main text with respect to θ yields (70) $\square$

C.1. Comparative statics: parameters σ_X, σ_A, ρ

Proposition 8 yields

Corollary 1. *For any $p > 1$ and $\phi > 0$, for any $c \in (0, c^*)$,*

$$\frac{\partial F}{\partial \sigma_X}(c, \sigma_X) = \mathbb{E}_c\left[\int_0^{\tau_0} e^{-(r-\mu)t}\left(-\rho\sigma_A C^*_{t^-} + \sigma_X\right)\frac{\partial^2 F}{\partial c^2}(C^*_{t^-}, \sigma_X)dt\right] \tag{72}$$

$$\frac{\partial F}{\partial \sigma_A}(c, \sigma_A) = \mathbb{E}_c\left[\int_0^{\tau_0} e^{-(r-\mu)t}\left(\sigma_A C^*_{t^-} - \rho\sigma_X\right)\frac{\partial^2 F}{\partial c^2}(C^*_{t^-}, \sigma_A)dt\right] \tag{73}$$

$$\frac{\partial F}{\partial \rho}(c, \rho) = \mathbb{E}_c\left[\int_0^{\tau_0} e^{-(r-\mu)t}\left(-\sigma_A\sigma_X\right)\frac{\partial^2 F}{\partial c^2}(C^*_{t^-}, \rho)dt\right] > 0 \tag{74}$$

and

$$\frac{dc^*(\theta)}{d\theta} = -\frac{r-\mu}{\lambda}\frac{\partial F}{\partial \theta}(c^*(\theta), \theta) \quad \text{for } \theta \in \{\sigma_X, \sigma_A, \rho\} \tag{75}$$

Equations (72)-(75) hold in the liquidation case and the refinancing case.

Proof of Corollary 1. We recall that, in the refinancing case $\tau_0 = \infty$ a.e. The proof follows directly from Proposition 8. It remains simply to remark that, for $\theta \in \{\sigma_X, \sigma_A, \rho\}$, we have

$$\frac{\partial F}{\partial \theta}(\bar{c}(\theta), \theta) - \frac{\partial F}{\partial \theta}(0, \theta) = 0. \tag{76}$$

Equation (76) results from differentiating $F(0, \theta) = F(\bar{c}(\theta), \theta) - p(\bar{c}(\theta) + \phi)$ with respect to θ and using the fact that $\frac{\partial F}{\partial c}(\bar{c}(\theta), \theta) = p$. □

C.2. Comparative statics: parameters p, ϕ

Corollary 2. *The following holds (refinancing case):*

1.

$$\frac{\partial F}{\partial p}(c, p) = -(\bar{c}(p) + \phi)\mathbb{E}_c\left[\sum_{n\geq 1} e^{-r\tau^*_n}\right] < 0, \quad \frac{dc^*(p)}{dp} = -\frac{r-\mu}{\lambda}\frac{\partial F}{\partial p}(c^*(p), p) > 0$$

2.

$$\frac{\partial F}{\partial \phi}(c, \phi) = -p\sum_{n\geq 1}\mathbb{E}_c\left[e^{-r\tau^*_n}\right] < 0, \quad \frac{dc^*(\phi)}{d\phi} = -\frac{r-\mu}{\lambda}\frac{\partial F}{\partial \phi}(c^*(\phi), \phi) > 0$$

Proof of Corollary 2. Direct implication of Proposition 8 □

C.3. Comparative statics: parameters α, μ

Corollary 3. *The following holds; in the refinancing case, for all $c \in [0, c^*)$,*

1.

$$\frac{\partial F}{\partial \alpha}(c, \alpha) = \mathbb{E}_c \left[\int_0^\infty e^{-(r-\mu)t} \frac{\partial F}{\partial c}(C^*_{t^-}, \alpha)\, dt \right] > 0,$$

$$\frac{dc^*(\alpha)}{d\alpha} = -\frac{r-\mu}{\lambda} \left(\frac{\partial F}{\partial \alpha}(c^*(\alpha), \alpha) - \frac{1}{r-\mu} \right) < 0.$$

2.

$$\frac{\partial F}{\partial \mu}(c, \mu) = \mathbb{E}_c \left[\int_0^\infty e^{-(r-\mu)t} \left(F(C^*_{t^-}, \mu) - C^*_{t^-} \frac{\partial F}{\partial c}(C^*_{t^-}, \mu) \right) dt \right] > 0,$$

$$\frac{dc^*(\mu)}{d\mu} = -\frac{r-\mu}{\lambda} \left(\frac{\partial F}{\partial \mu}(c^*(\mu), \mu) - \frac{\lambda}{(r-\mu)^2} (\frac{\alpha}{\lambda} - c^*(\mu)) \right) > 0$$

Proof of Corollary 3. Note that, Equation (76) holds for $\theta \in \{\alpha, \mu\}$. Then, formulas for $\frac{\partial F}{\partial \theta}(c, \theta)$ and $\frac{dc^*(\theta)}{d\theta}$ with $\theta \in \{\alpha, \mu\}$ follow from Proposition 8. Let us recall that $\frac{\partial F}{\partial c}(c, \theta) > 1$ over $[0, c^*)$ and $C^*_t \leq c^*$ $\mathbb{P}$ almost surely. Thus, $\frac{\partial F}{\partial \alpha}(c, \alpha) > 0$ and, for $c \in [0, c^*)$, we have

$$\frac{\partial F}{\partial \alpha}(c, \alpha) = \mathbb{E}_c \left[\int_0^\infty e^{-(r-\mu)t} \frac{\partial F}{\partial c}(C^*_{t^-}, \alpha)\, dt \right] > \mathbb{E} \left[\int_0^\infty e^{-(r-\mu)t} \right] = \frac{1}{r-\mu},$$

which implies $\frac{dc^*(\alpha)}{d\alpha} < 0$. Together with the concavity of F with respect to c, it follows also that, for all $c \in [0, c^*)$,

$$F(c, \mu) - c \frac{\partial F}{\partial c}(c, \mu) > F(c, \mu) - c > 0,$$

which leads to $\frac{\partial F}{\partial \mu}(c, \mu) > 0$. Lastly, noting that $c \longrightarrow F(c, \mu) - c\frac{\partial F}{\partial c}(c, \mu)$ is increasing over $[0, c^*]$, we get

$$\begin{aligned} \frac{\partial F}{\partial \mu}(c, \mu) &= \mathbb{E}_c \left[\int_0^\infty e^{-(r-\mu)t} \left(F(C^*_{t^-}, \mu) - C^*_{t^-} \frac{\partial F}{\partial c}(C^*_{t^-}, \mu) \right) dt \right] \\ &< \mathbb{E} \left[\int_0^\infty e^{-(r-\mu)t} \left(F(c^*, \mu) - c^* \right) dt \right] \\ &= \mathbb{E} \left[\int_0^\infty e^{-(r-\mu)t} \left(\frac{\alpha}{r-\mu} + (1 - \frac{\lambda}{r-\mu})c^* - c^* \right) dt \right] = \frac{\lambda}{(r-\mu)^2} (\frac{\alpha}{\lambda} - c^*), \end{aligned}$$

which implies that $\frac{dc^*(\mu)}{d\mu} > 0$. $\square$.

Corollary 4. *The following holds. In the liquidation case, for all $c \in [0, c^*)$,* *1.*

$$\frac{\partial F}{\partial \alpha}(c,\alpha) = \mathbb{E}_c\left[\int_0^{\tau_0} e^{-(r-\mu)t}\frac{\partial F}{\partial c}(C_t^*,\alpha)\,dt\right] + \mathbb{E}_c\left[e^{-(r-\mu)\tau_0}\frac{\omega}{r-\mu}\right] > 0.$$

The sign of $\dfrac{dc^*(\alpha)}{d\alpha}$ *is indeterminate.*

2.

$$\frac{\partial F}{\partial \mu}(c,\mu) = \mathbb{E}_c\left[\int_0^{\tau_0} e^{-(r-\mu)t}\left(F(C_{t^-}^*,\mu) - C_{t^-}^*\frac{\partial F}{\partial c}(C_{t^-}^*,\mu)\right)\,dt\right] + \mathbb{E}_c\left[e^{-(r-\mu)\tau_0}\frac{\omega\alpha}{(r-\mu)^2}\right] > 0.$$

The sign of $\dfrac{dc^*(\mu)}{d\mu}$ *is indeterminate.*

Proof of Corollary 4. Direct application of Proposition 8. □

D. Proof of Proposition 3

Remark that,

$$\begin{aligned} &\sup_{m_0\ge 0,\tau\in\mathcal{T}} \mathbb{E}\left[e^{-r\tau}(V(A_\tau,m_0) - p(m_0+I) - p\phi A_\tau)\right] \qquad (77)\\ = &\sup_{\tau\in\mathcal{T}}\mathbb{E}\left[\max_{m_0\ge 0}\mathbb{E}\left[e^{-r\tau}(V(A_\tau,m_0) - p(m_0+I) - p\phi A_\tau)|\mathcal{F}_\tau\right]\right].\end{aligned}$$

If issuance costs are low, then (65) is satisfied, $F(0) > \max_{c\in[-\phi,\infty)}(F(c) - p(c+\phi)) = F(\bar{c}) - p(\bar{c}+\phi)$ and the mapping $m \longrightarrow V(A_t,m) - p(m+I) - p\phi A_t$ reaches its maximum at $m_0 = \bar{c}A_t$. Thus, (77) can be written in the form

$$\begin{aligned}\sup_{\tau\in\mathcal{T}}\mathbb{E}\left[e^{-r\tau}\left(V(A_\tau,\bar{c}A_\tau) - p(\bar{c}A_\tau+I) - p\phi A_\tau\right)\right] &= \sup_{\tau\in\mathcal{T}}\mathbb{E}\left[e^{-r\tau}(F(\bar{c}) - p(\bar{c}+\phi))A_\tau - pI)\right]\\ &= \sup_{\tau\in\mathcal{T}}\mathbb{E}\left[e^{-r\tau}(F(0)A_\tau - pI)\right].\end{aligned}$$

Then, standard computations yield the result.

If issuance costs are high, then (64) is satisfied, $F(0) = \frac{\omega\alpha}{r-\mu}$, and the mapping $m \longrightarrow V(A_t,m) - p(m+I) - p\phi A_t$ is decreasing. Thus, no cash is raised at the time of investment (in addition to the investment cost I) and (77) can be written in the form

$$\sup_{\tau\in\mathcal{T}}\mathbb{E}\left[e^{-r\tau}\left(V(A_\tau,0) - pI - p\phi A_\tau\right)\right] = \sup_{\tau\in\mathcal{T}}\mathbb{E}\left[e^{-r\tau}(F(0) - p\phi)A_\tau - pI)\right].$$

If $F(0) > p\phi$, then standard computations leads to (31). Clearly, if $F(0) \le p\phi$, the option value to invest is worthless □

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