

Portfolio choices and asset prices: The  
comparative statics of ambiguity aversion

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## Abstract

This paper investigates the comparative statics of "more ambiguity aversion" as defined by Klibanoff, Marinacci and Mukerji (2005). The analysis uses the static two-asset portfolio problem with one safe asset and one uncertain one. While it is intuitive that more ambiguity aversion would reduce demand for the uncertain asset, this is not necessarily the case. We derive sufficient conditions for a reduction in the demand for the uncertain asset, and for an increase in the equity premium. An example which meets the sufficient conditions is when the set of plausible distributions for returns on the uncertain asset can be ranked according to their monotone likelihood ratio. It is also shown how ambiguity aversion distorts the price kernel in the alternative portfolio problem with complete markets for contingent claims.

**Keywords:** Smooth ambiguity aversion, monotone likelihood ratio, equity premium, portfolio choice, price kernel, central dominance.

# 1 Introduction

This paper examines the standard static portfolio problem with one safe asset and one uncertain asset. For the investor, the true probability distribution for the excess return on the uncertain asset is ambiguous. This ambiguity is expressed by a second order prior probability distribution over the set of plausible (first order) distributions for the excess return. Following Segal (1987) and Klibanoff, Marinacci and Mukerji (2005) (hereafter KMM), an investor's attitude towards ambiguity is introduced by relaxing the reduction of first and second order probabilities. In other words, the investor does not evaluate in the same way lottery 1 yielding a payoff  $a$  with probability  $p$  and lottery 2 yielding the same payoff  $a$  with an unknown probability whose expectation is  $p$ . This is in contrast to the standard Bayesian expected utility framework under which the two lotteries would be evaluated identically. It is assumed that the investor is ambiguity averse if they dislike any mean-preserving spread in the space of first order probability distributions for excess returns. For example, they prefer lottery 1 to lottery 2.

KMM propose a useful and elegant decision criterion called "smooth ambiguity aversion" that is compatible with ambiguity averse preferences under uncertainty. For a given portfolio allocation, the ex ante welfare of the investor is measured by computing the (second order) expectation of a concave function  $\phi$  of the (first order) expected utility  $u$  of final consumption conditional on each plausible distribution for the excess return. As usual, the concavity of the utility function  $u$  expresses risk aversion in the special case of risky acts, i.e. acts for which the plausible probability distributions for their consequences is unique. When  $\phi$  is linear, we are back to the standard expected utility model. The representation of the uncertain context can be reduced to a single compound probability distribution. However, when  $\phi$

is concave, the investor is ambiguity averse, and the reduction to a single compound distribution is not valid.

KMM also define the comparative notion of "more ambiguity aversion". Consider two agents, respectively with function  $\phi_1$  and  $\phi_2$ , who have the same beliefs expressed by the set of first and second order probability distributions. Suppose also that they have the same utility function  $u$  to evaluate risky acts. Agent  $\phi_2$  is more ambiguity averse than agent  $\phi_1$  if  $\phi_1$  prefers an uncertain act over a pure risky one whenever  $\phi_2$  does so. This is true if and only if function  $\phi_2$  is more concave than function  $\phi_1$ , in the Arrow-Pratt sense:  $-\phi_2''/\phi_2'$  is uniformly larger than  $-\phi_1''/\phi_1'$ . This paper examines the effect of such an increase in ambiguity aversion on the optimal level of demand for the uncertain asset. To do so, the first and second order beliefs, and the utility function  $u$  are fixed. The effect of a concave transformation of function  $\phi$  on the optimal portfolio can then be examined.

The KMM model has two attractive features in comparison to other models of ambiguity such as the pioneering maxmin expected utility (or multiple-prior) model of Gilboa and Schmeidler (1989). First, it creates a crisp separation between ambiguity aversion and ambiguity, i.e., between tastes and beliefs. Without this feature, the comparative static analysis of more ambiguity aversion could not be performed. Second, the KMM model applies the expected utility machinery sequentially, starting on the first order probability distributions before moving onto the second order distributions. This allows the huge range of techniques amassed over the years to tackle questions involving risk under the expected utility framework to be applied to the analysis of problems involving ambiguity aversion. A point illustrated by this paper.

The question of the comparative statics of ambiguity aversion for the portfolio problem is parallel to the one of risk aversion. It has been well

established since Arrow (1963) and Pratt (1964) that an increase in risk aversion reduces demand for the risky asset. It is therefore quite surprising that in general, as shown in this paper, it is not necessarily true that more ambiguity aversion reduces demand for an ambiguous asset. For a cleverly chosen - but not spurious - set of priors for the return on the ambiguous asset, it is shown that the introduction of ambiguity aversion *increases* the investor's demand. The intuition for why such counterexamples may exist can be explained as follows. The first-order condition of the portfolio problem with ambiguity aversion can be rewritten in the form that it takes under the expected utility model, but with a distortion in the way the different first order probability distributions for the excess return are compounded. Taking the expected utility approach, i.e., 'ambiguity neutrality', the compounding is made by using the true second order probability distribution. Under ambiguity aversion, this second order probability distribution is distorted by putting more weight on the plausible second order distributions yielding a smaller expected utility. This was first observed by Taboga (2005). In spite of the fact that the ambiguity averse investor's beliefs cannot be reduced to a single compound probability distribution over excess returns, the introduction of ambiguity aversion is observationally equivalent to the effect of distorting the compound distribution used by the ambiguity-neutral investor. This distortion is pessimistic, in a technical sense that is defined more precisely in the paper. It is well known from expected utility theory that pessimistic deteriorations in beliefs do not always reduce the demand for the risky asset.<sup>1</sup> As for Giffen goods in consumer theory, this deterioration in the terms of trade yields a wealth effect that may in fact raise demand.

The main objective of the paper is to characterize conditions under which

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<sup>1</sup>See Rothschild and Stiglitz (1971), Fishburn and Porter (1976), Meyer and Ormiston (1985), Hadar and Seo (1990), Gollier (1995), Eeckhoudt and Gollier (1995), Abel (2002) and Athey (2002), and the bibliographical references in these papers.

more ambiguity aversion reduces the optimal level of exposure to uncertainty. This can be done by restricting either the set of utility functions and/or the set of possible priors. If it is assumed that the set of priors can be ranked according to their order of first-degree or second-degree stochastic dominance (FSD/SSD), then some simple sufficient conditions on the utility function result in unambiguous comparative static properties after the introduction of ambiguity aversion. This is illustrated in the paper. These results are derived by using the following technique. Any increase in ambiguity aversion deteriorates the observationally equivalent second order distribution in a very specific way. More weight is transferred onto the worst priors in the sense of those with the lowest monotone likelihood ratio (MLR). This puts a very specific structure on the notion of pessimism caused by more ambiguity aversion. For example, if the plausible priors can be ranked according to FSD, then their compound MLR deteriorated second order probability distribution generated by more ambiguity aversion implies a FSD deterioration of the behaviorally equivalent changes of beliefs under expected utility. In turn this implies that the following two questions are linked :

1) under the EU model, what are the conditions on the utility function  $u$  that guarantee that any FSD deterioration in the distribution of excess return of the risky asset reduces the demand for it?; and

2) in the KMM model, what are the conditions that guarantee that any increase in ambiguity aversion reduces the demand for the ambiguous asset?

The same property holds when replacing FSD by SSD. More sophisticated methods are required when considering other stochastic orders to rank plausible priors for the excess return. The result which is easiest to express should be mentioned at this stage: if the plausible priors can be ranked according to MLR (a special case of FSD), then it is always true that more ambiguity aversion reduces the demand for the ambiguous asset.

It is easy to translate these results about the effect of comparative ambiguity aversion on the demand for the ambiguous asset into its effect on the equity premium. Therefore this work is related to recent research on the effect of ambiguity aversion on the equity premium.

Ju and Miao (2009) and Collard, Mukerji, Sheppard and Tallon (2009) examine a dynamic infinite-horizon portfolio problem in which the representative investor exhibits smooth ambiguity aversion and faces time-varying ambiguity about the second order distribution of the plausible probability distributions of consumption growth. These two papers consider different sets of risk-ambiguity attitudes  $(u, \phi)$ , and different stochastic processes for the first and second order probability distributions. They both use numerical analyses to solve the calibrated dynamic portfolio problem and they both conclude that ambiguity aversion raises the equity premium. Other papers draw a similar conclusion, but using other decision criterion for ambiguity aversion, either maxmin expected utility, Choquet expected utility, or robust control theory.<sup>2</sup> This paper demonstrates that these results are specific to the calibration under scrutiny.

The portfolio problem and two illustrations are presented in Section 2. The main results are presented in Section 3 in which sufficient conditions are derived for the comparative statics of more ambiguity aversion. Section 4 examines a Lucas economy with a representative agent facing ambiguous state probabilities. It is shown how the ambiguity aversion of the representative agent affects the equity premium, the price kernel of the economy, and individual asset prices.

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<sup>2</sup>See for example Dow and Werlang (1992), Chen and Epstein (2002), Epstein and Wang (1994), Hansen, Sargent and Tallarini (1999), Maenhout (2004), Hansen and Sargent (2008).

## 2 The smooth ambiguity model applied to the portfolio problem

The model is static with two assets. The first asset is safe with a rate of return that is normalized to zero. The uncertain asset has a return  $x$  whose distribution is ambiguous in the sense that it is sensitive to some parameter  $\theta$  whose true value is unknown. The investor is initially endowed with wealth  $w_0$ . If they invest  $\alpha$  in the uncertain asset, their final wealth will be  $w_0 + \alpha x$  conditional on a realized return  $x$  on the uncertain asset.

The ambiguity of the uncertain asset is characterized by a set  $\Pi = \{F_1, \dots, F_n\}$  of plausible cumulative probability distributions for  $\tilde{x}$ . Let  $\tilde{x}_\theta$  denote the random variable distributed as  $F_\theta$ . It is supposed that the support of all priors are bounded in  $[x_-, x_+]$ , with  $x_- < 0 < x_+$ . Based on their subjective information, the investor associates a second order probability distribution  $(q_1, \dots, q_n)$  over the set of priors  $\Pi$ , with  $\sum_{\theta=1}^n q_\theta = 1$ , where  $q_\theta \geq 0$  is the probability that  $F_\theta$  is the true probability distribution for excess returns. From now on  $\tilde{\theta}$  denotes the random variable  $(1, q_1; 2, q_2; \dots; n, q_n)$ . Following Klibanoff, Marinacci and Mukerji (2005), it is assumed that the investor's preferences exhibit smooth ambiguity aversion. For each plausible probability distribution  $F_\theta$ , the investor computes the expected utility  $U(\alpha, \theta) = Eu(w_0 + \alpha \tilde{x}_\theta) = \int u(w_0 + \alpha x) dF_\theta(x)$  conditional on  $F_\theta$  being the true distribution. It is assumed that  $u$  is increasing and concave, so that  $U(\cdot, \theta)$  is concave in the level of investment  $\alpha$  in the ambiguous asset, for all  $\theta$ . Ex ante, for a given portfolio allocation  $\alpha$ , the welfare of the agent is measured by  $V(\alpha)$  with:

$$V(\alpha) = \phi^{-1} \left( \sum_{\theta=1}^n q_\theta \phi(U(\alpha, \theta)) \right) = \phi^{-1} \left( \sum_{\theta=1}^n q_\theta \phi(Eu(w_0 + \alpha \tilde{x}_\theta)) \right),$$

$V(\alpha)$  can be interpreted as the certainty equivalent of the uncertain con-

ditional expected utility  $U(\alpha, \tilde{\theta})$ . The shape of  $\phi$  describes the investor's attitude towards ambiguity. A linear  $\phi$  means that the investor has a neutral attitude towards ambiguity, and that their compound probability distributions can be reduced to a single one  $\sum_{\theta} q_{\theta} F_{\theta}$ . In contrast, a concave  $\phi$  is synonymous with ambiguity aversion in the sense that the DM dislikes any mean-preserving spread of the conditional expected utility  $U(\alpha, \tilde{\theta})$ .

An interesting particular case arises when the absolute ambiguity aversion  $\eta(U) = -\phi''(U)/\phi'(U)$  is constant, so that  $\phi(U) = -\eta^{-1} \exp(-\eta U)$ . As proved by Klibanoff, Marinacci and Mukerji (2005), the ex ante welfare  $V(\alpha)$  essentially exhibits a maxmin expected utility functional  $V^{MEU}(\alpha) = \min_{\theta} Eu(w_0 + \alpha \tilde{x}_{\theta})$  à la Gilboa and Schmeidler (1989) when the degree  $\eta$  of absolute ambiguity aversion tends to infinity.

The optimal portfolio allocation  $\alpha^*$  maximizes the ex ante welfare of the investor  $V(\alpha)$ .  $\phi$  is increasing, therefore  $\alpha^*$  is the solution of the following program:

$$\alpha^* \in \arg \max_{\alpha} \sum_{\theta=1}^n q_{\theta} \phi(Eu(w_0 + \alpha \tilde{x}_{\theta})). \quad (1)$$

If  $\phi$  and  $u$  are strictly concave, then the objective function is concave in  $\alpha$  and the solution to program (1), when it exists, is unique. It can be observed that the demand for the ambiguous asset shares its sign with the equity premium  $E\tilde{x} = \sum_{\theta} q_{\theta} E\tilde{x}_{\theta}$ . All proofs are relegated to the Appendix.

**Lemma 1** *The demand for the ambiguous asset is positive (zero/negative) if the equity premium is positive (zero/negative).*

This means that ambiguity aversion, as is the case for risk aversion, has a second order nature, as defined by Segal and Spivak (1990). As soon as the equity premium is positive, the demand for the ambiguous asset is

positive, independent of the degree of ambiguity over the distribution of returns. From now on it is assumed that the equity premium is positive, so that  $\alpha^*$  is positive.

The remainder of this section, examines two illustrations. Consider first the following special case in which an analytical solution can be found for  $\alpha^*$ :

- Priors are normally distributed with the same variance  $\sigma^2$ , and with  $E\tilde{x}_\theta = \theta : \tilde{x}_\theta \sim N(\theta, \sigma)$ ;<sup>3</sup>
- The ambiguity over the equity premium  $\theta$  is itself normally distributed:  $\tilde{\theta} \sim N(\mu, \sigma_0)$ ;
- The investor's preferences exhibit constant absolute risk aversion:  $u(z) = -A^{-1} \exp -Az \in \mathbb{R}_-$ ;
- The investor's preferences exhibit constant relative ambiguity aversion:  $\phi(U) = -(-U)^{1+\gamma}/(1 + \gamma)$ . This function is increasing in  $\mathbb{R}_-$  and is concave in this domain if  $\gamma$  is positive.

As is well-known, the normality of the priors and the constancy of absolute risk aversion implies that the Arrow-Pratt approximation is exact.<sup>4</sup> This implies in turn that

$$U(\alpha, \theta) = -A^{-1} \exp -A (w_0 + \alpha\theta - 0.5A\alpha^2\sigma^2). \quad (2)$$

Because  $\phi(U)$  is an exponential function of  $\theta$ , and  $\tilde{\theta}$  is normally distributed, therefore the same trick can be used to compute  $E\phi$  yielding:

$$V(\alpha) = -A^{-1} \exp -A(w_0 + \alpha\mu - 0.5A\alpha^2(\sigma^2 + (1 + \gamma)\sigma_0^2)). \quad (3)$$

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<sup>3</sup>It is easy to extend this to the case of an ambiguous variance.

<sup>4</sup>For a simple proof, see for example Gollier (2001, page 57).

The optimal demand for the uncertain asset is thus equal to:

$$\alpha^* = \frac{\mu}{A(\sigma^2 + (1 + \gamma)\sigma_0^2)}. \quad (4)$$

It can be seen that under ambiguity ( $\sigma_0^2 > 0$ ), the demand for the uncertain asset is decreasing in the relative degree  $\gamma$  of ambiguity aversion of the investor. This exponential-power specification for  $(u, \phi)$  differs from the other three papers on this topic. Taboga (2005) examines an exponential-exponential specification. Ju and Miao (2009) used a power-power specification, whereas Collard, Mukerji, Sheppard and Tallon (2009) used a power-exponential specification. None of these three alternative problems can be solved analytically.

Alternatively, consider the following counterexample:

- $n = 2$ ,  $\tilde{x}_1 \sim (-1, 2/10; -0.25, 3/20; 0.75, 7/20; 1.25, 3/10)$  and  $\tilde{x}_2 \sim (-1, 1/5; 0, 1/5; 1, 3/5)$ ;
- $q_1 = 5\%$  and  $q_2 = 95\%$ ;
- $u(z) = \min(z, 3 + 0.3(z - 3))$  and  $w_0 = 2$ ;
- $\phi(U) = -\eta^{-1} \exp(-\eta U)$ .

It is easy to check that  $\tilde{x}_1$  is riskier than  $\tilde{x}_2$  in the sense of Rothschild and Stiglitz (1970) sense. The problem is solved numerically. Below a minimum threshold of around 20 for the degree  $\eta$  of ambiguity aversion, the optimal holding of the ambiguous asset equals  $\alpha^* = 1$ . However, above this threshold, the introduction of ambiguity aversion *increases*  $\alpha^*$  above the optimal investment of the ambiguity-neutral investor. For example,  $\alpha^*$  equals 1.204 when  $\eta = 50$ . When  $\eta$  tends to infinity, the optimal investment in the uncertain asset tends to  $\alpha^* = 4/3$ , which is the optimal holding of the ambiguous asset for an ambiguity-neutral investor who believes that the distribution of

excess return is  $\tilde{x}_1$  with certainty. In terms of portfolio allocation, it is observationally equivalent to increase absolute ambiguity aversion from zero to infinity or to replace beliefs  $\tilde{y}_1 \sim (\tilde{x}_1, 5\%; \tilde{x}_2, 95\%)$  by  $\tilde{y}_2 \sim (\tilde{x}_1, 100\%)$  under expected utility. Notice that because  $\tilde{x}_1$  is riskier than  $\tilde{x}_2$ , the extreme belief  $\tilde{y}_2$  is riskier than  $\tilde{y}_1$  in the Rothschild and Stiglitz sense. This example illustrates the fact – first observed by Rothschild and Stiglitz (1971) – that it is not true in general that a riskier distribution for excess returns reduces the demand for the risky asset in the expected utility model.

### 3 Effect of an increase of ambiguity aversion

The investor's beliefs are represented by the set of priors  $(\tilde{x}_1, \dots, \tilde{x}_n)$  for the excess return of the uncertain asset, together with the second order distribution  $(q_1, \dots, q_n)$  on these priors. A comparison is made between two agents with the same beliefs and the same concave utility function  $u$ , but with different attitudes toward ambiguity represented by the concave functions  $\phi_1$  and  $\phi_2$ . The demand for the uncertain asset by agent  $\phi_1$  is expressed by  $\alpha_1^*$  which must satisfy the following first-order condition:

$$\sum_{\theta=1}^n q_{\theta} \phi_1'(U(\alpha_1^*, \theta)) E \tilde{x}_{\theta} u'(w_0 + \alpha_1^* \tilde{x}_{\theta}) = 0. \quad (5)$$

Following Klibanoff, Marinacci and Mukerji (2005), it is assumed that the agent with function  $\phi_2$  is more ambiguity averse than agent  $\phi_1$  in the sense that there exists an increasing and concave transformation function  $k$  such that  $\phi_2(U) = k(\phi_1(U))$  for all  $U$  in the relevant domain. We would like to characterize the conditions under which the more ambiguity averse agent  $\phi_2$  has a smaller demand for the uncertain asset than agent  $\phi_1$ :  $\alpha_2^* \leq \alpha_1^*$ . This

would be the case if and only if:

$$\sum_{\theta=1}^n q_{\theta} \phi'_2(U(\alpha_1^*, \theta)) E \tilde{x}_{\theta} u'(w_0 + \alpha_1^* \tilde{x}_{\theta}) \leq 0. \quad (6)$$

The conditions under which it is always true that (5) implies (6) are sought. Notice that this implication can be rewritten as:

$$E \tilde{y}_1 u'(w_0 + \alpha_1^* \tilde{y}_1) = 0 \implies E \tilde{y}_2 u'(w_0 + \alpha_1^* \tilde{y}_2) \leq 0, \quad (7)$$

where  $\tilde{y}_i$  is a compound random variable which equals  $\tilde{x}_{\theta}$  with probability  $\hat{q}_{\theta}^i$ ,  $\theta = 1, \dots, n$ , such that

$$\hat{q}_{\theta}^i = \frac{q_{\theta} \phi'_i(U(\alpha_1^*, \theta))}{\sum_{t=1}^n q_t \phi'_i(U(\alpha_1^*, t))}. \quad (8)$$

Notice that the left equality on the left in (7) can be interpreted as the first-order condition of the problem  $\max_{\alpha} E u(w_0 + \alpha \tilde{y}_1)$  for an expected-utility-maximizing investor whose beliefs are represented by  $\tilde{y}_1 \sim (\tilde{x}_1, \hat{q}_1^1; \dots; \tilde{x}_n, \hat{q}_n^1)$ . Therefore, the ambiguity averse agent  $\phi_1$  behaves in the same way as an EU-maximizing agent who has distorted their second order beliefs from  $(q_1, \dots, q_n)$  to the "observationally equivalent probability distribution"  $\hat{q}^1 = (\hat{q}_1^1, \dots, \hat{q}_n^1)$ . The distortion factor  $\phi'_1(U(\alpha_1^*, \theta)) / \sum_t q_t \phi'_1(U(\alpha_1^*, t))$  is a Radon-Nikodym derivative, and the probability distribution  $\hat{q}^1$  is analogous to the risk-neutral probability distribution used in the theory of finance. Notice that the distortion functional described by equation (8) is endogenous, because it depends upon the portfolio allocation  $\alpha_1^*$  selected by the agent. The inequality on the right in (7) just means that shifting beliefs from  $\tilde{y}_1$  to  $\tilde{y}_2$  reduces the ambiguity-neutral investor's holding of the asset. These findings are summarized in the following lemma:

**Lemma 2** *The change in preferences from  $(u, \phi_1)$  to  $(u, \phi_2)$  reduces the demand for the ambiguous asset if the EU agent with utility function  $u$  reduces*

their demand for the uncertain asset when their beliefs about the excess return shift from  $\tilde{y}_1 \sim (\tilde{x}_1, \hat{q}_1^1; \dots; \tilde{x}_n, \hat{q}_n^1)$  to  $\tilde{y}_2 \sim (\tilde{x}_1, \hat{q}_1^2; \dots; \tilde{x}_n, \hat{q}_n^2)$ , where  $\hat{q}_\theta^i$  is defined by (8).

This result was initially due to Taboga (2005). It precisely expresses the observational equivalence property that has already been encountered in the counterexample presented in the previous section. It provides a test to determine whether more ambiguity aversion reduces demand. Observe that this test relies on two reduced probability distribution  $\tilde{y}_1$  and  $\tilde{y}_2$ . However, it is not true that the ambiguity averse investor  $(u, \phi_1)$  uses the corresponding reduced probability distribution  $\tilde{y}_1$  to evaluate the optimality of the different feasible portfolios. If they did so, they would re-evaluate the distribution of  $\tilde{y}_1$  for each portfolio, since vector  $\hat{q}_1$  is a function of  $\alpha$ . In the smooth ambiguity aversion model, beliefs cannot be reduced to a single probability distribution over the payoffs for each state of the world. However, this lemma builds a bridge between the comparative statics of increased ambiguity aversion and increases in risk in the classical EU model.

Let us now examine how changing function  $\phi_1$  into  $\phi_2$  modifies the observationally equivalent probability distribution for the excess return. A first answer to this question is provided by the following lemma.

**Lemma 3** *The following two conditions are equivalent:*

1. *Agent  $\phi_2$  is more ambiguity averse than agent  $\phi_1$ ;*
2. *Beliefs  $\hat{q}^2$  are dominated by  $\hat{q}^1$  in the sense of their monotone likelihood ratio order.*

Property 2 in the above lemma means that, assuming  $U(\alpha_1^*, 1) \leq U(\alpha_1^*, 2) \leq \dots \leq U(\alpha_1^*, n)$ ,  $\hat{q}_\theta^2/\hat{q}_\theta^1$  is decreasing in  $\theta$ . An increase in ambiguity aversion has

an effect on demand that is observationally equivalent to a MLR-dominated shift in the prior beliefs. In other words, it distorts beliefs by favouring the worst priors in a very specific sense: if agent  $\phi_1$  prefers prior  $\tilde{x}_\theta$  over prior  $\tilde{x}_{\theta'}$ , then, compared to agent  $\phi_1$ , the more ambiguity averse agent  $\phi_2$  increases the distorted probability  $\hat{q}_\theta^2$  relatively more than the probability  $\hat{q}_\theta^1$ . Lemma 3 provides a justification for saying that, in the case of the portfolio problem, more ambiguity aversion is observationally equivalent to more pessimism, i.e., to a MLR deterioration of beliefs. This result is central to proving the next proposition, in which three dominance orders are considered: first degree stochastic dominance (FSD), second-degree stochastic dominance (SSD), and Rothschild and Stiglitz's increase in risk (IR).

**Proposition 1** *Let  $D$  be one of the following stochastic orders: FSD, SSD or IR. Suppose that  $E\tilde{x} > 0$ , and that  $(\tilde{x}_1, \dots, \tilde{x}_n)$  can be ranked according to the stochastic order  $D$ . If there is a concave function  $k$  such that  $\phi_2 = k(\phi_1)$  and if  $\tilde{x}_1 \preceq_D \tilde{x}_2 \preceq_D \dots \preceq_D \tilde{x}_n$ , then*

$$\tilde{y}_2 \sim (\tilde{x}_1, \hat{q}_1^2; \dots; \tilde{x}_n, \hat{q}_n^2) \preceq_D (\tilde{x}_1, \hat{q}_1^1; \dots; \tilde{x}_n, \hat{q}_n^1) \sim \tilde{y}_1.$$

In words, if  $(\tilde{x}_1, \dots, \tilde{x}_n)$  can be ranked according to the stochastic order  $D$ , then an increase in ambiguity aversion deteriorates the observationally equivalent probability distribution for the excess return in the sense of the stochastic order  $D$ . Therefore, if priors can be ranked according to first-degree stochastic dominance, the increase in ambiguity aversion modifies the demand for the asset in the same direction as an FSD deterioration of the excess return in the expected utility model. The problem is that, in general, the comparative statics of an FSD deterioration in the excess return is ambiguous in the expected utility model. The intuition for this negative result is that a reduction in the return on an asset has a substitution effect and a wealth effect. As for the existence of Giffen goods in consumption theory,

the wealth effect may induce an increase in the asset demand. Technically, it is not true in general that condition (7) holds when  $\tilde{y}_2 \preceq_{FSD} \tilde{y}_1$ . It is easy to see why: by the definition of FSD, this would be true if and only if function  $f(y) = yu'(w_0 + \alpha y)$  was increasing, which is not true in general. As observed by Fishburn and Porter (1976), a sufficient condition for  $f$  to be increasing is that relative risk aversion  $R(z) = -zu''(z)/u'(z)$  is smaller than unity.<sup>5</sup> By implication, this is also a sufficient condition for an increase in ambiguity aversion to reduce the demand for the ambiguous asset when priors can be ranked according to FSD.

The same strategy can be used to examine the case when priors can be ranked in the sense of a Rothschild-Stiglitz increase in risk. In that case, the above proposition tells us that the observationally equivalent probability distribution  $\tilde{y}_2$  is an increase in risk compared to  $\tilde{y}_1$ . This does not in general imply that condition (7) holds because  $f$  is not necessarily concave. As initially shown by Rothschild and Stiglitz (1971), it is not true in general that an increase in risk for the excess return of the risky asset reduces its demand. Hadar and Seo (1990) provided a sufficient condition, which guarantees that  $f$  is concave. This condition is that relative prudence is positive and less than 2, where relative prudence is defined by  $P(z) = -zu'''(z)/u''(z)$  (Kimball (1990)). This proves the following result:

**Proposition 2** *Suppose that  $u \in C^3$  and  $E\tilde{x} > 0$ . Any increase in ambiguity aversion reduces the demand for the uncertain asset if one of the following two conditions is satisfied:*

1.  $(\tilde{x}_1, \dots, \tilde{x}_n)$  can be ranked according first-degree stochastic dominance, and  $R \leq 1$ ;

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<sup>5</sup>with  $A(z) = -u''(z)/u'(z)$ .

2.  $(\tilde{x}_1, \dots, \tilde{x}_n)$  can be ranked according to the Rothschild and Stiglitz's riskiness order, and  $0 \leq P^r \leq 2$ .

More generally, if the set of marginals can be ranked according to the SSD order, an increase in ambiguity aversion reduces the demand for the risky asset if relative risk aversion is less than unity, and relative prudence is positive and less than two. In the case of power utility function, relative prudence equals relative risk aversion plus one. This implies that when relative risk aversion is constant, and when priors can be ranked according to SSD, any increase in ambiguity aversion reduces the demand for the ambiguous asset if relative risk aversion is less than unity. This condition is not very convincing, since relative risk aversion is usually assumed to be larger than unity. Arguments have been provided based on introspection (Drèze (1981), Kandel and Stambaugh (1991), Gollier (2001)) or on the equity premium puzzle that can be solved in the canonical model only with a degree of relative risk aversion exceeding 40.

Rather than limiting the set of utility functions yielding an unambiguous effect, an alternative approach is to restrict the set of priors. To do this, let us first introduce the following concepts, which rely on the location-weighted-probability function  $T_\theta$  that is defined as follows:

$$T_\theta(x) = \int_{x_-}^x t dF_\theta(t). \quad (9)$$

Following Gollier (1995), it can be said that  $\tilde{x}_2$  is dominated by  $\tilde{x}_1$  in the sense of Central Dominance if there exists a nonnegative scalar  $m$  such that  $T_2(x) \leq mT_1(x)$  for all  $x \in [x_-, x_+]$ .<sup>6</sup> Gollier (1995) showed that  $\tilde{x}_2 \preceq_{CD} \tilde{x}_1$

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<sup>6</sup>There is no simple interpretation of this stochastic order in the literature. However, observe that replacing  $\tilde{x}_1$  by  $\tilde{x}_2 \sim (\tilde{x}_1, m; 0, 1 - m)$  implies that  $T_2 = mT_1$ . This proportional probability transfer to the zero excess return has no effect on the risk-averse investor's demand for the risky asset. This explains the presence of the arbitrary scalar

is necessary and sufficient to guarantee that all risk-averse investors reduce their demand for the risky asset whose distribution for excess returns goes from  $\tilde{x}_1$  to  $\tilde{x}_2$ . SSD-dominance is not sufficient for CD-dominance. Notice that  $\tilde{x}_1$  and  $\tilde{x}_2$  in the counterexample of the previous section violate the CD condition. It implies that there exists a concave utility function such that the demand for the asset is increased when beliefs go from  $(\tilde{x}_1, 5\%; \tilde{x}_2, 95\%)$  to the riskier  $\tilde{x}_1$ .

Here is a partial list of stochastic orders that have been shown to belong to the wide set of CD:

- Monotone Likelihood Ratio order (MLR) (Ormiston and Schlee (1993)). Notice that MLR is a subset of FSD.
- Strong Increase in Risk (Meyer and Ormiston (1985)): The excess return  $\tilde{x}_2$  is a strong increase in risk with respect to  $\tilde{x}_1$  if they have the same mean and if any probability mass taken out of some of the realizations of  $\tilde{x}_1$  is transferred out of the support of this random variable.
- Simple Increase in Risk (Dionne and Gollier (1992)): Random variable  $\tilde{x}_2$  is a simple increase in risk with respect to  $\tilde{x}_1$  if they have the same mean and  $x(F_1(x) - F_2(x))$  is nonnegative for all  $x$ .
- Monotone Probability Ratio order (MPR) (Eeckhoudt and Gollier (1995), Athey (2002)): When the two random variables have the same support, it can be said that  $\tilde{x}_2$  is dominated by  $\tilde{x}_1$  in the sense of MPR if the cumulative probability ratio  $F_2(x)/F_1(x)$  is nonincreasing. It can be

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$m$  in the definition of CD. Moreover a CD shift with  $m = 1$  requires a reduction of the location-weighted-probability function  $T$ . For example, if one divides a probability mass  $p$  at some return  $x = r < 0$  in two equal masses  $(p/2, p/2)$ , the transfer of mass to the left must be to the left of  $2r$ , which is a strong condition.

shown that MPR is more general than MLR, but is still a subset of FSD.

The next result allows the conditions on  $u$  to be relaxed, but at the cost of restricting the set of priors.

**Proposition 3** *Suppose that  $E\tilde{x} > 0$ . Any increase in ambiguity aversion reduces the demand for the uncertain asset if the set of priors  $(\tilde{x}_1, \dots, \tilde{x}_n)$  can be ranked according to both SSD dominance and central dominance, that is, if  $\tilde{x}_\theta \preceq_{SSD} \tilde{x}_{\theta+1}$  and  $\tilde{x}_\theta \preceq_{CD} \tilde{x}_{\theta+1}$  for all  $\theta = 1, \dots, n - 1$ .*

To illustrate, because it is known that MLR yields both first-degree stochastic dominance and central dominance, the following corollary is obtained directly:

**Corollary 1** *Suppose that  $E\tilde{x} > 0$  and that  $(\tilde{x}_1, \dots, \tilde{x}_n)$  can be ranked according to the monotone likelihood ratio order. Then, any increase in ambiguity aversion reduces the demand for the uncertain asset.*

In this case, it can be concluded that ambiguity aversion and risk aversion work in the same direction. A more general corollary holds when the MLR order is replaced by the more general MPR order.

It is noteworthy that the comparative statics of ambiguity aversion are much simpler when considering market participation. Of course, as observed in Lemma 1, the basic model is not well suited to examining this question, since all agents should have a positive demand for equity as soon as the equity premium is positive (second order risk aversion). Let us introduce a fixed cost  $C$  for market participation, so that the new model is with  $U(\alpha, \theta) = Eu(w_0 - C + \alpha\tilde{x}_\theta)$ ,  $V(\alpha) = \phi^{-1}(\sum_\theta q_\theta \phi(U(\alpha, \theta)))$ ,  $\bar{\alpha} = \arg \max V(\alpha)$ , and  $\alpha^* = \bar{\alpha}$  if  $V(\bar{\alpha}) \geq u(w_0)$ , and  $\alpha^* = 0$  otherwise. Obviously, because  $V$

is the certainty equivalent of  $U(\alpha, \tilde{\theta})$  under function  $\phi$ , an increase in the concavity of  $\phi$  reduces  $V(\alpha)$  for all  $\alpha$ . Therefore the condition for market participation  $V(\bar{\alpha}) \geq u(w_0)$  is less likely to hold when ambiguity aversion is increased. This means that ambiguity aversion may explain the market participation puzzle (Haliassos and Bertaut (1995)).<sup>7</sup>

## 4 Asset prices with complete markets

This section extends the focus of our analysis to the effect of ambiguity aversion on the price of contingent claims. Consider a Lucas tree economy with identical risk-averse and ambiguity averse representative agents whose preferences are characterized by increasing and concave functions  $(u, \phi_i)$ . Each agent is endowed with a tree producing an uncertain quantity of fruit at the end of the period. There are  $S$  possible states of nature, with  $c_s$  denoting the number of fruits produced by the trees in state  $s$ ,  $s = 1, \dots, S$ . The distribution of states is subject to some parametric uncertainty. Parameter  $\theta$  can take value  $1, \dots, n$  with probabilities  $(q_1, \dots, q_n)$ , and  $p_{s|\theta}$  is the probability of state  $s$  conditional on  $\theta$ . Let  $p_s = \sum_{\theta} q_{\theta} p_{s|\theta}$  denote the unconditional probability of state  $s$ . Ex ante, there is a market for contingent claims. Agents trade claims for fruit contingent on the harvest. Assuming complete markets, the ambiguity averse and risk-averse agent, whose preferences are given by the pair  $(u, \phi_i)$ , solves the following problem:

$$\max_{(x_1, \dots, x_S)} \sum_{\theta=1}^n q_{\theta} \phi_i \left( \sum_{s=1}^S p_{s|\theta} u(x_s) \right), \text{ s.t. } \sum_{s=1}^S \Pi_s (x_s - c_s) = 0, \quad (10)$$

where  $x_s - c_s$  is the demand for the Arrow-Debreu security associated with state  $s$ , and  $\Pi_s$  is the price of that contingent claim. The first-order conditions

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<sup>7</sup>Thus, our story of the role ambiguity aversion in explaining the market participation puzzle differs from the one by Dow and Werlang (1992) and Epstein and Schneider (2007), who consider a MEU model without participation cost.

for this program are written as:

$$u'(x_s) \left[ \sum_{\theta=1}^n q_{\theta} \phi'_i \left( \sum_{s'=1}^S p_{s'|\theta} u(x_{s'}) \right) p_{s|\theta} \right] = \lambda \Pi_s, \quad (11)$$

for all  $s$ , where  $\lambda$  is the Lagrange multiplier associated with the budget constraint. The market-clearing conditions impose that  $x_s = c_s$  for all  $s$ , which implies the following equilibrium state prices:

$$\Pi_s^i = \widehat{p}_s^i u'(c_s), \quad (12)$$

for all  $s$ , where the distorted state probability  $\widehat{p}_s^i$  is defined as follows:

$$\widehat{p}_s^i = \sum_{\theta=1}^n \widehat{q}_{\theta}^i p_{s|\theta} \text{ with } \widehat{q}_{\theta}^i = \frac{q_{\theta} \phi'_i(Eu(\tilde{c}_{\theta}))}{\sum_{t=1}^n q_t \phi'_i(Eu(\tilde{c}_t))}, \quad (13)$$

where  $\tilde{c}_{\theta}$  is distributed as  $(c_1, p_{1|\theta}; \dots, c_S, p_{S|\theta})$ . Under ambiguity neutrality, it follows that  $\widehat{q}_{\theta}^i = q_{\theta}$ , and  $\widehat{p}_s^i$  is the true probability of state  $s$  computed from the compound first and second order probabilities. The aversion to ambiguity of the representative agent affects prices in the equilibrium state in a way that is observationally equivalent to a distortion of beliefs in the EU model. This distortion takes the form of a transformation of the subjective prior distribution from  $(q_1, \dots, q_n)$  to  $(\widehat{q}_1^i, \dots, \widehat{q}_n^i)$  that is equivalent to the previous section with  $\tilde{c}_{\theta} = w_0 + \alpha_1^* \tilde{x}_{\theta}$ . Lemma 3 implies that  $\widehat{q}^2$  is dominated by  $\widehat{q}^1$  in the sense of MLR when  $\phi_2$  is more ambiguity averse than agent  $\phi_1$ . The next proposition is a direct consequence of this observation.

**Proposition 4** *Suppose that the set of priors  $(\tilde{c}_1, \dots, \tilde{c}_n)$  can be ranked according to the stochastic order  $D$  ( $D = FSD, SSD$  or  $IR$ ). If there is a concave  $k$  such that  $\phi_2 = k(\phi_1)$  and if  $\tilde{c}_1 \preceq_D \tilde{c}_2 \preceq_D \dots \preceq_D \tilde{c}_n$ , then  $(\tilde{c}_1, \widehat{q}_1^2; \dots; \tilde{c}_n, \widehat{q}_n^2) \preceq_D (\tilde{c}_1, \widehat{q}_1^1; \dots; \tilde{c}_n, \widehat{q}_n^1)$ .*

It is straightforward to reinterpret this result in terms of the impact of ambiguity aversion on the price kernel  $\pi_s = \Pi_s/p_s$ . Suppose that  $c_s \neq c_{s'}$  for all  $(s, s')$ , so that index  $s = 1, \dots, n$  can be substituted by another index  $s = c_1, \dots, c_S$ . Figure 1 draws the state price  $\pi_s = \hat{p}_s^i u'(c_s)/p_s$  as a function of  $c_s$ . Under ambiguity neutrality, this is a decreasing function, because  $u'$  is decreasing. The slope of the curve  $\pi(c)$  describes the degree of risk aversion of the agent. From Proposition 4, ambiguity aversion tends to reinforce risk aversion. Indeed, if the priors can be ranked by FSD, an increase in ambiguity aversion has an effect on asset prices that is observationally equivalent to a FSD-deteriorating shift in beliefs, that is it tends to transfer the distorted probability mass  $\hat{p}$  from the good states to the bad ones. The corresponding shift in  $\pi_s = \hat{p}_s u'(c_s)/p_s$  is described in Figure 1a. If the priors can be ranked according to their riskiness, an increase in ambiguity aversion tends to transfer the distorted probability mass to the extreme states. This implies convexifying the price kernel in the region of aggregate consumption where priors differ by mean-preserving spreads, as depicted in Figure 1b.

In the standard EU model, there is a decreasing relationship between the pricing kernel and aggregate consumption. However, there is some strong empirical evidence that this relation may be violated (Ait-Sahalia and Lo (2000), Rosenberg and Engle (2002), Yatchew and Härdle (2006), and Barone-Adesi and Dall'O (2009)). Typically, the empirical pricing kernel is "bump-shaped", as in Figure 2. Hens and Reichlin (2010) provide three plausible stories to explain this "pricing kernel puzzle": incomplete markets, risk-seeking behavior, and heterogeneous beliefs. This paper shows that ambiguity and ambiguity aversion are another plausible solution for this puzzle. This can be seen from Figure 1b, where the priors differ by a mean-preserving spread in the lower tail of the distribution. To illustrate, let us consider a four-state economy with  $c_1 = 0.75$ ,  $c_2 = 0.9$ ,  $c_3 = 1.05$  and  $c_4 = 1.2$ . Suppose also that there are

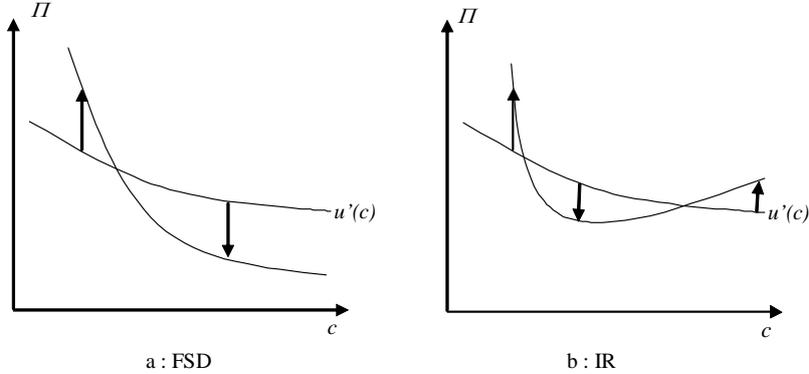


Figure 1: The effect of an increase in ambiguity aversion on the price kernel, when the priors can be ranked by the FSD order (a), or by the Rothschild-Stiglitz riskiness order (b).

two equally plausible distributions for the state probabilities. The two probability distributions are respectively  $(1/8, 1/2, 1/8, 1/4)$  and  $(3/8, 0, 3/8, 1/4)$ . It is easy to check that the second distribution is riskier than the first in the Rothschild-Stiglitz sense. Let us also assume that  $u'(c) = c^{-2}$ , and  $\phi'(U) = \exp(-\eta U)$ . The circled points in Figure 2 correspond to the pricing kernel when the representative agent is ambiguity-neutral ( $\eta = 0$ ), whereas the squared points correspond to the case of ambiguity aversion with  $\eta = 50$ . Using Proposition 4, many other distortions of the pricing kernel can be obtained by considering other sets of multiple priors.

## 5 Conclusion

This paper explores the determinants of the demand for uncertain assets and of asset prices when investors are ambiguity averse. The analysis was carried out using the standard static portfolio problem with one safe asset and one

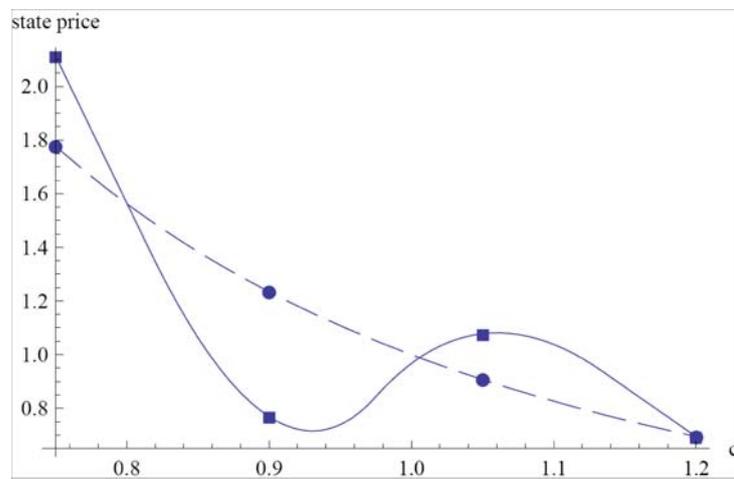


Figure 2: The effect of ambiguity aversion on the pricing kernel when priors differ by mean-preserving spreads in the lower tail of the distribution. The dashed curve corresponds to ambiguity neutrality. The four state prices are linked by quadratic interpolation.

uncertain asset. It was shown that, contrary to casual intuition, ambiguity aversion may yield an increase in demand for the risky and ambiguous asset, and a reduction in the demand for the safe one. In the same fashion, it is not necessarily true that ambiguity aversion raises the equity premium in the economy. It was shown first that the qualitative effect of an increase in ambiguity aversion in these settings is observationally equivalent to that of a shift in the beliefs of the investor in the standard EU model. If the set of plausible priors can be ranked according to the first degree stochastic dominance order, this shift is first degree stochastic deteriorating, whereas it is risk-increasing if these priors can be ranked according to the Rothschild-Stiglitz risk order. The problem originates from the observation already made by Rothschild and Stiglitz (1971) and Fishburn and Porter (1976) that a FSD/SSD deteriorating shift in the distribution for the return of the uncertain asset has an ambiguous effect on the demand for that asset in the EU framework. The literature that emerged from this negative result has been heavily relied upon to provide some sufficient conditions for any increase in ambiguity aversion to yield a reduction in the demand for the uncertain asset and therefore an increase in the equity premium. In most cases these conditions hold.

Two sets of findings confirm this view. First, the numerical analyses in the existing literature all show more ambiguity aversion reducing demand for the uncertain asset. It has also been shown that this will always be true when the first and second order probability distributions are normal, and the pair  $(u, \phi)$  are exponential-power functions. Second, the sufficient conditions cover a wide set of realistic situations. For example, if the set of priors can be ranked according to the well-known monotone likelihood ratio order, then it is always true that an increase in ambiguity aversion raises the equity premium. The conclusion is that the potential existence

of a counterintuitive demand effect arising from ambiguity aversion plays a role similar to the potential existence of Giffen goods in consumption theory. The observationally equivalent FSD deterioration of more ambiguity aversion has a wealth effect on the demand for the asset that may dominate the substitution effect. This is a rare event, but theoretical progress can rarely be made without understanding the mechanism that generates it. After all, the existence of Giffen goods is taught in Microeconomics 101.

## Appendix: Proofs

### Proof of Lemma 1

By concavity of the objective function in (1) with respect to  $\alpha$ , we have that  $\alpha^*$  is positive if the derivative of this objective function with respect to  $\alpha$  evaluated at  $\alpha = 0$  is positive. This derivative is written as

$$\sum_{\theta=1}^n q_{\theta} \phi'(u(w_0)) E \tilde{x}_{\theta} u'(w_0) = u'(w_0) \phi'(u(w_0)) E \tilde{x},$$

where  $E \tilde{x} = \sum_{\theta} q_{\theta} E x_{\theta}$  is the equity premium. This concludes the proof. ■

### Proof of Lemma 3

Because  $\phi_1$  and  $\phi_2$  are increasing in  $U$ , there exists an increasing function  $k$  such that  $\phi_2(U) = k(\phi_1(U))$ , or  $\phi_2'(U) = k'(\phi_1(U)) \phi_1'(U)$  for all  $U$ . Using definition (8), we obtain that

$$\frac{\hat{q}_{\theta}^2}{\hat{q}_{\theta}^1} = k'(\phi_1(U(\alpha_1^*, \theta))) \frac{\sum_{t=1}^n q_t \phi_1'(U(\alpha_1^*, t))}{\sum_{t=1}^n q_t \phi_2'(U(\alpha_1^*, t))} \quad (14)$$

for all  $\theta = 1, \dots, n$ . The Lemma is a direct consequence of (14), in the sense that the likelihood ratio  $\hat{q}_{\theta}^2/\hat{q}_{\theta}^1$  is decreasing in  $\theta$  if  $k'$  is decreasing in  $\phi_1$ . ■

### Proof of Proposition 1

Suppose that  $\tilde{x}_1 \preceq_D \tilde{x}_2 \preceq_D \dots \preceq_D \tilde{x}_n$ . It implies that  $U(\alpha_1^*, 1) \leq U(\alpha_1^*, 2) \leq \dots \leq U(\alpha_1^*, n)$ . We have to prove that  $(\tilde{x}_1, \hat{q}_1^1; \dots; \tilde{x}_n, \hat{q}_n^1)$  is preferred to  $(\tilde{x}_1, \hat{q}_1^2; \dots; \tilde{x}_n, \hat{q}_n^2)$  by all utility functions  $v$  in  $C$ , that is

$$\sum_{\theta=1}^n \hat{q}_{\theta}^2 E v(\tilde{x}_{\theta}) \leq \sum_{\theta=1}^n \hat{q}_{\theta}^1 E v(\tilde{x}_{\theta}),$$

where  $C$  is the set of increasing functions if  $D$ =FSD,  $C$  is the set of increasing and concave functions if  $D$ =SSD, and  $C$  is the set of concave functions if  $D$ =IR. Combining the conditions that  $\tilde{x}_{\theta} \preceq_D \tilde{x}_{\theta+1}$  and that  $v \in C$  implies

that  $Ev(\tilde{x}_\theta)$  is increasing in  $\theta$ . The above inequality is obtained by combining this property with the fact that  $\hat{q}^2$  is dominated by  $\hat{q}^1$  in the sense of MLR (Lemma 3), a special case of FSD. ■

### Proof of Proposition 3

The following lemma is useful to prove Proposition 3. Let  $K$  denote interval  $[\min_\theta \alpha_\theta^*, \max_\theta \alpha_\theta^*]$ , where  $\alpha_\theta^*$  is the maximand of  $Eu(w_0 + \alpha\tilde{x}_\theta)$ .

**Lemma 4** *Consider a specific set of priors  $(\tilde{x}_1, \dots, \tilde{x}_n)$  and a concave utility function  $u$ . They characterize function  $U$  defined by  $U(\alpha, \theta) = Eu(w_0 + \alpha\tilde{x}_\theta)$ . Consider a specific scalar  $\alpha_1^*$  in  $K$ . The following two conditions are equivalent:*

1. *Any agent  $\phi_2$  that is more ambiguity averse than agent  $\phi_1$  with demand  $\alpha_1^*$  for the ambiguous asset will have a demand for it that is smaller than  $\alpha_1^*$ ;*
2. *There exists  $\bar{\theta} \in \{1, \dots, n\}$  such that*

$$U(\alpha_1^*, \theta)U_\alpha(\alpha_1^*, \theta) \geq U(\alpha_1^*, \bar{\theta})U_\alpha(\alpha_1^*, \theta) \quad (15)$$

*for all  $\theta \in \{1, \dots, n\}$ .*

Proof: We first prove that condition 2 implies condition 1. Consider an agent  $\phi_2 = k(\phi_1)$  that is more ambiguity averse than agent  $\phi_1$ , so that the transformation function  $k$  is concave. The condition thus implies that

$$k'(\phi_1(U(\alpha_1^*, \theta)))U_\alpha(\alpha_1^*, \theta) \leq k'(\phi_1(U(\alpha_1^*, \bar{\theta})))U_\alpha(\alpha_1^*, \theta)$$

for all  $\theta$ . Multiplying both side of this inequality by  $q_\theta\phi_1'(U(\alpha_1^*, \theta)) \geq 0$  and summing up over all  $\theta$  yields

$$\sum_{\theta=1}^n q_\theta\phi_2'(U(\alpha_1^*, \theta))U_\alpha(\alpha_1^*, \theta) \leq k'(\phi_1(U(\alpha_1^*, \bar{\theta}))) \sum_{\theta=1}^n q_\theta\phi_1'(U(\alpha_1^*, \theta))U_\alpha(\alpha_1^*, \theta) = 0.$$

The last equality comes from the assumption that agent  $\phi_1$  selects portfolio  $\alpha_1^*$ . Thus, condition (6) is satisfied, thereby implying that  $\alpha_2^*$  is less than  $\alpha_1^*$ .

We then prove that condition 1 implies condition 2. Without loss of generality, rank the  $\theta$ s such that  $U(\alpha_1^*, \theta)$  is increasing in  $\theta$ . By contradiction, suppose that there exists a  $\theta_0 < n$  such that  $U_\alpha(\alpha_1^*, \theta_0) \geq 0$  and  $U_\alpha(\alpha_1^*, \theta_0 + 1) \leq 0$ . Select a prior distribution  $(q_1, \dots, q_n)$  so that  $q_\theta = 0$  for all  $\theta$  except for  $\theta_0$  and  $\theta_0 + 1$ . Select  $q_{\theta_0} = q \in [0, 1]$  so that

$$q\phi_1'(U(\alpha_1^*, \theta_0))U_\alpha(\alpha_1^*, \theta_0) + (1 - q)\phi_1'(U(\alpha_1^*, \theta_0 + 1))U_\alpha(\alpha_1^*, \theta_0 + 1) = 0, \quad (16)$$

so that agent  $\phi_1$  selects portfolio  $\alpha_1^*$ . Consider any concave transformation function  $k$ . It implies that

$$\begin{aligned} & \sum_{\theta=1}^n q_\theta \phi_2'(U(\alpha_1^*, \theta))U_\alpha(\alpha_1^*, \theta) \\ = & qk'(\phi_1(U(\alpha_1^*, \theta_0)))\phi_1'(U(\alpha_1^*, \theta_0))U_\alpha(\alpha_1^*, \theta_0) \\ & + (1 - q)k'(\phi_1(U(\alpha_1^*, \theta_0 + 1)))\phi_1'(U(\alpha_1^*, \theta_0 + 1))U_\alpha(\alpha_1^*, \theta_0 + 1). \end{aligned}$$

Because  $U_\alpha(\alpha_1^*, \theta_0 + 1) \leq 0$  and  $k'(\phi_1(U(\alpha_1^*, \theta_0 + 1))) \leq k'(\phi_1(U(\alpha_1^*, \theta_0)))$ , this is larger than

$$k'(\phi_1(U(\alpha_1^*, \theta_0))) [q\phi_1'(U(\alpha_1^*, \theta_0))U_\alpha(\alpha_1^*, \theta_0) + (1 - q)\phi_1'(U(\alpha_1^*, \theta_0 + 1))U_\alpha(\alpha_1^*, \theta_0 + 1)] = 0.$$

It implies that condition (6) is violated, implying in turn that  $\alpha_2^*$  is larger than  $\alpha_1^*$ , a contradiction. ■

If we rank the  $\theta$  in such a way that  $U(\alpha_1^*, \theta)$  is monotone in  $\theta$ , condition 2 is essentially a single-crossing property of function  $U_\alpha(\alpha_1^*, \theta)$ . To illustrate, suppose that  $u(z) = -A^{-1} \exp(-Az)$  and  $\tilde{x}_\theta \sim N(\theta, \sigma^2)$ , which implies that  $U(\alpha, \theta)$  is increasing in  $\theta$  and is given by equation (2). It implies that  $U_\alpha(\alpha, \theta)$  has the same sign as  $\theta - \alpha A\sigma^2$ . It implies in turn that condition 2 in Lemma 4 is satisfied with  $\bar{\theta} = \alpha A\sigma^2$ . Our Lemma implies that ambiguity aversion

reduces the demand for the uncertain asset in the exponential/normal case. This was shown in Section 2 in the special case of power  $\phi$  functions.

We need to prove a second lemma in order to prepare for the proof of Proposition 3.

**Lemma 5** *Suppose that  $\tilde{x}_2$  is centrally dominated by  $\tilde{x}_1$ . Then,  $E\tilde{x}_2u'(w_0 + \alpha\tilde{x}_2) \leq 0$  for any  $\alpha \geq 0$  such that  $E\tilde{x}_1u'(w_0 + \alpha\tilde{x}_1) \leq 0$ .*

Proof: By assumption, there exists a positive scalar  $m$  such that  $T_2(x) \leq mT_1(x)$ . Integrating by part, we have that

$$\begin{aligned} E\tilde{x}_2u'(w_0 + \alpha\tilde{x}_2) &= \int_{x_-}^{x_+} u'(w_0 + \alpha x)xdF_2(x) \\ &= u'(w_0 + \alpha x_+)T_2(x_+) - \alpha \int_{x_-}^{x_+} u''(w_0 + \alpha x)T_2(x)dx. \end{aligned}$$

This implies that

$$\begin{aligned} E\tilde{x}_2u'(w_0 + \alpha\tilde{x}_2) &\leq m \left[ u'(w_0 + \alpha x_+)T_1(x_+) - \alpha \int_{x_-}^{x_+} u''(w_0 + \alpha x)T_1(x)dx \right] \\ &= mE\tilde{x}_1u'(w_0 + \alpha\tilde{x}_1). \end{aligned}$$

By assumption, this is nonpositive. ■

We can now prove Proposition 3. Condition  $\tilde{x}_\theta \preceq_{SSD} \tilde{x}_{\theta+1}$  implies that  $U(\alpha, \theta + 1) \geq U(\alpha, \theta)$ , whereas, by Lemma 5, condition  $\tilde{x}_\theta \preceq_{CD} \tilde{x}_{\theta+1}$  implies that  $U_\alpha(\alpha, \theta) \leq 0$  whenever  $U_\alpha(\alpha, \theta + 1) \leq 0$ . This latter result implies that there exists a  $\bar{\theta}$  such that  $(\theta - \bar{\theta})U_\alpha(\alpha, \theta) \leq 0$  for all  $\theta$ . This immediately yields condition 2 in Lemma 4, which is sufficient for our comparative static property. ■

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